Enhanced Recovery Conditions for OMP/OLS by Exploiting both Coherence and Decay

Cédric Herzet, Charles Soussen

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Cédric Herzet[1], Charles Soussen[2].

[1]INRIA Centre Rennes - Bretagne Atlantique, Rennes, France.

[2] Centre de Recherche en Automatique de Nancy, Université de Lorraine, Nancy, France.

1 Introduction

In this paper, we focus on two popular instances of greedy algorithms, namely orthogonal matching pursuit (OMP) [1] and orthogonal least squares[1] (OLS) [5, 6].

The suboptimal nature of OMP and OLS has led many researchers to study conditions under which these procedures succeed in recovering the true sparse vector. This question has been widely addressed for OMP in the recent years, including worst-case uniform [7, 8] and probabilistic analyses [9]. Although OLS has been known in the literature for a few decades (under different names [10]), exact recovery analysis for OLS have only appeared very recently, see [11, 12, 13].

Most of the existing works deal with *uniform* guarantees: these conditions ensure the success of OMP/OLS for a given sparsity level (or a given support) *irrespective* of the magnitude of the non-zero coefficients. In contrast with these works, we derive new guarantees of success accounting for the decay of the non-zero elements of the sparse vector. Our conditions are expressed in terms of the mutual coherence of the dictionary μ and encompass, as particular cases, some well-known results of the literature. The proofs of the results are reported in our technical report [14].

2 Context and Main Results

Let us assume that $\mathbf{y} \in \mathbb{R}^m$ is a (noisy) linear combination of k columns of $\mathbf{A} \in \mathbb{R}^{m\times n}$ indexed by $\mathcal{Q}^\star$, that is

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{w} \quad \text{with} \left\{ \begin{array}{l} x_i \neq 0 \Leftrightarrow \ i \in \mathcal{Q}^\star \\ \mathrm{Card}\{\mathcal{Q}^\star\} = k \end{array} \right. \tag{1}$$

where $\mathbf{w} \in \mathbb{R}^m$ denotes some noise vector and $\mathrm{Card}\{\cdot\}$ stands for the cardinality operator. We assume that the columns $\mathbf{a}_i$ of the dictionary are normalized: $\|\mathbf{a}_i\|_2 = 1\ \forall i$.

Let us first consider the noiseless case ($\mathbf{w} = \mathbf{0}$). Theorem 1 provides sufficient conditions of success for OMP/OLS accounting for the possible decay of non-zero coefficients in $\mathbf{x}$. In our statement, we assume without loss of generality that

$$\mathcal{Q}^\star = \{1, 2, \ldots, k\}, \tag{2}$$

and

$$|x_1| \geq |x_2| \geq \ldots \geq |x_k| > 0. \tag{3}$$

Theorem 1. *If*

$$\mu < \frac{1}{k}, \tag{4}$$

and

$$|x_i| > \frac{2\mu(k-i)}{1-i\mu}|x_{i+1}| \qquad \forall i \in \{1, \ldots, k\}, \tag{5}$$

then Oxx selects atoms in $\mathcal{Q}^\star$ from noiseless data during the first k iterations.

In our technical report [14], we show that this kind of result can also be extended to the characterization of the success of OMP/OLS in the noisy setting ($\|\mathbf{w}\|_2 \leq \epsilon$) or the partial recovery of the support by the latter algorithms. In particular, we show that a well-known result by Donoho *et al.* [15, Th. 5.1] can be further relaxed as

Theorem 2. *If*

$$\mu < \frac{1}{2k-1}, \tag{6}$$

and

$$|x_i| > \frac{2\epsilon}{1-(2k-i)\mu} \qquad \forall i \in \{1, \ldots, k\}, \tag{7}$$

then Oxx selects atoms in $\mathcal{Q}^\star$ during the first k iterations.

More specifically, we see that the condition by Donoho *et al.* in [15, Th. 5.1] is sufficient for (7) to be satisfied and is therefore stronger than the conditions mentioned in Theorem 2. On the other hand, the two types of conditions become equivalent as soon as the nonzero elements of $\mathbf{x}$ have the same magnitude.

[1]The OLS algorithm is also known as forward selection [2], Order Recursive Matching Pursuit (ORMP) [3] and Optimized Orthogonal Matching Pursuit (OOMP) [4] in the literature.

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