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# ERRATUM: A CENTER MANIFOLD RESULT FOR DELAYED NEURAL FIELDS EQUATIONS

ROMAIN VELTZ\* AND OLIVIER FAUGERAS†

**Abstract.** Lemma C.1 in [R. Veltz and O. Faugeras, *SIAM J. Math. Anal.*, 45(3) (2013), pp. 1527-1562] is wrong. This lemma is used in the proof of the existence of a smooth center manifold, Theorem 4.4 in [5]. An additional assumption is required to prove this existence. We spell out this assumption, correct the proofs and show that the assumption is satisfied for a large class of delay functions  $\tau$ . We also weaken the general assumptions on  $\tau$ .

Lemma C.1 in [5] is wrong as shown by the counterexample  $\phi(\theta, \bar{\mathbf{r}}) = 1_{\tau(\mathbf{r}_0, \cdot)^{-1}(\{\theta\})}(\bar{\mathbf{r}})$ ,  $\theta \in [-\tau_m, 0]$ ,  $\bar{\mathbf{r}} \in \Omega$  for some  $\mathbf{r}_0 \in \Omega$  and sufficiently regular delay function  $\tau$ . This lemma is used in the proof of the regularity of  $\mathbf{R}$  in Lemma 4.2 of [5].

**1. Corrections to the paper.** To correct this problem requires choosing a slightly different functional setup from the one in the paper. We redefine the spaces  $\mathcal{X}^{(q)}$  and  $\mathcal{Y}^{(q)}$  for  $q > 2$  (definition 2.4 of [5]) as

$$\begin{cases} \mathcal{X}^{(q)} \equiv L^\infty \times L^q(-\tau_m, 0; L^\infty), & L^\infty \equiv L^\infty(\Omega, \mathbb{R}^p) \\ \mathcal{Y}^{(q)} \equiv \{u \in L^\infty \times W^{1,q}(-\tau_m, 0; L^\infty) \mid \pi_1 u = (\pi_2 u)(0)\} \end{cases}$$

and keep the original definition for  $q = 2$ :

$$\mathcal{X}^{(2)} \equiv L^2 \times L^2(-\tau_m, 0; L^2).$$

This choice does not alter the linear analysis (sections 1-3) in the paper but it affects a) Lemma B.2, b) Lemma 4.2, c) Theorem 4.4 and d) the main text in section 4.1, Lemma C.3 and Proposition C.4 as follows.

- a) Lemma B.2 needs to be proved for the new spaces  $\mathcal{X}^{(q)}$  as shown in section 3.
- b) Lemma 4.2 requires a different proof given in section 4 below. Lemma C.1 needs to be re-written in a way we also explain in section 4.
- c) In Theorem 4.4, the statement  $\Psi \in C^q(\mathcal{X}_c \times \mathbb{R}^{m_{par}}; \mathcal{Y}_h)$  becomes  $\Psi \in C^k(\mathcal{X}_c \times \mathbb{R}^{m_{par}}; \mathcal{Y}_h)$  (where  $S \in C^k(\mathbb{R}^p, \mathbb{R}^p)$ ).
- d) The main text in section 4.1 and Lemma C.3, Proposition C.4 (and their proofs) remain exactly the same modulo the change  $L^q \rightarrow L^\infty$  (i.e.  $\|\cdot\|_{L^q} \rightarrow \|\cdot\|_{L^\infty}$ ,  $L^q(-\tau_m, 0; L^q) \rightarrow L^q(-\tau_m, 0; L^\infty)$  and  $W^{1,q}(-\tau_m, 0; L^q) \rightarrow W^{1,q}(-\tau_m, 0; L^\infty)$ ).

**2. Preliminaries.** In order to modify and prove Lemma B.2 we need the following measure-theoretical preliminaries. We assume that  $\tau \in L^\infty(\Omega^2, \mathbb{R}^+)$ . For each  $\mathbf{x} \in \Omega$  define  $\tau_{\mathbf{x}} : \Omega \rightarrow [-\tau_m, 0]$  as  $\tau_{\mathbf{x}}(\mathbf{y}) = -\tau(\mathbf{x}, \mathbf{y})$  for all  $\mathbf{y} \in \Omega$ . We note  $\lambda_p$  the Lebesgue measure on  $\mathbb{R}^p$  and  $\tau_{\mathbf{x}} * \lambda_p$  the pushforward measure<sup>1</sup> of  $\lambda_p$  by  $\tau_{\mathbf{x}}$ , i.e. the measure on  $[-\tau_m, 0]$  such that for each Borelian  $B$  of  $[-\tau_m, 0]$ ,  $\tau_{\mathbf{x}} * \lambda_p(B)$  is equal to  $\lambda_p(\tau_{\mathbf{x}}^{-1}(B))$ .

Note that  $\tau_{\mathbf{x}} * \lambda_p([-\tau_m, 0]) \leq \lambda_p(\Omega)$  for all  $\mathbf{x} \in \Omega$  and that for all measurable function  $f$  on  $[-\tau_m, 0]$  we have the equality:

$$\int_{\Omega} f(-\tau(\mathbf{x}, \mathbf{y})) d\lambda_p(\mathbf{y}) = \int_{-\tau_m}^0 f(\theta) d(\tau_{\mathbf{x}} * \lambda_p)(\theta)$$

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<sup>1</sup> Legitimate because  $\tau_{\mathbf{x}}$  is measurable

whenever  $f \circ \tau_{\mathbf{x}}$  is  $\lambda_p$  integrable [2][th. 3.6.1]. As  $\tau_{\mathbf{x}} * \lambda_p$  and  $\lambda_1$  are  $\sigma$ -finite, the Lebesgue-Radon-Nikodým theorem give the following decomposition

$$\forall \mathbf{x} \in \Omega, \quad \frac{1}{\lambda_p(\Omega)} \tau_{\mathbf{x}} * \lambda_p = \mu_{\mathbf{x}}^{(abs)} + \mu_{\mathbf{x}}^{(at)} + \mu_{\mathbf{x}}^{(sing)} \quad (2.1)$$

where  $\mu_{\mathbf{x}}^{(abs)}$ ,  $\mu_{\mathbf{x}}^{(at)}$  and  $\mu_{\mathbf{x}}^{(sing)}$  are three measures on  $[-\tau_m, 0]$  such that  $\mu_{\mathbf{x}}^{(abs)}$  is absolutely continuous w.r.t  $\lambda_1$ , with density  $g_{\mathbf{x}}$ ,  $\mu_{\mathbf{x}}^{(at)}$  is atomic and  $\mu_{\mathbf{x}}^{(sing)}$  is continuous singular. We have:

$$\mu_{\mathbf{x}}^{(at)} = \sum_n a_n(\mathbf{x}) \delta_{-D_n(\mathbf{x})}.$$

If we define  $D_{\mathbf{x}} = \{\theta \in [-\tau_m, 0] \mid \lambda_p(\tau_{\mathbf{x}}^{-1}(\theta)) > 0\}$ , then  $D_{\mathbf{x}}$  is at most countable [3] [XIII.18.6], hence we write  $D_{\mathbf{x}} = (D_n(\mathbf{x}))_n$ . We make the following hypothesis (justified in section 3.1)

$$\boxed{(H1) \quad \forall \mathbf{x} \in \Omega \quad \mu_{\mathbf{x}}^{(sing)} = 0, D_n(\mathbf{x}) = D_n \text{ and } a_n(\mathbf{x}) = a_n.}$$

In this case, we can write for  $f \tau_{\mathbf{x}} * \lambda_p$ -integrable

$$\forall \mathbf{x}, \quad \frac{1}{\lambda_p(\Omega)} \int_{\Omega} f(-\tau(\mathbf{x}, \mathbf{y})) d\lambda_p(\mathbf{y}) = \int_{-\tau_m}^0 f(\theta) g_{\mathbf{x}}(\theta) d\lambda_1(\theta) + \sum_n a_n f(-D_n).$$

LEMMA 2.1. *The two components  $\mu_{\mathbf{x}}^{(abs)}$  and  $\mu_{\mathbf{x}}^{(at)}$  in the decomposition of the measure  $\tau_{\mathbf{x}} * \lambda_p$  satisfy:*

- $\forall \mathbf{x} \in \Omega, 0 \leq g_{\mathbf{x}} \leq 1$  a.e. and  $\sup_{\mathbf{x}} \|g_{\mathbf{x}}\|_1 \leq 1$ . It implies  $\sup_{\mathbf{x}} \|g_{\mathbf{x}}\|_q \leq 1$  for all  $q \geq 1$ .
- $0 \leq a_n \leq 1$  and  $\sum_n a_n \leq 1$ .

*Proof.* This is a consequence of  $\tau_{\mathbf{x}} * \lambda_p$  being finite and positive.

**3. Correction of Lemma B.2 for the new spaces  $\mathcal{X}^{(q)}$ .** The domain of  $S_t, T_0(t)$  is changed from  $L^q$  to  $L^\infty$  (see [1]).

LEMMA 3.1. *(Lemma B.2 of [5]) Assume that (H1) is satisfied and that  $\mathbf{J} \in L^\infty(\Omega^2, \mathbb{R}^{p \times p})$ . Then for each space  $\mathcal{X}^{(q)}$  with  $2 \leq q < \infty$ , there exists  $Q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\lim_{t \rightarrow 0^+} Q(t) = 0$  such that*

$$\forall \begin{bmatrix} x \\ \phi \end{bmatrix} \in D(\mathbf{A}_{(q)}) \quad \int_0^t \|\mathbf{L}_1(S_s x + T_0(s)\phi)\|_{L^\infty} ds \leq Q(t) \left\| \begin{bmatrix} x \\ \phi \end{bmatrix} \right\|_{\mathcal{X}^{(q)}} \quad (M).$$

*Proof.* The case  $q = 2$  is proved in the paper. Let us focus on the case  $q > 2$ . We first note that for  $\phi \in W^{1,q}(-\tau_m, 0; L^\infty)$ :

$$\|\phi(s - \tau(\mathbf{x}, \mathbf{y}), \mathbf{y})\|_{\mathbb{R}^p} \leq \|\phi(s - \tau(\mathbf{x}, \mathbf{y}))\|_{L^\infty}. \quad (3.1)$$

We focus on the second term  $\int_0^t \|\mathbf{L}_1(T_0(s)\phi)\|_{L^\infty} ds$  which is the most difficult to handle. Indeed, the first term  $\int_0^t \|\mathbf{L}_1(S_s x)\|_{L^\infty} ds$  is bounded by  $Kt \|x\|$  for some constant  $K$  because the norm of  $\mathbf{L}_1 S_s$  on  $L^\infty$  is bounded by a constant independent of  $s$ . By definition of  $\mathbf{L}_1, T_0$ :

$$(\mathbf{L}_1(T_0(s)\phi))(\mathbf{x}) = \int_{\Omega} \mathbf{1}_{[0, \tau(\mathbf{x}, \mathbf{y})]}(s) \mathbf{J}(\mathbf{x}, \mathbf{y}) \phi(s - \tau(\mathbf{x}, \mathbf{y}), \mathbf{y}) d\lambda_p(\mathbf{y}),$$

which gives:

$$\|(\mathbf{L}_1(T_0(s)\phi))(\mathbf{x})\|_{\mathbb{R}^p} \leq \| \mathbf{J} \|_{\infty} \int_{\Omega} \mathbf{1}_{[0, \tau(\mathbf{x}, \mathbf{y})]}(s) \|\phi(s - \tau(\mathbf{x}, \mathbf{y}))\|_{\mathbb{R}^p} d\lambda_p(\mathbf{y}).$$

For any given function  $f \in L^1([-\tau_m, 0], \mathbb{R})$ , we extend it to a function of  $L^1([-\tau_m, \tau_m], \mathbb{R})$  by setting  $f = 0$  on  $[0, \tau_m]$  so that we do not have to worry about the integral bounds. According to (2.1) we have

$$\begin{aligned} \frac{1}{\lambda_p(\Omega)} \int_{\Omega} \mathbf{1}_{[0, \tau(\mathbf{x}, \mathbf{y})]}(s) \|\phi(s - \tau(\mathbf{x}, \mathbf{y}))\|_{L^\infty} d\lambda_p(\mathbf{y}) &= \\ \frac{1}{\lambda_p(\Omega)} \int_{\Omega} \mathbf{1}_{[0, -\tau_{\mathbf{x}}(\mathbf{y})]}(s) \|\phi(s + \tau_{\mathbf{x}}(\mathbf{y}))\|_{L^\infty} d\lambda_p(\mathbf{y}) &= \\ \frac{1}{\lambda_p(\Omega)} \int_{\Omega} \mathbf{1}_{[0, -\theta]}(s) \|\phi(s + \theta)\|_{L^\infty} d(\tau_{\mathbf{x}} * \lambda_p)(\theta) &= \\ \int_{-\tau_m}^0 \mathbf{1}_{[0, -\theta]}(s) \|\phi(s + \theta)\|_{L^\infty} g_{\mathbf{x}}(s + \theta) d\lambda_1(\theta) + \sum_n a_n \mathbf{1}_{[0, D_n]}(s) \|\phi(s - D_n)\|_{L^\infty}. \end{aligned} \quad (3.2)$$

From Lemma 2.1

$$\sup_{\mathbf{x}} \int_{-\tau_m}^0 \mathbf{1}_{[0, -\theta]}(s) \|\phi(s + \theta)\|_{L^\infty} g_{\mathbf{x}}(s + \theta) d\lambda_1(\theta) \stackrel{\text{H\"older}}{\leq} \|\phi\|_{L^q(-\tau_m, 0; L^\infty)}.$$

As  $\phi \in D(\mathbf{A}_{(q)})$ , it belongs to  $W^{1,q}(-\tau_m, 0; L^\infty)$ . Hence, the function  $\theta \rightarrow \|\phi(\theta)\|_{L^\infty}$  is continuous on  $[-\tau_m, 0]$  and its supremum is a max attained at  $\theta = -D_{max}$ . This gives:

$$\sum_n a_n \mathbf{1}_{[0, D_n]}(s) \|\phi(s - D_n)\|_{L^\infty} \leq \|\phi(-D_{max})\|_{L^\infty}$$

It then follows from the Beppo-Levi's theorem and Lemma 2.1 that

$$\begin{aligned} \int_0^{\min(t, \tau_m)} \sum_n a_n \mathbf{1}_{[0, D_n]}(s) \|\phi(s - D_n)\|_{L^\infty} ds &= \\ \sum_n a_n \int_0^{\min(t, \tau_m)} \mathbf{1}_{[0, D_n]}(s) \|\phi(s - D_n)\|_{L^\infty} ds & \\ &\stackrel{\text{H\"older}}{\leq} (\min(t, \tau_m))^{\frac{1}{\bar{q}}} \|\phi\|_{L^q(-\tau_m, 0; L^\infty)}, \end{aligned}$$

where  $\bar{q}$  is the Hölder conjugate integer of  $q$ . Summing up, we have found that:

$$\begin{aligned} \int_0^t \|\mathbf{L}_1(T_0(s)\phi)\|_{L^\infty} ds \stackrel{\text{def. of } T_0}{=} \int_0^{\min(t, \tau_m)} \|\mathbf{L}_1(T_0(s)\phi)\|_{L^\infty} ds \leq \\ \lambda_p(\Omega) \| \mathbf{J} \|_{\infty} \left( t + \min(\tau_m, t)^{1/\bar{q}} \right) \|\phi\|_{L^q(-\tau_m, 0; L^\infty)} \end{aligned} \quad (3.3)$$

which gives

$$\int_0^t ds \|\mathbf{L}_1(S_s x + T_0(s)\phi)\|_{L^\infty} \leq K_2 \max\left(t, \min(\tau_m, t)^{1/\bar{q}}\right) \left\| \begin{bmatrix} x \\ \phi \end{bmatrix} \right\|_{\mathcal{X}^{(q)}}.$$

for some constant  $K_2$ . This ends the proof.  $\square$

**3.1. Example of possible delay functions.** Let us show that the assumptions of lemma 3.1 are satisfied for some realistic delay function. Apart from (H1), the only requirement has been that

$$\tau \in L^\infty(\Omega^2, \mathbb{R}^+).$$

The next lemma shows that (H1) holds for a large class of delay functions that includes a combination of constant and propagation delays,

LEMMA 3.2. *Let us consider  $\tau(\mathbf{x}, \mathbf{y}) = D + c\kappa(\mathbf{x}, \mathbf{y})$  with  $c, \kappa \geq 0$ . We assume that  $\forall \mathbf{x} \in \Omega$ ,  $\kappa(\mathbf{x}, \cdot) \in C^1(\overline{\Omega}, \mathbb{R}^+)$  and that  $\forall \mathbf{x}$ , the gradient of  $\kappa(\mathbf{x}, \mathbf{y})$  w.r.t.  $\mathbf{y}$  is non zero almost everywhere. Then  $\tau$  satisfies (H1).*

*Proof.* Straightforward application of integration theory on submanifolds.

□

**4. Correction of Lemmas C.1 and 4.2.** The new lemma C.1 reads as follows.

LEMMA 4.1. *(Lemmas B.1, C.1 of [5]) Assume that  $\tau \in L^\infty(\Omega^2, \mathbb{R}^+)$  and  $\mathbf{J} \in L^\infty(\Omega^2, \mathbb{R}^{p \times p})$ , then we have the following results:*

1. Define  $\mathbf{J}[s]$  by  $\forall s \in [-\tau_m, 0]$ ,  $J_{ij}(\mathbf{r}, \mathbf{r}')[s] \equiv J_{ij}(\mathbf{r}, \mathbf{r}')H(s + \tau_{ij}(\mathbf{r}, \mathbf{r}'))$  where  $H$  is the Heaviside function. Then:

$$\mathbf{L}_1\phi = \mathbf{J}\phi(0) - \int_{-\tau_m}^0 \mathbf{J}[s]\dot{\phi}(s)ds \quad \forall \phi \in W^{1,q}(-\tau_m, 0; L^\infty), \text{ for all } 2 \leq q < \infty. \quad (4.1)$$

2.  $\mathbf{L}_1 \in \mathcal{L}(W^{1,q}(-\tau_m, 0; L^\infty), L^\infty)$  for all  $2 \leq q < \infty$ .
3.  $\forall l \in \mathbb{N}$ ,  $(\phi_1, \dots, \phi_l) \rightarrow \mathbf{L}_1(\phi_1 \cdots \phi_l)$  is linear continuous from  $(W^{1,q}(-\tau_m, 0; L^\infty))^l$  to  $L^\infty$ .

*Proof.*

1. Let us consider  $\phi \in W^{1,q}(-\tau_m, 0; L^\infty)$  with  $2 \leq q < \infty$ . From the definition of Bochner spaces, we have

$$\phi_j(\bar{\mathbf{r}}, \theta) = - \int_{\theta}^0 \dot{\phi}_j(\bar{\mathbf{r}}, s)ds + \phi_j(\bar{\mathbf{r}}, 0) \text{ for almost all } \bar{\mathbf{r}} \in \Omega$$

which gives:

$$\phi_j(\bar{\mathbf{r}}, \theta) = - \int_{-\tau_m}^0 \dot{\phi}_j(\bar{\mathbf{r}}, s)H(s - \theta)ds + \phi_j(\bar{\mathbf{r}}, 0) \text{ for almost all } \bar{\mathbf{r}} \in \Omega \quad (4.2)$$

Moreover,  $\forall \mathbf{r} \in \Omega$ ,  $\theta \rightarrow H(\theta + \tau(\mathbf{r}, \cdot)) \in L^\infty(-\tau_m, 0; \mathbb{R}^+) \subset L^\infty(-\tau_m, 0; L^\infty)$ . This shows that  $\forall \mathbf{r} \in \Omega$ ,  $\theta \rightarrow \dot{\phi}_j(\cdot, \theta)H(\theta + \tau(\mathbf{r}, \cdot)) \in L^q(-\tau_m, 0; L^\infty)$  with  $q \geq 2$ . From (4.2) and the definition of the Bochner integral, it follows that  $\phi(\cdot, -\tau(\mathbf{r}, \cdot)) \in L^\infty$ . As  $\mathbf{J} \in L^\infty(\Omega^2, \mathbb{R}^{p \times p})$ , it implies that  $\mathbf{L}_1$  is well defined and  $\mathbf{L}_1\phi \in L^\infty$ .

Plugging (4.2) in the expression of  $\mathbf{L}_1$  and using<sup>2</sup> [4][proposition C.4] gives the equality of the lemma with  $\mathbf{J}(\mathbf{r}, \mathbf{r}')[s] \in L^\infty(\Omega^2, \mathbb{R}^{p \times p})$ .

2. If  $\mathbf{J} \in L^\infty(\Omega^2, \mathbb{R}^{p \times p})$ , it is straightforward to show that  $\mathbf{L}_1 \in \mathcal{L}(W^{1,q}(-\tau_m, 0; L^\infty), L^\infty)$  by using (4.1).
3. This is a consequence of 2. and the fact that  $W^{1,q}(-\tau_m, 0; L^\infty)$  is a Banach algebra.

<sup>2</sup>Basically  $\mathbf{J} \int_0^\theta \dot{\phi} = \int_0^\theta \mathbf{J} \dot{\phi}$

□

This allows us to obtain a corrected version of Lemma 4.2 of [5].

LEMMA 4.2. (Lemma 4.2 of [5]) If  $\mathbf{J} \in L^\infty(\Omega^2, \mathbb{R}^{p \times p})$ ,  $\mathbf{S} \in C^k(\mathbb{R}^p, \mathbb{R}^p)$  and  $2 < q < \infty$ . Then

$$\begin{aligned} \mathbf{A}_{(q)} &\in \mathcal{L}(\mathcal{Y}^{(q)}, \mathcal{X}^{(q)}), \\ \mathbf{R} &\in C^k(\mathcal{Y}^{(q)} \times \mathbb{R}^{m_{par}}, \mathcal{X}^{(q)}), \end{aligned}$$

and

$$D_u^l \mathbf{R}(u_0, \mu)[u_1, \dots, u_l] = \begin{bmatrix} \mathbf{L}_1(\mu) \mathbf{S}^{(l)}(\mathbf{V}^f + \pi_2 u_0) \pi_2(u_1 \cdots u_l) \\ 0 \end{bmatrix}, \quad l = 1, \dots, k$$

where  $u_1 \cdots u_l$  is the component-wise product of the functions  $u_i$  in  $\mathcal{Y}^{(q)}$ .

*Proof.*

**Case of  $\mathbf{A}_{(q)}$ .** The only "difficulty" is showing that  $\phi \rightarrow \mathbf{L}_1 \cdot (DS(\mathbf{V}^f)\phi)$  belongs to  $\mathcal{L}(W^{1,q}(-\tau_m, 0; L^\infty), L^\infty)$ . This was done in Lemma 4.1.2.

**Case of  $\mathbf{R}$ .** Recall that  $\pi_2 \mathbf{R} = 0$  while

$$\pi_1 \mathbf{R}(u, \mu) \stackrel{def}{=} \mathbf{L}_1(\mu) \cdot [\mathbf{S}(\mathbf{V}^f + \pi_2 u) - \mathbf{S}(\mathbf{V}^f)] - \mathbf{L}_1(\mu_c) \cdot DS(\mathbf{V}^f) \pi_2 u$$

We focus on the differentiability at  $u = 0$  and ignore the differentiability w.r.t. the parameter  $\mu$ . The differentiability at  $u \neq 0$  follows from the same argument. It is easy to see from the definition of  $\mathbf{R}$  that, since  $\mathbf{S}$  is  $C^k$  and  $\mathbf{L}_1$  is bounded (lemma 4.1.3)

$$\pi_1 D^l \mathbf{R}(0, \mu)[u_1, \dots, u_l] = \begin{cases} (\mathbf{L}_1(\mu) - \mathbf{L}_1(\mu_c)) \cdot \mathbf{S}^{(1)}(\mathbf{V}^f) \pi_2 u_1 & l = 1 \\ \mathbf{L}_1(\mu) \cdot \mathbf{S}^{(l)}(\mathbf{V}^f) \pi_2(u_1 \cdots u_l) & l = 2, \dots, k \end{cases}$$

The proof that  $\pi_1 \mathbf{R}(u, \mu)$  is  $C^k$  at  $u = 0$  then follows from the fact that  $\mathbf{S}$  is  $C^k$  and  $\mathbf{L}_1$  is a bounded operator:

$$\pi_1 \mathbf{R}(h, \mu) - \sum_{l=1}^k \frac{1}{l!} \pi_1 D^l \mathbf{R}(0, \mu) \pi_2 h^l = \mathbf{L}_1(\mu) \cdot \left( \mathbf{S}(\mathbf{V}^f + \pi_2 h) - \mathbf{S}(\mathbf{V}^f) - \sum_{l=1}^k \frac{\mathbf{S}^{(l)}(\mathbf{V}^f)}{l!} \pi_2 h^l \right)$$

□

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