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Quantification of the unique continuation property for the nonstationary Stokes problem

Muriel Boulakia*

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Abstract

The purpose of this work is to establish stability estimates for the unique continuation property of the nonstationary Stokes problem. These estimates hold without prescribing boundary conditions and are of logarithmic type. They are obtained thanks to Carleman estimates for parabolic and elliptic equations. Then, these estimates are applied to an inverse problem where we want to identify a Robin coefficient defined on some part of the boundary from measurements available on another part of the boundary.

1 Introduction

Let Ω be a regular bounded connected open set of class C^2 in dimension 3. For any fixed final time $T > 0$, we define $Q = (0,T) \times \Omega$. We consider the nonstationary Stokes problem

$$\left\{ \begin{array}{rcll} u_t - \Delta u + \nabla p &=& 0, & \text{in } Q, \\ \text{div } u &=& 0, & \text{in } Q, \end{array} \right. \tag{1}$$

where u and p denote respectively the fluid velocity and the fluid pressure. Since the work made by Fabre and Lebeau in [14], the unique continuation property of this system is a well-known property. It is given by the following result

Proposition 1. *Let ω be a nonempty open subset of Ω. If (u,p) is a solution of (1) which belongs to $L^2(0,T;H^1_{loc}(\Omega)) \times L^2_{loc}(Q)$ and if $u = 0$ in $(0,T) \times \omega$, then*

$$u = 0 \textit{ and } p \textit{ is constant in } Q.$$

This property directly implies

Corollary 1. *Let $\Gamma \subset \partial\Omega$ be a nonempty open subset of $\partial\Omega$. If (u,p) is a solution of (1) which belongs to $L^2(0,T;H^2(\Omega)) \times L^2(0,T;H^1(\Omega))$ and if*

$$u = 0, \nabla u \cdot n - pn = 0 \textit{ on } (0,T) \times \Gamma,$$

then

$$u = 0,\, p = 0 \textit{ in } Q.$$

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In this work, we want to quantify these unique continuation properties. More precisely, we want to derive stability inequalities which assert that, if the measurements of u and p made on an interior domain ω or on a boundary part Γ are small, then u and p stay small on the whole domain Ω. In what follows, we will prove the following result:

Theorem 1. *1. Let $\Gamma \subset \partial\Omega$ be a nonempty open subset of $\partial\Omega$. There exist a constant $\alpha > 0$ and, for all $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$\|u\|_{C([\varepsilon,T-\varepsilon];C^1(\overline{\Omega}))} + \|p\|_{C([\varepsilon,T-\varepsilon]\times\overline{\Omega})} \leq \frac{CM}{\left(\log\left(\frac{CM}{G}\right)\right)^{\alpha}} \tag{2}$$

for all (u,p) solution of (1) in $\big(H^1(0,T;H^3(\Omega)) \cap H^2(0,T;H^1(\Omega))\big) \times H^1(0,T;H^2(\Omega))$. In this inequality, M is defined by

$$M := \|u\|_{H^1(0,T;H^3(\Omega))} + \|u\|_{H^2(0,T;H^1(\Omega))} + \|p\|_{H^1(0,T;H^2(\Omega))} \tag{3}$$

and G is defined by

$$G = \|u\|_{L^2((0,T)\times\Gamma)} + \|\nabla u \cdot n\|_{L^2((0,T)\times\Gamma)} + \|p\|_{L^2((0,T)\times\Gamma)} + \|\nabla p \cdot n\|_{L^2((0,T)\times\Gamma)}. \tag{4}$$

2. Let $\hat{\omega}$ be an open subset of Ω relatively compact in Ω. There exist a constant $\alpha > 0$ and, for all $\varepsilon > 0$, there exists a constant $C > 0$ such that we have

$$\|u\|_{C([\varepsilon,T-\varepsilon];C^1(\overline{\Omega}))} + \|p\|_{C([\varepsilon,T-\varepsilon]\times\overline{\Omega})} \leq \frac{CM}{\left(\log\left(\frac{CM}{\|u\|_{L^2((0,T)\times\hat{\omega})} + \|p\|_{L^2((0,T)\times\hat{\omega})}}\right)\right)^{\alpha}} \tag{5}$$

for all (u,p) solution of (1) in $\big(H^1(0,T;H^3(\Omega)) \cap H^2(0,T;H^1(\Omega))\big) \times H^1(0,T;H^2(\Omega))$. Here, M is again given by (3).

Let us emphasize that these stability inequalities hold without prescribing any boundary conditions on the solution. The logarithmic nature of the inequalities comes from the fact that we estimate the norm of u and p on the whole domain Ω. If we are interested by interior estimates of u and p, we get inequalities of Hölder type (Propositions 2 and 3).

If we compare this result to its counterpart for the steady equation in [9] (we refer to Theorem 1.4 in this paper), we notice that we need the same kind of measurements on u and p. In particular, in both cases, extra measurements are necessary compared to the unique continuation property proved by Fabre and Lebeau. This is linked to the fact that we need global measurements on the velocity and the pressure. In [22] and [19], local estimates are proved which only require measurements on u. In [22], a three-balls inequality for the stationary Stokes problem is proved which only involves the L^2-norm of the velocity. It leads to a quantification of the unique continuation property like in Theorem 1 of the following type:

$$\|u\|_{L^2(A)} \leq C\|u\|^{1-\theta}_{L^2(\Omega)}\|u\|^{\theta}_{L^2(\omega)}$$

where A is a compact subset of Ω and $0 < \theta < 1$. In [19], the authors prove a local stability estimate which only involves the velocity for the solution of the Navier-Stokes equation. They assume that the data belong to a Gevrey class and enforce specific conditions on the solution which are satisfied if periodic boundary conditions are prescribed.

For the quantification of the unique continuation property of the Laplace equation, let us quote among others the works [1], [10], [23]. In this case, it is well established that the best possible rate for the global stability is logarithmic. We refer to the overview [2] and the references therein for works on the stability estimates for elliptic equations.

The proof of our main result relies on estimates of propagation of smallness in the interior and up to the boundary. These estimates are stated in section 2 and give simultaneous estimates on

u and p. Another method could have been to prove a global estimate on the velocity alone with the help of adapted Carleman estimates. According to Stokes equation, this would allow to directly get an estimate on ∇p and, thanks to an adapted Poincaré inequality, this leads to an estimate on p if we also have measurements of p on an arbitrary sub-domain. This alternative method seems to lead to similar measurements as the ones in the inequalities given by Theorem 1.

In a second step, we will be interested in applying this quantification result to get a stability estimate for an inverse problem which has already been studied in [8] and in [9]. Our objective will be to identify a Robin coefficient defined on some part of the boundary from measurements available on another part of the boundary. To study this inverse problem, it is capital to have an estimate of the pressure and the velocity on the whole domain like the one given by Theorem 1.

More precisely, we assume that the boundary $\partial\Omega$ is composed of two open non-empty parts Γ_0 and Γ_e such that $\Gamma_e \cup \Gamma_0 = \partial\Omega$ and $\overline{\Gamma}_e \cap \overline{\Gamma}_0 = \emptyset$ and we consider the following problem

$$\left\{ \begin{array}{rcll} u_t - \Delta u + \nabla p & = & 0, & \text{in } Q, \\ \text{div } u & = & 0, & \text{in } Q, \\ \nabla u \cdot n - pn & = & g, & \text{on } (0,T) \times \Gamma_e, \\ \nabla u \cdot n - pn + qu & = & 0, & \text{on } (0,T) \times \Gamma_0, \\ u(0,\cdot) & = & u_0, & \text{in } \Omega. \end{array} \right. \tag{6}$$

Such system may be viewed as a simple model of the blood flow in the cardiovascular system (see [24] and [26]) or of the airflow in the respiratory tract (see [4]). We refer to [13] for a presentation in this last area of application. In these contexts, the real geometry is truncated and the properties of the upstream domain are condensed on the boundary conditions which are prescribed on the artificial boundary. The boundary part Γ_e corresponds to the external boundary on which measurements are available and the boundary part Γ_0 corresponds to an artificial boundary on which Robin boundary conditions are prescribed. For similar studies with the identification of a Robin coefficient with the Laplace equation, we refer to [3] and [12] and with the heat equation to [6] (see also the references therein).

In [8], we proved a stability result (see Theorem 4.18 in this reference) which holds for a parameter q which does not depend on time and for measurements made on the interval $[0, +\infty[$. As in [6] for the Laplace equation, this result relies on properties satisfied by the semigroup generated by the operator associated to the problem and is proved by comparing the solution of the non-stationary problem with the solution of the stationary problem. The quantification of the unique continuation property given by Theorem 1 allows to generalize the result given in [8] to a parameter q which depends on time and to measurements made on a finite interval. More precisely, we have the following result:

Theorem 2. *Let Ω be of class $C^{2,1}$and $\Gamma \subseteq \Gamma_e$ be a nonempty open subset of the boundary of Ω. Let $\nu_0 > 0$ and $N_0 > 0$.*
Let $u_0 \in H^4(\Omega) \cap V$, $g \in H^2(0,T; L^2(\Gamma_e)) \cap H^1(0,T; H^{\frac{3}{2}}(\Gamma_e))$ be non identically zero and $q_1, q_2 \in H^2(0,T; H^2(\Gamma_0))$ such that $q_1, q_2 \geq \nu_0$ on $(0,T) \times \Gamma_0$. We assume that

$$\|u_0\|_{H^4(\Omega)} + \|g\|_{H^2(0,T;L^2(\Gamma_e)) \cap H^1(0,T;H^{\frac{3}{2}}(\Gamma_e))} + \sum_{j=1}^{2} \|q_j\|_{H^2(0,T;H^2(\Gamma_0))} \leq N_0.$$

We denote by (u_j, p_j) the solution of system (6) with $q = q_j$ for $j = 1, 2$. Let K be a compact subset of $\{(t,x) \in (\varepsilon, T-\varepsilon) \times \Gamma_0 \,/\, u_1 \neq 0\}$ for some $\varepsilon > 0$ and $m > 0$ be such that $|u_1| \geq m$ on K.

Then, there exists $\alpha > 0$ independent of ε, $C > 0$ which depends on ε, ν_0 and N_0 such that

$$\|q_1 - q_2\|_{C(K)} \leq \frac{1}{m} \frac{C}{\left(\log \left(\dfrac{C}{\|u_1 - u_2\|_{L^2((0,T)\times\Gamma)} + \|p_1 - p_2\|_{L^2((0,T)\times\Gamma)} + \|\nabla(p_1 - p_2) \cdot n\|_{L^2((0,T)\times\Gamma)}} \right) \right)^{\alpha}}. \tag{7}$$

We have used the following notation

$$V = \{v \in H^1(\Omega) / \text{ div } v = 0 \text{ in } \Omega\}.$$

In the hypotheses of this theorem, the existence of the constant m and of the compact K is ensured by the continuity of u_1 and the fact that u_1 can not be identically null on $(0,T)\times\Gamma_0$. This last property is due to the unique continuation property (Corollary 1) and the hypothesis that g is non identically null. Through m and K, the estimate given in this theorem depends on u_1. To get an estimate on the whole set $(0,T)\times\Gamma_0$, it would be necessary to prove a lower bound on the velocity obtained thanks to a doubling inequality on the boundary ([1]). This sometimes may lead to estimates of log-log type like in [5] or [7]. In our case, the interior doubling inequality obtained in [19] (Theorem 2.1) with an exponential rate with respect to the radius of the ball leads us to believe that we could obtain a log-log inequality.

In the next section, we present local estimates of u and p in the interior of the domain or near the boundary. We then gather these inequalities to prove Theorem 1. Section 3 is dedicated to the proof of these local estimates. At last, in Section 4, we apply our estimates to the identification problem of a Robin coefficient and prove Theorem 2.

2 Local estimates of the solution

In what follows, we will use the following notation: for $t_1 < t_2$, we define

$$H^{1,0}((t_1,t_2)\times\Omega) = \{u \in L^2((t_1,t_2)\times\Omega)/\nabla u \in L^2((t_1,t_2)\times\Omega)\}.$$

Theorem 1 will be proved with the help of three propositions that we state now. The proofs of these propositions rely on local Carleman estimates for parabolic and elliptic equations. In [9], our quantification result was based on local Carleman inequalities ([18], [21] and [25]) obtained thanks to Gårding inequalities involving pseudodifferential computation. The same inequalities were used in [23] to quantify the unique continuation property for the Laplace equation. We refer to the survey [20] (and the references therein) for a general presentation of these local Carleman estimates in the elliptic and parabolic cases. Here, the local Carleman estimates that we will use are derived through direct computations. Like the global Carleman inequalities, they are obtained thanks to the method of Fursikov and Imanuvilov [15]. We call them local Carleman estimates because they are stated on a subdomain of $(0,T)\times\Omega$ where we do not prescribe boundary conditions on the solutions. Regarding the Carleman inequalities that we will use, the inequality for the parabolic case is stated in [27] and the inequality in the elliptic case can be proved with the methods presented in [15].

The first proposition gives an estimate of u and p in the interior of Ω with respect to measurements on a part of the boundary of Ω:

Proposition 2. *Let $\Gamma \subset \partial\Omega$ be a nonempty open subset of $\partial\Omega$ and let Ω_0 be a nonempty open set such that $\overline{\Omega}_0 \subset \Omega\cup\Gamma$ and $\partial\Omega_0\cap\partial\Omega \subsetneq \Gamma$. There exists $\theta\in(0,1)$ and, for any $\varepsilon>0$, there exists $C>0$ such that*

$$\|u\|_{H^{1,0}((\varepsilon,T-\varepsilon)\times\Omega_0)} + \|p\|_{H^{1,0}((\varepsilon,T-\varepsilon)\times\Omega_0)} \le C\left(\|u\|_{H^{1,0}(Q)} + \|p\|_{H^{1,0}(Q)}\right)^{1-\theta} F^{\theta}, \tag{8}$$

for all (u,p) solution of (1) in $H^{1,0}((0,T)\times\Omega)\times H^{1,0}((0,T)\times\Omega)$. In this inequality, F is defined by

$$F = \|u\|_{H^1((0,T)\times\Gamma)} + \|\nabla u\cdot n\|_{L^2((0,T)\times\Gamma)} + \|p\|_{L^2(0,T;H^1(\Gamma))} + \|\nabla p\cdot n\|_{L^2((0,T)\times\Gamma)}. \tag{9}$$

If we compare estimate (8) with the equivalent estimate proved for the stationary Stokes equation in [9] (see Proposition 2.6 in this reference), we see that the norms of the measurements are similar (the norms of the measurements in (8) correspond to the L^2-norms in time of the norms of the measurements in Proposition 2.6 in [9]) except that we need an additional measurement of u in $H^1(0,T;L^2(\Gamma))$ for the estimate (8). For parabolic equations like heat equation, it is proved in [27] that this norm can not be removed, otherwise the estimate fails.

The second proposition gives an estimate of u and p in the interior of Ω with respect to measurements in the interior:

Proposition 3. *Let $\hat{\omega}$ be a nonempty open subset of Ω and let $\Omega_0 \subset \Omega$ be a nonempty open set relatively compact in Ω. There exists $\theta \in (0,1)$ and, for any $\varepsilon > 0$, there exists $C > 0$ such that*

$$\|u\|_{H^{1,0}((\varepsilon,T-\varepsilon)\times\Omega_0)}+\|p\|_{H^{1,0}((\varepsilon,T-\varepsilon)\times\Omega_0)} \leq C\left(\|u\|_{H^{1,0}(Q)}+\|p\|_{L^2(Q)}\right)^{1-\theta}\left(\|u\|_{L^2((0,T)\times\hat{\omega})}+\|p\|_{L^2((0,T)\times\hat{\omega})}\right)^{\theta}, \tag{10}$$

for all (u,p) solution of (1) in $L^2(0,T;H^1(\Omega))\times L^2(Q)$.

In these two propositions, the exponent θ only depends on the geometry of the domain, whereas the constants C also depend on ϵ.

And the last proposition gives an estimate of u and p on the boundary with respect to measurements in the interior:

Proposition 4. *There exists a neighborhood $\tilde{\Omega}$ of $\partial\Omega$, a nonempty open subset $\tilde{\omega} \subset \Omega$ relatively compact in Ω, a constant $\alpha > 0$ and, for all $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$\|u\|_{C([\varepsilon,T-\varepsilon];C^1(\tilde{\Omega}\cap\overline{\Omega}))} + \|p\|_{C([\varepsilon,T-\varepsilon]\times(\tilde{\Omega}\cap\overline{\Omega}))} \leq \frac{CM}{\left(\log\left(\dfrac{CM}{\|u\|_{L^2((0,T)\times\tilde{\omega})}+\|p\|_{L^2((0,T)\times\tilde{\omega})}}\right)\right)^{\alpha}} \tag{11}$$

for all (u,p) solution of (1) in $\left(H^1(0,T;H^3(\Omega))\cap H^2(0,T;H^1(\Omega))\right)\times H^1(0,T;H^2(\Omega))$, where

$$M := \|u\|_{H^1(0,T;H^3(\Omega))}+\|u\|_{H^2(0,T;H^1(\Omega))}+\|p\|_{H^1(0,T;H^2(\Omega))}. \tag{12}$$

Again, in this proposition, the exponent α only depends on the geometry of the domain, whereas the constants C also depend on ϵ.

Remark 1. *In Theorem 1, the hypotheses of regularity on the solution come from the hypotheses of regularity made in Proposition 4. In Propositions 2 and 3, the regularity of the solutions is much weaker (even if, we have to give a sense to the norms which appear in the measurements on the boundary given by (9)). In Proposition 4, if we do not assume that u belongs to $H^2(0,T;H^1(\Omega))$ and remove the norm of u in $H^2(0,T;H^1(\Omega))$ in M, we can prove inequality (11) with the norm of u in $H^1(0,T;L^2(\tilde{\omega}))$ instead of $L^2((0,T)\times\tilde{\omega})$ in the right hand-side.*

Remark 2. *In Proposition 3, we only assume that p belongs to $L^2(Q)$ and we get an estimate of p in $H^{1,0}((\varepsilon,T-\varepsilon)\times\Omega_0)$. Since, for all $t\in(0,T)$, $p(t,\cdot)$ is a solution of the Laplace equation in Ω, results on the interior regularity for elliptic problem ([16]) directly implies that p belongs to $H^{1,0}((0,T)\times\Omega_0)$.*

These three propositions will allow to prove the quantification of the unique continuation property given in Theorem 1:

Proof of Theorem 1. 1. We first apply Proposition 4 and we obtain the existence of a neighborhood $\tilde{\Omega}$ of $\partial\Omega$, an open subset $\tilde{\omega}\subset\Omega$ relatively compact in Ω and a constant $\alpha>0$ such that, for all $\varepsilon>0$

$$\|u\|_{C([2\varepsilon,T-2\varepsilon];C^1(\tilde{\Omega}\cap\overline{\Omega}))}+\|p\|_{C([2\varepsilon,T-2\varepsilon]\times(\tilde{\Omega}\cap\overline{\Omega}))} \leq \frac{CM}{\left(\log\left(\dfrac{CM}{\|u\|_{L^2((\epsilon,T-\epsilon)\times\tilde{\omega})}+\|p\|_{L^2((\epsilon,T-\epsilon)\times\tilde{\omega})}}\right)\right)^{\alpha}}, \tag{13}$$

for some $C>0$. Let us now apply Proposition 2 on $\tilde{\omega}$. We get the existence of constants $C>0$ and $0<\theta<1$ such that

$$\|u\|_{L^2((\varepsilon,T-\varepsilon)\times\tilde{\omega})}+\|p\|_{L^2((\varepsilon,T-\varepsilon)\times\tilde{\omega})} \leq C\left(\|p\|_{H^{1,0}(Q)}+\|u\|_{H^{1,0}(Q)}\right)^{1-\theta}F^{\theta} \leq CM^{1-\theta}F^{\theta}$$

where F is given by (9). Using this estimate in the right hand-side of (13), we get

$$\|u\|_{C([2\varepsilon,T-2\varepsilon];C^1(\tilde{\Omega}\cap\overline{\Omega}))}+\|p\|_{C([2\varepsilon,T-2\varepsilon]\times(\tilde{\Omega}\cap\overline{\Omega}))}\leq\frac{CM}{\left(\log\left(\frac{CM}{F}\right)\right)^{\alpha}}. \tag{14}$$

Let us introduce an open set Ω_0 such that $\Omega\setminus\left(\tilde{\Omega}\cap\Omega\right)\subset\subset\Omega_0\subset\subset\Omega$. We have, according to interpolation inequalities,

$$\begin{aligned}&\|u\|_{C([2\varepsilon,T-2\varepsilon];C^1(\Omega_0))}+\|p\|_{C([2\varepsilon,T-2\varepsilon]\times\Omega_0)}\\&\qquad\leq CM^{7/8}(\|u\|_{L^2(2\varepsilon,T-2\varepsilon;H^1(\Omega_0))}+\|p\|_{L^2(2\varepsilon,T-2\varepsilon;H^1(\Omega_0))})^{1/8}.\end{aligned}$$

We apply again Proposition 2 and we get

$$\|u\|_{C([2\varepsilon,T-2\varepsilon];C^1(\Omega_0))}+\|p\|_{C([2\varepsilon,T-2\varepsilon]\times\Omega_0)}\leq CM^{7/8}(M^{1-\theta}F^{\theta})^{1/8}=CM\left(\frac{F}{M}\right)^{\theta/8}$$

We gather this inequality with (14) and we get that there exists a constant $C>0$ such that

$$\|u\|_{C([2\varepsilon,T-2\varepsilon];C^1(\overline{\Omega}))}+\|p\|_{C([2\varepsilon,T-2\varepsilon]\times\overline{\Omega})}\leq\frac{CM}{\left(\log\left(\frac{CM}{F}\right)\right)^{\alpha}}.$$

To conclude the proof, we notice that

$$\begin{aligned}&\|u\|_{L^2(0,T;H^1(\Gamma))}+\|p\|_{L^2(0,T;H^1(\Gamma))}\\&\qquad\leq C\|u\|^{1/3}_{L^2((0,T)\times\Gamma)}\|u\|^{2/3}_{L^2(0,T;H^2(\Omega))}+C\|p\|^{1/3}_{L^2((0,T)\times\Gamma)}\|p\|^{2/3}_{L^2(0,T;H^2(\Omega))}\leq CG^{1/3}M^{2/3}\end{aligned}$$

and

$$\|u\|_{H^1(0,T;L^2(\Gamma))}\leq C\|u\|^{1/2}_{L^2((0,T)\times\Gamma)}\|u\|^{1/2}_{H^2(0,T;H^1(\Omega))}\leq CG^{1/2}M^{1/2}.$$

Thus, $F\leq CG^{1/3}M^{2/3}$ which leads to (2).

2. We proceed in the same way as in the first step except that we apply Proposition 3 instead of Proposition 2.

□

3 Proof of the local estimates

3.1 Estimates in the interior of the domain: proof of Propositions 2 and 3

Let us first define some well-chosen weight functions which will be useful in the proof of Proposition 2. To do so, we take again the setting of Proposition 2: we introduce $\Gamma\subset\partial\Omega$ a nonempty open subset of $\partial\Omega$ and Ω_0 a nonempty open set such that $\overline{\Omega}_0\subset\Omega\cup\Gamma$ and $\partial\Omega_0\cap\partial\Omega\subsetneq\Gamma$. Then, we consider $\widetilde{\Omega}$ a domain of class C^2 such that $\Omega\subset\widetilde{\Omega}$, $\overline{\Gamma}=\overline{\partial\Omega\cap\widetilde{\Omega}}$ and $\partial\Omega\backslash\Gamma\subset\partial\widetilde{\Omega}$. Let $d\in C^2(\overline{\widetilde{\Omega}})$ be such that $d(x)>0$ for all $x\in\widetilde{\Omega}$, $d(x)=0$ for $x\in\partial\widetilde{\Omega}$ and $|\nabla d(x)|>0$ for $x\in\overline{\Omega}$. We define

$$\delta=\|d\|_{C(\overline{\widetilde{\Omega}})}.$$

Since $\overline{\Omega}_0\subset\widetilde{\Omega}$, we can choose a sufficiently large $N>5$ such that

$$\Omega_0\subset\left\{x\in\widetilde{\Omega}/d(x)>\frac{5}{N}\delta\right\}. \tag{15}$$

These constants δ and N only depend on the domains Γ and Ω. Let $\epsilon > 0$ be fixed and choose $\beta > 0$ such that

$$\frac{3\beta}{2}\varepsilon^2 < \delta < 2\beta\varepsilon^2. \tag{16}$$

We arbitrarily fix $t_0 \in (\sqrt{2}\varepsilon, T - \sqrt{2}\varepsilon)$ and set, for all $(t,x) \in \mathbb{R} \times \mathbb{R}^3$

$$\psi(t,x) = d(x) - \beta(t-t_0)^2 \tag{17}$$

and

$$\varphi(t,x) = e^{\lambda\psi(t,x)}, \tag{18}$$

where λ is a large enough fixed positive parameter. For $1 \leq i \leq 5$, we define

$$\mu_i = e^{\lambda\left(\frac{i}{N}\delta - \frac{\beta\varepsilon^2}{N}\right)}. \tag{19}$$

We then define Ω_1 a sub-domain of Ω such that $\Omega_0 \subset \Omega_1$, $\partial\Omega_1 \cap \partial\Omega \neq \emptyset$, $\partial\Omega_1 \cap \partial\Omega \subsetneq \Gamma$ and

$$\|d\|_{C(\overline{\Omega}\setminus\Omega_1)} \leq \frac{1}{3N}\delta. \tag{20}$$

For $1 \leq i \leq 5$, we denote by

$$\widetilde{D}_i = \left\{ x \in \overline{\Omega}_1 / d(x) > \frac{i}{N}\delta \right\}$$

$$D_i = \left\{ (t,x) \in \mathbb{R} \times \overline{\Omega}_1 / \varphi(t,x) > \mu_i \right\}.$$

For these domains, the following lemma holds:

Lemma 1. *The sets $(\widetilde{D}_i)$ and (D_i) satisfy the following properties:*

(i) $\left(t_0 - \frac{\varepsilon}{\sqrt{N}}, t_0 + \frac{\varepsilon}{\sqrt{N}}\right) \times \Omega_0 \subset D_5 \subset \ldots \subset D_1$.

(ii) For all $2 \leq i \leq 5$,

$$D_i \subset \left(t_0 - \sqrt{2}\varepsilon, t_0 + \sqrt{2}\varepsilon\right) \times \widetilde{D}_{i-1}.$$

(iii) For all $1 \leq i \leq 5$,

$$\partial D_i \subset \Sigma_{1,i} \cup \Sigma_{2,i} \tag{21}$$

with $\Sigma_{1,i} \subset \left(t_0 - \sqrt{2}\varepsilon, t_0 + \sqrt{2}\varepsilon\right) \times \Gamma$ *and* $\Sigma_{2,i} = \left\{(t,x) \in \left(t_0 - \sqrt{2}\varepsilon, t_0 + \sqrt{2}\varepsilon\right) \times \overline{\Omega_1} / \varphi(t,x) = \mu_i\right\}$.

(iv) For all $1 \leq i \leq 5$,

$$\partial\widetilde{D}_i \subset \widetilde{\Sigma}_{1,i} \cup \widetilde{\Sigma}_{2,i} \tag{22}$$

with $\widetilde{\Sigma}_{1,i} \subset \Gamma \cap \partial\Omega_1$ *and* $\widetilde{\Sigma}_{2,i} = \left\{ x \in \overline{\Omega}_1 / d(x) = \frac{i}{N}\delta \right\}$.

Proof of Lemma 1. (i) Let $(t,x) \in \left(t_0 - \frac{\varepsilon}{\sqrt{N}}, t_0 + \frac{\varepsilon}{\sqrt{N}}\right) \times \Omega_0$. We notice that

$$\psi(t,x) = d(x) - \beta(t-t_0)^2 > d(x) - \frac{\beta\epsilon^2}{N} > \frac{5}{N}\delta - \frac{\beta\epsilon^2}{N}$$

according to (15). This implies that $\varphi(t,x) > \mu_5$ which shows the first inclusion.

The fact that $D_{i+1} \subset D_i$ for $1 \leq i \leq 4$ is obvious.

(ii) Now, let (t,x) belong to D_i, for $2 \le i \le 5$. Since $\varphi(t,x) > \mu_i$,

$$d(x) - \beta(t-t_0)^2 > \frac{i}{N}\delta - \frac{\beta\epsilon^2}{N}. \tag{23}$$

Thus

$$\beta(t-t_0)^2 < \left(1-\frac{i}{N}\right)\delta + \frac{\beta\epsilon^2}{N} < 2\beta\epsilon^2\left(1-\frac{i}{N}\right) + \frac{\beta\epsilon^2}{N} < 2\beta\epsilon^2$$

thanks to (16). This implies that $|t-t_0| < \sqrt{2}\epsilon$. On the other hand, we deduce from (23) that

$$d(x) > \frac{i}{N}\delta - \frac{\beta\epsilon^2}{N} > \frac{i}{N}\delta - \frac{2}{3N}\delta > \frac{i-1}{N}\delta$$

using again (16). Thus, $D_i \subset \left(t_0-\sqrt{2}\varepsilon, t_0+\sqrt{2}\varepsilon\right) \times \widetilde{D}_{i-1}$.

(iii) Let (t,x) belong to ∂D_i. If x belongs to Ω_1, then $\varphi(t,x) = \mu_i$ (otherwise, $\varphi(t,x) > \mu_i$ and thus $(t,x) \in \mathring{D}_i$). Thus, (21) holds with $\Sigma_{1,i} \subset \left(t_0-\sqrt{2}\varepsilon, t_0+\sqrt{2}\varepsilon\right) \times \partial\Omega_1$ and $\Sigma_{2,i} = \left\{(t,x) \in \left(t_0-\sqrt{2}\varepsilon, t_0+\sqrt{2}\varepsilon\right) \times \Omega / \varphi(t,x) = \mu_i\right\}$.
Let $(t,x) \in \Sigma_{1,i}$. We have two cases. Either x belongs to $\partial\Omega_1 \cap \partial\Omega \subset \Gamma$ or x belongs to $\partial\Omega_1 \setminus (\partial\Omega_1 \cap \partial\Omega)$. In the second case, according to (20),

$$d(x) \le \frac{1}{3N}\delta.$$

Moreover, since $\varphi(t,x) \ge \mu_i \ge \mu_1$ we have that

$$\beta(t-t_0)^2 \le d(x) + \frac{\beta\epsilon^2}{N} - \frac{1}{N}\delta \le -\frac{2}{3N}\delta + \frac{\beta\epsilon^2}{N} < 0$$

according to (16). We get a contradiction and this allows to conclude that $\Sigma_{1,i} \subset \left(t_0-\sqrt{2}\varepsilon, t_0+\sqrt{2}\varepsilon\right) \times \Gamma$.

(iv) According to the definition of $\widetilde{D}_i$, (22) holds for $\widetilde{\Sigma}_{1,i} \subset \partial\Omega_1$ and $\widetilde{\Sigma}_{2,i} = \left\{x \in \overline{\Omega}_1 / d(x) = \frac{i}{N}\delta\right\}$.
If $x \in \partial\Omega_1 \setminus (\Gamma \cap \partial\Omega_1)$, then according to (20), $d(x) \le \frac{1}{3N}\delta$. Thus, $x \notin \widetilde{D}_i$. This implies that $\widetilde{\Sigma}_{1,i} \subset \Gamma \cap \partial\Omega_1$.

□

Before starting the proof of Proposition 2, we give the following classical lemma

Lemma 2. *Let $A > 0$, $B > 0$, $C_1 > 0$, $C_2 > 0$ and $D > 0$. We assume that there exists $c_0 > 0$ and $\gamma_1 > 0$ such that $D \le c_0 B$ and for all $\gamma \ge \gamma_1$,*

$$D \le Ae^{C_1\gamma} + Be^{-C_2\gamma}.$$

Then, there exists $C > 0$ such that:

$$D \le CA^{\frac{C_2}{C_1+C_2}} B^{\frac{C_1}{C_1+C_2}}.$$

Proof of Proposition 2. In this proof, $C > 0$ stands for a generic constant which may depend on Ω, Γ, T, λ and ϵ but which is independent of s and t_0.

Let $\chi \in C^2(\mathbb{R} \times \Omega)$ be a cut-off function such that $0 \le \chi \le 1$ and

$$\chi(t,x) = \begin{cases} 0, & \text{if } \varphi(t,x) \le \mu_3, \\ 1, & \text{if } \varphi(t,x) \ge \mu_4. \end{cases} \tag{24}$$

To define this function, we can take $\chi(t,x) = \bar{\chi}\left(\frac{\varphi(t,x)-\mu_3}{\mu_4-\mu_3}\right)$ where $\bar{\chi} \in C^2(\mathbb{R})$ is such that

$$0 \le \bar{\chi}(t,x) \le 1 \quad \text{and} \quad \bar{\chi}(\xi) = \begin{cases} 0, & \text{if } \xi \le 0, \\ 1, & \text{if } \xi \ge 1. \end{cases} \tag{25}$$

We have the following estimate: for all $1 \le i,j \le 3$, for all $(t,x) \in \mathbb{R} \times \mathbb{R}^3$,

$$|\partial_t \chi(t,x)| + |\partial_i \chi(t,x)| + |\partial^2_{ij}\chi(t,x)| \le C. \tag{26}$$

Then $v = \chi u$ satisfies

$$\partial_t v - \Delta v = \partial_t \chi u - \Delta \chi u - 2\nabla\chi \cdot \nabla u - \chi \nabla p \quad \text{in } Q.$$

We apply the Carleman inequality for parabolic equations on the domain D_2 (see Theorem 3.2 in [27]) with the weight φ: for all fixed λ large enough, there exist a constant $s_0 > 0$ and a constant C such that, for all $s > s_0$

$$\begin{aligned} &\iint_{D_2} \left(s|\nabla v|^2 + s^3|v|^2\right) e^{2s\varphi} dxdt \\ &\le C \iint_{D_2} |\partial_t\chi u - \Delta\chi u - 2\nabla\chi\cdot\nabla u|^2 e^{2s\varphi} dxdt + C \iint_{D_3} |\nabla p|^2 e^{2s\varphi} dxdt + C \iint_{\partial D_2} e^{2s\varphi}(s|\nabla_{t,x} v|^2 + s^3|v|^2) d\sigma dt \end{aligned}$$

Let us mention that the constants C and s_0 do not depend on t_0 since, if we look at the dependence of the domain D_2 with respect to t_0, we see that the domains D_2 are in translation with each other with respect to t_0.
By the definition of χ (24), the first term in the right hand-side of this inequality is in fact an integral on $\{(t,x) \in Q / \mu_3 \le \varphi(t,x) \le \mu_4\}$. Thus, using (26), we get the existence of a constant $C > 0$ such that:

$$\iint_{D_2} |\partial_t\chi u - \Delta\chi u - 2\nabla\chi \cdot \nabla u|^2 e^{2s\varphi} dxdt \le C e^{2s\mu_4} \|u\|^2_{H^{1,0}(Q)}.$$

Moreover, for the boundary integral in the right hand-side, we use Lemma 1 (iii) for $i = 2$. We obtain that there exists a constant $C > 0$ such that:

$$\iint_{\partial D_2} e^{2s\varphi}(s|\nabla_{t,x} v|^2 + s^3|v|^2) d\sigma dt \le C e^{C_0 s} \int_0^T \int_\Gamma |\partial_t u|^2 + |\nabla u|^2 + |u|^2 d\sigma dt$$

where C_0 does not depend on ε. Then, since by Lemma 1 (i), $\left(t_0 - \dfrac{\varepsilon}{\sqrt{N}}, t_0 + \dfrac{\varepsilon}{\sqrt{N}}\right) \times \Omega_0 \subset D_5 \subset D_2$, we obtain:

$$\begin{aligned} &e^{2s\mu_5} \int_{t_0 - \frac{\varepsilon}{\sqrt{N}}}^{t_0 + \frac{\varepsilon}{\sqrt{N}}} \int_{\Omega_0} s|\nabla u|^2 + s^3|u|^2 dxdt \\ &\qquad \le C e^{2s\mu_4} \|u\|^2_{H^{1,0}(Q)} + C \iint_{D_3} |\nabla p|^2 e^{2s\varphi} dxdt + C e^{C_0 s} \int_0^T \int_\Gamma |\partial_t u|^2 + |\nabla u|^2 + |u|^2 d\sigma dt. \end{aligned} \tag{27}$$

Let us now obtain estimates on the pressure p to estimate the second term in the right hand-side. According to Lemma 1 (ii), we have

$$\iint_{D_3} (s|\nabla p|^2 + s^3|p|^2) e^{2s\varphi} dxdt \le \int_{t_0 - \sqrt{2}\varepsilon}^{t_0 + \sqrt{2}\varepsilon} \int_{\widetilde{D}_2} (s|\nabla p|^2 + s^3|p|^2) e^{2s\varphi} dxdt. \tag{28}$$

We introduce a cut-off function $\tilde{\chi}$ in $C^2(\Omega)$ such that $0 \le \tilde{\chi} \le 1$ and

$$\tilde{\chi}(x) = \begin{cases} 1, & \text{if } d(x) \ge \frac{2}{N}\delta, \\ 0, & \text{if } d(x) \le \frac{3}{2N}\delta. \end{cases} \tag{29}$$

As previously for χ, this function can be defined explicitly with the help of $\bar{\chi}$:

$$\tilde{\chi}(x) = \bar{\chi}\left(\frac{2N}{\delta}\left(d(x) - \frac{3}{2N}\delta\right)\right).$$

We have the following estimate: for all $1 \leq i, j \leq 3$, for all $x \in \Omega$,

$$|\partial_i \tilde{\chi}(x)| + |\partial_{ij}\chi(x)| \leq C. \tag{30}$$

Let us define $\pi = \tilde{\chi}\, p$. Using that $\widetilde{D}_2 \subset \widetilde{D}_1$ and $\tilde{\chi} = 1$ on $\widetilde{D}_2$, inequality (28) becomes

$$\iint_{D_3} (s|\nabla p|^2 + s^3|p|^2)e^{2s\varphi}dxdt \leq \int_{t_0-\sqrt{2}\varepsilon}^{t_0+\sqrt{2}\varepsilon}\int_{\widetilde{D}_1}(s|\nabla\pi|^2 + s^3|\pi|^2)e^{2s\varphi}dxdt. \tag{31}$$

By taking the divergence of the first equation of (1), we obtain that $\Delta p = 0$ in Q. Thus, π is solution of

$$\Delta\pi = p\Delta\tilde{\chi} + 2\nabla p\cdot\nabla\tilde{\chi} \quad \text{in } Q.$$

We apply to π the classical Carleman inequality for elliptic equations (which can be proved as in [15]) on $\widetilde{D}_1$ with $\tilde{\varphi} = e^{\lambda d}$: for all fixed λ large enough, there exist constants $\tilde{s}_0$, C and C_1 such that, for all $\tilde{s} > \tilde{s}_0$,

$$\int_{\widetilde{D}_1}(\tilde{s}|\nabla\pi|^2 + \tilde{s}^3|\pi|^2)e^{2\tilde{s}\tilde{\varphi}}dx \leq C\int_{\widetilde{D}_1}|p\Delta\tilde{\chi} + 2\nabla p\cdot\nabla\tilde{\chi}|^2 e^{2\tilde{s}\tilde{\varphi}}dx + C\int_{\partial\widetilde{D}_1}(\tilde{s}|\nabla\pi|^2 + \tilde{s}^3|\pi|^2)e^{2\tilde{s}\tilde{\varphi}}d\sigma \tag{32}$$

By using Lemma 1 (iv), there exists $C > 0$ such that

$$\int_{\partial\widetilde{D}_1}|\nabla\pi|^2 + |\pi|^2 d\sigma \leq C\int_\Gamma |p|^2 + |\nabla p|^2 d\sigma.$$

Thus, if we take $\tilde{s} = se^{-\lambda\beta(t-t_0)^2}$ and if we integrate inequality (32) over $(t_0 - \sqrt{2}\epsilon, t_0 + \sqrt{2}\epsilon)$, thanks to the properties (29) and (30) satisfied by $\tilde{\chi}$, we deduce from inequality (31) that there exist a constant s_0 and a constant C such that, for all $s > s_0$,

$$\iint_{D_3}(s|\nabla p|^2 + s^3|p|^2)e^{2s\varphi}dxdt \leq C\int_0^T\int_B(|p|^2 + |\nabla p|^2)e^{2s\varphi}dxdt + Ce^{C_1 s}\int_0^T\int_\Gamma |p|^2 + |\nabla p|^2 d\sigma dt \tag{33}$$

where $B := \left\{x \in \Omega / \frac{3}{2N}\delta < d(x) < \frac{2}{N}\delta\right\}$. Let us remark that, thanks to inequality (16), we have for $(t, x) \in (0, T) \times B$:

$$\psi(t,x) = d(x) - \beta(t-t_0)^2 < \frac{2}{N}\delta \leq \frac{3}{N}\delta - \frac{\beta\epsilon^2}{N},$$

which implies that $\varphi \leq \mu_3$ on $(0, T) \times B$. Thus, inequality (33) implies that:

$$\iint_{D_3}\left(s|\nabla p|^2 + s^3|p|^2\right)e^{2s\varphi}dxdt \leq Ce^{2s\mu_3}\|p\|^2_{H^{1,0}((0,T)\times B)} + Ce^{C_1 s}\int_0^T\int_\Gamma |p|^2 + |\nabla p|^2 d\sigma dt. \tag{34}$$

We sum up inequalities (27) and (34). The second term in the right hand-side of inequality (27) with the gradient of p is absorbed by the left hand-side of inequality (34), for s large enough. Then, since by Lemma 1 (i), $\left(t_0 - \frac{\varepsilon}{\sqrt{N}}, t_0 + \frac{\varepsilon}{\sqrt{N}}\right) \times \Omega_0 \subset D_5 \subset D_3$, we obtain that, for all $s \geq s_0$,

$$\begin{aligned} &e^{2s\mu_5}\left(\int_{t_0-\frac{\varepsilon}{\sqrt{N}}}^{t_0+\frac{\varepsilon}{\sqrt{N}}}\int_{\Omega_0} s|\nabla p|^2 + s^3|p|^2 + s|\nabla u|^2 + s^3|u|^2 dxdt\right) \\ &\qquad \leq Ce^{2s\mu_4}\left(\|p\|^2_{H^{1,0}(Q)} + \|u\|^2_{H^{1,0}(Q)}\right) + Ce^{C_2 s}\int_0^T\int_\Gamma |\partial_t u|^2 + |\nabla u|^2 + |u|^2 + |p|^2 + |\nabla p|^2 d\sigma dt, \end{aligned}$$

for $C_2 = \max(C_0, C_1)$ which is independent of ε. This implies that, for all $s \geq s_0$:

$$\begin{aligned}\|u\|^2_{H^{1,0}((t_0-\frac{\varepsilon}{\sqrt{N}},t_0+\frac{\varepsilon}{\sqrt{N}})\times\Omega_0)} + \|p\|^2_{H^{1,0}((t_0-\frac{\varepsilon}{\sqrt{N}},t_0+\frac{\varepsilon}{\sqrt{N}})\times\Omega_0)}& \\ \leq Ce^{-2s(\mu_5-\mu_4)}\left(\|p\|^2_{H^{1,0}(Q)} + \|u\|^2_{H^{1,0}(Q)}\right) + Ce^{C_2 s}F^2& \\ \leq Ce^{-C_3 s}\left(\|p\|^2_{H^{1,0}(Q)} + \|u\|^2_{H^{1,0}(Q)}\right) + Ce^{C_2 s}F^2& \qquad (35)\end{aligned}$$

where C_3 only depends on δ, N and λ and where F is given by (9).

As already noticed, the constants in the right hand-side of (35) are independent of t_0. Let us take the following values for t_0:

$$t_{0,j} = \sqrt{2}\epsilon + \frac{j\epsilon}{\sqrt{N}}, \text{ with } j = 0, \dots, m$$

where $m \in \mathbb{N}$ is such that

$$\sqrt{2}\epsilon + \frac{m\epsilon}{\sqrt{N}} \leq T - \sqrt{2}\epsilon \leq \sqrt{2}\epsilon + \frac{(m+1)\epsilon}{\sqrt{N}} \leq T.$$

If we sum up over j the estimates (35) obtained with $t_0 = t_{0,j}$ for $0 \leq j \leq m$, we obtain estimates of u and p in $(\sqrt{2}\varepsilon, T - \sqrt{2}\varepsilon) \times \Omega_0$. Thus, replacing $\sqrt{2}\epsilon$ by ϵ, we obtain, for all $s \geq s_0$:

$$\|u\|_{H^{1,0}((\varepsilon,T-\varepsilon)\times\Omega_0)} + \|p\|_{H^{1,0}((\varepsilon,T-\varepsilon)\times\Omega_0)} \leq Ce^{-sC_3/2}\left(\|p\|_{H^{1,0}(Q)} + \|u\|_{H^{1,0}(Q)}\right) + Ce^{sC_2/2}F. \qquad (36)$$

Thus, thanks to Lemma 2, we have proved estimate (8). □

The proof of Proposition 3 follows exactly the same steps as the proof of Proposition 2, so we will only explain the main arguments and stress the main differences with the previous proof.
We introduce a function d_0 which belongs to $C^2(\overline{\Omega})$ and which satisfies

$$d_0 > 0 \text{ in } \Omega, \quad d_0 = 0 \text{ on } \partial\Omega, \quad |\nabla d_0| > 0 \text{ in } \overline{\Omega} \setminus \omega_0$$

where ω_0 is a nonempty open subset of Ω such that $\overline{\omega_0} \subset \hat{\omega}$. Next, we take $N > 5$ large enough so that

$$\Omega_0 \subset \left\{x \in \Omega / d_0(x) > \frac{5}{N}\delta\right\}.$$

where δ is now defined by $\delta = \|d_0\|_{C(\overline{\Omega})}$ and we choose $\beta > 0$ which satisfies (16). We keep the same definitions (17), (18) and (19) for, respectively, ψ, φ and μ_i with d_0 instead of d. Moreover, we define

$$\widetilde{D}_i = \left\{x \in \overline{\Omega} / d_0(x) > \frac{i}{N}\delta\right\}$$

and

$$D_i = \left\{(t,x) \in \mathbb{R} \times \overline{\Omega} / \varphi(t,x) > \mu_i\right\}.$$

Points (i) and (ii) of Lemma 1 still hold with these new definitions.

Proof of Proposition 3. If we adapt the proof of Theorem 3.1 in [27] to our new weight φ, we get the following local Carleman estimate:
Let $D \subset (0,T) \times \Omega$ be a domain of class C^2 such that, for all $t \in [0,T]$, the boundary of the domain $D \cap \{t\}$ is composed of a finite number of smooth surfaces. For all fixed λ large enough, there exist a constant $s_0 > 0$ and a constant C such that, for all $s > s_0$, for all v in $H^{1,0}(Q)$ satisfying $\partial_t v - \Delta v \in L^2(Q)$ and $\operatorname{supp} v \subset\subset D$

$$\iint_D \left(s|\nabla v|^2 + s^3|v|^2\right) e^{2s\varphi} dxdt \ \leq \ C \iint_D |\partial_t v \ - \ \Delta v|^2 e^{2s\varphi} dxdt \ + \ C \iint_{((0,T)\times\hat{\omega})\cap D} s^3|v|^2 e^{2s\varphi} dxdt.$$

Let us define χ by (24) with the new definition of φ and take $v = \chi u$. Since the support of v is relatively compact in D_2, we can apply this inequality to v in D_2. This implies that

$$e^{2s\mu_5} \int_{t_0-\frac{\varepsilon}{\sqrt{N}}}^{t_0+\frac{\varepsilon}{\sqrt{N}}} \int_{\Omega_0} s|\nabla u|^2 + s^3|u|^2 dxdt$$
$$\leq Ce^{2s\mu_4}\|u\|^2_{H^{1,0}(Q)} + C\iint_{D_3} |\nabla p|^2 e^{2s\varphi} dxdt + C\int_0^T\int_{\hat{\omega}} s^3|u|^2 e^{2s\varphi} dxdt.$$

To estimate the second term in the right hand-side, we notice that (31) still holds with $\pi = \tilde{\chi}p$ where $\tilde{\chi}$ is defined by (29) with d_0 instead of d. Then, we apply to π the standard elliptic Carleman estimate ([15]) in $\widetilde{D}_1 \subset\subset \Omega$ with our new weight $\tilde{\varphi} = e^{\lambda d_0}$ for λ fixed large enough. Arguing in a similar way as in the proof of Proposition 2, we get that there exist a constant $\tilde{s}_0 > 0$ and a constant C such that, for all $\tilde{s} > \tilde{s}_0$,

$$\iint_{D_3} (s|\nabla p|^2 + s^3|p|^2)e^{2s\varphi} dxdt \leq Ce^{2s\mu_3}\|p\|^2_{H^{1,0}((0,T)\times B)} + Cs^3\int_0^T\int_{\hat{\omega}} |p|^2 e^{2s\varphi} dxdt$$

where $B := \left\{x \in \Omega / \frac{3}{2N}\delta < d_0(x) < \frac{2}{N}\delta\right\} \subset\subset \Omega$. Moreover, according to Cacciopoli inequality ([17]), since $\Delta p = 0$ in $(0,T)\times\Omega$, we have

$$\|p\|^2_{H^{1,0}((0,T)\times B)} \leq C\|p\|^2_{L^2((0,T)\times\Omega)}.$$

We then proceed exactly as in the proof of Proposition 2 to conclude the proof. $\square$

3.2 Estimates on the boundary of the domain: proof of Proposition 4

For any $x \in \mathbb{R}^3$, we use the following notation $x = (x_1, x')$ where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^2$. Moreover, for all $R > 0$, we denote by

$$\mathcal{B}(0,R)^+ = \{x = (x_1,x') \in \mathcal{B}(0,R)/x_1 > 0\},$$

where $\mathcal{B}(0,R)$ is the open ball of center 0 and of radius R, and by

$$D_R = \{x = (x_1,x') \in \mathbb{R}^3/0 < x_1 < R \text{ and } |x'| < R\}.$$

Let (u,p) be a solution of (1). Thanks to a change of coordinates, we can straighten locally the boundary of Ω and go back to the upper half-plane. For all $P \in \partial\Omega$, let ϕ_P be such a change of variables in a neighborhood of P. The function ϕ_P is a C^2-diffeomorphism on $\mathcal{B}(0,r_P)$ for some $r_P > 0$ and satisfies

$$\phi_P(0) = P, \quad \phi_P(\mathcal{B}(0,r_P)^+) = \Omega \cap \phi_P(\mathcal{B}(0,r_P))$$

and

$$\phi_P\big(\{(x_1,x') \in \mathcal{B}(0,r_P), x_1 = 0\}\big) = \partial\Omega \cap \phi_P(\mathcal{B}(0,r_P)).$$

Moreover, due to the regularity and compactness of Ω, there exists $R > 0$ such that $\forall P \in \partial\Omega$, $r_P \geq 3R$ and we can always assume that $R < 1$. Next, since $\partial\Omega \subset \bigcup_{P\in\partial\Omega} \phi_P(\mathcal{B}(0,R/2))$, by compactness of $\partial\Omega$, there exist N points $(P_i)_{1\leq i\leq N}$ of $\partial\Omega$ such that $\partial\Omega \subset \bigcup_{i=1}^{N} \phi_{P_i}(\mathcal{B}(0,R/2))$.

In the following, we fix $1 \leq i \leq N$ and, to simplify the notations, we set $\phi = \phi_{P_i}$. Let us define, for all $(t,x) \in (0,T)\times\mathcal{B}(0,3R)^+$

$$v(t,x) = u(t,\phi(x)), \quad q(t,x) = p(t,\phi(x)). \tag{37}$$

These functions satisfy the following problem:

$$\begin{cases} \partial_t v - \operatorname{div}(\nabla v A_\phi) + B_\phi \nabla q &= 0, \quad \text{in } (0,T)\times\mathcal{B}(0,3R)^+, \\ \operatorname{div}(A_\phi \nabla q) &= 0, \quad \text{in } (0,T)\times\mathcal{B}(0,3R)^+, \end{cases} \tag{38}$$

with

$$A_\phi = |\det(\nabla\phi)|(\nabla\phi)^{-1}(\nabla\phi)^{-t},$$

and

$$B_\phi = |\det(\nabla\phi)|(\nabla\phi)^{-t}.$$

Let us define the operator P_ϕ by:

$$P_\phi f = -\operatorname{div}(A_\phi \nabla f) \tag{39}$$

for a regular scalar function f, and by

$$P_\phi F = (P_\phi F_1, P_\phi F_2, P_\phi F_3),$$

for a regular vector-valued function $F = (F_1, F_2, F_3)$. We can rewrite system (38) as follows:

$$\begin{cases} \partial_t v + P_\phi v + B_\phi \nabla q &= 0, \quad \text{in } (0,T) \times \mathcal{B}(0,3R)^+, \\ P_\phi q &= 0, \quad \text{in } (0,T) \times \mathcal{B}(0,3R)^+. \end{cases} \tag{40}$$

Let $\varepsilon > 0$ be given. We consider $t_0 \in (\varepsilon, T-\varepsilon)$ and $x_0' \in \mathbb{R}^2$ such that $|x_0'| \leq R$. We choose $\beta > 0$ and $\gamma > 0$ such that

$$R < \beta\varepsilon^2 < 2R \text{ and } \frac{\gamma R}{4} > 1 \tag{41}$$

and we define

$$\begin{aligned} d(x) &= x_1 - \gamma|x' - x_0'|^2, \\ \psi(t,x) &= d(x) - \beta(t-t_0)^2 \end{aligned} \tag{42}$$

and

$$Q(\eta) = \{(t,x) \in \mathbb{R} \times \mathbb{R}^3 / x_1 < R + \eta \text{ and } \psi(t,x) > \eta\}.$$

Lemma 3. *For $0 < \eta < \frac{R}{2}$, we have $Q(\eta) \subset (t_0 - \varepsilon, t_0 + \varepsilon) \times D_{\frac{3R}{2}} \subset (t_0 - \varepsilon, t_0 + \varepsilon) \times \mathcal{B}(0,3R)^+$.*

Proof of Lemma 3. Let $(t,x) \in Q(\eta)$. First, we have that $x_1 < R + \eta < \frac{3R}{2}$. Moreover, since $\psi(t,x) > \eta$

$$x_1 > \eta + \gamma|x' - x_0'|^2 + \beta(t-t_0)^2$$

This implies that $x_1 > 0$ and that $\gamma|x' - x_0'|^2 + \beta(t-t_0)^2 < R$. According to the conditions (41) satisfied by β and γ, we obtain the first inclusion. The second inclusion is readily proved. □

In order to apply local Carleman inequality, we need to introduce a cut-off function

Lemma 4. *Let $0 < \eta < \dfrac{R}{2}$ be given. We can define a function $\chi_\eta \in C^\infty(\mathbb{R} \times \mathbb{R}^3)$ such that $0 \leq \chi_\eta \leq 1$,*

$$\chi_\eta(t,x) = \begin{cases} 1, & \text{if } \psi(t,x) \geq 3\eta, \\ 0, & \text{if } \psi(t,x) \leq 2\eta, \end{cases} \tag{43}$$

and, for all $(t,x) \in (0,T) \times D_{\frac{3R}{2}}$,

$$|\partial_i \chi_\eta(t,x)| + |\partial_t \chi_\eta(t,x)| \leq \frac{C}{\eta}, \quad \left|\partial_{ij}^2 \chi_\eta(t,x)\right| \leq \frac{C}{\eta^2}, \text{ for all } 1 \leq i,j \leq n,$$

where $C > 0$ is a constant which only depends on R, T, ε and γ.

To prove this lemma, we can define χ thanks to the definition (25) of $\bar{\chi}$ by $\chi_\eta(t,x) = \bar{\chi}\left(\dfrac{\psi(t,x) - 2\eta}{\eta}\right)$.

Proof of Proposition 4. Let us define $(v_\eta, q_\eta) = (\chi_\eta v, \chi_\eta q)$. The function v_η satisfies the following equation

$$\partial_t v_\eta + P_\phi v_\eta = -\chi_\eta B_\phi \nabla q + \partial_t \chi_\eta v + [P_\phi, \chi_\eta] v, \text{ in } (0,T) \times \mathcal{B}(0,2R)^+,$$

where the operator $[P_\phi, \chi_\eta]$ is defined by

$$[P_\phi, \chi_\eta] v = -\operatorname{div}(A_\phi v \nabla \chi_\eta^t) - A_\phi \nabla v \nabla \chi_\eta$$

for all vector-valued function v. We denote by

$$\widetilde{D}_R = \left\{ x \in \mathbb{R}^3 / R < x_1 < \frac{3R}{2}, |x'| < \frac{3R}{2} \right\}.$$

In the following, we consider that $\eta \in \left(0, \frac{R}{8}\right)$ is given. We apply the Carleman estimate for parabolic equations (Theorem 3.2 in [27]) in $Q(\eta)$ with the weight $\varphi = e^{\lambda\psi}$ where ψ is given by (42): for all fixed λ large enough, there exists a constant $s_0 > 0$ and a constant C such that, for all $s > s_0$,

$$\iint_{Q(\eta)} \left(\frac{1}{s} |\partial_t v_\eta|^2 + s|\nabla v_\eta|^2 + s^3 |v_\eta|^2 \right) e^{2s\varphi} dxdt \leq C \iint_{Q(\eta)} |-\chi_\eta B_\phi \nabla q + \partial_t \chi_\eta v + [P_\phi, \chi_\eta] v|^2 e^{2s\varphi} dxdt$$
$$+ C \iint_{\partial Q(\eta)} (s|\nabla_{t,x} v_\eta|^2 + s^3 |v_\eta|^2) e^{2s\varphi} d\sigma dt. \quad (44)$$

Notice that, since the domain $Q(\eta)$ is a translation of $Q(0)$ in the direction x_1 for any η, the constants s_0 and C are independent of η.
Let us first estimate the last term in this inequality. We remark that

$$\partial Q(\eta) = \{(t,x)/x_1 = R + \eta,\, \psi(t,x) \geq \eta\} \cup \{(t,x)/x_1 \leq R + \eta,\, \psi(t,x) = \eta\}.$$

Since χ_η satisfies (43), $v_\eta = |\nabla v_\eta| = 0$ on $\{(t,x)/x_1 \leq R + \eta,\, \psi(t,x) = \eta\}$. Moreover, according to Lemma 3, $\{(t,x)/x_1 = R + \eta,\, \psi(t,x) \geq \eta\} \subset \{(t,x) \in (0,T) \times D_{\frac{3R}{2}} / x_1 = R + \eta\}$. Hence, there exists a constant $C > 0$ which does not depend on η such that

$$\iint_{\partial Q(\eta)} (s|\nabla_{t,x} v_\eta|^2 + s^3 |v_\eta|^2) e^{2s\varphi} d\sigma dt \leq C e^{Cs} J_1^2$$

where

$$J_1 = \|v\|_{H^1(0,T;H^2(\widetilde{D}_R))}. \quad (45)$$

For the first term in the right hand-side of (44), we first notice that there exists $C > 0$ such that

$$\iint_{Q(\eta)} |\chi_\eta B_\phi \nabla q|^2 e^{2s\varphi} dxdt \leq C \iint_{Q(\eta) \cap Q(2\eta)} |\nabla q|^2 e^{2s\varphi}\, dx\, dt$$

according to (43) and that there exists $C > 0$ such that

$$\iint_{Q(\eta)} |\partial_t \chi_\eta v + [P_\phi, \chi_\eta] v|^2 e^{2s\varphi}\, dx\, dt \leq \frac{C}{\eta^4} e^{2s\alpha_1} \|v\|^2_{H^{1,0}((0,T) \times D_{\frac{3R}{2}})}$$

where we have denoted $\alpha_1 = e^{3\lambda\eta}$ and used Lemmas 3 and 4. By this way, if we denote $\alpha_2 = e^{4\lambda\eta}$, inequality (44) becomes

$$e^{2s\alpha_2} \iint_{Q(\eta) \cap Q(4\eta)} \left(s|\nabla v|^2 + s^3 |v|^2 \right) dxdt$$
$$\leq C \iint_{Q(\eta) \cap Q(2\eta)} |\nabla q|^2 e^{2s\varphi}\, dx\, dt + \frac{C}{\eta^4} e^{2s\alpha_1} \|v\|^2_{H^{1,0}((0,T) \times D_{\frac{3R}{2}})} + C e^{Cs} J_1^2 \quad (46)$$

where the constants C do not depend on η. We now want to estimate the term with the pressure (first term in the right hand-side). To do so, we will use the fact that q satisfies an elliptic equation and apply a Carleman estimate. First, we denote by

$$E(\eta) = \{x \in \mathbb{R}^3 / d(x) > \eta \text{ and } x_1 < R + \eta\}.$$

Note that, according to (41), $E(\eta) \subset D_{\frac{3R}{2}}$ and

$$Q(\eta) \subseteq (t_0 - \epsilon, t_0 + \epsilon) \times E(\eta). \tag{47}$$

In a similar way as in Lemma 4, let us introduce a cut-off function which satisfies the following properties: For a given $\eta > 0$, there exists $\zeta_\eta \in C^2(\mathbb{R}^3)$ such that $0 \leq \zeta_\eta \leq 1$ and

$$\zeta_\eta(x) = \begin{cases} 1, & \text{if } d(x) \geq 2\eta, \\ 0, & \text{if } d(x) \leq \dfrac{3\eta}{2}. \end{cases} \tag{48}$$

Moreover, there exists a constant $C > 0$ depending only on R and γ such that, for all $x \in D_{\frac{3R}{2}}$, we have

$$|\partial_i \zeta_\eta(x)| \leq \frac{C}{\eta} \text{ and } \left|\partial^2_{ij} \zeta_\eta(x)\right| \leq \frac{C}{\eta^2}, \text{ for all } 1 \leq i, j \leq 3.$$

We denote by $\pi_\eta = \zeta_\eta q$. According to (47) and (48), we will thus be able to estimate the first term in the right hand-side of (46) thanks to the following inequality

$$\iint_{Q(\eta)\cap Q(2\eta)} (s|\nabla q|^2 + s^3|q|^2) e^{2s\varphi} \, dx \, dt \leq \int_{t_0-\epsilon}^{t_0+\epsilon} \int_{E(\eta)} (s|\nabla \pi_\eta|^2 + s^3|\pi_\eta|^2) e^{2s\varphi} \, dx \, dt. \tag{49}$$

The function π_η is solution of

$$P_\phi \pi_\eta = [P_\phi, \zeta_\eta] q \text{ in } (0,T) \times \mathcal{B}(0, 2R)^+,$$

where the operator $[P_\phi, \zeta_\eta]$ is defined by

$$[P_\phi, \zeta_\eta] q = -\nabla \zeta_\eta^t A_\phi \nabla q - \operatorname{div}(q A_\phi \nabla \zeta_\eta)$$

for all scalar function q. We then apply the elliptic Carleman estimate to π_η in $E(\eta)$ with the weight $\tilde{\phi} = e^{\lambda d}$: for all fixed $\lambda > 0$ large enough, there exists a constant $\tilde{s}_0 > 0$ and a constant C such that, for all $\tilde{s} > \tilde{s}_0$,

$$\int_{E(\eta)} \left(\tilde{s}|\nabla \pi_\eta|^2 + \tilde{s}^3|\pi_\eta|^2\right) e^{2\tilde{s}\tilde{\phi}} \, dx \leq C \int_{E(\eta)} |[P_\phi, \zeta_\eta] q|^2 e^{2\tilde{s}\tilde{\phi}} \, dx + C \int_{\partial E(\eta)} (\tilde{s}|\nabla \pi_\eta|^2 + \tilde{s}^3|\pi_\eta|^2) e^{2\tilde{s}\tilde{\phi}} d\sigma. \tag{50}$$

The domain $E(\eta)$ is a translation of $E(0)$ in the direction x_1, thus the constants $\tilde{s}_0$ and C do not depend on η. Moreover, we notice that

$$\partial E(\eta) = \left\{x \in \mathbb{R}^3 / d(x) = \eta \text{ and } 0 < x_1 \leq R + \eta\right\} \cup \left\{x \in \mathbb{R}^3 / d(x) \geq \eta \text{ and } x_1 = R + \eta\right\}$$

and, by construction, $|\pi_\eta| = |\nabla \pi_\eta| = 0$ on $\left\{x \in \mathbb{R}^3 / d(x) = \eta \text{ and } 0 < x_1 \leq R + \eta\right\}$. Thus, if we set $\tilde{s} = s e^{-\lambda \beta (t - t_0)^2}$ in inequality (50) and if we integrate in time over $(t_0 - \epsilon, t_0 + \epsilon)$, we obtain that there exists a constant $s_0 > 0$ and a constant C such that, for all $s > s_0$,

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \int_{E(\eta)} \left(s|\nabla \pi_\eta|^2 + s^3|\pi_\eta|^2\right) e^{2s\varphi} \, dx \, dt \leq \frac{C}{\eta^4} e^{2s\alpha_1} \|q\|^2_{H^{1,0}((0,T)\times D_{\frac{3R}{2}})} + C e^{Cs} J_2^2$$

where J_2 is given by

$$J_2 = \|q\|_{L^2(0,T;H^{7/4}(\widetilde{D}_R))}. \tag{51}$$

Thus, inequality (49) becomes

$$\iint_{Q(\eta)\cap Q(2\eta)} (s|\nabla q|^2 + s^3|q|^2)e^{2s\varphi}\,dx\,dt \le \frac{C}{\eta^4} e^{2s\alpha_1}\|q\|^2_{H^{1,0}((0,T)\times D_{\frac{3R}{2}})} + Ce^{Cs}J_2^2.$$

If we add up this inequality with inequality (46), the left hand-side of this inequality allows to absorb the first term in the right hand-side of (46) if we take s large enough. Since $Q(\eta)\cap Q(4\eta) \subset Q(\eta)\cap Q(2\eta)$, we obtain that, for all $s \ge s_0$,

$$\|v\|^2_{H^{1,0}(Q(4\eta)\cap Q(\eta))} + \|q\|^2_{H^{1,0}(Q(4\eta)\cap Q(\eta))} \le \frac{C}{\eta^4} e^{-2sC_\eta} M^2 + Ce^{Cs}(J_1^2 + J_2^2)$$

where $C_\eta = \alpha_2 - \alpha_1 > 0$ and M is given by (12). Since $Q(4\eta) = \big(Q(4\eta)\cap Q(\eta)\big) \cup \big(Q(4\eta)\cap\{R+\eta \le x_1 < R+4\eta\}\big)$ and since $\|v\|^2_{H^{1,0}(Q(4\eta)\cap\{R+\eta<x_1<R+4\eta\})} + \|q\|^2_{H^{1,0}(Q(4\eta)\cap\{R+\eta<x_1<R+4\eta\})}$ is bounded by $J_1^2 + J_2^2$, we can in fact estimate v and q on the whole set $Q(4\eta)$: for all $s \ge s_0$

$$\|v\|^2_{H^{1,0}(Q(4\eta))} + \|q\|^2_{H^{1,0}(Q(4\eta))} \le \frac{C}{\eta^4} e^{-2sC_\eta} M^2 + Ce^{Cs}(J_1^2 + J_2^2).$$

Since $C_\eta = e^{4\lambda\eta} - e^{3\lambda\eta} = e^{3\lambda\eta}(e^{\lambda\eta} - 1) \ge e^{\lambda\eta} - 1 \ge \lambda\eta$, we get the existence of s_0, $C > 0$ and $C_1 > 0$ which are independent of η such that, for all $s \ge s_0$,

$$\|v\|_{H^{1,0}(Q(4\eta))} + \|q\|_{H^{1,0}(Q(4\eta))} \le \frac{C}{\eta^2} e^{-s\lambda\eta} M + Ce^{C_1 s}(J_1 + J_2).$$

Note that the previous inequality is in fact valid for all $s \ge 0$ since $\|v\|_{H^{1,0}(Q(4\eta))} + \|q\|_{H^{1,0}(Q(4\eta))} \le M$. If $J_1^2 + J_2^2 = 0$, letting $s \to +\infty$, we see that $v = 0$ and $q = 0$ in $Q(4\eta)$. Assume now that $J_1^2 + J_2^2 \ne 0$ and choose s such that the first term in the right hand-side has the same value as the second term in the right hand-side: we take $s = \dfrac{1}{C_1 + \lambda\eta} \log\left(\dfrac{M}{(J_1 + J_2)\eta^2}\right)$. Then we obtain that there exists a constant $C > 0$ such that, for all $\eta \in \big(0, \frac{R}{8}\big)$

$$\|v\|_{H^{1,0}(Q(4\eta))} + \|q\|_{H^{1,0}(Q(4\eta))} \le C\left(\frac{M}{\eta^2}\right)^{\frac{C_1}{C_1+\lambda\eta}} (J_1 + J_2)^{\frac{\lambda\eta}{C_1+\lambda\eta}} \le \frac{C}{\eta^2} M^{\frac{C_1}{C_1+\lambda\eta}} (J_1 + J_2)^{\frac{\lambda\eta}{C_1+\lambda\eta}}. \tag{52}$$

Moreover, this inequality still holds if $J_1 + J_2 = 0$. Now, let us introduce the following set

$$Q_1(4\eta) = \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^3 / |t - t_0| < \frac{\epsilon}{2},\, 4\eta + \gamma|x' - x_0'|^2 < x_1 - \beta(t-t_0)^2 < 4\eta + \frac{R}{2}\right\}.$$

Thanks to the property (41), for all $\eta \in \big(0, \frac{R}{8}\big)$, $Q_1(4\eta) \subset Q(4\eta)$. For all $(t,x) \in Q_1(4\eta)$, let us set

$$\tilde{x}_1 = x_1 - \beta(t - t_0)^2.$$

Then

$$(t,x) \in Q_1(4\eta) \quad \Leftrightarrow \quad t \in I_\epsilon,\ (\tilde{x}_1, x') \in B(4\eta)$$

where

$$I_\epsilon = \left(t_0 - \frac{\epsilon}{2}, t_0 + \frac{\epsilon}{2}\right),\ B(4\eta) = \left\{(\tilde{x}_1, x')/4\eta + \gamma|x' - x_0'|^2 < \tilde{x}_1 < 4\eta + \frac{R}{2}\right\}.$$

Let us set, for all $t \in I_\epsilon$ and $(\tilde{x}_1, x') \in B(4\eta)$,

$$w(t, \tilde{x}_1, x') = v(t, \tilde{x}_1 + \beta(t-t_0)^2, x').$$

Then,

$$\|v\|_{C\left(\overline{Q_1(4\eta)}\right)} = \|w\|_{C\left(\overline{(I_\epsilon \times B(4\eta))}\right)} \le C\|w\|^{\frac{7}{8}}_{H^1(I_\epsilon;H^2(B(4\eta)))}\|w\|^{\frac{1}{8}}_{L^2((I_\epsilon\times B(4\eta))}.$$

Since the domain $B(4\eta)$ is the translation of $B(0)$ in the direction x_1, the constants in these inequalities are independent of η. Coming back to the function v in the right hand-side, we get that

$$\|v\|_{C\left(\overline{Q_1(4\eta)}\right)} \leq C\|v\|^{\frac{7}{8}}_{H^1(0,T;H^2(\mathcal{B}(0,3R)^+))}\|v\|^{\frac{1}{8}}_{L^2(Q_1(4\eta))} \leq CM^{\frac{7}{8}}\|v\|^{\frac{1}{8}}_{L^2(Q(4\eta))}.$$

The same inequalities hold for ∇v and q. Consequently, thanks to (52), we have

$$\begin{aligned}\|v\|_{C\left(\overline{Q_1(4\eta)}\right)} + \|\nabla v\|_{C\left(\overline{Q_1(4\eta)}\right)} + \|q\|_{C\left(\overline{Q_1(4\eta)}\right)} &\leq CM^{\frac{7}{8}}\left(\frac{C}{\eta^2}M^{\frac{C_1}{C_1+\lambda\eta}}(J_1+J_2)^{\frac{\lambda\eta}{C_1+\lambda\eta}}\right)^{\frac{1}{8}} \\ &\leq \frac{C}{\eta^{\frac{1}{4}}}M\left(\frac{J_1+J_2}{M}\right)^{\frac{\lambda\eta}{8(C_1+\lambda\eta)}}. \qquad (53)\end{aligned}$$

Since $\eta \in \left(0, \frac{R}{8}\right)$, we have that $\left(\frac{J_1+J_2}{M}\right)^{\frac{\lambda\eta}{8(C_1+\lambda\eta)}} < \left(\frac{J_1+J_2}{M}\right)^{C\eta}$, where $C > 0$ is a constant which does not depend on η. Thus,

$$\|v\|_{C\left(\overline{Q_1(4\eta)}\right)} + \|\nabla v\|_{C\left(\overline{Q_1(4\eta)}\right)} + \|q\|_{C\left(\overline{Q_1(4\eta)}\right)} \leq \frac{C}{\eta^{\frac{1}{4}}}M\left(\frac{J_1+J_2}{M}\right)^{C\eta}.$$

Let $\xi' = (\xi_2, \xi_3) \in \mathbb{R}^2$. In particular, for all $|\xi' - x_0'| < \sqrt{\frac{R}{2\gamma}}$, $0 < \eta < \frac{R}{8}$, we have:

$$\begin{aligned}&\left|v\left(t_0, 4\eta + \gamma|\xi' - x_0'|^2, \xi'\right)\right| + \left|\nabla v\left(t_0, 4\eta + \gamma|\xi' - x_0'|^2, \xi'\right)\right| \\ &\qquad + \left|q\left(t_0, 4\eta + \gamma|\xi' - x_0'|^2, \xi'\right)\right| \leq \frac{C}{\eta^{\frac{1}{4}}}M\left(\frac{J_1+J_2}{M}\right)^{C\eta}. \qquad (54)\end{aligned}$$

Let us define

$$Q_{x_0'} = \left\{x \in \mathbb{R}^3 /\, \gamma|x' - x_0'|^2 < x_1 < \frac{R}{2}, |x' - x_0'| < \sqrt{\frac{R}{2\gamma}}\right\}.$$

Thanks to the following change of variables:

$$(\eta, \xi') \to (4\eta + \gamma|\xi' - x_0'|^2, \xi'),$$

we obtain:

$$\int_{Q_{x_0'}} |v(t_0,x)|^2 dx = 4\int_{|\xi'-x_0'|<\sqrt{\frac{R}{2\gamma}}}\int_0^{\frac{R}{8}-\frac{\gamma}{4}|\xi'-x_0'|^2} \left|v(t_0, 4\eta + \gamma|\xi' - x_0'|^2, \xi')\right|^2 d\eta d\xi'.$$

By performing the same calculation for ∇v and q, we obtain thanks to (54),

$$\int_{Q_{x_0'}} |v(t_0,x)|^2 + |\nabla v(t_0,x)|^2 + |q(t_0,x)|^2 dx \leq CM^2\int_0^{\frac{R}{8}} \eta^{-\frac{1}{2}}\left(\frac{J_1+J_2}{M}\right)^{C\eta} d\eta. \qquad (55)$$

According to the definitions (45), (51) and (12) of J_1, J_2 and M, there exists a constant $C_2 > 0$ such that $J_1 + J_2 \leq C_2 M$. We notice that

$$\begin{aligned}\int_0^{\frac{R}{8}} \eta^{-\frac{1}{2}}\left(\frac{J_1+J_2}{C_2M}\right)^{C\eta} d\eta &\leq \int_0^{+\infty} \eta^{-\frac{1}{2}}e^{-C\log(\frac{C_2M}{J_1+J_2})\eta}d\eta \\ &\leq \left(\frac{1}{C\log(\frac{C_2M}{J_1+J_2})}\right)^{-\frac{1}{2}}\int_0^{+\infty}\left(C\log\left(\frac{C_2M}{J_1+J_2}\right)\eta\right)^{-\frac{1}{2}} e^{-C\log(\frac{C_2M}{J_1+J_2})\eta}d\eta.\end{aligned}$$

By performing the change of variables $\eta_1 = C\log\left(\frac{C_2 M}{J_1+J_2}\right)\eta$, we finally obtain that

$$\int_0^{\frac{R}{8}} \eta^{-\frac{1}{2}} \left(\frac{J_1+J_2}{C_2M}\right)^{C\eta} d\eta \leq C \frac{\Gamma\left(\frac{1}{2}\right)}{\left(\log\left(\frac{C_2M}{J_1+J_2}\right)\right)^{\frac{1}{2}}},$$

where Γ is the gamma function. Thus, if we come back to (55), we obtain that, for all $x_0' \in \mathbb{R}^2$ such that $|x_0'| \leq R$

$$\|v(t_0,.)\|_{H^1(Q_{x_0'})} + \|q(t_0,.)\|_{L^2(Q_{x_0'})} \leq \frac{CM}{\left(\log\left(\frac{C_2M}{J_1+J_2}\right)\right)^{\frac{1}{4}}}.$$

Then, by applying an interpolation inequality, we obtain

$$\sup_{x\in\overline{Q_{x_0'}}} |v(t_0,x)| \quad \leq \quad C\|v(t_0,.)\|^{\frac{7}{8}}_{H^2(Q_{x_0'})}\|v(t_0,.)\|^{\frac{1}{8}}_{L^2(Q_{x_0'})} \leq CM^{\frac{7}{8}}\|v(t_0,.)\|^{\frac{1}{8}}_{L^2(Q_{x_0'})}$$

where the constant $C > 0$ does not depend on x_0' since the domains $Q_{x_0'}$ are in translation with each other in the direction x'. Similar inequalities hold for ∇v and q.
Thus, for all $t_0 \in (\epsilon, T-\epsilon)$, for all x_0' such that $|x_0'| \leq R$

$$\sup_{x\in\overline{Q_{x_0'}}} |v(t_0,x)| + |\nabla v(t_0,x)| + |q(t_0,x)| \leq \frac{CM}{\left(\log\left(\frac{C_2M}{J_1+J_2}\right)\right)^{\frac{1}{32}}}.$$

Since $\overline{\mathcal{B}(0,R/2)^+} \subset \{x\in\mathbb{R}^3/0\leq x_1\leq R/2,\ |x'|\leq R\} \subset \bigcup_{\{x_0'/|x_0'|\leq R\}} Q_{x_0'}$, this implies that

$$\sup_{(t_0,x)\in(\epsilon,T-\epsilon)\times\overline{\mathcal{B}(0,R/2)^+}} |v(t_0,x)| + |\nabla v(t_0,x)| + |q(t_0,x)| \leq \frac{CM}{\left(\log\left(\frac{C_2M}{J_1+J_2}\right)\right)^{\frac{1}{32}}}.$$

We notice that

$$J_1 + J_2 \leq \left(\|v\|_{L^2((0,T)\times\widetilde{D}_R)} + \|q\|_{L^2((0,T)\times\widetilde{D}_R)}\right)^{1/8} M^{7/8}.$$

Thus,

$$\|v\|_{C\left(\varepsilon,T-\varepsilon;C^1\left(\overline{\mathcal{B}(0,R/2)^+}\right)\right)} + \|q\|_{C\left((\varepsilon,T-\varepsilon)\times\overline{\mathcal{B}(0,R/2)^+}\right)} \leq \frac{CM}{\left(\log\left(\frac{CM}{\|v\|_{L^2((0,T)\times\widetilde{D}_R)}+\|q\|_{L^2((0,T)\times\widetilde{D}_R)}}\right)\right)^{\frac{1}{32}}}.$$

The last step consists of coming back to an estimate on $(\varepsilon, T-\varepsilon)\times\phi_{P_i}(\mathcal{B}(0,R/2)^+)$ of u and p related to v and q by the change of variables (37). Then, if we sum up the obtained inequality for $1\leq i\leq N$, we get estimate (11). $\square$

4 Stability estimate for the identification of the Robin coefficient

In this section, we come back to the inverse problem presented in the introduction. In particular, we consider system (6) and we assume that $\partial\Omega$ is the union of two disjoint pars Γ_0 and Γ_e. We will prove the stability result on the Robin coefficient q given by Theorem 2. Let us first state regularity results for the following problem

$$\left\{\begin{array}{rcll} u_t - \Delta u + \nabla p & = & 0, & \text{in } (0,T)\times\Omega, \\ div\ u & = & 0, & \text{in } (0,T)\times\Omega, \\ \nabla u\cdot n - pn & = & g, & \text{on } (0,T)\times\Gamma_e, \\ \nabla u\cdot n - pn + qu & = & h, & \text{on } (0,T)\times\Gamma_0, \\ u(0,\cdot) & = & u_0, & \text{in } \Omega. \end{array}\right. \tag{56}$$

Using regularity results for the stationary Stokes problem with Neumann boundary conditions [11] and the same kind of arguments as in the proof of Theorem 2.6 in [8], we can prove that the solution of system (56) satisfies the two following propositions:

Proposition 5. *Let Ω be a bounded and connected open set in $\mathbb{R}^3$ of class $C^{1,1}$ and let $\nu_0 > 0$, $\overline{N}_0 > 0$ be given. We assume that $u_0 \in V$, $g \in H^1(0,T;L^2(\Gamma_e)) \cap L^2(0,T;H^{\frac{1}{2}}(\Gamma_e))$, $h \in H^1(0,T;L^2(\Gamma_0)) \cap L^2(0,T;H^{\frac{1}{2}}(\Gamma_0))$, $q \in W^{1,\infty}(0,T;L^\infty(\Gamma_0)) \cap L^2(0,T;H^s(\Gamma_0))$ with $s > 1$ and that*

$$q \geq \nu_0 \text{ on } (0,T) \times \Gamma_0, \quad \|q\|_{L^2(0,T;H^s(\Gamma_0))} + \|q\|_{W^{1,\infty}(0,T;L^\infty(\Gamma_0))} \leq \overline{N}_0.$$

Then problem (56) admits a unique solution $(u,p) \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \times L^2(0,T;H^1(\Omega))$. Moreover, this solution satisfies:

$$\begin{aligned}&\|u\|_{L^2(0,T;H^2(\Omega))} + \|u\|_{H^1(0,T;L^2(\Omega))} + \|u\|_{L^\infty(0,T;H^1(\Omega))} + \|p\|_{L^2(0,T;H^1(\Omega))} \\ &\leq C(\|u_0\|_{H^1(\Omega)} + \|g\|_{H^1(0,T;L^2(\Gamma_e))} + \|g\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma_e))} + \|h\|_{H^1(0,T;L^2(\Gamma_0))} + \|h\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma_0))})\end{aligned}$$

where C depends on ν_0 and $\overline{N}_0$.

Proposition 6. *Let Ω be a bounded and connected open set in $\mathbb{R}^3$ of class $C^{2,1}$ and let $\nu_0 > 0$, $\overline{N}_0 > 0$ be given. We assume that $u_0 \in H^4(\Omega) \cap V$, $g \in H^2(0,T;L^2(\Gamma_e)) \cap H^1(0,T;H^{\frac{3}{2}}(\Gamma_e))$, $h \in H^2(0,T;L^2(\Gamma_0)) \cap H^1(0,T;H^{\frac{3}{2}}(\Gamma_0))$, $q \in H^2(0,T;H^2(\Gamma_0))$ and*

$$q \geq \nu_0 \text{ on } (0,T) \times \Gamma_0, \quad \|q\|_{H^2(0,T;H^2(\Gamma_0))} \leq \overline{N}_0.$$

Then problem (56) admits a unique solution $(u,p) \in H^2(0,T;H^1(\Omega)) \cap W^{2,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^3(\Omega)) \times H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^2(\Omega))$. Moreover, this solution satisfies:

$$\begin{aligned}&\|u\|_{H^2(0,T;H^1(\Omega))} + \|u\|_{H^1(0,T;H^3(\Omega))} + \|u\|_{W^{2,\infty}(0,T;L^2(\Omega))} + \|p\|_{H^2(0,T;L^2(\Omega))} + \|p\|_{H^1(0,T;H^2(\Omega))} \\ &\leq C(\|u_0\|_{H^4(\Omega)} + \|g\|_{H^2(0,T;L^2(\Gamma_e))} + \|g\|_{H^1(0,T;H^{\frac{3}{2}}(\Gamma_e))} + \|h\|_{H^2(0,T;L^2(\Gamma_0))} + \|h\|_{H^1(0,T;H^{\frac{3}{2}}(\Gamma_0))})\end{aligned}$$

where C depends on ν_0 and $\overline{N}_0$.

We are now able to prove Theorem 2. Under the hypotheses made in the statement of this theorem, according to Proposition 6, (u_1, p_1) and (u_2, p_2) belong to $H^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^3(\Omega)) \times H^1(0,T;H^2(\Omega))$ and we have, for $i = 1, 2$,

$$\|u_i\|_{H^2(0,T;H^1(\Omega))} + \|u_i\|_{H^1(0,T;H^3(\Omega))} + \|p_i\|_{H^1(0,T;H^2(\Omega))} \leq C(\nu_0, N_0). \tag{57}$$

The functions $u = u_1 - u_2$ and $p = p_1 - p_2$ are solutions of

$$\left\{\begin{array}{rcll} u_t - \Delta u + \nabla p &=& 0, & \text{in } (0,T) \times \Omega, \\ div\ u &=& 0, & \text{in } (0,T) \times \Omega, \\ \nabla u \cdot n - pn &=& 0, & \text{on } (0,T) \times \Gamma_e, \\ \nabla u \cdot n - pn + q_2 u &=& u_1(q_2 - q_1), & \text{on } (0,T) \times \Gamma_0, \\ u(0,\cdot) &=& 0, & \text{in } \Omega. \end{array}\right. \tag{58}$$

Thus, on $K \subset \{(t,x) \in (\varepsilon, T-\varepsilon) \times \Gamma_0 \,/\, u_1 \neq 0\}$, we have

$$m\|q_2 - q_1\|_{C(K)} \leq C(N_0)(\|\nabla u\|_{C((\varepsilon,T-\varepsilon)\times\Gamma_0)} + \|p\|_{C((\varepsilon,T-\varepsilon)\times\Gamma_0)} + \|u\|_{C((\varepsilon,T-\varepsilon)\times\Gamma_0)}).$$

Then, we can apply the first part of Theorem 1 and we get that there exists $\alpha > 0$ and $C > 0$

$$m\|q_2 - q_1\|_{C(K)} \leq C(N_0) \frac{CM}{\left(\log\left(\dfrac{CM}{G}\right)\right)^\alpha}$$

where M is defined by (3) and G is defined by (4). According to Proposition 6, applied with $u_0 = 0$, $g = 0$ and $h = u_1(q_2 - q_1)$, we have

$$M \leq C(\|u_1(q_2 - q_1)\|_{H^2(0,T;L^2(\Gamma_0))} + \|u_1(q_2 - q_1)\|_{H^1(0,T;H^{\frac{3}{2}}(\Gamma_0))}) \leq C(\nu_0, N_0),$$

according to (57). This proves Theorem 2.

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References

[1] G. Alessandrini, E. Beretta, E. Rosset, and S. Vessella. Optimal stability for inverse elliptic boundary value problems with unknown boundaries. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 29(4):755–806, 2000.

[2] G. Alessandrini, L. Rondi, E. Rosset, and S. Vessella. The stability for the Cauchy problem for elliptic equations. *Inverse Problems*, 25(12):123004, 47, 2009.

[3] G. Alessandrini and E. Sincich. Detecting nonlinear corrosion by electrostatic measurements. *Appl. Anal.*, 85(1-3):107–128, 2006.

[4] L. Baffico, C. Grandmont, and B. Maury. Multiscale modeling of the respiratory tract. *Math. Models Methods Appl. Sci.*, 20(1):59–93, 2010.

[5] A. Ballerini. Stable determination of an immersed body in a stationary Stokes fluid. *Inverse Problems*, 26(12):125015, 25, 2010.

[6] M. Bellassoued, J. Cheng, and M. Choulli. Stability estimate for an inverse boundary coefficient problem in thermal imaging. *J. Math. Anal. Appl.*, 343(1):328–336, 2008.

[7] M. Bellassoued, M. Choulli, and A. Jbalia. Stability of the determination of the surface impedance of an obstacle from the scattering amplitude. *Math. Methods Appl. Sci.*, 36(18):2429–2448, 2013.

[8] M. Boulakia, A. Egloffe, and C. Grandmont. Stability estimates for a Robin coefficient in the two-dimensional Stokes system. *Math. Control Relat. Fields*, 3(1):21–49, 2013.

[9] M. Boulakia, A. Egloffe, and C. Grandmont. Stability estimates for the unique continuation property of the Stokes system and for an inverse boundary coefficient problem. *Inverse Problems*, 29(11):115001, 21, 2013.

[10] L. Bourgeois and J. Dardé. About stability and regularization of ill-posed elliptic Cauchy problems: the case of Lipschitz domains. *Appl. Anal.*, 89(11):1745–1768, 2010.

[11] F. Boyer and P. Fabrie. *Éléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles*, volume 52 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2006.

[12] S. Chaabane, I. Fellah, M. Jaoua, and J. Leblond. Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems. *Inverse Problems*, 20(1):47–59, 2004.

[13] A.-C. Egloffe. *Étude de quelques problèmes inverses pour le système de Stokes. Application aux poumons.* PhD thesis, Université Paris VI, 2012.

[14] C. Fabre and G. Lebeau. Prolongement unique des solutions de l'equation de Stokes. *Comm. Partial Differential Equations*, 21(3-4):573–596, 1996.

[15] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[16] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[17] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[18] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985. Pseudodifferential operators.

[19] M. Ignatova and I. Kukavica. Strong unique continuation for the Navier-Stokes equation with non-analytic forcing. *J. Dynam. Differential Equations*, 25(1):1–15, 2013.

[20] J. Le Rousseau and G. Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Control Optim. Calc. Var.*, 18(3):712–747, 2012.

[21] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.

[22] C.-L. Lin, G. Uhlmann, and J.-N. Wang. Optimal three-ball inequalities and quantitative uniqueness for the Stokes system. *Discrete Contin. Dyn. Syst.*, 28(3):1273–1290, 2010.

[23] K.-D. Phung. Remarques sur l'observabilité pour l'équation de laplace. *ESAIM: Control, Optimisation and Calculus of Variations*, 9:621–635, 2003.

[24] A. Quarteroni and A. Veneziani. Analysis of a geometrical multiscale model based on the coupling of ODEs and PDEs for blood flow simulations. *Multiscale Model. Simul.*, 1(2):173–195 (electronic), 2003.

[25] L. Robbiano. Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques. *Comm. Partial Differential Equations*, 16(4-5):789–800, 1991.

[26] I. E. Vignon-Clementel, C. A. Figueroa, K. E. Jansen, and C. A. Taylor. Outflow boundary conditions for three-dimensional finite element modeling of blood flow and pressure in arteries. *Comput. Methods Appl. Mech. Engrg.*, 195(29-32):3776–3796, 2006.

[27] M. Yamamoto. Carleman estimates for parabolic equations and applications. *Inverse Problems*, 2009.