Genericity of weakly computable objects

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Abstract

In computability theory many results state the existence of objects that in many respects lack algorithmic structure but at the same time are effective in some sense. Friedberg and Muchnik's answer to Post's problem is one of the most celebrated results in this form. The main goal of the paper is to develop a general result that embodies a large number of these particular constructions, capturing the essential idea that is common to all of them, and expressing it in topological terms.

To do so, we introduce the effective topological notions of *irreversible function* and *directional genericity* and provide two main results that identify situations when such constructions are possible, clarifying the role of topology in many arguments from computability theory. We apply these abstract results to particular situations, illustrating their strength and deriving new results.

This paper is an extended version of the conference paper [Hoy14] with detailed proofs and new results.

Contents

1 Introduction

One of the main goals of computability theory is to understand and classify the algorithmic content of infinite objects, which can be expressed as the difficulty of computing them or as their ability to help solving problems. In establishing this classification one is often led to separate classes of algorithmic complexity and the construction of counter-examples is usually a hard task that requires the use of advanced techniques, one of the simplest being the so-called priority method with finite injury. The difficulty in carrying out such constructions is that the built object should meet two types of requirements going in opposite directions: (i) it should lack algorithmic content but at the same time (ii) it should be constructible in some way. In other words, these objects live somewhere between *generic* objects (objects with no structure) and *computable* objects (the most constructible objects). While computability theory provides formal notions of genericity, these ones are always incompatible with computability, so the two prescriptions are conflicting.

In this paper we propose an abstract view on this topic by providing general results that, when applied properly, implement the sought constructions, and provide a shortcut for producing objects with prescribed properties. We expect that checking the conditions of these theorems is much easier than writing down the complete construction which is often merely an adaptation of known techniques to new situations.

The first main result (Theorem 3.3.1) enables one to construct objects that (i) are not computable, but (ii) have a computable image under some given function. As an illustration, we prove a negative result about the computability of the ergodic decomposition. This part is developed in Section 3 and is the version with detailed proofs of the results already presented in [Hoy14].

The second main result (Theorem 4.1.1) enables one to construct objects that (i) are generic in some sense, but (ii) are computable in a weak sense. It is done by introducing a new notion of genericity, called *directional generic-*

ity, that has two advantages: (i) it is formally close to plain genericity, being defined as a slight variation of 1-genericity, and (ii) it is compatible with various weak forms of computability. For instance for subsets of $\mathbb{N}$, directional genericity is called genericity from above and we prove the existence of c.e. sets that are generic from above. Such sets are in particular solutions to Friedberg-Muchnik's theorem. Theorem 4.1.1 has important consequences:

- It subsumes and unifies many ad hoc existing constructions,
- It can be applied to obtain new results,
- It clarifies the role of topology in computability theory.

We then illustrate these notions of genericity in several contexts, showing evidence that they clarify existing results, and we apply our main theorem to derive a separation result. This part is developed in Section 4 and is a significant extension of what appears in [Hoy14].

The proofs of our main results use the priority method with finite injury. We observe that while this technique has been massively used in many situations, no result has convincingly clarified its scope and the contexts in which it can be applied, and each particular construction is yet another adaptation of the same argument, based on the same idea. We hope that the results presented here shed light on this subject.

The paper is organized as follows. In Section 2 we present the needed background on computable analysis, providing along the way a few results of independent interest on effective Polish spaces. In Section 3 we introduce the notion of irreversible function and prove the first main result: if a computable function is effectively irreversible then it maps a non-computable point to a computable image. The results of this section come from [Hoy14], we present here their detailed proofs. In Section 4 we introduce a notion of genericity. We prove the second main result of the paper: under suitable assumptions, a point exists that is at the same time "generic" and "weakly computable". We then apply this result to four classes of objects: c.e. sets, left-c.e. reals, Π^0_1-classes and regular Π^0_1-classes.

2 Background and notations

We assume familiarity with basic computability theory on the natural numbers. We implicitly use Weihrauch's notions of computability on effective topological spaces, based on the standard representation (see [Wei00] for more details), however we do not express them in terms of representations.

2.1 Notations

In a metric space (X, d), if $x \in X$ and $r \in (0, +\infty)$ then we denote the open ball with center x and radius r by $B(x, r) = \{x' \in X : d(x, x') < r\}$. We denote the corresponding closed ball by $\overline{B}(x, r) = \{x' \in X : d(x, x') \leq r\}$. The Cantor space of infinite binary sequences, or equivalently subsets of $\mathbb{N}$, is denoted by $2^{\mathbb{N}}$. The halting set, denoted $\emptyset'$, is the set of numbers of Turing machines that halt upon input of their own number. It is a noncomputable set that is computably enumerable (c.e.).

2.2 Effective topology

An ***effective topological space*** $(X, \tau, \mathcal{B})$ consists of a topological space[1] (X, τ) together with a countable basis $\mathcal{B} = \{B_0, B_1, \ldots\}$ numbered in such a way that the finite intersection operator is computable: for each i, j there exists a set $W \subseteq \mathbb{N}$ that is c.e. uniformly in (i, j), such that $B_i \cap B_j = \bigcup_{k \in W} B_k$. An open subset $U \subseteq X$ is ***effectively open*** if $U = \bigcup_{k \in W} B_k$ for some c.e. set $W \subseteq \mathbb{N}$.

To a point $x \in X$ we associate $N(x) = \{n \in \mathbb{N} : x \in B_n\}$. By an ***enumeration of*** $N(x)$ we mean a total function $f : \mathbb{N} \to \mathbb{N}$ whose range is $N(x)$. A point x is ***computable*** if $N(x)$ is c.e., i.e. if $N(x)$ has a computable enumeration.

Given points x, y in effective topological spaces X, Y respectively, we say that y is ***computable relative to*** x if there is an oracle Turing machine M that, given any enumeration of $N(x)$ as an oracle, outputs an enumeration of $N(y)$. We denote it by $M^x = y$. In other words, y is computable relative to x if $N(y)$ is enumeration reducible to $N(x)$. As proved by Selman [Sel71] and pointed out by Miller [Mil04], y is computable relative to x if and only if every enumeration of $N(x)$ computes an enumeration of $N(y)$ (uniformity is not explicitly required, but is a consequence).

A (possibly partial) function $f : X \to Y$ is ***computable*** if there is a machine M such that for every $x \in \mathrm{dom}(f)$, $M^x = f(x)$. A computable function is always continuous.

2.3 Effective Polish spaces

An ***effective Polish space*** is a topological space such that there exists a dense sequence $s_0, s_1, \ldots$ of points, called ***simple*** points and a complete metric d inducing the topology, such that all the reals numbers $d(s_i, s_j)$ are

[1] Such spaces are usually assumed to be T_0, however this assumption is not necessary.

computable uniformly in (i, j). Every effective Polish space can be made an effective topological space, taking as canonical basis the open balls $B(s, r)$ with s as a simple point and with r as a positive rational number together with a standard effective numbering. It is sometimes useful to use another basis: we say that a family $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ of open sets is an ***effective basis*** if the identity between the effective topological spaces induced by $\mathcal{B}$ and by the canonical basis is computable in the two directions. For instance,

Proposition 2.3.1. *There always exists an effective basis $\mathcal{B}$ that makes the predicate $s_i \in B_j$ computable in i, j.*

Proof. By a standard diagonalization there exists a computable dense sequence of positive real numbers r_n such that $d(s_i, s_j) \neq r_n$ for all n. Take the basis of metric balls centered at simple points with radii in the sequence r_n. $\square$

In an effective Polish space, a point x is computable if and only if for every $\epsilon > 0$ a simple point s can be computed, uniformly in ϵ, such that $d(s, x) < \epsilon$.

We present a few simple observations on effective Polish spaces which seem to be new and are of independent interest. Let X be an effective Polish space. $X' \subseteq X$ is an ***effective Polish subspace*** if it is an effective Polish space under the induced topology and such that the canonical injection from X' to X is a computable homeomorphism between X' as a space and X' as a subset of X. Alexandrov's theorem gives a way to obtain Polish subspaces of a Polish space, and has an effective version, which we present now.

A set A is an ***effective*** G_δ***-set*** if there exists a family of uniformly effective open sets U_n such that $A = \bigcap_n U_n$. The following result unifies two known results: every closed set whose hit set (see statement below) is c.e. contains a dense computable sequence; every dense effective G_δ-set contains a dense computable sequence (computable Baire Category theorem [YMT99, Bra01]).

Proposition 2.3.2. *Let A be an effective G_δ-subset of an effective Polish space. The hit set $\{i \in \mathbb{N} : B_i \cap A \neq \emptyset\}$ is c.e. if and only if A contains a computable sequence dense in A. We then say that A is* **c.e.**

Proof. We prove the non-trivial direction. Let U_n be uniformly effective open sets and $A = \bigcap_n U_n$, such that $\{i \in \mathbb{N} : B_i \cap A \neq \emptyset\}$ is c.e. Given a ball B_0 intersecting A, one can compute a sequence of balls B_n of radius $< 2^{-n}$ such that $\overline{B}_{n+1} \subseteq B_n \subseteq U_n$ ($\overline{B}_{n+1}$ is the closed ball rather than the closure of B_{n+1}). The intersection $\bigcap_n B_n$ contains one point that is computable and belongs to A. The induction hypothesis on B_n is that it intersects A. Once B_n has been built, there must be some ball B of radius $< 2^{-n-1}$ such that $\overline{B} \subseteq B_n \cap U_n$ and B intersects A, and such a B can be effectively computed. Let $B_{n+1} = B$. To obtain a dense computable sequence, start from any ball B_0 intersecting A. □

We will often use the same technic to build a point, by means of a ***shrinking*** sequence of balls. A sequence B_n of balls is shrinking if the radius of B_n converges to 0 and $\overline{B}_{n+1} \subseteq B_n$. By completeness of the metric, $\bigcap_n B_n$ always contains a point.

Proposition 2.3.3 (Effective Alexandrov Theorem)**.** *Let X be an effective Polish space. Every c.e. effective G_δ-set is an effective Polish subspace of X.*

Proof. The proof is a simple effective version of the proof that can be found in [HY61] (Theorem 2-76 therein). Let $A = \bigcap_n U_n$ be a c.e. effective G_δ-set. Let d be a complete computable metric on X. Let $d_n : X \to [0, +\infty)$ be uniformly computable functions such that $d_n(x) > 0 \iff x \in U_n$. One can think of d_n as a computable version of the distance to the complement of U_n. If $U_n = \bigcup_i B(x_i, r_i)$ then one can define d_n as $d_n(x) = \sum_i 2^{-i} \max(1 - d(x, x_i)/r_i, 0)$. For $x \in U_n$, let $f_n(x) = \frac{1}{d_n(x)}$. For $x, y \in A$, let

$$d'(x, y) = d(x, y) + \sum_n 2^{-n} \frac{|f_n(x) - f_n(y)|}{1 + |f_n(x) - f_n(y)|}.$$

The function $d' : A \times A \to \mathbb{R}$ is computable. On A it is a complete metric that induces the same topology as d. The computable sequence which is dense in A can serve as special points in A. The canonical injection is 1-Lipschitz, hence has a computable modulus of uniform continuity and is computable on the special points of A, so it is computable. □

The statement in the theorem can actually be extended to an equivalence.

We will be concerned with computability and Baire category, so we will naturally meet the notion of a 1-generic point: a point that does not belong to any "effectively meager set" in the following sense.

Definition 2.1. $x \in X$ is 1-***generic*** if x does not belong to the boundary of any effective open set. In other words, for every effective open set U, either $x \in U$ or there exists a neighborhood B of x disjoint from U.

By the Baire category theorem, every Polish space is a Baire space so 1-generic points exist and form a co-meager set. One can relativize this notion by considering all the open sets that are effective relative to a given oracle.

A weaker notion of genericity is useful.

Definition 2.2. $x \in X$ is ***weakly* 1*-generic*** if x belongs to every dense effective open set.

3 Irreversible functions

3.1 A non-uniform result

Let X be an effective Polish space, Y an effective topological space and $f : X \to Y$ a (total) computable function.

To introduce the results of this section informally, assume temporarily that f is one-to-one. If f^{-1} is computable, i.e. if every x is computable relative to $f(x)$ *uniformly* in $f(x)$, then f^{-1} is continuous. As mentioned earlier uniformity is crucial here: that some x is computable relative to $f(x)$ does not imply in general that f^{-1} is continuous at $f(x)$. Theorem 3.1.1 below surprisingly shows that a non-uniform version can still be obtained, valid at most points.

Let us now make it precise and formal. We do not assume anymore that f is one-to-one.

When focusing on the problem of inverting a function, one comes naturally to the following basic notions:

- f is *invertible* at x if x is the only pre-image of $f(x)$,
- f is *locally invertible* at x if x is isolated in the pre-image of $f(x)$.

If one has access to x via its image only, then x is determined unambiguously in the first case, with the help of a discrete advice (a basic open set isolating x) in the second case. However, "being uniquely determined" is not sufficient in practice: physically or computationally, one cannot entirely know $f(x)$ in one step, but progressively as a limit of finite approximations. We need to consider stronger, topological versions of the two basic notions of invertibility, expressing that x can be recovered from the knowledge of its image given by finer and finer neighborhoods.

Definition 3.1. Let $f : X \to Y$ be a function.

We say that f is ***continuously invertible at*** x if the pre-images of the neighborhoods of $f(x)$ form a neighborhood basis of x, i.e. for every neighborhood U of x there exists a neighborhood V of $f(x)$ such that $f^{-1}(V) \subseteq U$.

We say that f is ***locally continuously invertible at*** x if there exists a neighborhood B of x such that the restriction of f to B is continuously invertible at x, i.e. for every neighborhood U of x there exists a neighborhood V of $f(x)$ such that $B \cap f^{-1}(V) \subseteq U$.

Observe that these notions are very natural when investigating the problem of inverting a function: we think that they are not technical *ad hoc* conditions.

Every effective topological space Y has a countable basis hence is sequential, i.e. continuity notions can be expressed in terms of sequences, which may be more intuitive. We will be particularly interested in the negations of these notions, which we characterize now, using any metric d generating the topology.

Proposition 3.1.1. *f is not continuously invertible at x if and only if there exists $\delta > 0$ and a sequence x_n such that $d(x, x_n) > \delta$ and $f(x_n)$ converges to $f(x)$.*

f is not locally continuously invertible at x if and only if for every $\epsilon > 0$ there exists $\delta > 0$ and a sequence x_n such that $\epsilon > d(x, x_n) > \delta$ and $f(x_n)$ converges to $f(x)$.

We now come to our first result.

Theorem 3.1.1 (Computability implies continuity, pointwise)**.** *Let $f : X \to Y$ be a computable function and $x \in X$ a 1-generic point.*

If x is computable relative to $f(x)$ then f is locally continuously invertible at x.

Proof. Assume that x is computable relative to $f(x)$ and f is not locally continuously invertible at x. We show that x belongs to the boundary of an effective open set U, i.e. that x is not 1-generic.

Intuitively, for a point y, there are two possible ways in which a Turing machine M may fail to compute y from $f(y)$: either it diverges, or it outputs something that is incompatible with y. The latter can be recognized in finite time: we then say that $M^{f(y)}$ *positively* fails to compute y. Our effective open set U will be the set of points y such that $M^{f(y)}$ positively fails to compute y.

Let us make it more precise. As x is computable relative to $f(x)$, there exist uniformly effective open sets $V_n \subseteq Y$ such that for all n, $x \in B_n \iff f(x) \in V_n$, where B_n is the canonical basis induced by a complete effective metric on X. We then define

$$U = \bigcup_n f^{-1}(V_n) \setminus \overline{B}_n$$

which is an effective open set (if $B_n = B(x,r)$ then $\overline{B}_n = \overline{B}(x,r)$ is the corresponding closed ball). By definition of V_n, x does not belong to U.

Let us show that x belongs to the closure of U. Let B be a neighborhood of x and U_B another neighborhood coming from the fact that f is not locally continuously invertible at x. Let B_n be a neighborhood of x such that $\overline{B}_n \subseteq U_B$. The set V_n is a neighborhood of $f(x)$, so $f^{-1}(V_n)$ intersects $B \setminus U_B \subseteq B \setminus \overline{B}_n$. As a result, B intersects U. This is true for every neighborhood B of x, so x belongs to the closure of U. □

In the sequel we introduce a condition on f which roughly means that f is "almost nowhere" locally continuously invertible and that entails (i) the existence of an x that is not computable relative to $f(x)$ (Theorem 3.2.1) and, better, (ii) the existence of a non-computable x such that $f(x)$ is computable (Theorem 3.3.1).

3.2 Irreversible functions

We now consider the following notion: an *irreversible* function is locally continuously invertible at almost no point, in the sense of Baire category.

Definition 3.2. $f : X \to Y$ is ***irreversible*** if for every non-empty open set $B \subseteq X$ there exists a non-empty open set $U_B \subseteq B$ such that there is no open set $V \subset Y$ satisfying $\emptyset \neq f^{-1}(V) \cap B \subseteq U_B$.

In other words, if the pre-image of an open set intersects B then it intersects $B \setminus U_B$.

Intuitively, in a game between a player progressively describing $f(x)$ for some $x \in U_B$ and an opponent trying to progressively guess x, the opponent can never guess that $x \in U_B$ even knowing that $x \in B$.

An application of an irreversible function f to x comes with a loss of information about x, that can hardly be recovered. Being irreversible is orthogonal to not being one-to-one: the function $x \mapsto x^2$ is not one-to-one but not irreversible: x can be (continuously or computably) recovered from x^2; a one-to-one function can be irreversible if its inverse is dramatically discontinuous (examples of such functions will be encountered in the sequel).

In terms of sequences, f is irreversible if and only if for every B there exists a non-empty open set $U_B \subseteq B$ such that for every $x \in U_B$ there is a sequence $x_n \in B \setminus U_B$ such that $f(x_n)$ converges to $f(x)$.

As announced, the set of points at which an irreversible function is locally continuously invertible is small in the sense of Baire category.

Proposition 3.2.1. *Let f be irreversible. There is a dense G_δ-set D such that f is not locally continuously invertible at any $x \in D$.*

Proof. Let W_n be the union of U_B for all basic open sets B of radius $< 2^{-n}$. W_n is a dense open set. Let $x \in D := \bigcap_n W_n$. For each n there is a ball B of radius $< 2^{-n}$ such that $x \in U_B$. For every neighborhood V of $f(x)$, $x \in f^{-1}(V) \cap B \neq \emptyset$ so $f^{-1}(V) \cap B \nsubseteq U_B$. $\square$

In other words, for almost every x the application of f to x comes with a "topological information" loss.

The preceding proposition does not rule out the possibility that the restriction of f to a "large" set have a continuous inverse (for instance, the characteristic function of the rational numbers is nowhere continuous, but its restriction to the co-meager set of irrational numbers is continuous). The next assertion shows that this is not possible.

Proposition 3.2.2. *Let f be irreversible and $C \subseteq X$ be such that $f_{|C} : C \to f(C)$ is a homeomorphism. Then C is nowhere dense.*

Proof. Assume that the closure of C contains a ball B. Let $x \in U_B \cap C$. There exists a sequence $x_n \in B \setminus U_B$ such that $f(x_n)$ converges to $f(x)$. By density of C in B, x_n can be taken in C. As $f_{|C}$ is a homeomorphism and $f(x_n)$ converges to $f(x)$, x_n should converge to x and eventually enter U_B, which gives a contradiction. $\square$

In the definition of an irreversible function (Definition 3.2), B and U_B can be assumed w.l.o.g. to be basic balls, in some fixed effective basis. We can make Definition 3.2 effective by requiring that given an index for B in the numbered basis, one can compute an index for U_B.

Definition 3.3. f is ***effectively irreversible*** if an index for U_B can be computed from an index for B in Definition 3.2.

Proposition 3.2.3. *Being effectively irreversible does not depend on the effective basis.*

Proof. Let $\mathcal{B}$ and $\mathcal{B}'$ be effective bases, and assume that f is effectively irreversible w.r.t. $\mathcal{B}$. Let us show that f is effectively irreversible w.r.t. $\mathcal{B}'$. Given i, we want to compute j such that B'_j plays the role of $U_{B'_i}$, i.e. $B'_j \subseteq B'_i$ and no open set $V \subseteq Y$ satisfies $\emptyset \neq f^{-1}(V) \cap B'_i \subseteq B'_j$. First, B'_i is effectively open in the basis $\mathcal{B}$ so one can find k such that $B_k \subseteq B'_i$. As f is effectively irreversible w.r.t. $\mathcal{B}$ one can compute l such that B_l plays the role of U_{B_k}. As B_l is effectively open in the basis $\mathcal{B}'$ one can find j such that $B'_j \subseteq$

B_l. This is the sought j. Indeed, if there is an open set $V \subseteq Y$ such that $\emptyset \neq f^{-1}(V) \cap B'_i \subseteq B'_j$, then as $B'_j \subseteq B_l \subseteq B_k$, $\emptyset \neq f^{-1}(V) \cap B_k \subseteq B_l$ contradicting the choice of B_l. □

The following result is the effective version of Proposition 3.2.2.

Theorem 3.2.1. *If f is computable and effectively irreversible then for every 1-generic x, x is not computable relative to $f(x)$.*

Proof. The dense G_δ-set provided by Proposition 3.2.1 is effective when f is effectively irreversible so it contains every 1-generic point (even every weakly-1-generic point). Hence for every 1-generic x, f is not locally continuously invertible at x. We now apply Theorem 3.1.1. □

In other words, if x is 1-generic then the application of f to x comes with an "algorithmic information" loss. So if f is effectively irreversible then there exists some x that is not computable relative to $f(x)$.

3.3 The constructive result

We now present the main result of the paper. It is the constructive version of Theorem 3.2.1 as it makes $f(x)$ computable. The construction uses a priority argument with finite injury.

Theorem 3.3.1. *If f is computable and effectively irreversible then there exists a non-computable x such that $f(x)$ is computable.*

The proof uses the priority method with finite injury, which can be seen as a game between a player, computing $f(x)$, and infinitely many opponents (all the Turing machines) trying to compute x. The remainder of this section is devoted to the detailed proof of Theorem 3.3.1.

Here we take as effective basis the balls induced by a complete effective metric of X, so that every shrinking sequence of open sets has non-empty intersection. We fix a one-to-one computable enumeration $n_0, n_1, \dots$ of the halting set $\emptyset'$. We construct $x \in X$ such that $f(x)$ is computable and $\emptyset'$ is computable relative to x. We construct a shrinking sequence of metric balls B_n and define x as the unique member of their intersection. Of course, the sequence B_n must not be computable otherwise x would be computable. The sequence B_n is constructed in stages: at stage s we define $B_n[s]$ and for each n the sequence $B_n[s]$ is eventually constant, with limit B_n. For each s, the sequence $B_n[s]$ is shrinking, so the limiting sequence B_n will be shrinking as well. One may imagine, for each s, the sequence $B_n[s]$ as an infinite path in a tree. At stage $s+1$, n_s is enumerated into $\emptyset'$ and the current path branches at depth n_s.

In order to make $f(x)$ computable we enumerate along the construction the indices of all its basic neighborhoods into a list $L \subseteq \mathbb{N}$. L is the union of a computable growing sequence of finite lists L_s. At stage s, the current neighborhood of $f(x)$, denoted by V_s, is the (finite) intersection of the basic open sets indexed by L_s. As $L_s \subseteq L_{s+1}$, $V_{s+1} \subseteq V_s$.

We need to consider two technical points. First we use a particular set of special points, induced by the effective irreversibility of f, obtained as follows. We can assume w.l.o.g. that the radius of U_B is at most half the radius of B, that the closure of U_B is contained in B and that there is no open set $V \subseteq Y$ such that $\emptyset \neq f^{-1}(V) \cap B \subseteq \overline{U}_B$ (if $U_B = B(x,r)$ then replace it by $B(x, r/2)$). Given a basic ball B, consider the computable sequence $U_B^{(n)}$ defined inductively by $U_B^{(0)} = B$ and $U_B^{(n+1)} = U_{U_B^{(n)}}$. $U_B^{(n)}$ is a computable shrinking sequence and the unique member a of $\bigcap_n U_B^{(n)}$ is computable, uniformly in B. The canonical enumeration B_j of basic balls induces a computable dense sequence a_j, which will serve as simple points.

We then come to the second technical point. Let $(B'_k)_{k\in\mathbb{N}}$ be the canonical enumeration of the basic open subsets of Y. We assume that the effective open sets $f^{-1}(B'_k)$ come with growing enumerations $f^{-1}(B'_k)[s]$ such that the predicate $a_i \in f^{-1}(B'_k)[s]$ is decidable in i, k, s (use the effective basis given by Proposition 2.3.1).

We now proceed to the construction of the sequence $B_n[s]$ for each stage s. For each s, $B_n[s]$ will be a shrinking sequence, $x[s]$ will be defined as the unique member of their intersection and will be one of the points $\{a_j : j \in \mathbb{N}\}$.

Stage 0. We start with a ball $B_0[0]$ of radius 1, $B_{n+1}[0] = U_{B_n[0]}$ and $\{x[0]\} = \bigcap_n B_n[0]$. Start with $L_0 = \emptyset$ and $V_0 = Y$. Observe that for each n, $B_n[0] \cap f^{-1}(V_0)$ is non-empty as it contains $x[0]$.

Stage $s+1$. First, L_{s+1} is obtained by adding to L_s all the numbers $k \leq s$ such that $x[s] \in f^{-1}(B'_k)[s]$. Let V_{s+1} be the intersection of the open sets B'_k with $k \in L_{s+1}$.

Let $n = n_s$ be the next element enumerated into the halting set. Let $B_{n+1}[s+1]$ be a ball satisfying $\overline{B}_{n+1}[s+1] \subseteq f^{-1}(V_{s+1}) \cap B_n[s] \setminus \overline{B}_{n+1}[s]$. Such a ball exists: $f^{-1}(V_{s+1}) \cap B_{n+1}[s]$ is non-empty as it contains $x[s]$, f is irreversible and $B_{n+1}[s] = U_{B_n[s]}$. For $n' \leq n$, let $B_{n'}[s+1] = B_{n'}[s]$. For $n' > n$ define by induction $B_{n'+1}[s+1] = U_{B_{n'}[s+1]}$. Let $\{x[s+1]\} = \bigcap_n B_n[s+1]$.

Verification. By construction one has $\overline{B}_{n+1}[s] \subseteq B_n[s]$ and $B_{n+1}[s] = U_{B_n[s]}$

for sufficiently large n so $B_n[s]$ is a shrinking sequence.

We call the *settling time* of n the minimal number s such that $n_{s'} \geq n$ for all $s' \geq s$.

We say that $n \in \emptyset'$ is a *forward element* if no element $m < n$ is enumerated into $\emptyset'$ after the enumeration stage of n: in other words, the settling time of n coincides with its enumeration stage. As $\emptyset'$ is infinite, it has infinitely many forward elements.

Claim 1. For each n, $B_n[s]$ is eventually constant.

Proof. Let s_0 be the settling time of n: $B_n[s] = B_n[s_0]$ for all $s \geq s_0$. $\square$

Let B_n be its limit. B_n is a shrinking sequence as well, let x be the member of its intersection. Observe that the sequence $x[s]$ converges to x. Indeed, given ϵ, let n be such that B_n has radius $< \epsilon$ and s_0 be the settling time of n: for all $s \geq s_0$, $x[s] \in B_n[s] = B_n$ so $d(x[s], x) \leq \epsilon$.

Claim 2. $f(x)$ is computable.

Proof. We prove that a basic open set B'_k contains $f(x)$ if and only if k is enumerated into the list $L = \bigcup_s L_s$.

If $k \in L_s$ for some s, let n be a forward element which is enumerated at some stage $s' \geq s$. $x \in \overline{B}_{n+1} = \overline{B}_{n+1}[s'+1] \subseteq f^{-1}(V_{s'+1}) \subseteq f^{-1}(V_s) \subseteq f^{-1}(B'_k)$.

Now let B'_k be a basic neighborhood of $f(x)$. Let i_0 be such that $x \in f^{-1}(B'_k)[i_0]$. As $x[s]$ converges to x there is s such that $x[s] \in f^{-1}(B'_k)[i_0]$ for all $s' \geq s$. Let $t = \max(s, i_0)$: $x[t] \in f^{-1}(B'_k)[i_0] \subseteq f^{-1}(B'_k)[t]$ so k must be added to the list at stage $t+1$ or earlier. $\square$

Claim 3. $\emptyset'$ is computable relative to x.

Proof. Let $(p_i)_{i\in\mathbb{N}}$ be the increasing sequence of forward elements. $\emptyset'$ can be computed from the sequence p_i and the (computable) enumeration of $\emptyset'$.

From x one can inductively compute the sequence p_i. First, p_0 is the minimal n such that $x \notin B_{n+1}[0]$. Once p_i is known, let s be the stage at which p_i is enumerated into $\emptyset'$, i.e. $n_s = p_i$. p_{i+1} is the minimal $n > p_i$ such that $x \notin B_{n+1}[s+1]$. $\square$

In the proof $\emptyset'$ is encoded into x. The argument relativizes: given a set $A \subseteq \mathbb{N}$, there exists x_A such that A computes $f(x_A)$ and the pair (x_A, A) computes A'. All the reductions are uniform, so computing the Turing jump operator can be reduced to computing the inverse of f (when f is one-to-one). The notion capturing this idea is Weihrauch reducibility [Wei92, BG11].

Corollary 3.3.1. *If f is one-to-one, computable and effectively irreversible then the jump operator is Weihrauch reducible to f^{-1}.*

3.4 Application to the ergodic decomposition

We now present an application of Theorem 3.3.1. Let P be a Borel probability measure P over the Cantor space. P is ***computable*** if the real numbers $P[w]$ are uniformly computable. P is ***shift-invariant*** if $P[w] = P[0w] + P[1w]$ for each finite string w. P is ***ergodic*** if it cannot be written as $P = \frac{1}{2}(P_1 + P_2)$ with $P_1 \neq P_2$ both shift-invariant.

The ergodic decomposition theorem [Phe01] says that every shift-invariant measure can be uniquely decomposed into a convex combination (possibly uncountable) of ergodic measures. Our question is: given a computable shift-invariant measure, can we compute in a sense its ergodic decomposition? This question was implicitly addressed by V'yugin [V'y97] who constructed a counter example: a countably infinite combination of ergodic measures which is computable but not effectively decomposable. In [Hoy11] we raised the following question: does the ergodic decomposition become computable when restricting to finite combinations? As an application of Theorem 3.3.1, we solve the problem and prove that it is already non-effective in the finite case:

Theorem 3.4.1. *There exist two ergodic shift-invariant measures P and Q such that neither P nor Q is computable but $P + Q$ is computable.*

The strategy is as follows: the mapping $(P, Q) \mapsto P + Q$ is computable, two-to-one on the space $\mathcal{E} \times \mathcal{E}$ of pairs of ergodic measures and we prove

Theorem 3.4.2. *The function $(P, Q) \mapsto P+Q$ defined on $\mathcal{E}\times\mathcal{E}$ is effectively irreversible.*

which implies Theorem 3.4.1 by applying Theorem 3.3.1.

Before proving the theorem, we need some preliminaries so show that $\mathcal{E} \times \mathcal{E}$ is an effective Polish space.

We consider the space $\mathcal{P}(2^{\mathbb{N}})$ of Borel probability measures over the Cantor space together with the complete metric

$$d(P, Q) = \sum_{w\in\{0,1\}^*} 2^{-|w|}|P[w] - Q[w]|.$$

The finite rational combinations of Dirac measures are dense in $\mathcal{P}(2^{\mathbb{N}})$ and d is computable over them, so $\mathcal{P}(2^{\mathbb{N}})$ is an effective Polish space. The subset $\mathcal{I}$ of shift-invariant measures is closed so d is complete over $\mathcal{I}$ as well. $\mathcal{I}$ easily contains a dense computable sequence (take the Markovian measures with rational coefficients), so $\mathcal{I}$ is an effective Polish subspace of $\mathcal{P}(2^{\mathbb{N}})$. Let $\mathcal{E} \subseteq \mathcal{I}$ be the set of ergodic shift-invariant measures. The metric d is no longer complete over $\mathcal{E}$, but $\mathcal{E}$ is an effective G_δ-set that is c.e., so Proposition 2.3.3 implies that $\mathcal{E}$ is an effective Polish subspace (see [Par61] for results on the Baire category of the set of ergodic measures). We work with the basis given by the intersection of the canonical metric basis of $\mathcal{I}$ with $\mathcal{E}$, which is an effective basis of $\mathcal{E}$.

We now present the proof of Theorem 3.4.2.

Proof of Theorem 3.4.2. Let $B \subseteq \mathcal{I} \times \mathcal{I}$ be an open set and $(P, Q) \in B$ with $P \neq Q$. Let $\epsilon > 0$ be such that $d(P, Q) > \epsilon$ and $B(P, \epsilon) \times B(Q, \epsilon) \subseteq B$. Let $\delta = \epsilon/4$ and $U_B = B(P, \delta) \times B(Q, \delta) \subseteq B$. Observe that U_B can be effectively obtained from B.

We now show that if $V \subseteq \mathcal{I}$ is open and $f^{-1}(V)$ intersects B then it intersects $B \setminus U_B$, where $f : \mathcal{E} \times \mathcal{E} \to \mathcal{I}$ is the sum mapping. Let $(P_1, Q_1) \in f^{-1}(V) \cap B$. If $(P_1, Q_1) \notin U_B$ then we are done. Assume then that $(P_1, Q_1) \in U_B$, which implies $d(P_1, Q_1) > 2\delta$. For $\lambda \in [0, 1]$, define

$$P_\lambda = \lambda P_1 + (1 - \lambda)Q_1,$$
$$Q_\lambda = \lambda Q_1 + (1 - \lambda)P_1.$$

First observe that $P_\lambda + Q_\lambda = P_1 + Q_1 \in V$. There is some $\lambda \in (0, 1)$ such that $(P_\lambda, Q_\lambda) \in B \setminus U_B$, more precisely $\delta < d(P, P_\lambda) < \epsilon$ and $\delta < d(Q, Q_\lambda) < \epsilon$. Indeed,

$$d(P_1, P_\lambda) = d(Q_1, Q_\lambda) = (1 - \lambda)d(P_1, Q_1).$$

and as $d(P_1, Q_1) > 2\delta$ there exists $\lambda \in (0, 1)$ such that $(1-\lambda)d(P_1, Q_1) = 2\delta$. One then has

$$d(P, P_\lambda) \leq d(P, P_1) + d(P_1, P_\lambda) < 3\delta < \epsilon$$

and

$$d(P, P_\lambda) \geq d(P_1, P_\lambda) - d(P, P_1) > \delta,$$

and similarly $\delta < d(Q, Q_\lambda) < \epsilon$. Observe that the shift-invariant measures P_λ and Q_λ are not ergodic. However the ergodic measures are dense in the set of shift-invariant measures so there exist two ergodic measures P', Q' close to P_λ and Q_λ respectively, such that

$$\begin{gathered}\delta < d(P, P') < \epsilon, \\ \delta < d(Q, Q') < \epsilon, \\ P' + Q' \in V\end{gathered}$$

which implies that $(P', Q') \in f^{-1}(V) \cap B \setminus U_B$, which is then non-empty. $\square$

4 Directional genericity

Given an effectively irreversible function f,

- Theorem 3.2.1 tells us that if x is 1-generic then x is not computable relative to $f(x)$,
- Theorem 3.3.1 tells us that there exist non-computable x such that $f(x)$ is computable.

The two results are "disjoint" in the sense that in general a single x cannot at the same time be 1-generic and have a computable image, except for some particular functions like constant functions. We raise the following question: is it possible to bring the two results closer together? How far can x be from being computable, given that $f(x)$ is computable? How *generic* can x be? In this section we give an answer to these questions, introducing a notion of genericity that is compatible with a weak form of computability.

For the sake of simplicity, we will assume that f is the identity. We fix a set X, endowed with an effective Polish topology τ and a weaker effective topology τ'. In doing so we lose no generality, as a function $f : (X, \tau_X) \to (Y, \tau_Y)$ can always be thought as the identity from (X, τ_X) to (X, τ') where τ' is the initial topology of f whose open sets are the preimages of τ_Y-open sets.

4.1 Being generic from above

Let (X, τ) be an effective Polish space and τ' be an effective topology on X that is effectively weaker than τ: the basic τ'-open sets are effective τ-open sets, uniformly. In other words, we require the identity function from (X, τ) to (X, τ') to be computable.

Our general goal is to build elements of X that are to some extent generic in the topology τ but still computable in the topology τ'. The latter condition is usually weaker than being computable in the topology τ, and will be our weak notion of computability. We now have to define a suitable notion of genericity.

The topology τ' induces a pre-order on X, called the *specialization* pre-order $\leq$:

$$x \leq y \iff \text{every } \tau'\text{-neighborhood of } x \text{ is a } \tau'\text{-neighborhood of } y.$$

$x \leq y$ means that if one describes x by listing its basic neighborhoods then one can never distinguish x from y. Observe that when τ' is T_0, $\leq$ is an order, and when τ' is T_1, $\leq$ is the trivial ordering (equality).

Definition 4.1. To $x \in X$ we associate

$$S_x = \{y \in X : x \leq y\}$$

which is the intersection of all the τ'-neighborhoods of x.

S_x is the set of elements that cannot be distinguished from x when describing x in the topology τ'. If τ' is T_1 then $S_x = \{x\}$ for all x.

In a game where the player describes an element x in the τ'-topology, the player enumerates the basic τ'-neighborhoods of x. Each enumerated basic open set is a commitment: if V is enumerated then x must belong to V. Each commitment reduces the degrees of freedom of the player to fool the opponent. However some free space is always left, and this space is precisely S_x: at any moment the player is allowed to move into S_x (and then change x at the same time). As a consequence, during the computation of x in the topology τ', the player is able to make x as generic as possible, *inside the subset* S_x. This motivates the following definition.

Definition 4.2. Let (X, τ) be an effective Polish space, $A \subseteq X$ and $x \in A$. We say that x is ***generic inside*** A if for every effective open set $U \subseteq X$,

- either $x \in U$,
- or there exists a neighborhood B of x such that $B \cap U \cap A = \emptyset$.

If τ' is a weaker topology on X then we say that x is ***generic from above*** if x is generic inside $S_x = \{y \in X : x \leq y\}$, where $\leq$ is the specialization pre-order induced by τ'.

Every 1-generic element is generic inside any set containing it. Let us give a few examples illustrating these notions. For $A = X$, being generic inside A is the same as being 1-generic. Every element x is vacuously generic inside $\{x\}$. In the product space $X \times X$, (x, y) is generic in $\{x\} \times X$ if and only if y is 1-generic relative to x, i.e. y does not belong to the boundary of any open set that is effective relative to x (effective open subsets of X relative to x are the same as the sections at x of effective open subsets of $X \times X$).

Informally, x is generic from above means that x belongs to every effective open set that is dense *above* it, for the specialization order induced by τ' (while a 1-generic elements belongs to every effective open set that is dense *along* it).

With this notion in hand we obtain the sought combination of Theorems 3.2.1 and 3.3.1. For this we need a reasonable technical assumption on the bases $\mathcal{B}$ and $\mathcal{B}'$ of the topologies τ and τ' respectively.

Assumption 1 There is an algorithm that given a finite number of basic open sets from $\mathcal{B}$ and $\mathcal{B}'$, decides whether their intersection is non-empty, where $\mathcal{B}$ is a basis associated with a complete effective metric.

In practice this assumption is often satisfied. We do not know how to prove the next theorem without this assumption, and we do not know whether the theorem fails without this assumption.

Theorem 4.1.1 (Generic and weakly computable)**.** *Let (X, τ) be an effective Polish space and τ' and effectively weaker topology, satisfying assumption 1. Let U_n be uniformly effective dense τ-open sets. There exists $x \in \bigcap_n U_n$ such that*

- *x is generic inside S_x,*
- *x is τ'-computable.*

Observe that the theorem is only interesting when τ' is not T_1, otherwise $S_x = \{x\}$ and the first condition is vacuously satisfied for every x.

Observe that x is not in general weakly 1-generic, so it does not belong to every dense effective open set. However, if an *effective sequence* of such sets U_n is given in advance then x can be taken in their intersection, as stated by the theorem. This is possible as the family of dense effective open sets is not enumerable in general.

Proof. Let a_i be a dense computable sequence of simple points in (X, τ). Again we can assume w.l.o.g. that the basic open sets $B'_k \in \mathcal{B}'$ come with a computable enumeration $B'_k[s]$ ($B'_k[s]$ is an effective open set) such that the predicate $a_i \in B'_k[s]$ is decidable in i, k, s (using Proposition 2.3.1).

The proof is a finite injury argument. We want to satisfy the requirements

$$R_e : \ x \in W_e \text{ or } \exists \epsilon, B(x, \epsilon) \cap W_e \cap S_x = \emptyset,$$

where W_e is the effective open set with number e (in the sequel, $W_{e,s}$ will be computable growing finite unions of elements of $\mathcal{B}$ with union W_e). At stage s, each requirement R_e is assigned a ball $B_e[s]$. They satisfy $\overline{B}_{e+1}[s] \subseteq B_e[s] \cap U_e$. For each e, the sequence $B_e[s]$ is eventually constant when s grows. At the same time, a list $L \subseteq \mathbb{N}$ is enumerated containing exactly the indices of the basic $\mathcal{B}'$-neighborhoods of x. L is obtained as the union of a growing computable sequence of finite sets L_s. We denote by V_s the finite intersection of elements of $\mathcal{B}'$ whose indices are given by L_s, i.e. $V_s = \bigcap_{k \in L_s} B'_k$. V_s will be a neighborhood of x in the topology τ'. In order to satisfy the requirement R_e, one tests whether $B_e[s]$ intersects $W_{e,s} \cap V_s$ and if it is so, forces x to belong to the intersection.

Stage 0**.** Let $B_0[0]$ be any ball of radius $< 2^{-0}$ and inductively choose $\overline{B}_{e+1}[0] \subseteq B_e[0] \cap U_e$ of radius $< 2^{-e-1}$. Let x_0 be the center of $B_0[0]$ and $L_0 = \emptyset$.

Stage $s+1$**.** Let $e \leq s$ be minimal such that $B_e[s] \cap W_{e,s} \cap V_s \neq \emptyset$ (decidable by Assumption 1) and R_e is not already declared satisfied, if it exists (decidable property). Let $B_{e'}[s+1] = B_{e'}[s]$ for $e' \leq e$, let $\overline{B}_{e+1}[s+1] \subseteq B_e[s] \cap W_{e,s} \cap V_s \cap U_e$ have radius $< 2^{-e-1}$ and x_{s+1} be the center of $B_{e+1}[s+1]$. Define inductively $\overline{B}_{e'+1}[s+1] \subseteq B_{e'}[s+1] \cap U_{e'}$ of radius $< 2^{-e'-1}$ for $e' > e$. We say that R_e acts and R_e is declared satisfied. All $R_{e'}$ with $e' > e$ are initialized, which means that all of them are declared unsatisfied.

If such an $e \leq s$ does not exist then let $B_e[s+1] = B_e[s]$ for all e and $x_{s+1} = x_s$.

Let L_{s+1} contain L_s together with every $k \leq s$ such that $x_{s+1} \in B'_k[s]$.

Verification. By the usual analysis of finite injury arguments, each requirement acts finitely many times, so for each e there is s_0 such that $B_e[s] = B_e[s_0]$ for all $s \geq s_0$. Let $B_e = B_e[s_0]$. One has $\overline{B}_{e+1} \subseteq B_e$ and B_e has radius $< 2^{-e}$. Let x be the unique member of $\bigcap_e B_e$.

Claim 4. The sequence x_s converges to x.

For each k, and all sufficiently large s, only requirements R_e with $e > k$ act, so $x_{s+1} \in B_{e+1}[s+1] \subseteq B_k[s+1] = B_k$. As $x \in B_k$ and B_k has radius $< 2^{-k}$, $d(x_{s+1}, x) < 2^{-k+1}$ for all sufficiently large s.

Claim 5. $x \in \bigcap_e U_e$.

By construction, $B_{e+1} \subseteq U_e$ for all e, hence $x \in \bigcap_e B_{e+1} \subseteq \bigcap_e U_e$.

Claim 6. L lists exactly the elements of $\mathcal{B}'$ containing x, hence x is τ'-computable.

First, x belongs to each V_s by construction of the balls $B_e[s]$, so L lists only (indices of) neighborhoods of x. Conversely, assume that B'_k is a neighborhood of x. It implies that some $B'_k[s_0]$ is a neighborhood of x. As x_s converges to x, x_s belongs to $B'_k[s_0]$ for all s larger that some s_1. As a result, for $s > \max(s_0, s_1)$, k is listed in L_s.

Claim 7. x is generic inside S_x.

Let $e \in \mathbb{N}$ be such that $x \notin W_e$. Let s be such that no requirement $e' < e$ acts from stage s on. R_e cannot act at a stage $s' \geq s$, otherwise $x \in B_{e+1} = B_{e+1}[s'+1] \subseteq W_e$ which contradicts the assumption $x \notin W_e$. In the same way, R_e cannot be declared satisfied at stage s, otherwise $x \in B_{e+1} = B_{e+1}[s'+1] \subseteq W_e$. As R_e never acts after stage s, it means that $x \in B_e = B_e[s]$ and $B_e[s] \cap W_e \cap S_x = \emptyset$, otherwise there exists $s' \geq s$ such that $B_e[s] \cap W_{e,s'} \cap S_x \neq \emptyset$, hence $B_e[s'] \cap W_{e,s'} \cap V_{s'} \neq \emptyset$ as $B_e[s'] = B_e[s]$ and $S_x \subseteq V_{s'}$, and then R_e must act at stage s' as it is not declared satisfied. This is a contradiction. $\square$

The point x provided by Theorem 4.1.1 actually satisfies a stronger notion of genericity than being generic inside S_x.

Lemma 4.1.1. *For each effective open set $W_e = \bigcup_s W_e[s]$,*

- *either $x \in W_e$,*
- *or there exists a neighborhood B of x such that $B \cap (\bigcup_s W_e[s] \cap V_s) = \emptyset$.*

Proof. Indeed, if $x \notin W_e$ then when all $R_{e'}$ with $e' \leq e$ have settled, $B_e[s] = B_e$ and $B_e[s] \cap W_e[s] \cap V_s = \emptyset$ otherwise R_e will act and force x to fall into W_e, so one can take $B = B_e$. $\square$

In other words, x satisfies the condition of 1-genericity not for every effective open set W_e, but for the effective open set $\bigcup_s W_e[s] \cap V_s$. This genericity condition has the consequence that x behaves in some respects as a 1-generic point, as illustrated by the following two results.

Corollary 4.1.1. *The point x provided by Theorem 4.1.1 is low, i.e. the set $\{e \in \mathbb{N} : x \in W_e\}$ is Δ^0_2 (or $\emptyset'$-computable, or limit of a computable sequence).*

Proof. For each e, the computable predicate $B_e[s]\cap W_e[s]\cap V_s \neq \emptyset$ converges to the predicate $x \in W_e$. Indeed, if $B_e[s] \cap W_e[s] \cap V_s \neq \emptyset$ for infinitely many s then R_e will eventually act and never be injured later, so x will be forced to fall into W_e. If $x \in W_e$ then for sufficiently large s, $B_e[s] = B_e$ and $x \in W_e[s]$ so $B_e[s] \cap W_e[s] \cap V_s \neq \emptyset$. □

On compact spaces satisfying assumption 1, Theorem 4.1.1 is indeed a strengthening of Theorem 3.3.1.

Corollary 4.1.2. *Assume that (X, τ) is compact. If* $\mathrm{id} : (X, \tau) \to (X, \tau')$ *is effectively irreversible then the point x provided by Theorem 4.1.1 can be taken to be non-computable.*

Proof. One can take x in the dense effective G_δ-set given by Proposition 3.2.1, so that id is not locally continuously invertible at x.

If x is computable then there exists e such that $W_e = X\setminus\{x\}$. As $x \notin W_e$ there exists a neighborhood B of x such that $B\cap\bigcup_s W_e[s]\cap V_s = \emptyset$. Let $\overline{U}_B \subseteq B$ come from the local continuous non-invertibility of *id* at x. As W_e covers the compact set $X \setminus U_B$, there exists s such that $W_e[s]$ already covers that set. As V_s is a τ'-neighborhood of x, $B \cap V_s \setminus U_B \neq \emptyset$ so $B \cap V_s \cap W_e[s] \neq \emptyset$, which contradicts the choice of B. □

We now illustrate directional genericity in several situations and show how Theorem 4.1.1 embodies many constructions encountered in computability theory. It means that in many situations, in order to construct an object satisfying a given set of requirements, one only has to find the suitable topologies τ and τ' that make directionally generic objects have the sought properties. Theorem 4.1.1 can then be directly applied, instead of explicitly constructing the object by means of a finite injury argument.

4.2 Genericity for c.e. sets

We consider the Cantor space X of subsets of $\mathbb{N}$. Here τ is the Cantor topology and τ' is the Scott topology generated by the sets $\{A \subseteq \mathbb{N} : F \subseteq A\}$, where F ranges among the finite subsets of $\mathbb{N}$. For a set $A \subseteq \mathbb{N}$, $S_A = \{B \subseteq \mathbb{N} : A \subseteq B\}$ is the class of supersets of A.

Definition 4.3. A set $A \subseteq \mathbb{N}$ is ***generic from above*** if it is 1-generic inside S_A, which means that for every effective open class $\mathcal{U}$, either $A \in \mathcal{U}$ or there exists n such that $[A{\restriction_n}] \cap \mathcal{U} \cap S_A = \emptyset$.

In other words, A is generic from above if it belongs to every effective open class that is dense *above* A in the subset ordering. This should be compared to the notion of 1-generic set, which belongs to every effective open class that is dense *along* or *at* it. Observe that a co-finite set A is vacuously generic from above as there is an n such that $[A{\upharpoonright}_n] \cap S_A = \{A\}$. Hence only co-infinite sets that are generic from above are interesting.

As a direct application of Theorem 4.1.1 we obtain:

Corollary 4.2.1. *There exists a co-infinite c.e. set $A \subseteq \mathbb{N}$ that is generic from above.*

Proof. The class of co-infinite sets is a dense effective G_δ-set. □

As the next result shows, Theorem 4.1.1 embodies simple finite injury arguments such as the Friedberg-Muchnik theorem for instance. Interestingly many arguments showing that 1-generic sets satisfy some property already apply to sets that are generic from above. Indeed, in these arguments, an effective open class is shown to be dense at the set and it often happens that it is even dense *above* the set. We now give illustrations of this phenomenon.

Proposition 4.2.1. *Let A be co-infinite and generic from above. $\mathbb{N} \setminus A$ is hyperimmune, $A = A_1 \oplus A_2$ where A_1 and A_2 are Turing incomparable, A is not autoreducible.*

Proof. The simple arguments showing the results for 1-generic sets actually give this stronger result, observing that the involved open set is not only dense *along* A, but even *above* A. For instance, to prove that $A_2 \not\leq_T A_1$, given a Turing functional ϕ, let $U = \{A_1 \oplus A_2 : \exists n, \phi^{A_1}(n) = 0 \wedge A_2(n) = 1\}$. If $\phi^{A_1} = A_2$ then replacing a 0 in A_2 by a 1 arbitrarily far gives an element of U arbitrarily close to $A_1 \oplus A_2$ that is *above* (i.e. is a superset of) $A_1 \oplus A_2$. □

It happens that the co-infinite sets that are generic from above are exactly the p-generic sets defined by Ingrassia [Ing81].

4.3 Genericity for left-c.e. reals

We consider the unit real interval $[0, 1]$. τ is the Euclidean topology, τ' is the topology induced by the semi-lines $(x, 1]$. The specialization order is the natural ordering on real numbers. For a real $x \in [0, 1]$, $S_x = [x, 1]$.

Definition 4.4. A real $x \in [0, 1]$ is ***generic from the right*** if it is 1-generic inside $[x, 1]$, which means that for every effective open set $\mathcal{U} \subseteq [0, 1]$, either $x \in \mathcal{U}$ or there exists $\epsilon > 0$ such that $[x, x + \epsilon) \cap \mathcal{U} = \emptyset$.

Again x is generic from the right if it belongs to every effective open set that is dense *above* x in the real ordering.

Kurtz built a left-c.e. weakly 1-generic real (see Theorem 1.8.49 in [Nie09]). The construction even gives a left-c.e. real that is generic from the right. One can think of the proof of Theorem 4.1.1 as a kind of generalization of this argument (replacing the lexicographic order used in the proof appearing in [Nie09] by the specialization pre-order induced by τ').

Genericity from the right easily lies between two classical notions of genericity.

Proposition 4.3.1. *Every* 1*-generic is generic from the right. Every generic from the right is weakly-*1*-generic. The implications are strict.*

Proof. An open set that is dense is dense on the right of x. An open set that is dense on the right of x is dense along x. Right-c.e. reals cannot be generic on the right, but there exists a right-c.e. weakly-1-generic real. Left-c.e. reals cannot be 1-generic, but there exists a left-c.e. real that is generic on the right. $\square$

In Section 4.5 we will separate genericity from the right from weakly-1-genericity among the left-c.e. reals.

Solovay reducibility vs. cl-reducibility. If A is a subset of $\mathbb{N}$ then we denote by x_A the real number whose binary expansion is A. In [DHL04] it is proved that there exist two sets A, B such that A is a c.e. set, x_B is a left-c.e. real and $A \leq_{\mathrm{cl}} B$ but $x_A \not\leq_{\mathrm{S}} x_B$. Here $\leq_{\mathrm{cl}}$ stands for computably Lipschitz reducibility and $\leq_{\mathrm{S}}$ stand for Solovay reducibility. The construction is a finite injury argument, which again is captured by Theorem 4.1.1.

Let us recall that $A \leq_{\mathrm{cl}} B$ means that there is a Turing functional computing A with oracle B, reading the first $n + c$ bits of B to compute the n first bits of A, for some constant c and all n. $x_A \leq_{\mathrm{S}} x_B$ means that there exists a constant $c \in \mathbb{N}$ and computable sequences $a_i \nearrow x_A$ and $b_i \nearrow x_B$ such that $x_A - a_i \leq c(x_B - b_i)$ for all i, or equivalently that $cx_B - x_A$ is left-c.e.

Theorem 4.3.1. *Let x_B be left-c.e. and generic on the right and $A = \{w \in 2^{<\mathbb{N}} : w <_{lex} B\}$. One has $A \leq_{cl} B$ but $x_A \not\leq_S x_B$.*

Note that we identify A with a subset of $\mathbb{N}$ by using a computable bijection between $2^{<\mathbb{N}}$ and $\mathbb{N}$ (we will assume that the string represented by a number n has length at most n).

Proof. The reduction $A \leq_{\mathrm{cl}} B$ is obvious: to know whether $w <_{\mathrm{lex}} B$, one only needs to know the $|w|$ first bits of B.

Assume that $x_A \leq_{\mathrm{S}} x_B$. It implies the existence of a one-to-one computable enumeration w_i of A, a computable sequence $b_i \nearrow x_B$ and a computable order $h : \mathbb{N} \to \mathbb{N}$ such that for all $i \in \mathbb{N}$, if $x_B - b_i < 2^{-h(n)}$ then $|w_i| > n$. Let B_i be the maximal element of $\{w_0, \dots, w_i\}$ in the lexicographic ordering. If $x_B - b_i < 2^{-h(n)}$ then $B_i\restriction_n = B\restriction_n$: indeed, the string $B\restriction_n$ belongs to A so it must be w_j for some j, which must be less than i.

For $w \in 2^{<\mathbb{N}}$, let $[w]$ be the interval containing all real numbers having a binary expansion starting with w, namely $[w] = [0.w, 0.w + 2^{-|w|}]$.

Let $U = \bigcup_{n,i}(b_i, b_i + 2^{-h(n)}) \setminus [B_i\restriction_n]$. U is an effective open set. U does not contain x_B. Indeed, if $x_B - b_i < 2^{-h(n)}$ then $B_i\restriction_n = B\restriction_n$ so $x_B \in [B_i\restriction_n]$. We now prove that U is dense on the right of x_B, which contradicts the assumption that x_B is generic on the right. As x_B is generic on the right it is weakly 1-generic, so there exist infinitely many n such that B contains all the natural numbers from n to $h(n)$. In other words for infinitely many n, x_B is very close to the right endpoint of $[B\restriction_n]$, namely at distance $< 2^{-h(n)}$. For such n and sufficiently large i, $(b_i, b_i + 2^{-h(n)}) \setminus [B_i\restriction_n]$ is a non-empty subset of the interval $(x_B, x_B + 2^{-h(n)})$. □

We actually prove more: there is no computable order h and no computable sequences $a_i \nearrow x_A$, $b_i \nearrow x_B$ such that $x_B - b_i \leq 2^{-h(n)}$ implies $x_A - a_i \leq 2^{-n}$, which would be a generalization of Solovay reducibility.

Left-c.e. reals with only computable presentations. Downey and LaForte [DL02] proved the existence of non-computable left-c.e. reals x all of whose presentations are computable: each prefix-free c.e. set A of finite binary strings satisfying $\sum_{w \in A} 2^{-|w|} = x$ is actually a computable set. A corollary of a result of Stephan and Wu [SW05] is that any such real is weakly 1-random, i.e. it must belong to every effective open set of measure one. Actually it must be weakly-1-generic and even generic from the right.

Proposition 4.3.2. *If x is a non-computable left-c.e. real all of whose presentations are computable then x is generic from the right.*

Proof. Let U be an effective open set that does not contain x: we must find $y > x$ such that $[x, y)$ is disjoint from U. First replace U by $V = U \cup [0, x)$. Let A be a prefix-free c.e. set such that $V = \bigcup_{w \in A}[w]$. The set $A_{<x} = \{w \in A : w <_{\mathrm{lex}} x\}$ is a presentation of x hence it is computable, so $A_{>x} = \{w \in A : w >_{\mathrm{lex}} x\} = A \setminus A_{<x}$ is c.e. As a result, $y := \inf \bigcup_{w \in A_{>x}}[w]$ is right-c.e. As x is not computable and $x \leq y$, one has $x < y$ and we get the result as $[x, y)$ is disjoint from U. $\square$

4.4 Genericity for Π^0_1-classes

We work on the set $\mathrm{CL}(2^{\mathbb{N}})$ of non-empty closed subsets of the Cantor space, endowed with the so-called *hit-or-miss* topology τ_{hm} [Mic51]. τ_{hm} is generated by the *miss* sets $\mathcal{U}_u = \{P \in \mathrm{CL}(2^{\mathbb{N}}) : P \cap [u] = \emptyset\}$ where $u \in 2^{<\mathbb{N}}$, together with their complements (the *hit* sets). We obtain an effective Polish space $(\mathrm{CL}(2^{\mathbb{N}}), \tau_{hm})$. A computable element of this space is usually called a computable closed set, and is the set of infinite branches of a computable tree without dead-ends.

Proposition 4.4.1. *In the space $(CL(2^{\mathbb{N}}), \tau_{hm})$, every weakly-1-generic element contains only weakly-1-generic sequences.*

Proof. Let $U \subseteq 2^{\mathbb{N}}$ be a dense effective open set. Let $\mathcal{U} = \{P \in \mathrm{CL}(2^{\mathbb{N}}) : P \subseteq U\}$. $\mathcal{U}$ is a dense effective open set in the space $\mathrm{CL}(2^{\mathbb{N}})$. Hence every weakly-1-generic closed set P belongs to $\mathcal{U}$, i.e. is contained in U. $\square$

We consider a weaker topology τ_m called the *miss* topology, generated by the miss sets $\mathcal{U}_u$ with $u \in 2^{<\mathbb{N}}$. A Π^0_1-class is a computable member of $(\mathrm{CL}(2^{\mathbb{N}}), \tau_m)$. The specialization pre-order induced by τ_m is the reverse inclusion, so that for each non-empty closed set P one has $\mathcal{S}_P = \{Q \in \mathrm{CL}(2^{\mathbb{N}}) : Q \subseteq P\}$ (being "above" P in this pre-order means being *inside* P). Definition 4.2 is instantiated as follows.

Definition 4.5. A non-empty closed set $P \subseteq 2^{\mathbb{N}}$ is ***generic from inside*** if P is 1-generic inside S_P, which means that for every effective τ_{hm}-open set $\mathcal{U} \subseteq \mathrm{CL}(2^{\mathbb{N}})$, either $P \in \mathcal{U}$ or there exists a τ_{hm}-neighborhood $\mathcal{N}$ of P such that $\mathcal{N} \cap \mathcal{U} \cap \mathcal{S}_P = \emptyset$.

Proposition 4.4.2. *Every closed set P that is generic from inside has empty interior, i.e. has a dense complement.*

Proof. Given a cylinder $[u]$, the class $\mathcal{U}$ of non-empty closet sets that do not contain $[u]$ is an effective open class: it is the union over $[v] \subseteq [u]$ of the miss sets $\mathcal{U}_v$. If $[v] \subseteq [u]$ then $P \setminus [v]$ belongs to $\mathcal{S}_P \cap \mathcal{U}$ and is arbitrarily close to P in the topology τ_{hm} as the length of v tends to infinity. As P is generic from inside, P must belong to $\mathcal{U}$ so P does not contain $[u]$. $\square$

As a result, no member of a Π^0_1-class P that is generic from inside can be weakly-1-generic. However all the elements of P are weakly-1-generic *inside* P: if U is an effective open subset of the Cantor space such that $P \cap U$ is dense in P then U contains every member of P, i.e. U contains P.

Proposition 4.4.3. *If P is generic from inside then every member of P is weakly-1-generic inside P.*

Proof. Let $U \subseteq 2^{\mathbb{N}}$ be an effective open set. Consider the set $\mathcal{U} = \{P : P \subseteq U\}$. $\mathcal{U}$ is an effective open set in the space $(\mathrm{CL}(2^{\mathbb{N}}), \tau_{hm})$ (and even in the topology τ_m). If $P \cap U$ is dense in P then there exists $Q \subseteq P \cap U$ arbitrarily τ_{hm}-close to P (let Q be a finite set of points from $P \cap U$ whose ϵ-neighborhood covers P, for arbitrarily small ϵ), so P belongs to the closure of $\mathcal{U} \cap \mathcal{S}_P$. If P is generic from inside then P must belong to $\mathcal{U}$, so $P \subseteq U$. □

In particular,

Corollary 4.4.1. *A perfect closed set that is generic from inside has no computable member.*

Proof. If x is computable then $U = 2^{\mathbb{N}} \setminus \{x\}$ is an effective open set. If P has no isolated point then $P \cap U$ is dense in P. By the previous result, P is then contained in U, i.e. P does not contain x. □

Now, Theorem 4.1.1 can be instantiated as follows.

Corollary 4.4.2. *There exists a perfect Π^0_1-class that is generic from inside.*

Proof. Being perfect, or having no isolated point is a dense effective G_δ-property in the space $(\mathrm{CL}(2^{\mathbb{N}}), \tau_{hm})$. □

4.5 Genericity for regular Π^0_1-classes

We know from Proposition 4.3.1 that every real $x \in [0,1]$ that is generic on the right is weakly-1-generic, but not the converse. Here we prove the existence of *left-c.e.* reals that are weakly-1-generic but not generic from the right.

To this end we need to construct a Π^0_1-set P such that (i) its leftmost element x is weakly-1-generic, and (ii) the complement of P is dense on the right of x. The first condition requires the class to have non-empty interior, and even that the interior of P be dense along x. Together with the second condition, it implies that x should not be isolated in the boundary of P.

The class P that we build will actually satisfy these conditions *at every point of its boundary*: P is regular (it coincides with the closure of its interior), its boundary is perfect (has no isolated point) and contains only weakly-1-generic points.

A suitable way of describing a regular closed set C is by giving approximations of C in the hit-or-miss topology and at the same time enumerating its interior. This can be formalized by introducing a new topology τ on $\mathrm{CL}([0,1])$ that is stronger than the hit-or-miss topology τ_{hm}. First, the hit-or-miss topology τ_{hm} is generated by the *hit sets* $\{C \in \mathrm{CL}([0,1]) : C \cap (a,b) \neq \emptyset\}$ where $a < b$ are rational, and the *miss sets* $\{C \in \mathrm{CL}([0,1]) : C \cap [a,b] = \emptyset\}$ where $a < b$ are again rational. The stronger topology τ is generated by the hit-or-miss open sets together with the sets

$$\{C \in \mathrm{CL}([0,1]) : [a,b] \subseteq \mathrm{int}(C)\},$$

where $a < b$ are rational numbers and $\mathrm{int}(C)$ is the interior of C. A canonical enumeration of the rational numbers gives a numbered basis for the topology τ, which makes $(\mathrm{CL}([0,1]), \tau)$ an effective topological space.

Intuitively, describing a closed set C in the topology τ amounts to giving approximations of C in the hit-or-miss topology and at the same time enumerating the interior of C, which is equivalent to giving approximations of both C and $(\mathrm{int}(C))^c$ in the hit-or-miss topology.

Proposition 4.5.1. *The space $(CL([0,1]), \tau)$ is an effective Polish space.*

Proof. The space can be embedded into $\mathrm{CL}([0,1]) \times \mathrm{CL}([0,1])$ endowed with the product of the hit-or-miss topology, which is an effective Polish space. Indeed, the space is computably homeomorphic to the subset $\{(C, (\mathrm{int}(C))^c) : C \in \mathrm{CL}([0,1])\}$ of $\mathrm{CL}([0,1]) \times \mathrm{CL}([0,1])$. We show that this subset is a c.e. effective G_δ-set, which implies that it is an effective Polish space by Proposition 2.3.3.

Claim 8. The set $\{(A,B) : (\mathrm{int}(A))^c \subseteq B\}$ is Π^0_1.

Indeed, $(\mathrm{int}(A))^c \subseteq B$ is equivalent to $A^c \subseteq B$, which holds iff every rational interval $[a,b]$ that intersects A^c intersects B.

Claim 9. The set $\{(A,B) : B \subseteq (\mathrm{int}(A))^c\}$ is Π^0_2.

Indeed, $B \subseteq (\mathrm{int}(A))^c$ iff every rational interval (a,b) that intersects B also intersects A^c.

Now it is c.e. The collection of pairs $(C, (\mathrm{int}(C))^c)$ where C ranges over the finite unions of closed rational intervals is dense in it. □

The Polish topology τ induces a notion of 1-genericity that fits with our objectives.

Proposition 4.5.2. *In the space $(CL([0,1]),\tau)$, every 1-generic element is regular and its boundary contains only weakly 1-generic reals.*

Proof. First, the set of regular closed sets is a dense effective G_δ-set. Indeed, C is regular iff every rational interval (a,b) is disjoint from C or intersects the interior of C. The collection of finite unions of closed rational intervals (regular sets) is dense.

Let $U \subseteq [0,1]$ be a dense effective open set. Let $\mathcal{U} \subseteq \mathrm{CL}([0,1])$ be the collection of closed sets whose boundary is contained in U. It is an effective open set in the topology τ: the boundary of C is contained in U iff $\mathrm{int}(C) \cup C^c \cup U$ covers $[0,1]$, which is semi-decidable from a description of C. $\mathcal{U}$ is moreover dense. □

Theorem 4.5.1. *Let P be a non-empty regular closed set that is τ-generic from inside. The boundary of P contains only weakly-1-generic points.*

Proof. Let $U \subseteq [0,1]$ be a dense effective open set. The class $\mathcal{U}$ of regular closed sets whose boundary is contained in U is an effective open class in the topology τ. Now let P be a regular closed set. There exists a sequence P_n of finite unions of closed intervals contained in $\mathrm{int}(P)$ and converging to P in the topology τ. As U is dense, the endpoints of the intervals constituting P_n can be taken in U. Each P_n is contained in P (i.e. P_n belongs so S_P), and belongs to $\mathcal{U}$, so $\mathcal{U}$ is dense below P. If P is τ-generic from inside then P must belong to $\mathcal{U}$, i.e. its boundary must be contained in U. □

Theorem 4.1.1 directly gives the following result.

Corollary 4.5.1. *In the space $(CL([0,1]),\tau)$ there exists a regular Π^0_1-class that is generic from inside and whose boundary is perfect.*

Proof. Having a perfect boundary is again a dense effective G_δ-property in the topology τ. □

As a result, the leftmost element of this set is a left-c.e. real that is weakly-1-generic but not generic from the right.

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