ERRATUM: A Center Manifold Result for Delayed Neural Fields Equations

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ERRATUM: A CENTER MANIFOLD RESULT FOR DELAYED NEURAL FIELDS EQUATIONS

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Abstract. Lemma C.1 in [R. Veltz and O. Faugeras, *SIAM J. Math. Anal.*, 45(3) (2013), pp. 1527-1562] is wrong. This lemma is used in the proof of the existence of a smooth center manifold, Theorem 4.4 in [5]. An additional assumption is required to prove this existence. We spell out this assumption, correct the proofs and show that the assumption is satisfied for a large class of delay functions τ. We also weaken the general assumptions on τ.

Lemma C.1 in [5] is wrong as shown by the counterexample $\phi(\theta,\bar{\mathbf{r}}) = 1_{\tau(\mathbf{r}_0,\cdot)^{-1}(\{\theta\})}(\bar{\mathbf{r}})$, $\theta \in [-\tau_m, 0]$, $\bar{\mathbf{r}} \in \Omega$ for some $\mathbf{r}_0 \in \Omega$ and sufficiently regular delay function τ. This lemma is used in the proof of the regularity of $\mathbf{R}$ in Lemma 4.2 of [5].

1. Corrections to the paper. To correct this problem requires choosing a slightly different functional setup from the one in the paper. We redefine the spaces $\mathcal{X}^{(q)}$ and $\mathcal{Y}^{(q)}$ for $q > 2$ (definition 2.4 of [5]) as

$$\begin{cases} \mathcal{X}^{(q)} \equiv \mathrm{L}^\infty \times \mathrm{L}^\mathrm{q}(-\tau_\mathrm{m}, 0; \mathrm{L}^\infty), \qquad \mathrm{L}^\infty \equiv \mathrm{L}^\infty(\Omega, \mathbb{R}^\mathrm{p}) \\ \mathcal{Y}^{(q)} \equiv \left\{ u \in \mathrm{L}^\infty \times \mathrm{W}^{1,\mathrm{q}}(-\tau_\mathrm{m}, 0; \mathrm{L}^\infty) \mid \pi_1 \mathrm{u} = (\pi_2 \mathrm{u})(0) \right\} \end{cases}$$

and keep the original definition for $q = 2$:

$$\mathcal{X}^{(2)} \equiv \mathrm{L}^2 \times \mathrm{L}^2(-\tau_\mathrm{m}, 0; \mathrm{L}^2).$$

This choice does not alter the linear analysis (sections 1-3) in the paper but it affects a) Lemma B.2, b) Lemma 4.2, c) Theorem 4.4 and d) the main text in section 4.1, Lemma C.3 and Proposition C.4 as follows.

a) Lemma B.2 needs to be proved for the new spaces $\mathcal{X}^{(q)}$ as shown in section 3.
b) Lemma 4.2 requires a different proof given in section 4 below. Lemma C.1 needs to be re-written in a way we also explain in section 4.
c) In Theorem 4.4, the statement $\Psi \in C^q(\mathcal{X}_c \times \mathbb{R}^{m_{par}}; \mathcal{Y}_h)$ becomes $\Psi \in C^k(\mathcal{X}_c \times \mathbb{R}^{m_{par}}; \mathcal{Y}_h)$ (where $S \in C^k(\mathbb{R}^p, \mathbb{R}^p)$).
d) The main text in section 4.1 and Lemma C.3, Proposition C.4 (and their proofs) remain exactly the same modulo the change $\mathrm{L}^\mathrm{q} \to \mathrm{L}^\infty$ (*i.e.* $\|\cdot\|_{\mathrm{L}^\mathrm{q}} \to \|\cdot\|_{\mathrm{L}^\infty}$, $\mathrm{L}^\mathrm{q}(-\tau_\mathrm{m}, 0; \mathrm{L}^\mathrm{q}) \to \mathrm{L}^\mathrm{q}(-\tau_\mathrm{m}, 0; \mathrm{L}^\infty)$ and $\mathrm{W}^{1,\mathrm{q}}(-\tau_\mathrm{m}, 0; \mathrm{L}^\mathrm{q}) \to \mathrm{W}^{1,\mathrm{q}}(-\tau_\mathrm{m}, 0; \mathrm{L}^\infty)$).

2. Preliminaries. In order to modify and prove Lemma B.2 we need the following measure-theoretical preliminaries. We assume that $\tau \in \mathrm{L}^\infty(\Omega^2, \mathbb{R}^+)$. For each $\mathbf{x} \in \Omega$ define $\tau_\mathbf{x} : \Omega \to [-\tau_m, 0]$ as $\tau_\mathbf{x}(\mathbf{y}) = -\tau(\mathbf{x}, \mathbf{y})$ for all $\mathbf{y} \in \Omega$. We note λ_p the Lebesgue measure on $\mathbb{R}^p$ and $\tau_\mathbf{x} * \lambda_p$ the pushforward measure[1] of λ_p by $\tau_\mathbf{x}$, i.e. the measure on $[-\tau_m, 0]$ such that for each Borelian B of $[-\tau_m, 0]$, $\tau_\mathbf{x} * \lambda_p(B)$ is equal to $\lambda_p(\tau_\mathbf{x}^{-1}(B))$.

Note that $\tau_\mathbf{x} * \lambda_p([-\tau_m, 0]) \leq \lambda_p(\Omega)$ for all $\mathbf{x} \in \Omega$ and that for all measurable function f on $[-\tau_m, 0]$ we have the equality:

$$\int_\Omega f(-\tau(\mathbf{x}, \mathbf{y})) d\lambda_p(\mathbf{y}) = \int_{-\tau_m}^0 f(\theta) d\left(\tau_\mathbf{x} * \lambda_p\right)(\theta)$$

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[1] Legitimate because $\tau_\mathbf{x}$ is measurable

whenever $f \circ \tau_{\mathbf{x}}$ is λ_p integrable [2][th. 3.6.1]. As $\tau_{\mathbf{x}} * \lambda_p$ and λ_1 are σ-finite, the Lebesgue-Radon-Nikodým theorem give the following decomposition

$$\forall \mathbf{x} \in \Omega, \quad \frac{1}{\lambda_p(\Omega)} \tau_{\mathbf{x}} * \lambda_p = \mu_{\mathbf{x}}^{(abs)} + \mu_{\mathbf{x}}^{(at)} + \mu_{\mathbf{x}}^{(sing)} \tag{2.1}$$

where $\mu_{\mathbf{x}}^{(abs)}$, $\mu_{\mathbf{x}}^{(at)}$ and $\mu_{\mathbf{x}}^{(sing)}$ are three measures on $[-\tau_m, 0]$ such that $\mu_{\mathbf{x}}^{(abs)}$ is absolutely continuous w.r.t λ_1, with density $g_{\mathbf{x}}$, $\mu_{\mathbf{x}}^{(at)}$ is atomic and $\mu_{\mathbf{x}}^{(sing)}$ is continuous singular. We have:

$$\mu_{\mathbf{x}}^{(at)} = \sum_n a_n(\mathbf{x}) \delta_{-D_n(\mathbf{x})}.$$

If we define $D_{\mathbf{x}} = \left\{\theta \in [-\tau_m, 0] \mid \lambda_p\left(\tau_{\mathbf{x}}^{-1}(\theta)\right) > 0\right\}$, then $D_{\mathbf{x}}$ is at most countable [3] [XIII.18.6], hence we write $D_{\mathbf{x}} = \left(D_n(\mathbf{x})\right)_n$. We make the following hypothesis (justified in section 3.1)

$$\boxed{\text{(H1) } \forall \mathbf{x} \in \Omega \quad \mu_{\mathbf{x}}^{(sing)} = 0,\ D_n(\mathbf{x}) = D_n \text{ and } a_n(\mathbf{x}) = a_n.}$$

In this case, we can write for f $\tau_{\mathbf{x}} * \lambda_p$−integrable

$$\forall \mathbf{x}, \quad \frac{1}{\lambda_p(\Omega)} \int_\Omega f(-\tau(\mathbf{x}, \mathbf{y})) d\lambda_p(\mathbf{y}) = \int_{-\tau_m}^0 f(\theta) g_{\mathbf{x}}(\theta) d\lambda_1(\theta) + \sum_n a_n f\left(-D_n\right).$$

Lemma 2.1. *The two components $\mu_{\mathbf{x}}^{(abs)}$ and $\mu_{\mathbf{x}}^{(at)}$ in the decomposition of the measure $\tau_{\mathbf{x}} * \lambda_p$ satisfy:*

- $\forall \mathbf{x} \in \Omega$, $0 \le g_{\mathbf{x}} \le 1$ *a.e. and* $\sup_{\mathbf{x}} \|g_{\mathbf{x}}\|_1 \le 1$*. It implies* $\sup_{\mathbf{x}} \|g_{\mathbf{x}}\|_q \le 1$ *for all* $q \ge 1$.
- $0 \le a_n \le 1$ *and* $\sum_n a_n \le 1$.

Proof. This is a consequence of $\tau_{\mathbf{x}} * \lambda_p$ being finite and positive.

3. Correction of Lemma B.2 for the new spaces $\mathcal{X}^{(q)}$.

The domain of $S_t, T_0(t)$ is changed from $\mathrm{L^q}$ to L^∞ (see [1]).

Lemma 3.1. *(Lemma B.2 of [5]) Assume that (H1) is satisfied and that $\mathbf{J} \in \mathrm{L}^\infty(\Omega^2, \mathbb{R}^{\mathrm{p}\times\mathrm{p}})$. Then for each space $\mathcal{X}^{(q)}$ with $2 \le q < \infty$, there exists $Q : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to 0^+} Q(t) = 0$ such that*

$$\forall \begin{bmatrix} x \\ \phi \end{bmatrix} \in D(\mathbf{A}_{(q)}) \quad \int_0^t \|\mathbf{L}_1(S_s x + T_0(s)\phi)\|_{\mathrm{L}^\infty} ds \le Q(t) \left\| \begin{bmatrix} x \\ \phi \end{bmatrix} \right\|_{\mathcal{X}^{(q)}} \quad (M).$$

Proof. The case $q = 2$ is proved in the paper. Let us focus on the case $q > 2$. We first note that for $\phi \in \mathrm{W}^{1,\mathrm{q}}(-\tau_\mathrm{m}, 0; \mathrm{L}^\infty)$:

$$\|\phi(s - \tau(\mathbf{x}, \mathbf{y}), \mathbf{y})\|_{\mathbb{R}^p} \le \|\phi(s - \tau(\mathbf{x}, \mathbf{y}))\|_{\mathrm{L}^\infty}. \tag{3.1}$$

We focus on the second term $\int_0^t \|\mathbf{L}_1(T_0(s)\phi)\|_{\mathrm{L}^\infty} ds$ which is the most difficult to handle. Indeed, the first term $\int_0^t \|\mathbf{L}_1(S_s x)\|_{\mathrm{L}^\infty} ds$ is bounded by $Kt\|x\|$ for some constant K because the norm of $\mathbf{L}_1 S_s$ on L^∞ is bounded by a constant independent of s. By definition of $\mathrm{L}_1, \mathrm{T}_0$:

$$\left(\mathbf{L}_1(T_0(s)\phi)\right)(\mathbf{x}) = \int_\Omega \mathbf{1}_{[0,\tau(\mathbf{x},\mathbf{y})]}(s) \mathbf{J}(\mathbf{x}, \mathbf{y}) \phi(s - \tau(\mathbf{x}, \mathbf{y}), \mathbf{y}) d\lambda_p(\mathbf{y}),$$

which gives:

$$\|(\mathbf{L}_1(T_0(s)\phi))(\mathbf{x})\|_{\mathbb{R}^p} \leq |||\mathbf{J}|||_\infty \int_\Omega \mathbf{1}_{[0,\tau(\mathbf{x},\mathbf{y})]}(s)\,\|\phi(s-\tau(\mathbf{x},\mathbf{y}))\|_{\mathbb{R}^p}\,d\lambda_p(\mathbf{y}).$$

For any given function $f \in \mathrm{L}^1([-\tau_\mathrm{m}, 0], \mathbb{R})$, we extend it to a function of $\mathrm{L}^1([-\tau_\mathrm{m}, \tau_\mathrm{m}], \mathbb{R})$ by setting $f = 0$ on $[0, \tau_m]$ so that we do not have to worry about the integral bounds. According to (2.1) we have

$$\begin{aligned}
&\frac{1}{\lambda_p(\Omega)}\int_\Omega \mathbf{1}_{[0,\tau(\mathbf{x},\mathbf{y})]}(s)\,\|\phi(s-\tau(\mathbf{x},\mathbf{y}))\|_{\mathrm{L}^\infty}\,d\lambda_p(\mathbf{y}) = \\
&\qquad\frac{1}{\lambda_p(\Omega)}\int_\Omega \mathbf{1}_{[0,-\tau_\mathbf{x}(\mathbf{y})]}(s)\,\|\phi(s+\tau_\mathbf{x}(\mathbf{y}))\|_{\mathrm{L}^\infty}\,d\lambda_p(\mathbf{y}) = \\
&\qquad\frac{1}{\lambda_p(\Omega)}\int_\Omega \mathbf{1}_{[0,-\theta]}(s)\,\|\phi(s+\theta)\|_{\mathrm{L}^\infty}\,d(\tau_\mathbf{x} * \lambda_p)(\theta) = \\
&\int_{-\tau_m}^0 1_{[0,-\theta]}(s)\,\|\phi(s+\theta)\|_{\mathrm{L}^\infty}\,g_\mathbf{x}(s+\theta)d\lambda_1(\theta) + \sum_n a_n 1_{[0,D_n]}(s)\,\|\phi(s-D_n)\|_{\mathrm{L}^\infty}\,.
\end{aligned} \tag{3.2}$$

From Lemma 2.1

$$\sup_\mathbf{x} \int_{-\tau_m}^0 1_{[0,-\theta]}(s)\,\|\phi(s+\theta)\|_{\mathrm{L}^\infty}\,g_\mathbf{x}(s+\theta)d\lambda_1(\theta) \overset{\text{Hölder}}{\leq} \|\phi\|_{\mathrm{L}^q(-\tau_\mathrm{m},0;\mathrm{L}^\infty)}\,.$$

As $\phi \in D(\mathbf{A}_{(q)})$, it belongs to $W^{1,q}(-\tau_m, 0; \mathrm{L}^\infty)$. Hence, the function $\theta \to \|\phi(\theta)\|_{\mathrm{L}^\infty}$ is continuous on $[-\tau_m, 0]$ and its supremum is a max attained at $\theta = -D_{max}$. This gives:

$$\sum_n a_n 1_{[0,D_n]}(s)\,\|\phi(s-D_n)\|_{\mathrm{L}^\infty} \leq \|\phi(-D_{max})\|_{\mathrm{L}^\infty}$$

It then follows from the Beppo-Levi's theorem and Lemma 2.1 that

$$\begin{aligned}
&\int_0^{\min(t,\tau_m)} \sum_n a_n 1_{[0,D_n]}(s)\,\|\phi(s-D_n)\|_{\mathrm{L}^\infty}\,ds = \\
&\qquad\sum_n a_n \int_0^{\min(t,\tau_m)} 1_{[0,D_n]}(s)\,\|\phi(s-D_n)\|_{\mathrm{L}^\infty}\,ds \\
&\qquad\qquad\overset{\text{Hölder}}{\leq} (\min(t,\tau_m))^{\frac{1}{\bar q}}\,\|\phi\|_{\mathrm{L}^q(-\tau_\mathrm{m},0;\mathrm{L}^\infty)}\,,
\end{aligned}$$

where $\bar q$ is the Hölder conjugate integer of q. Summing up, we have found that:

$$\begin{aligned}
&\int_0^t \|\mathbf{L}_1(T_0(s)\phi)\|_{\mathrm{L}^\infty}\,ds \overset{\text{def. of } T_0}{=} \int_0^{\min(t,\tau_m)} \|\mathbf{L}_1(T_0(s)\phi)\|_{\mathrm{L}^\infty}\,ds \leq \\
&\qquad\lambda_p(\Omega)\,|||\mathbf{J}|||_\infty \left(t + \min(\tau_m, t)^{1/\bar q}\right)\|\phi\|_{\mathrm{L}^q(-\tau_\mathrm{m},0;\mathrm{L}^\infty)}
\end{aligned} \tag{3.3}$$

which gives

$$\int_0^t ds\,\|\mathbf{L}_1(S_s x + T_0(s)\phi)\|_{\mathrm{L}^\infty} \leq K_2 \max\left(t, \min(\tau_m, t)^{1/\bar q}\right)\left\|\begin{bmatrix} x \\ \phi \end{bmatrix}\right\|_{\mathcal{X}^{(q)}}.$$

for some constant K_2. This ends the proof. □

3.1. Example of possible delay functions. Let us show that the assumptions of lemma 3.1 are satisfied for some realistic delay function. Apart from (H1), the only requirement has been that

$$\tau \in \mathrm{L}^{\infty}(\Omega^2, \mathbb{R}^+).$$

The next lemma shows that (H1) holds for a large class of delay functions that includes a combination of constant and propagation delays,

LEMMA 3.2. *Let us consider $\tau(\mathbf{x},\mathbf{y}) = D + c\kappa(\mathbf{x},\mathbf{y})$ with $c,\kappa \geq 0$. We assume that $\forall \mathbf{x} \in \Omega$, $\kappa(\mathbf{x},\cdot) \in C^1(\overline{\Omega}, \mathbb{R}^+)$ and that $\forall \mathbf{x}$, the gradient of $\kappa(\mathbf{x},\mathbf{y})$ w.r.t. $\mathbf{y}$ is non zero almost everywhere. Then τ satisfies (H1).*

Proof. Straightforward application of integration theory on submanifolds.

□

4. Correction of Lemmas C.1 and 4.2. The new lemma C.1 reads as follows.

LEMMA 4.1. *(Lemmas B.1, C.1 of [5]) Assume that $\tau \in \mathrm{L}^{\infty}(\Omega^2, \mathbb{R}^+)$ and $\mathbf{J} \in \mathrm{L}^{\infty}(\Omega^2, \mathbb{R}^{\mathrm{p}\times\mathrm{p}})$, then we have the following results:*

1. *Define $\mathbf{J}[s]$ by $\forall s \in [-\tau_m, 0]$, $J_{ij}(\mathbf{r},\mathbf{r}')[s] \equiv J_{ij}(\mathbf{r},\mathbf{r}')H(s+\tau_{ij}(\mathbf{r},\mathbf{r}'))$ where H is the Heaviside function. Then:*

$$\mathbf{L}_1\phi = \mathbf{J}\phi(0) - \int_{-\tau_m}^{0} \mathbf{J}[s]\dot{\phi}(s)ds \quad \forall \phi \in \mathrm{W}^{1,\mathrm{q}}(-\tau_{\mathrm{m}}, 0; \mathrm{L}^{\infty}), \textit{for all } 2 \leq \mathrm{q} < \infty. \tag{4.1}$$

2. *$\mathbf{L}_1 \in \mathcal{L}\left(\mathrm{W}^{1,\mathrm{q}}(-\tau_{\mathrm{m}}, 0; \mathrm{L}^{\infty}), \mathrm{L}^{\infty}\right)$ for all $2 \leq q < \infty$.*
3. *$\forall l \in \mathbb{N}$, $(\phi_1, \cdots, \phi_l) \to \mathbf{L}_1(\phi_1 \cdots \phi_l)$ is linear continuous from $\left(\mathrm{W}^{1,\mathrm{q}}(-\tau_{\mathrm{m}}, 0; \mathrm{L}^{\infty})\right)^l$ to L^{∞}.*

Proof.

1. Let us consider $\phi \in \mathrm{W}^{1,\mathrm{q}}(-\tau_{\mathrm{m}}, 0; \mathrm{L}^{\infty})$ with $2 \leq q < \infty$. From the definition of Bochner spaces, we have

$$\phi_j(\bar{\mathbf{r}}, \theta) = -\int_{\theta}^{0} \dot{\phi}_j(\bar{\mathbf{r}}, s)ds + \phi_j(\bar{\mathbf{r}}, 0) \text{ for almost all } \bar{\mathbf{r}} \in \Omega$$

   which gives:

$$\phi_j(\bar{\mathbf{r}}, \theta) = -\int_{-\tau_m}^{0} \dot{\phi}_j(\bar{\mathbf{r}}, s)H(s-\theta)ds + \phi_j(\bar{\mathbf{r}}, 0) \text{ for almost all } \bar{\mathbf{r}} \in \Omega \tag{4.2}$$

   Moreover, $\forall \mathbf{r} \in \Omega$, $\theta \to H(\theta + \tau(\mathbf{r},\cdot)) \in \mathrm{L}^{\infty}(-\tau_{\mathrm{m}}, 0; \mathbb{R}^+) \subset \mathrm{L}^{\infty}(-\tau_{\mathrm{m}}, 0; \mathrm{L}^{\infty})$. This shows that $\forall \mathbf{r} \in \Omega, \theta \to \dot{\phi}_j(\cdot,\theta)H(\theta + \tau(\mathbf{r},\cdot)) \in \mathrm{L}^{\mathrm{q}}(-\tau_{\mathrm{m}}, 0; \mathrm{L}^{\infty})$ with $q \geq 2$. From (4.2) and the definition of the Bochner integral, it follows that $\phi(\cdot, -\tau(\mathbf{r},\cdot)) \in \mathrm{L}^{\infty}$. As $\mathbf{J} \in \mathrm{L}^{\infty}(\Omega^2, \mathbb{R}^{\mathrm{p}\times\mathrm{p}})$, it implies that $\mathbf{L}_1$ is well defined and $\mathbf{L}_1\phi \in \mathrm{L}^{\infty}$.
   Plugging (4.2) in the expression of $\mathbf{L}_1$ and using[2] [4][proposition C.4] gives the equality of the lemma with $\mathbf{J}(\mathbf{r},\mathbf{r}')[s] \in \mathrm{L}^{\infty}(\Omega^2, \mathbb{R}^{\mathrm{p}\times\mathrm{p}})$.
2. If $\mathbf{J} \in \mathrm{L}^{\infty}(\Omega^2, \mathbb{R}^{\mathrm{p}\times\mathrm{p}})$, it is straightforward to show that $\mathbf{L}_1 \in \mathcal{L}\left(\mathrm{W}^{1,\mathrm{q}}(-\tau_{\mathrm{m}}, 0; \mathrm{L}^{\infty}), \mathrm{L}^{\infty}\right)$ by using (4.1).
3. This is a consequence of 2. and the fact that $\mathrm{W}^{1,\mathrm{q}}(-\tau_{\mathrm{m}}, 0; \mathrm{L}^{\infty})$ is a Banach algebra.

[2]Basically $\mathbf{J}\int_0^{\theta} \dot{\phi} = \int_0^{\theta} \mathbf{J}\dot{\phi}$

□

This allows us to obtain a corrected version of Lemma 4.2 of [5].

LEMMA 4.2. *(Lemma 4.2 of [5]) If* $\mathbf{J} \in \mathrm{L}^\infty(\Omega^2, \mathbb{R}^{\mathrm{p}\times\mathrm{p}})$, $\mathbf{S} \in \mathrm{C}^k(\mathbb{R}^p, \mathbb{R}^p)$ *and* $2 < q < \infty$. *Then*

$$\begin{array}{rcl} \mathbf{A}_{(q)} & \in & \mathcal{L}(\mathcal{Y}^{(q)}, \mathcal{X}^{(q)}), \\ \mathbf{R} & \in & \mathrm{C}^k(\mathcal{Y}^{(q)} \times \mathbb{R}^{m_{par}}, \mathcal{X}^{(q)}), \end{array}$$

and

$$D^l_u \mathbf{R}(u_0, \mu)[u_1, \cdots, u_l] = \begin{bmatrix} \mathbf{L}_1(\mu)\mathbf{S}^{(l)}(\mathbf{V}^f + \pi_2 u_0)\pi_2(u_1 \cdots u_l) \\ 0 \end{bmatrix}, \ l = 1, \cdots, k$$

where $u_1 \cdots u_l$ *is the component-wise product of the functions* u_i *in* $\mathcal{Y}^{(q)}$.

Proof.

Case of $\mathbf{A}_{(q)}$. The only "difficulty" is showing that $\phi \to \mathbf{L}_1 \cdot (D\mathbf{S}(\mathbf{V}^f)\phi)$ belongs to $\mathcal{L}\left(W^{1,q}(-\tau_m, 0; \mathrm{L}^\infty), \mathrm{L}^\infty\right)$. This was done in Lemma 4.1.2.

Case of R. Recall that $\pi_2 \mathbf{R} = 0$ while

$$\pi_1 \mathbf{R}(u, \mu) \stackrel{def}{=} \mathbf{L}_1(\mu) \cdot \left[\mathbf{S}\left(\mathbf{V}^f + \pi_2 u\right) - \mathbf{S}\left(\mathbf{V}^f\right)\right] - \mathbf{L}_1(\mu_c) \cdot D\mathbf{S}\left(\mathbf{V}^f\right)\pi_2 u$$

We focus on the differentiability at $u = 0$ and ignore the differentiability w.r.t. the parameter μ. The differentiability at $u \neq 0$ follows from the same argument. It is easy to see from the definition of $\mathbf{R}$ that, since $\mathbf{S}$ is C^k and $\mathbf{L}_1$ is bounded (lemma 4.1.3)

$$\pi_1 D^l \mathbf{R}(0, \mu)[u_1, \cdots, u_l] = \begin{cases} (\mathbf{L}_1(\mu) - \mathbf{L}_1(\mu_c)) \cdot \mathbf{S}^{(1)}(\mathbf{V}^f)\pi_2 u_1 & l = 1 \\ \mathbf{L}_1(\mu) \cdot \mathbf{S}^{(l)}(\mathbf{V}^f)\pi_2(u_1 \cdots u_l) & l = 2, \cdots, k \end{cases}$$

The proof that $\pi_1 R(u, \mu)$ is C^k at $u = 0$ then follows from the fact that $\mathbf{S}$ is C^k and $\mathbf{L}_1$ is a bounded operator:

$$\pi_1 \mathbf{R}(h, \mu) - \sum_{l=1}^k \frac{1}{l!}\pi_1 D^l \mathbf{R}(0, \mu)\pi_2 h^l = \mathbf{L}_1(\mu) \cdot \left(\mathbf{S}(\mathbf{V}^f + \pi_2 h) - \mathbf{S}(\mathbf{V}^f) - \sum_{l=1}^k \frac{\mathbf{S}^{(l)}(\mathbf{V}^f)}{l!}\pi_2 h^l\right)$$

□

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