Public Announcements, Topology and Paraconsistency

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Abstract

In this paper, we discuss public announcement logic in topological context. Then, as an interesting application, we consider public announcement logic in a paraconsistent topological model.

Key words Public announcement logic, topological semantics, paraconsistent logic.

1 Motivation

Public announcement logic is generally discussed within the framework of classical logic using Kripke semantics. In this case, there are two issues that need to be addressed. First, what is the role of Kripke semantics on the logic? Is there any advantages or disadvantages of using it? Second, what changes if we adopt non-classical logics, particularly paraconsistent logic, as the underlying logical framework? Would we need non-classical logics to express some epistemic phenomena?

Public announcement logic (PAL, henceforth), as a dynamic epistemic logic, describes how an epistemic model is updated after a truthful and external announcement (Plaza, 1989). The semantics of PAL relies on possible worlds. We can define possible worlds as the states that never satisfies the contradiction ($\perp$). Then, *impossible worlds* are the worlds that *satisfies the contradiction*.

This paper deals with how public announcement logic works in a model with impossible worlds by using a semantical structure different than Kripke semantics.

A motivating informal example might be in order.

Example 1.1. Consider a situation where an agent *knows* an inconsistent theory such as the naive set theory. Clearly, this is not the knowledge we are familiar from epistemic normal modal logic. The epistemic attitudes that create inconsistencies can easily be represented in a model with impossible worlds. Now, assume that the agent/knower receives a public announcement - a newly discovered theorem in the naive set theory. In this case, the agent will update his model keeping some of the inconsistencies perhaps as they may not be related to the new theorem. Therefore, public announcement logic needs to be extended to express the updates in *inconsistent theories*.

More examples on impossible worlds can be found in (Barwise, 1997; Nolan, 1997).

An analysis of such situations where the external real universe is conceived of including inconsistent worlds, requires a framework that can describe dynamic epistemic actions in paraconsistent models. This is one of our goals in this work.

Also, we mentioned that PAL heavily and traditionally relies on Kripkean semantics. But, this choice is arbitrary and is exercised for rather non-logical reasons: Kripkean semantics is well-known and well-studied, easy to depict visually and simple to grasp. However, the oldest semantics for modal languages is topological semantics (van Benthem & Bezhanishvili, 2007; McKinsey, 1945; Goldblatt, 1975). Apart from its historical significance, the topological semantics relies on topology which is an even better studied field in exact sciences, which brings along its own methodology and tools. Also, recently, there has been an increasing interest towards topological semantics in modal logic (Artemov *et al.*, 1997; Awodey & Kishida, 2008; Başkent, 2013; van Benthem *et al.*, 2006; Bezhanishvili *et al.*, 2005; Dabrowski *et al.*, 1996; Georgatos, 1994; Konev *et al.*, 2006; Kremer & Mints, 2005).

The connection between PAL and topological semantics, however, was established only quite recently. In an earlier paper, we gave a topological semantics for PAL, proved the immediate completeness and decidability results (Başkent, 2012a). PAL with topological semantics distinguishes itself from the standard Kripke models for PAL. For example, public announcements may stabilize in more than ω steps in topological models. Moreover, under the assumption of rationality, the backward induction procedure can take more than ω steps (ibid). These results do not hold in PAL with Kripkean semantics. In other words, in PAL with Kripke semantics, announcement stabilize less than ω step, and the backward induction procedure take less than ω step (van Benthem & Gheerbrant, 2010). Therefore, it would not be wrong to suggest that topological semantics presents itself as a more suitable formalism to deal with infinatary cases.

Another advantage of topological semantics is that it can easily carry over to non-classical logics, including intuitionistic and paraconsistent systems. In this work, we take another step forward and investigate the relation between topology, public announcements and inconsistency-friendly logics, particularly paraconsistent logic. By paraconsistent logic, we mean the logical systems in which the explosion principle (which says that from a contradiction, everything follows) fails. Therefore, in paraconsistent systems, there are some formulas that do *not* follow from a contradiction. Paraconsistent logics help us build inconsistent but non-trivial theories.

Paraconsistent logic is an active research field with various applications in philosophy, logic and computer science (da Costa *et al.*, 2007; Grant & Subrahmanian, 2000; Priest, 2002; Rahman & Carnielli, 2000; Weber, 119). However, the literature on paraconsistency so far has not addressed the dynamic epistemic concerns adequately to the best of our knowledge. Priest in his work, focused on paraconsistent belief revision (Priest, 2001), Villadsen discussed it from a database theory point of view (Villadsen, 2002). Our research fits in the current discussions on dynamic epistemologies and supplements the work on public announcement logics as there does not seem many work discussing the connections between dynamic epistemologies and non-classical logics (van Benthem, 2009; Tennant, 1979).

The logical and epistemic motivation for our approach comes from the examples we discuss in Section 3.4. Briefly, paraconsistent public announcements enable us to formalize epistemic updates in *impossible worlds* and in *paradoxical situations*, and more importantly, epistemic aspects of such events and ontologies call for an inconsistency friendly dynamic epistemic logical framework. This is what we discuss in this article.

2 Public Announcement Logic

As we underlined earlier, our focus on this work is paraconsistent PAL with topological semantics. But, first, we introduce classical PAL with topological semantics.

Let us start with basic definitions within the framework of topological semantics for *the classical* modal logic to make this work more self contained. Given a set S, a topology σ is a collection of subsets of S satisfying the following conditions.

- The empty set and S are in σ,
- The collection σ is closed under finite intersection and arbitrary unions.

We call the tuple (S, σ) a topological space. The members of the topology is called *opens*. Complement of an open set (with respect to the classical set theoretical complement) is called a *closed set*. However, it is also possible to consider the dual of this definition and construct a topology with closed sets where each set in σ is closed. We call a topological space where the members of the topology are closed sets as a *closed set topology*.

Let us now define the basic concepts of our framework. Let $M = (S, \sigma, v)$ be a topological model where (S, σ) is a topology and v is a valuation function assigning subsets of S to propositional variables. Denote the extension of φ in a model M with $|\varphi|^M$, and define it as follows $|\varphi|^M = \{s \in S : s, M \models \varphi\}$. When it is obvious, we will drop the superscript which denotes the model. Then, for an announcement φ, we define the *updated* model $M'_\varphi = (S', \sigma', v')$ as follows. Set $S' = S \cap |\varphi|$, $\sigma' = \{O \cap S' : O \in \sigma\}$, and $v' = v \cap S'$. When no confusion arises, we will drop the subscript and simply write M' for the updated model. Notice that the new topology σ', which we obtained by relativizing σ, is a familiar one, and is called the *induced topology*. The language of topological PAL includes the epistemic modality K and the public announcement modality $[\cdot]$, and they are defined in the standard way. We denote the dual of K as L. It is also straightforward to introduce multi-agent version of topological semantics, yet we will keep our focus on single-agent case for simplicity.

In a topology, for a given set, we define the *interior* operator Int and the *closure* operator Clo as the operators which return the largest open set contained in the given set, and the smallest closed set containing the given set respectively. The extensions of modal/epistemic formulas depend on such operators. We put $|\mathsf{K}\varphi| = \mathsf{Int}(|\varphi|)$. Dually, we have $|\mathsf{L}\varphi| = \mathsf{Clo}(|\varphi|)$. Intuitively, extension of a modal formula is the interior (or the closure) of the extension of the formula. Notice that in the classical case, epistemic modal operators necessarily produce topological entities. However, it is not necessary that $|p|$ for a proposition p will be open or closed. Also, it does not follow from this definition that to each topological object O in σ in a model M, there corresponds to a formula φ in the language such that $|\varphi|^M = O$. The topology (in the topological model) can be bigger than the extensions of the modal formulas.

The semantics of propositional variables and Booleans are standard within this context of classical modal logic. Let us give the semantics of the modalities here.

$$
\begin{array}{lll}
w, M \models \mathsf{K}\varphi & \text{iff} & \exists O \in \sigma.(w \in O \wedge \forall w' \in O, w', M \models \varphi) \\
w, M \models [\varphi]\psi & \text{iff} & w, M \models \varphi \ \text{ implies } \ w, M' \models \psi
\end{array}
$$

The axiomatization of the topological PAL does not differ from the traditional PAL with Kripke semantics. The axioms of topological PAL are given as follows (Başkent, 2012a).

1. All the substitutional instances of the tautologies of the *classical* propositional logic
2. $\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$
3. $\mathsf{K}\varphi \to \varphi$
4. $\mathsf{K}\varphi \to \mathsf{K}\mathsf{K}\varphi$
5. $\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi$
6. $[\varphi]p \leftrightarrow (\varphi \to p)$
7. $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
8. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
9. $[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \to \mathsf{K}[\varphi]\psi)$

The rules of deduction in topological PAL are as expected: normalization and modus ponens. Based on this axiomatization and the semantics, we observe the following.

Theorem 2.1 ((Başkent, 2012a)). *PAL in topological models is complete and decidable.*

Notice that, in this paper, for simplicity we only consider the single-agent case. Multi-agent PAL with topological semantics is not entirely straight-forward as it refers to various methods to combine topological spaces.

3 Paraconsistent Public Announcements

3.1 Motivation

Use of topological semantics enriches the theory of public announcement logic by introducing various topological tools. Topological tools (such as homeomorphisms and homotopies) provide us with a unifying framework for classical and non-classical logics (Başkent, 2012b; Başkent, 2013). For that reason, topological semantics make it possible and easier to give semantics for various non-classical logics, and provide a broader framework to reason about topological and epistemic notions (ibid). In this section, we will focus on paraconsistent public announcement logics. First, we will mention the logical (and topological) motivations for our approach, and then supplement it with various philosophical and epistemic justifications for the study of public announcement logic in non-classical frameworks.

Let us now assume that we have a closed set topology where the we take closed sets as the objects in the topological space, and stipulate further that the extension of propositional variables are also closed sets. Notice that when we discussed the (classical) topological semantics, we underlined that the topological operators are introduced when the semantics of the *modal* formulas are considered. The topological semantics for the classical (modal) logic, as it stands, does not impose any condition on the topological qualities on the extension of propositional variables. However, the imposition that forces the extensions of propositional variables to be closed sets makes an important difference for negation as the compliment of a closed set is not necessarily a closed set. For that reason, we cannot use the standard definition of negation as the set theoretical complement on the extension of the formula. As a consequence, we need to redefine the semantics of negation carefully in this framework. We define it as the "closure of the complement" (Goodman, 1981; Mortensen, 2000; Başkent, 2013). In this case, boundary points, the points that are shared by the closure of a given set and the closure of its complement, are the points that satisfy the contradictions. Let us denote this paraconsistent negation by $-$.

Notice that in this way, it is also possible to obtain the standard topological semantics for the intuitionistic logic. In this case, we work with an open set topology and further impose that the extension of propositional variables will be open sets. Similarly, then, the negation will be defined as the "interior of the complement" (Mints, 2000). In this case, the boundary points, again, will be the points that have no truth values. The reason for that is very similar: the boundary is not a part of an open set, and it is also not a part of the interior of its complement. Therefore, neither a formula nor its negation can be satisfied at boundary points. In the light of these observations, our approach can be considered as a dual-intuitionistic method to obtain paraconsistent models.

Let us give a simple example to illustrate our point. Take the formula $p \wedge -p$. Call the extension of p as $K \in \sigma$ where σ is a closed set topology, K is a closed set. Then the extension of $p \wedge -p$ is $K \cap \mathsf{Clo}(\overline{K})$ which is $\partial(K)$ where $\partial(\cdot)$ is the boundary operator which is defined as $\partial K := \mathsf{Clo}(K) - \mathsf{Int}(K)$. Therefore, the contradictions hold on the boundary points. Thus, we now have a paraconsistent logic in which contradictions do not trivialize the system. Recall that paraconsistent logic is an umbrella term for the logical systems where inconsistencies do not trivialize the logic. In other words, in paraconsistent logics, the law of explosion fails. The reason why explosion fails is because for some formula φ, the extension of $\varphi \wedge -\varphi$ is not necessarily an empty set, but it is $\partial(K)$ for some set K. Thus, it is not necessarily a subset of every set, so not every formula follows from a contradiction in this system. There exist a wide variety of paraconsistent logics suggested for different philosophical and technical reasons. We refer the reader to the following references for a more comprehensive and up-to-date overview of paraconsistency (Priest, 2002; Priest, 2007; da Costa *et al.*, 2007).

However, we need to elaborate a bit more on the philosophical meaning of the use of paraconsistent spaces in the context of public announcement logic. PAL heavily depends on the law of non-contradiction. Recall now how PAL operates. An external and truthful announcement is made. Then, the agents update their epistemic models by eliminating the states in their model which do *not* agree with the announcement. Afterwards, accordingly, the epistemic relation and the valuation are also relativized. Therefore, the classical PAL does not *control* the inconsistencies, it completely eliminates them. Yet, in paraconsistent spaces, some contradictions need not be eliminated as they do not trivialize the theory. This is how we control inconsistencies in paraconsistent PAL.

In paraconsistent spaces, public announcements obtain a broader meaning. Namely, when φ is announced in a paraconsistent space, it simply means “Keep φ”. It can very well be the case that some of the possible worlds that satisfy φ may also satisfy $-\varphi$, namely, those states may be impossible worlds. Clearly, this stems from the fact that negation $-$ in paraconsistent PAL is not classical, thus the methods of “eliminating the states that do not satisfy the announcement” and “keeping the states that satisfy the announcement” are not identical, unlike in classical logic. This distinction surfaces very clearly in paraconsistent PAL.

In short, we provide a broader reading of PAL. PAL eliminates the states that contradict the announcement not just because they create an inconsistency. The main problem caused by the inconsistencies is, of course, that they trivialize the theory and collapse the model. Therefore, if there exists some contradictions that do not trivialize the theory, there seems to be no need to eliminate them. Yet, on the other hand, we only keep the states that agree with the announcement - by definition, even if some of such states may satisfy some other propositions, including the negation of the announcement. This is our pivotal point for paraconsistent PAL.

Here, notice that we do not focus on inconsistent announcements *per se*. Our framework is more radical, and allows inconsistent possible worlds (or *impossible* worlds). Moreover, we also follow the standard "state elimination based" paradigm for PAL. Model theoretically, we can also eliminate the accessibility relation arrows and keep the states. In a topological setting, this would amount to reducing the topology to an induced topology and keeping the space as it was initially given. From modal logical perspective, there seems to be no model theoretical difference (Kooi & Renne, 2011).

3.2 Model

Let us now give a precise meaning to public announcements in paraconsistent framework. First, we define the updated model M' after the announcement the same way. There could exist, however, some other ways to define the model after the announcements. Namely, one may wish to exclude the states that do not agree with the announcement from the space. This is the way it is done in the classical PAL.

Let $M = (S, \sigma, v)$ be a topological model where (S, σ) is a closed set topology where every $K \in \sigma$ is a closed set. In order to make the formal matters of this work more self-contained, let us spell out the updated model in ParaPAL after an announcement φ. For an announcement φ, we obtain an *updated* model $M'_\varphi = (S', \sigma', v')$ where $S' = S \cap |\varphi|^M$, $\sigma' = \{K \cap S' : K \in \sigma\}$, and $v' = v \cap S'$. We will remove the subscript when it is clear from the context. As mentioned before, we stipulate that the extension of each propositional variable is closed. The intention here is to impose that the extension of each *formula* must be a closed set as closedness is preserved with the logical connectives in this framework.

Now, define $M^-_\varphi := (S^-, \sigma^-, v^-)$ as the model obtained after the announcement of φ where $S^- = S \setminus |-\varphi|$, $\sigma^- = \{O \cap S^- : O \in \sigma\}$, $v^- = v \cap S^-$. We will call M^-_φ the *reduced model*. Provided no confusion arises, we will drop the subscript. For a given model M, and a formula φ, we have $M^-_\varphi = M'_\varphi$ in classical PAL for all models and formulas. But, in paraconsistent space, the reduced model is a subset of the updated model. If we define an intuitionistic variant of PAL, then the updated model would be a subset of the reduced model. We combine our observations in the following lemma.

Lemma 3.1. *In classical PAL, for a model M, the updated model M'_φ, and the reduced model M^-_φ are identical for any formula φ. In paraconsistent PAL, $M^-_\varphi \subseteq M'_\varphi$.*

Proof. Follows immediately. ■

Therefore, the traditional way to obtain updated model returns rather big models in paraconsistent systems. This is due to the impossible worlds.

Let us now present the formal aspects of paraconsistent public announcement logic, which we will call ParaPAL for short. We define the syntax of Para-

PAL as follows where p is a propositional variable, and $-$ is a negation symbol.

$$p \mid -\varphi \mid \varphi \wedge \varphi \mid \mathsf{K}_i\varphi \mid [\varphi]\varphi$$

As expected, K_i is the knowledge operator for an agent i, and $[\varphi]$ denotes the public announcement of φ. We define disjunction and implication in the usual way as abbriviations. The dual operator L_i is defined as expected: $\mathsf{L}_i p := -\mathsf{K}_i - p$. We will call the logic obtained from the *static* modal part (i.e. without the public announcement operator but with the epistemic operator) of this language *Paraconsistent Topological Logic* (PTL, for short). In PTL, the semantics of the modal operator and the Booleans is given topologically in a similar way. The language or ParaPAL (and PTL) is given for a multi-agent setting. Yet, for simplicity both in notation and exposition, we will consider the single-agent version in this paper.

Let us now give the semantics of the modalities here. Note that in ParaPAL, we have $|-p| = \mathsf{Clo}(S \setminus |p|)$. In this respect, the semantics for propositional variables and Booleans are as usual. Let us reinstate the semantics of the modal and dynamic operators for a given ParaPAL model $M = (S, \sigma, v)$ where σ is a closed set topology.

$$\begin{array}{lll} |-\varphi|^M & = & \mathsf{Clo}(S \setminus |\varphi|^M) \\ w, M \models \mathsf{K}\varphi & \text{iff} & \exists K \in \sigma.(w \in K \wedge \forall w' \in K : w', M \models \varphi) \\ w, M \models [\varphi]\psi & \text{iff} & w, M \models \varphi \quad \text{implies} \quad w, M' \models \psi \end{array}$$

In ParaPAL, after the announcement, the updated model will keep the states that satisfy the announcement. However, some of such states may also satisfy the *negation* of the announcements. This reflects the basic dictum of paraconsistent logic. Paraconsistent logic distinguishes two different types of *true*s and *false*s. The trues that are only true and the trues that are also false; and similarly falses that are only false and the falses that are also true (Priest, 1979). This intuition is reflected on our distinction of possible and impossible worlds within the framework of PAL. Thus, ParaPAL presents a fine tuning for truth in dynamic epistemic contexts.

Notice that in ParaPAL, since the extension of *each* propositional variable is a closed set, we have $\mathsf{L}p \leftrightarrow p$. Therefore, if we stipulate that the extension of each and every propositional variable is a closed set, then we reduce the unimodal epistemic logic to propositional logic. This is a straight-forward results which is transferred from PTL. If the extension of each formula is a closed set already, its extension under the epistemic modal operator L, for instance, will take the closure of the extension of the given formula. But, the closure of a closed set is already itself, therefore, the modal operator will not change the extension of a given formula. Nevertheless, for expressivity purposes, we will keep the modal operators. This is a design decision similar to the classical PAL where the public announcement operator is not more expressive, yet provides succinctness (Kooi, 2007). Nevertheless, we observe the following.

Lemma 3.2. *ParaPAL and PTL are equi-expressible.*

Yet, when compared to the classical PAL, ParaPAL provides a more expressive framework as some contradictions can be true in some models.

Lemma 3.3. *ParaPAL is more expressive than PAL.*

In ParaPAL, we can have true statements such as $[p](q \wedge -q)$ or $[\top]\bot$ as we discussed earlier.

Before proceeding further, we need to make sure that the updated topology in ParaPAL is indeed a topology.

Lemma 3.4. *Given a closed set topology (S, σ). Then, for any formula φ with a closed set extension, the updated space (S', σ') where $S' = S \cap |\varphi|$ and $\sigma' = \{C \cap S' : C \in \sigma\}$ is also a topological space.*

Proof. The tuple (S', σ') is called an induced topology, and indeed a topological space. ■

We now define the public announcements in the usual way in ParaPAL.

$$w, M \models [\varphi]\psi \text{ iff } w, M \models \varphi \text{ implies } w, M'_{\varphi} \models \psi$$

3.3 Reduction Axioms

Let us see whether the standard reduction axioms of public announcements works in ParaPAL. This is not straight-forward as we are now in inconsistency-friendly logical systems.

Consider the axiom $[\varphi]p \leftrightarrow (\varphi \to p)$ on a ParaPAL model $M = (S, \sigma, v)$ where $w \in S$, and p is a propositional variable. Suppose further that $w, M \models \varphi$.

$$\begin{array}{lll} w, M \models [\varphi]p & \text{iff} & w, M' \models p \\ & \text{iff} & w, M \models p \\ & & \text{as } w, M \models \varphi \text{ is assumed,} \\ & \text{iff} & w, M \models (\varphi \to p) \end{array}$$

Notice that the above result simply depends on the fact that the valuation of the propositional variables are independent from the topology.

ParaPAL presents a new negation. Thus, it is more important now to consider the reduction axiom for negation: $[\varphi]-\psi \leftrightarrow (\varphi \to -[\varphi]\psi)$. Similarly, take a ParaPAL model $M = (S, \sigma, v)$ where $w \in S$, and p is a propositional variable. Suppose further that $w, M \models \varphi$.

$$\begin{array}{lll} w, M \models [\varphi]-\psi & \text{iff} & w, M' \models -\psi \\ & \text{iff} & w \in \mathsf{Clo}(S' \setminus |\psi|) \\ & \text{iff} & w \in \mathsf{Clo}((S \cap |\varphi|) \setminus |\psi|) \\ & & \text{as } w \in |\varphi| \text{ is assumed,} \\ & \text{iff} & w \in \mathsf{Clo}(S \setminus (|\varphi| \cap |psi|)) \\ & \text{iff} & w, M \models -[\varphi]\psi \end{array}$$

As we already pointed out, the reduction axioms for the epistemic modal operator holds vacuously. Thus, we obtain the following result which follows directly.

Theorem 3.5. *ParaPAL reduces to PTL by the following reduction axioms:*

- $[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
- $[\varphi]-\psi \leftrightarrow (\varphi \rightarrow -[\varphi]\psi)$
- $[\varphi]\psi \wedge [\varphi]\chi \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
- $[\varphi]\mathsf{K}\psi \leftrightarrow (\varphi \rightarrow \mathsf{K}[\varphi]\psi)$

Proof. We already showed the soundness of the first two axioms. The third one on conjunction follows immediately, and the fourth one on the epistemic modality follows almost trivially as the epistemic modality becomes redundant in ParaPAL and PTL due to the properties of the closure operator (Başkent, 2013). ■

3.4 Some Examples and Further Results

We motivated ParaPAL from a topological perspective which may seem rather technical. However, this does not mean that ParaPAL lacks philosophical and epistemological motivations. As paradoxes and inconsistencies have always been interesting for philosophers and logicians, we now offer various examples that justify the introduction of our system. In the following examples, we will only offer a conceptual analysis.

Example 3.6. An interesting conceptual and philosophical application involves *impossible worlds* as we already mentioned. Define the set of impossible worlds as $X = \{x : x \models \varphi \wedge -\varphi \text{ for some } \varphi\}$.The states that are not impossible then will be the possible worlds. A way to conceptualize impossible worlds would be to consider things that do not exist. Such things are, but they do not exist. An alternative way to imagine this would be to consider a mathematical system where Cantor's paradox (about the incommensurability of natural numbers and reals) is a *theorem*.

In ParaPAL, after the announcement of $[\varphi]$, we eliminate the possible worlds that satisfy $-\varphi$, but keep the impossible worlds that satisfy $\varphi \wedge -\varphi$.

This is interesting. In ParaPAL (and in PTL), we have way to conceptualize and formalize impossible worlds, which is the boundary. Boundary of a set K, denoted by $\partial(K)$, is defined as the difference of $\mathsf{Clo}(K) - \mathsf{Int}(K)$. Therefore, the set of impossible worlds X is a subset of boundary points in the topological model. In short, given a topological model (S, σ, v), $x \in X$ implies that $x \in \partial(O)$, for some set $O \in \sigma$ where X is the set of impossible worlds.

Then, what about the converse. Given a point $w \in \partial(O)$, can we claim that $w \in X$? The answer to this question is a "No". The reason is that the topology σ in the given model may contain sets that are not extensions of any formula. Namely, for a given formula in the language of ParaPAL, we have a topological object in the given model that corresponds to it. Yet, this does not entail, by itself, that for every topological object in the given model, there exists a formula in the language. Therefore, set of impossible worlds is not necessarily identical to the set of boundary points in ParaPAL.[1]

Therefore, conceptually, ParaPAL offers a way to formalize and normalize impossible worlds. Even if it goes beyond the limits and the scope of this paper, this approach gives rise to the possibility of studying *counterfactuals* in this setting. $\boxtimes$

Law and norms provide various philosophical motivations to introduce paraconsistency. In the following example, we approach the subject from a dynamic epistemic paradigm.

Example 3.7. We now give an example from law.

> Suppose that there is a certain country which has a constitutional parliamentary system of government. And suppose that its constitution contains the following clauses. In a parliamentary election:
>
> (1) no person of the female sex shall have the right to vote;
>
> (2) all property holders shall have the right to vote.
>
> (Priest, 2006, p. 184)

Let us denote the above rules as public announcements φ_1, φ_2 respectively. This is far from being unrealistic. Ideally, when new laws are made, they are introduced to both the society and the legal system which are then *updated* accordingly reflecting the new laws and what they have brought about. The law must be truthful, it must be communicated publicly, and the legal system and the society must follow it, at least in theory.

Therefore, when the Law (1) was introduced, we can consider it as $[\varphi_1]$, and similarly Law (2) as $[\varphi_2]$. The Laws(1) and (2) are contradictory, so are their announcement. For simplicity, consider the simultaneous announcement of $[\varphi_1 \wedge \varphi_2]$.

When $[\varphi_1 \wedge \varphi_2]$ is announced, the states that satisfy the contradictory statement will be kept - which is the set of propositions about female property holders, in this example. This announcement does not trivialize the model in ParaPAL. $\boxtimes$

Even if we stick to rather conceptual examples in this work, let us now discuss an interesting application of ParaPAL by using various topological notions. Let us first give the definitions we need. A function defined on a topological space is called *continuous* if the inverse image of an open is an open; *open* if the

[1]We can *generate* a submodel of a given model that only contain points and sets that are extensions of some subformula. In other words, by removing the *unnecessary* points and sets from the points, we can identify the set of impossible worlds and boundary points.

image of an open is an open. A function is called *homeomorphism* if it is a continuous function between topological spaces with a continuous inverse. Notice that these definitions can easily be dualized and be stated by using closed sets.

Now we can discuss the following case an interesting application of PAL in topological semantics.

Let us define *functional representation* as follows.

Definition 3.8. Given a formula φ, and a model $M = (S, \sigma, v)$, we call φ "functionally representable in M" if there is an open and continuous function $f^M_\varphi : (S, \sigma) \mapsto (S', \sigma')$ where $M' = (S', \sigma', v')$ is the updated model which is obtained after the public announcment of φ.

The idea behind such a definition is that functional representation will give us more tools in topologies.

Proposition 3.9. *Every public announcement is functionally representable.*

Proof. The proof is rather immediate. Given $M = (S, \sigma, v)$, construct $M' = (S', \sigma', v')$ with respect to the public announcement φ. Then, for every open $O \in \sigma$ in M, assign $f(O) = O'$ where $O' = O \cap S'$ in σ' in M'. Here, notice that O' can be the empty set for some $O \in \sigma$ which is perfectly OK as f is not imposed to be an one-to-one function. We claim f functionally represents φ.

Note that modal formulas necessarily produce open (or dually closed) sets as their extensions, and they are taken care of by the given function f. However, we may still have Boolean formulas which do not have open or closed extensions in the model. However, notice that they do not violate functional representation as the definition of functional representation quantifies over open sets.

Now, since both, O and O' are open, so f is an open map. Take $U' \in \sigma'$. Since, $U' = U \cap S'$ for some $U \in \sigma$, the inverse of image of U' under f is U which is an open in σ showing that f is continuous.

Thus, we conclude that f functionally represents φ. ∎

We can take one more step and introduce *homotopies*.

Definition 3.10. Let S and S' be two topological spaces with continuous functions $f, f' : S \mapsto S'$. A homotopy between f and f' is a continuous function $H : S \times [0, 1] \mapsto S'$ such that for $s \in S$, $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$.

Homotopies help us understand the connection between updated models. Standard PAL studies the connection between given model and the updated model. However, a model can produce various updated models with various announcements. In order to reason about the connection between updated models, we need homotopies.

Proposition 3.11. *Given M, consider a family of updated homeomorphic models $\{M_i\}_{i<\omega}$ each of which is obtained by an announcement φ_i representable by f_i. Then f_is are homotopic.*

Proof. Immediate. The only seemingly unnatural condition we imposed is the homeomorphism. This amounts to the fact that each epistemic state is updated in a unique way without ending up the same. We call this epistemic condition of *uniqueness of updated epistemic states*. ■

The converse of the above statement is not always true. Clearly, not each pair updated models in a class of homotopic models are obtainable from one another by an update. Given M, consider the updated models M_1 and M_2 where the prior is obtained by an announcement of p while the latter $\neg p$. Even if there is a continuous transformation between M_1 and M_2, this transformation is not a public announcement.

4 Conclusion

Our contribution in this work provides a *fine tuning* for the announcements and how the models are updated after announcements. ParaPAL clarifies how state elimination works, and how impossible worlds can be incorporated into the model from a wider point of view. Even if there is not much work on the very same field, there seems to be some unpublished work on similar ideas by Girard and Tanaka, which surely is relevant to our narrowly focused approach here.

Public announcement logic is an interesting playground to observe how paraconsistent reasoning works epistemically. Agents in ParaPAL can reason soundly in a world of inconsistencies. Our system is based on an inconsistent universe, yet takes announcements as honest and truthful epistemic operations.

The field is rich, and there can be considered a variety of future work possibilities including the algebraic connection between paraconsistency and public announcements, and paradoxical announcements. We leave it to future work.

Another interesting direction is the relation between mereology and public announcements. Mereology is the research area that studies the connection between parts and wholes, and exhibits intriguing algebraic qualities. Therefore, the question that how the relation between parts and wholes change after a public announcement is yet another interesting research direction to pursue, especially in an inconsistent universe.

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