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Jean-Daniel Boissonnat, Ramsay Dyer, Arijit Ghosh, Steve Y. Oudot

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Jean-Daniel Boissonnat * Ramsay Dyer † Arijit Ghosh ‡
Steve Y. Oudot §

December 18, 2014

Abstract

In this paper, we give the first algorithm that outputs a faithful reconstruction of a submanifold of Euclidean space without maintaining or even constructing complicated data structures such as Voronoi diagrams or Delaunay complexes. Our algorithm uses the witness complex and relies on the stability of *power protection*, a notion introduced in this paper. The complexity of the algorithm depends exponentially on the intrinsic dimension of the manifold, rather than the dimension of ambient space, and linearly on the dimension of the ambient space. Another interesting feature of this work is that no explicit coordinates of the points in the point sample is needed. The algorithm only needs the *distance matrix* as input, i.e., only distance between points in the point sample as input.

Keywords. Witness complex, power protection, sampling, manifold reconstruction

1 Introduction

We present an algorithm for reconstructing a submanifold of Euclidean space, from an input point sample, that does not require Delaunay complexes, unlike previous algorithms, which either had to maintain a subset of the Delaunay complex in the ambient space [CDR05, BGO09], or a family of m-dimensional Delaunay complexes [BG14]. Maintaining these highly structured data structures is challenging and in addition, the methods are limited as they require explicit coordinates of the points in the input point sample. One of the goals of this work was to develop a procedure to reconstruct submanifolds that only uses elementary data structures.

We use the witness complex to achieve this goal. The witness complex was introduced by Carlsson and de Silva [CdS04]. Given a point cloud W, their idea was to carefully select a subset L of landmarks on top of which the witness complex would be built, and to use the remaining data points to drive the complex construction. More precisely, a point $w \in W$ is called a *witness* for a simplex $\sigma \in 2^L$ if no point of $L \setminus \sigma$ is closer to w than are the

*Geometrica, INRIA, Sophia Antipolis, France Jean-Daniel.Boissonnat@inria.fr
†Johann Bernoulli Institute, University of Groningen, Groningen, The Netherlands, r.h.dyer@rug.nl
‡D1: Algorithms & Complexity, Max-Planck-Institut für Informatik, Germany agosh@mpi-inf.mpg.de
§Geometrica, INRIA, Saclay, France Steve.Oudot@inria.fr

vertices of σ, i.e., if there is a ball centered at w that includes the vertices of σ, but no other points of L. The witness complex is then the largest abstract simplicial complex that can be assembled using only witnessed simplices. The geometric test for being a witness can be viewed as a simplified version of the classical Delaunay predicate, and its great advantage is to only require mere comparisons of (squared) distances. As a result, witness complexes can be built in arbitrary metric spaces, and the construction time is bound to the size of the input point cloud rather than to the dimension d of the ambient space.

Since its introduction, the witness complex has attracted interest, which can be explained by its close connection to the Delaunay triangulation and the restricted Delaunay complex [AEM07, BGO09, CIDSZ06, CO08, CdS04, GO08]. In his seminal paper [dS08], de Silva showed that the witness complex is always a subcomplex of the Delaunay triangulation $\mathrm{Del}(L)$, provided that the data points lie in some Euclidean space or more generally in some Riemannian manifold of constant sectional curvature. With applications to reconstruction in mind, Attali *et al.* [AEM07] and Guibas and Oudot [GO08] considered the case where the data points lie on or close to some m-submanifold of $\mathbb{R}^d$. They showed that the witness complex is equal to the restricted Delaunay complex when $m = 1$, and a subset of it when $m = 2$. Unfortunately, the case of 3-manifolds is once again problematic, and it is now a well-known fact that the restricted Delaunay and witness complexes may differ significantly (no respective inclusion, different topological types, etc) when $m \geq 3$ [BGO09]. To overcome this issue, Boissonnat, Guibas and Oudot [BGO09] resorted to the sliver removal technique on some superset of the witness complex, whose construction incurs an exponential dependence on d. The state of affairs as of now is that the complexity of witness complex based manifold reconstruction is exponential in d, and whether it could be made only polynomial in d (while still exponential in m) was an open question, which this paper answers affirmatively.

Our contributions

Our paper relies on recent results on the stability of Delaunay triangulations [BDG13c] which we extend in the context of Laguerre geometry where points are weighted. We introduce the notion of power protection of Delaunay simplices and show that the weighting mechanism already used in [CDE+00, CDR05] and [BGO09] can be adapted to our context. As a result, we get an algorithm that constructs a (weighted) witness complex that is a *faithful reconstruction*, i.e. homeomorphic and a close geometric approximation, of the manifold. Differently from previous reconstruction algorithms [CDR05, BGO09, BG14], our algorithm can be simply adapted to work when we don't have explicit coordinates of the points but just the interpoint distance matrix.

2 Definitions and preliminaries

For the notations used in this paper, but not given in this section, refer to Section D.

2.1 General notations

We will mainly work in d-dimensional Euclidean space $\mathbb{R}^d$ with the standard norm, $\|\cdot\|$. The distance between $p \in \mathbb{R}^d$ and a set $X \subset \mathbb{R}^m$, is

$$d(p, X) = \inf_{x \in X} \|x - p\|.$$

We refer to the distance between two points a and b as $\|b - a\|$ or $d(a, b)$ as convenient.

A ball $B(c, r) = \{x : d(x, c)^2 < r^2\}$ is open, and $\overline{B}(c, r) = \{x : d(x, c)^2 \leq r^2\}$ is closed. A sphere is $S(c, r) = \overline{B}(c, r) \setminus B(c, r)$. Note that in this paper, we only assume that $r^2 \in \mathbb{R}$ and does not necessarily have to be ≥ 0.

For $X \subseteq \mathbb{R}^d$, $\mathsf{P} \subseteq \mathbb{R}^d$ is called an *ϵ-sample* of X if forall $x \in X$, $d(x, \mathsf{P}) < \epsilon$. The set P is called *ν-sparse* if for all p, $q\,(\neq p) \in \mathsf{P}$, $\|p - q\| \geq \nu$. The set P is called a *(ν, ϵ)-net* if P is ν-sparse ϵ-sample. When $\mu = \epsilon$, P is called an ϵ-net.

Generally, we denote the convex hull of a set X by $\mathrm{conv}(X)$, and the affine hull by $\mathrm{aff}(X)$. When we talk about $\dim X$, we mean $\dim \mathrm{aff}(X)$. The cardinality of X, and not its measure, is denoted by $\#X$. If $X \subseteq \mathbb{R}$, $\mu(X)$ denotes the standard Lesbesgue measure of X.

For given vectors u and v in $\mathbb{R}^d$, $\langle u, v\rangle$ denotes the *Euclidean inner product* of the vectors u and v.

For given U and V vector spaces of $\mathbb{R}^d$, with $\dim U \leq V$, the *angle* between them is defined by

$$\angle(U, V) = \max_{u \in U} \min_{v \in V} \angle(u, v).$$

By angle between affine spaces, we mean the angle between corresponding parallel vector spaces.

The following result is simple consequence of the above defintion. For a proof refer to [BG14].

Lemma 1 *Let U and V be affine spaces of $\mathbb{R}^d$ with $\dim(U) \leq \dim(V)$, and let $U^\perp$ and $V^\perp$ are affine spaces of $\mathbb{R}^d$ with $\dim(U^\perp) = d - \dim(U)$ and $\dim(V^\perp) = d - \dim(V)$.*

1. *If $U^\perp$ and $V^\perp$ are the orthogonal complements of U and V in $\mathbb{R}^d$, then $\angle(U, V) = \angle(V^\perp, U^\perp)$.*

2. *If $\dim(U) = \dim(V)$ then $\angle(U, V) = \angle(V, U)$.*

2.2 Manifolds and reach

For a given submanifold $\mathcal{M}$ of $\mathbb{R}^d$, the *medial axis* $\mathcal{O}_\mathcal{M}$ of $\mathcal{M}$ is defined as the closure of the set of points in $\mathbb{R}^d$ that have more than one closest points in $\mathcal{M}$. The *reach* of $\mathcal{M}$ is defined as the

$$rch(\mathcal{M}) = \inf_{x \in \mathcal{M}} d(x, \mathcal{O}_\mathcal{M}).$$

Federer [Fed59] proved that $rch(\mathcal{M})$ is (strictly) positive when $\mathcal{M}$ is of class C^2 or even $C^{1,1}$, i.e. the normal bundle is defined everywhere on $\mathcal{M}$ and is Lipschitz continuous. For simplicity, we are anyway assuming that $\mathcal{M}$ is a smooth submanifold.

For a point p in $\mathcal{M}$, $T_p\mathcal{M}$ and $N_p\mathcal{M}$ denotes the tangent and normal space at the point p of $\mathcal{M}$ respectively.

We will use the following results from [Fed59, GW04, BDG13a]. See [GW04, Lem. 6 & 7] and [BDG13a, Lem. B.3].

Lemma 2 *Let p and q be points on the manifold $\mathcal{M}$.*

1. *If $\|p-q\| < rch(\mathcal{M})$, $\sin\angle(T_p\mathcal{M}, [p\,q]) \leq \frac{\|p-q\|}{2rch(\mathcal{M})}$.*

If $\|p-q\| < \frac{rch(\mathcal{M})}{4}$, then

2. *$d(q,\mathcal{M}) \leq \frac{2\|p-q\|^2}{rch(\mathcal{M})}$, and*

3. *$\sin\angle(T_p\mathcal{M}, T_q\mathcal{M}) < \frac{6\|x-y\|}{rch(\mathcal{M})}$.*

2.3 Simplices

Given a set of $j+1$ points $p_0, \ldots, p_j$ in $\mathbb{R}^d$, a *j-simplex*, or just simplex, $\sigma = [p_0, \ldots, p_j]$ denotes the set $\{p_0, \ldots, p_j\}$. The points p_i are called the *vertices* of σ and j denotes the *combinatorial dimension* of the simplex σ. Sometimes we will use an additional superscript, like σ^j, to denote a j-simplex. A simplex σ^j is called *degenerate* if $j > \dim \operatorname{aff}(\sigma)$.

We will denote by $R(\sigma)$, $L(\sigma)$, $\Delta(\sigma)$ the lengths of the smallest circumradius, the smallest edge, and the longest edge of the simplex σ respectively. The circumcentre of the simplex σ will be denoted by $C(\sigma)$. , and $N(\sigma)$ denotes the affine space, passing through $C(\sigma)$ and of dimension $d - \dim \operatorname{aff}(\sigma)$, orthogonal to $\operatorname{aff}(\sigma)$.

Any subset $\{p_{i_0}, \ldots, p_{i_k}\}$ of $\{p_0, \ldots, p_j\}$ defines a k-simplex which we call a *face* of σ. We will write $\tau \leq \sigma$ if τ is a face of σ, and $\tau < \sigma$ if τ is a *proper face* of σ.

For a given vertex p of σ, σ_p denotes the subsimplex of σ with the vertex set $\{p_0, \ldots, p_j\} \setminus p$. If τ is a j-simplex, and p is not a vertex of τ, we can get a $(j+1)$-simplex $\sigma = p * \tau$, called the *join* of p and τ. We will denote τ by σ_p.

The *altitude* of a vertex p in σ is $D(p,\sigma) = d(p, \operatorname{aff}(\sigma_p))$, where σ denotes the simplex opposite to p in σ. We also write $\sigma = \sigma_p * p$. A poorly-shaped simplex can be characterized by the existence of a relatively small altitude. The *thickness* of a j-simplex σ of diameter $\Delta(\sigma)$ is defined as

$$\Upsilon(\sigma) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{p\in\sigma} \frac{D(p,\sigma)}{j\Delta(\sigma)} & \text{otherwise.} \end{cases}$$

A simplex that is not thick has a relatively small altitude, but we want to characterize bad simplices for which *all* the altitudes are relatively small.This motivates the definition of Γ_0-slivers.

Definition 3 (Γ_0-good simplices and Γ_0-slivers) *Let Γ_0 be a positive real number smaller than one. A simplex σ is* Γ_0-good *if $\Upsilon(\sigma^j) \geq \Gamma_0^j$ for all j-simplices $\sigma^j \leq \sigma$. A simplex is* Γ_0-bad *if it is not Γ_0-good. A* Γ_0-sliver *is a Γ_0-bad simplex in which all the proper faces are Γ_0-good.*

Observe that a sliver must have dimension at least 2, since $\Upsilon(\sigma^j) = 1$ for $j < 2$. Observe also that our definition departs from the standard one since the slivers we consider have no upper bound on their circumradius, and in fact may be degenerate and not even have a circumradius.

Ensuring that all simplices are Γ_0-good is the same as ensuring that there are no slivers. Indeed, if σ is Γ_0-bad, then it has a j-face σ^j that is not Γ_0^j-thick. By considering such a face with minimal dimension we arrive at the following important observation:

Lemma 4 *A simplex is Γ_0-bad if and only if it has a face that is a Γ_0-sliver.*

The following result is due to [BDG13b].

Corollary 5 *Let σ be a k-simplex with $k \leq m$ and the vertices of σ are on the submanifold $\mathcal{M}$. If σ is Γ_0-good and $\Delta(\sigma) < rch(\mathcal{M})$, then*

$$\sin \angle(\mathrm{aff}(\sigma), T_p\mathcal{M}) \leq \frac{\Delta(\sigma)}{\Gamma_0^m rch(\mathcal{M})}.$$

2.4 Weighted points and weighted Delaunay complex

For a finite set of points P in $\mathbb{R}^d$, a weight assignment of L is a non-negative real function from L to $[0, \infty)$, i.e., $\omega : L \to [0, \infty)$. A pair $(p, \omega(p))$, $p \in L$, is called a *weighted point*. For simplicity, we denote the weighted point $(p, \omega(p))$ as p^ω. The *relative amplitude* of ω is defined as

$$\tilde{\omega} = \max_{p \in L} \max_{q \in L \setminus p} \frac{\omega(p)}{\|p - q\|}.$$

Given a point $x \in \mathbb{R}^d$, the *weighted distance* of x from a weighted point $(p, \omega(p))$ is defined as

$$d(x, p^\omega) = \|x - p\|^2 - \omega(p)^2.$$

The following structural result is due to Boissonnat, Guibas and Oudot [BGO09, Lem. 4.3 & 4.4].

Lemma 6 *Let $L \subseteq \mathcal{M}$ be a ϵ-sample of $\mathcal{M}$ with $\epsilon < rch(\mathcal{M})$, and $\omega : L \to [0, \infty)$ be a weight assignment with $\tilde{\omega} < \frac{1}{2}$.*

1. *For all $p \in L$, $\omega(p) \leq 2\tilde{\omega}\epsilon$.*

2. *If k is a non-negative integer and $\epsilon \leq \frac{rch(\mathcal{M})}{4}$, then, for all $x \in \mathcal{M}$ and $k \in \{0, 1\}$, the Euclidean distance between x and its $k+1$-nearest weighted neighbor in L is at most $(1 + 2\tilde{\omega} + 2k(1 + 3\tilde{\omega}))\epsilon$.*

It is well known that, given a finite point set $L \subset \mathbb{R}^d$ and a weight assignment $\omega : L \to [0, \infty)$, one can define a decomposition of $\mathbb{R}^d$ called the *weighted Voronoi diagram* of L. We denote it by $\mathrm{Vor}_\omega(L)$. For more details, see the appendix on notations given at the end of this paper.

The *weighted Delaunay complex* $\mathrm{Del}_\omega(L)$ is defined as the *nerve* of $\mathrm{Vor}_\omega(L)$, i.e.,

$$\sigma \in \mathrm{Del}_\omega(L) \;\text{ iff }\; \mathrm{Vor}_\omega(\sigma) \neq \emptyset,$$

where $\mathrm{Vor}_\omega(\sigma)$ is the intersection of the cells of the vertices of σ.

2.5 Witness, cocone and tangential complex

We now recall the definition of the weighted *witness complex* introduced by de Silva [dS08]. Let $W \subset \mathbb{R}^d$, and let $L \subseteq W$ be a finite set, and $\omega : L \to [0, \infty)$ be a weight assignment of L.

- We say $w \in W$ is a ω-*witness* of a simplex $\sigma = [p_0, \ldots, p_k]$ with vertices in L, if the $p_0, \ldots, p_k$ are among the $k+1$ nearest neighbors of w in the weighted distance, i.e., $p \in \sigma$, $q \in L \setminus \sigma$, $d(w, p^\omega) \leq d(w, q^\omega)$.
- The ω-*witness complex* $\mathrm{Wit}_\omega(L, W)$ is the maximum abstract simplicial complex with vertices in L, whose faces are ω-witnessed by points of W. When there is no ambiguity, we will call $\mathrm{Wit}_\omega(L, W)$ just witness complex for simplicity.

For any point p on a smooth submanifold $\mathcal{M}$ and $\theta \in [0, \frac{\pi}{2}]$, we call the θ-*cocone* of $\mathcal{M}$ at p, or $\mathrm{K}^{\theta_0}(p)$ for short, the cocone of semi-aperture θ around the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at p:

$$\mathrm{K}^\theta(p) = \left\{ x \in \mathbb{R}^d : \angle(px, T_p\mathcal{M}) \leq \theta \right\}.$$

Given an angle $\theta \in [0, \frac{\pi}{2}]$, a finite point set $\mathsf{P} \subset \mathcal{M}$, and a weight assignment $\omega : \mathsf{P} \to [0, \infty)$, the *weighted θ-cocone complex* of P, denoted by $\mathrm{K}^\theta_\omega(\mathsf{P})$, is defined as

$$\mathrm{K}^\theta_\omega(\mathsf{P}) = \left\{ \sigma \in \mathrm{Del}_\omega(\mathsf{P}) : \mathrm{Vor}_\omega(\sigma) \cap \left(\bigcup_{p \in \sigma} \mathrm{K}^\theta(p) \right) \neq \emptyset \right\}.$$

The cocone complex was first introduced by Amenta *et al.* [ACDL02] in $\mathbb{R}^3$ for reconstructing surfaces and was generalized by Cheng *et al.* [CDR05] for reconstructing submanifolds.

The *weighted tangential complex*, or just tangential complex, of P is the weighted θ-cocone complex $\mathrm{K}^\theta_\omega(\mathsf{P})$ with θ equal to "zero" and will be denoted by $\mathrm{Del}_\omega(\mathsf{P}, T\mathcal{M})$.

Hypothesis 7 *For the rest of this paper, we take*

$$\begin{aligned} \theta_0 &\stackrel{\text{def}}{=} \frac{\pi}{32}, \\ \mathrm{K}(p) &\stackrel{\text{def}}{=} \mathrm{K}^{\theta_0}(p), \text{ and} \\ \mathrm{K}_\omega(L) &\stackrel{\text{def}}{=} \mathrm{K}^{\theta_0}_\omega(L). \end{aligned}$$

3 Power protection

Let $\mathsf{P} \subset \mathbb{R}^d$ be a point sample with $\dim \mathrm{aff}(\mathsf{P}) = d$. A simplex $\sigma \in \mathrm{Del}_\omega(\mathsf{P})$ is δ^2-*power protected* at $c \in \mathrm{Vor}_\omega(\sigma)$ if for all $q \in L \setminus \sigma$ and $p \in \sigma$, then

$$\|q - c\|^2 - \omega(q)^2 > \|p - c\|^2 - \omega(p)^2 + \delta^2.$$

The following result shows that power protecting d-simplices implies power protecting lower dimensional subsimplices as well.

Lemma 8 *Let $\mathsf{P} \subset \mathbb{R}^d$ be a set of points, and let $\omega : \mathsf{P} \to [0, \infty)$ be a weight assignment. In addition, let p be a point of P whose Voronoi cell $\mathrm{Vor}_\omega(p)$ is bounded. Then, if all the d-simplices incident to p in $\mathrm{Del}_\omega(\mathsf{P})$ are δ^2-power protected, with $\delta > 0$, then any j-simplex incident to p is*

$$\frac{\delta^2}{d-j+1}\text{-power protected.}$$

The above result implies that if $\mathrm{aff}\,\mathsf{P} = \mathbb{R}^d$ and if all the d-simplices in $\mathrm{Del}_\omega(\mathsf{P})$ are δ^2-power protected then all the simplices, not on the boundary of $\mathrm{Del}_\omega(\mathsf{P})$, are also power protected. Another interesting aspect of this result is the fact that the decay in power protection depends linearly on the dimension of the ambient space. The proof of the theorem is done in the lifted $\mathbb{R}^{d+1}$ space where power protection translates to *vertical distance* of points in from hyperplanes.

To prove Lemma 8 we need the following lemma on power protection.

Lemma 9 *Let $\mathsf{P} \subset \mathbb{R}^d$ and $\omega : \mathsf{P} \to [0,\infty)$ be a weight distribution. Let $p \in \mathsf{P}$ such that $\mathrm{Vor}_\omega(p)$ is bounded and all the d-simplices in $\mathrm{Del}_\omega(\mathsf{P})$ incident to p are δ^2-power for some $\delta > 0$. Then*

1. *the dimension of maximal simplices in $\mathrm{Del}_\omega(\mathsf{P})$ incident to p is equal to d; and*
2. *for all j-simplex $\sigma^j \in \mathrm{Del}_\omega(\mathsf{P})$ incident to p, $\dim \mathrm{Vor}_\omega(\sigma^j) = d - j$.*

3.1 Proof of Lemma 9

For the rest of this section we will assume the following hypothesis:

Hypothesis 10 *Let $\mathsf{P} \subset \mathbb{R}^d$ be a finite point sample, and let $\omega : \mathsf{P} \to [0,\infty)$ be a weight assignment such that there exists $p \in \mathsf{P}$ with $\mathrm{Vor}_\omega(p)$ is bounded.*

Since $\mathrm{Vor}_\omega(p)$ is bounded, we have $\dim \mathrm{aff}(\mathsf{P}) = d$.

The following lemma is analogous to [BDG13c, Lem. 3.2], and the proof is exactly samce as that lemma.

Lemma 11 (Maximal simplices) *Every $\sigma \in \mathrm{Del}_\omega(\mathsf{P})$ incident to p is a face of a simplex $\sigma' \in \mathrm{Del}_\omega(\mathsf{P})$ with $\dim \mathrm{aff}(\sigma') = d$.*

Following lemma is a direct consequence is the above result.

Lemma 12 (No degeneracies) *If every d-simplex in $\mathrm{Del}_\omega(\mathsf{P})$ incident to p is δ^2-power protected for some $\delta > 0$, then there are no degenerate simplices in $\mathrm{Del}_\omega(\mathsf{P})$ incident to p.*

Like in the case of Lemma 11, following result is analogous to [BDG13c, Lem. 3.3] and can be proved exactly along the same lines.

Lemma 13 (Separation) *If $\sigma^j \in \mathrm{Del}_\omega(\mathsf{P})$ is a j-simplex incident to p, and $q \in \mathsf{P} \setminus \sigma^j$, then there is a d-simplex $\sigma^d \in \mathrm{Del}_\omega(\mathsf{P})$ incident to p such that $\sigma^j \leq \sigma^d$ and $q \notin \sigma^d$.*

We will need the definition of $N_\omega(\sigma)$ defined in Section D.

Proof of Lemma 9 Let $\sigma^k \in \mathrm{Del}_\omega(\mathsf{P})$ be a maximal k-simplex incident to p such that $\dim \mathrm{Vor}_\omega(\sigma^k) < d-k$. From Lemma 12, we have $k < d$. We can generate, from Lemma 11, a sequence of simplices

$$\sigma^k < \sigma^{k+1} < \cdots < \sigma^d$$

where $\sigma^j \in \mathrm{Del}_\omega(\mathsf{P})$ and $j \in \{k, \ldots, d\}$.

Let $q = \sigma^{k+1} \setminus \sigma^k$. From Lemma 13, there exists a d-simplex $\sigma \in \mathrm{Del}_\omega(\mathsf{P})$ such that $\sigma^k < \sigma$ and $q \notin \sigma$. Let $c \in \mathrm{Vor}_\omega(\sigma)$. Since the d-simplices incident to p is δ^2-power protected, it is easy to see that $\exists\, \delta' > 0$ such that $\forall\, x \in B(c, \delta')$, we have

$$d(x, q^\omega) < d(x, r^\omega),\ \forall\, q \in \sigma \text{ and } r \in \mathsf{P} \setminus \sigma \tag{1}$$

This implies $N_\omega(\sigma^k) \cap B(c, \delta') \subseteq \mathrm{Vor}_\omega(\sigma^k)$, and since $c \in N_\omega(\sigma^k)$, we get a contradiction with the initial assumption that $\dim \mathrm{Vor}_\omega(\sigma^k) < d-k$. □

3.2 Lifting map, space of spheres and Voronoi diagram

We are going to argue about the power protection of Delaunay simplices in the "space of spheres" or "lifting space". For our purposes we will be working primarily from the Voronoi perspective. We will give a self-contained summary of the properties of the space of spheres that we will use. Full details can be found in [BY98, Chap. 17].

The lifting map ϕ takes a sphere $S(c, r) \subset \mathbb{R}^d$, with centre c and radius r, to the point $(c, \|c\|^2 - r^2) \in \mathbb{R}^{d+1}$. We consider the points in $\mathbb{R}^d$ to be spheres with $r = 0$, and thus $\mathbb{R}^d$ itself is represented as a (hyper-) paraboloid in $\mathbb{R}^{d+1}$.

Let P be a point set and $\omega : \mathsf{P} \to [0, \infty)$ be a weight distribution. The set of spheres that are orthogonal to point p, with weight $\omega(p)$, are represented by a hyperplane $\mathcal{H}_p^\omega \subset \mathbb{R}^{d+1}$ that passes through $\phi(S(p, \omega(p)))$. Indeed, for any such sphere $S(c, r)$ we have

$$r^2 + \omega(p)^2 = \|c - p\|^2$$

and so

$$\mathcal{H}_p^\omega \quad = \quad \big(\underbrace{\ c\ }_{d},\ \underbrace{2c \cdot p + \omega(p)^2 - \|p\|^2}_{1}\big)$$

For any $p \in \mathsf{P} \subset \mathbb{R}^d$, we represent its Voronoi cell $\mathrm{Vor}_\omega(p)$ in the space of spheres by associating to each $c \in \mathrm{Vor}_\omega(p)$ the unique sphere $S(c, r)$, where $r^2 = \|p - c\|^2 - \omega(p)^2$. Thus $\phi(\mathrm{Vor}_\omega(p))$ lies on the hyperplane $\mathcal{H}_p^\omega$.

For any Delaunay simplex $\sigma \in \mathrm{Del}(\mathsf{P})$, its Voronoi cell $\mathrm{Vor}_\omega(\sigma) = \bigcap_{p \in \sigma} \mathrm{Vor}_\omega(p)$ is mapped in the space of spheres lies to the intersection of the hyperplanes that support the lifted Voronoi cells of its vertices:

$$\phi(\mathrm{Vor}_\omega(\sigma)) \subset \bigcap_{p \in \sigma} \mathcal{H}_p^\omega.$$

If P is generic and σ is a k-simplex, then $\phi(\mathrm{Vor}_\omega(\sigma))$ lies in a $(m-k)$-dimensional affine space.

We can say more. The lifted Voronoi cell $\phi(\mathrm{Vor}_\omega(\sigma))$ is a convex polytope. Any two points $z, z' \in \phi(\mathrm{Vor}_\omega(\sigma))$ have corresponding points $c, c' \in \mathrm{Vor}_\omega(\sigma) \subset \mathbb{R}^d$, and a line segment between c and c' gets lifted to a line segment between z and z' in $\phi(\mathrm{Vor}_\omega(\sigma))$.

3.3 Power protection in the "space of spheres" framework

We can talk about the power-protection at a point $c \in \mathrm{Vor}_\omega(\sigma)$: it is the power-protection enjoyed by the Delaunay sphere $S(c, r)$ centred at c. For a point $q \in \mathsf{P} \setminus \sigma$, we say that c is $\check{\delta}^2$*-power-protected from* q if

$$\|q - c\|^2 - \omega(q)^2 - r^2 > \check{\delta}^2.$$

In the lifting space, if $z = \phi(S(c, r))$, then the power protection of c from q is given by the "vertical" distance of z above $\mathcal{H}_q^\omega$, which we will refer to as the *clearance* of z above $\mathcal{H}_q^\omega$.

Thus for any $q \in \mathsf{P} \setminus \sigma$ we have a function $f_q : \phi(\mathrm{Vor}_\omega(\sigma)) \to \mathbb{R}$ which associates to each $z \in \phi(\mathrm{Vor}_\omega(\sigma))$ the clearance of z above $\mathcal{H}_q^\omega$. This is a linear function of the sphere centres. Indeed, if $p \in \sigma$ and $z = \phi(S(c, r))$, then $r^2 + \omega(p)^2 = \|p - c\|^2$, and

$$f_q(z) = 2c \cdot (p - q) - (\|p\|^2 - \|q\|^2) + (\omega(p)^2 - \omega(q)^2).$$

Finally, we have everything in place to give the proof of Lemma 8.

Proof of Lemma 8 We wish to find a bound $h_j(\delta)$ such that if all the d-simplices in $\mathrm{Del}_\omega(\mathsf{P})$ incident to p are δ^2-power-protected, then the Delaunay j-simplices will be $h_j(\delta)$-power-protected. We observe that for any j-simplex σ its Voronoi cell $\mathrm{Vor}_\omega(\sigma)$, as $\mathrm{Vor}_\omega(p)$ ($\supset \mathrm{Vor}_\omega(\sigma)$) is bounded, is the convex hull of Voronoi vertices: the circumcentres of the Delaunay d-simplices that have σ as a face. It follows that $\phi(\mathrm{Vor}_\omega(\sigma))$ is the convex hull of a finite set of points which correspond to these d-simplices. We choose an affinely independent set $\{z_i\}$, $i \in \{0, 1, \ldots, k\}$, of $k + 1$ of these points, where $k = m - j$.

Let

$$z^* = \frac{1}{k+1} \sum_{i=0}^{k} z_i$$

be the barycentre of these lifted Delaunay spheres, and consider the clearance, $f_q(z^*)$, of z^* above $\mathcal{H}_q^\omega$, where $q \in \mathsf{P} \setminus \sigma$. Let σ_i be the Delaunay d-simplex corresponding to z_i. There must be a σ_ℓ, $l \in \{0, \ldots, k\}$, which does not contain q, since otherwise the k-simplex defined by the set $\{z_i\}_{i \in \{0,\ldots,k\}}$ would lie in $\phi(\mathrm{Vor}_\omega(q * \sigma))$, contradicting Lemma 9. Since σ_ℓ is δ^2-power-protected, we have $f_q(z_\ell) > \delta^2$, and by the linearity of f_q we get a bound on the clearance of z^* above $\mathcal{H}_q^\omega$:

$$f_q(z^*) = \frac{1}{k+1} \sum_{i=0}^{k} f_q(z_i) \geq \frac{f_q(z_\ell)}{k+1} > \frac{\delta^2}{k+1}$$

Since q was chosen arbitrarily from $\mathsf{P} \setminus \sigma$, this provides a lower bound on the power protection at $c^* \in \mathrm{Vor}_\omega(\sigma)$, where $z^* = \phi(S(c^*, r^*))$, and hence a lower bound on the power protection of σ. □

Remark 14 *We remark that if we could find two lifted Voronoi vertices z_1 and z_2 such that the line segment between them lies in the relative interior of $\phi(\mathrm{Vor}_\omega(\sigma))$, then the midpoint of that segment would have a power protection of $\frac{\delta^2}{2}$. However, this isn't possible in general, for $\mathrm{Vor}_\omega(\sigma)$ could be a j-simplex, when σ is not a maximal shared face of any two Delaunay d-simplices.*

4 Stability, protection and witness complex

Let $\alpha_0 < \frac{1}{2}$ be an absolute constant, and $\Gamma_0 < 1$ and $\delta_0 < \alpha_0$ be paramters to the algorithm satisfying Eq. (5). We define

$$\tilde{\alpha}_0 \stackrel{\text{def}}{=} \sqrt{\alpha_0^2 - \delta_0^2}.$$

Let $W \subset \mathcal{M}$ be an ε-sample of $\mathcal{M}$, $L \subset W$ a λ-net of W with $\varepsilon \leq \lambda$, and $\omega : L \to [0, \infty)$ a weight assignment with $\tilde{\omega} \leq \tilde{\alpha}_0$ (to be defined later). A weight assignment $\xi : L \to [0, \infty)$ will be called an *elementary weight perturbation* of ω (*ewp* for short) if

$$\exists\, p \in L,\ \xi(p) \in \left[\omega(p),\ \sqrt{\omega(p)^2 + \delta_0^2\lambda^2}\right]$$

and

$$\xi(q) = \omega(q) \ \text{ if } \ q \in L \setminus p.$$

We call the weight assignment $\omega : L \to [0, \infty)$ *stable* (resp., *locally stable at* $p \in L$) if for all ewp ξ of ω, $\mathrm{K}_\xi(L)$ contains no Γ_0-slivers of dimension $\leq m+1$ (resp., no such slivers incident to p).

The main structural result in this section is the following theorem:

Theorem 15 *Let $W \subset \mathcal{M}$ be an ε-sample of $\mathcal{M}$, $L \subset W$ be a λ-net of W with $\varepsilon \leq \lambda$, and $\omega : L \to [0, \infty)$ be a stable weight assignment. If*

$$\lambda < \min\left\{\frac{3\sin\theta_0}{2^{11}(m+1)}, \frac{\sin\theta_0\Gamma_0^m}{2^{10}}, \frac{\Gamma_0^{2m+1}}{24}, \frac{\delta_0^2}{2^{15}(m+1)}\right\} rch(\mathcal{M}),$$

and

$$\varepsilon < \frac{\lambda}{24}\left(\frac{\delta_0^2}{m+1} - \frac{2^{15}\lambda}{rch(\mathcal{M})}\right)$$

then,

$$\mathrm{Del}_\omega(L, T\mathcal{M}) = \mathrm{Wit}_\omega(L, W).$$

In addition, if λ is sufficiently small, $\mathrm{Wit}_\omega(L, W)$ *is homeomorphic and a close geometric approximation of $\mathcal{M}$.*

The rest of this section is devoted to give an outline of the proof of the above theorem. For full details refer to Appendix A.

Since ω is a stable weight assignment, $\mathrm{K}_\omega(L)$ contains no Γ_0-slivers of dimension $\leq m+1$. Moreover, for λ sufficiently small, we can show the following:

P1 $\forall\, \sigma \in \mathrm{K}_\omega(L)$, $\angle(\mathrm{aff}\,\sigma, T_p\mathcal{M}) = O(\lambda)$

P2 the simplices of $\mathrm{Del}_\omega(L, T\mathcal{M})$ have dimension at most m, and the dimension of maximal simplices in $\mathrm{K}_\omega(L) \leq m$

P3 $\forall\, \sigma \in \mathrm{Del}_\omega(L, T\mathcal{M})$ and $p \in \sigma$, $\mathrm{Vor}_\omega(\sigma) \cap T_p\mathcal{M} \neq \emptyset$

P4 $\mathrm{Wit}_\omega(L, W) \subseteq \mathrm{Del}_\omega(L, T\mathcal{M})$

P5 $\mathrm{Del}_\omega(L, T\mathcal{M})$ is a faithful reconstruction of $\mathcal{M}$

P1, P2, P3 and P5 are direct consequences of results from from [CDR05, BGO09, BG14].

We prove P4 by contradiction. Let $\sigma^k \in \mathrm{Wit}_\omega(L, W)$ be a k-simplex with $\sigma^k \notin \mathrm{Del}_\omega(L, T\mathcal{M})$ and p a vertex of σ^k. Using the sampling assumptions on L and W, we can show that for any $w \in W$ that is a ω-witness of σ^k or of its subfaces, $\|p - w\| = O(\lambda)$ [BGO09, Lem. 4.4]. This implies, from [GW04, Lem. 6],

$$d(w, T_p\mathcal{M}) = O\left(\frac{\lambda^2}{rch(\mathcal{M})}\right).$$

From [dS08, Thm. 4.1], we know that $\mathrm{Vor}_\omega(\sigma^k)$ intersects the convex hull of the ω-witnesses of σ^k and its subfaces. Let $c_k \in \mathrm{Vor}_\omega(\sigma^k)$ be a point in this intersection. We have

$$d(c_k, T_p\mathcal{M}) = O(\frac{\lambda^2}{rch(\mathcal{M})})$$

and, since L is λ-sparse,

$$\|p - c_k\| = \Omega(\lambda).$$

Therefore, using the sampling assumption on λ, we get

$$\frac{d(c_k, T_p\mathcal{M})}{d(c_k, p)} = O\left(\frac{\lambda}{rch(\mathcal{M})}\right) < \sin\theta_0.$$

This implies that $\sigma^k \in \mathrm{K}_\omega(L)$. As $\angle(\mathrm{aff}\,\sigma^k, T_p\mathcal{M})$ is small (property P4), $\exists\ c'_k \in T_p\mathcal{M}$ such that the line segment $[c_k, c'_k]$ is orthogonal to σ^k and

$$d(c_k, c'_k) = O\left(\frac{\lambda^2}{rch(\mathcal{M})}\right).$$

Again, as λ is small, the line segment $[c_k, c'_k]$ is contained in $\mathrm{K}(p)$. Since $\sigma^k \notin \mathrm{Del}_\omega(L, T\mathcal{M})$, $\exists\ c_{k+1} \in [c_k, c'_k]$ and a $(k+1)$-simplex σ^{k+1} such that

$$c_{k+1} \in \mathrm{Vor}_\omega(\sigma^{k+1}) \quad \text{with} \quad \sigma^k < \sigma^{k+1}.$$

Therefore, $\sigma^{k+1} \in \mathrm{K}_\omega(L)$. If $k = m$, we have reached a contradiction with property P2. Otherwise, using the facts that $\angle(\sigma^{k+1}, T_p\mathcal{M})$ is small, $d(c_{k+1}, T_p\mathcal{M}) = O(\frac{\lambda^2}{rch(\mathcal{M})})$ and $d(p, c_{k+1}) = \Omega(\lambda)$, we will find a $c'_{k+1} \in T_p\mathcal{M}$ such that $[c_{k+1}, c'_{k+1}] \in \mathrm{K}(p)$. Since $\sigma^{k+1} \notin \mathrm{K}_\omega(L)$, $\exists\ c_{k+2} \in [c_{k+1}, c'_{k+1}]$ and $k+2$-simplex $\sigma^{k+2} \in \mathrm{K}_\omega(L)$ such that

$$c_{k+2} \in \mathrm{Vor}_\omega(\sigma^{k+2}) \quad \text{and} \quad \sigma^{k+1} < \sigma^{k+2}.$$

Continuing this procedure of walking on the Voronoi cell of the simplex from a point, like c_{k+1}, in the intersection the Voronoi cell of the simplex and $\mathrm{K}(P)$ towards $T_p\mathcal{M}$, we will get a sequence of points

$$c_k, \ldots, c_{m+1}$$

and simplies

$$\sigma^k < \cdots < \sigma^{m+1}$$

with

$$c_j \in \mathrm{Vor}_\omega(\sigma^j) \cap \mathrm{K}(p) \quad \text{and} \quad \sigma^j \in \mathrm{K}_\omega(L).$$

We have now reached a contradiction via property P2.

To complete the proof, we have to show $\mathrm{Del}_\omega(L, T\mathcal{M}) \subseteq \mathrm{Wit}_\omega(L, W)$.

We first show that all m-simplices in $\mathrm{Del}_\omega(L, T\mathcal{M})$ are δ^2-power protected on $T_p\mathcal{M}$, where $\delta = \delta_0\lambda$. To reach a contradiction, let us assume that there exists a m-simplex $\sigma \in \mathrm{Del}_\omega(L, T\mathcal{M})$ that is not power protected on $T_p\mathcal{M}$ for some $p \in \sigma$. Then there exists $c \in \mathrm{Vor}_\omega(\sigma) \cap T_p\mathcal{M}$ and $q \in L \setminus \sigma$ such that

$$d(c, p^\omega) \geq d(c, q^\omega) - \delta^2.$$

Consider now the following weight assignment:

$$\xi(x) = \begin{cases} \omega(x) & \text{if } x \neq q \\ \sqrt{\omega(q)^2 + \beta^2} & \text{if } x = q \end{cases}$$

where

$$\beta^2 = d(c, q^\omega) - d(c, p^\omega).$$

It is easy to see that ξ is an ewp of ω and $\tilde{\xi} < 1/2$. Let $\sigma' = q * \sigma$. Since λ is sufficiently small, σ' is a Γ_0-bad $(m+1)$-simplex. Broadly, the idea behind the proof is the following (see also the proofs of [CDR05, Lem. 13] and [BG14, Lem. 4.9]): the thickness of any $(m+1)$-simplex embedded in $\mathbb{R}^m$ is zero. Here σ' is a $(m+1)$-simplex embedded in $\mathbb{R}^d$ whose vertices belong to a small neighborhood of $\mathcal{M} \subset \mathbb{R}^d$. It follows that its thickness is expected to be small. Specifically, we prove that σ' is Γ_0-bad, which contradicts Property P2.

We now prove that all simplices (of all dimensions) in $\mathrm{Del}_\omega(L, T\mathcal{M})$ are $\frac{\delta^2}{m+1}$-power protected on $T_p\mathcal{M}$ for all $p \in \sigma$. To establish this result, we want to use Lemma 8 but we cannot use the lemma directly since it only holds for d-simplices of $\mathbb{R}^d$. To overcome this issue, we resort to Lemma 2.2 of [BG14] which states that $\mathrm{Vor}_\omega(L) \cap T_p\mathcal{M}$ is identical to a weighted Voronoi diagram $\mathrm{Vor}_\psi(L')$ where L' is the orthogonal projection of L onto $T_p\mathcal{M}$, i.e., $\mathrm{Vor}_\omega(\sigma) \cap T_p\mathcal{M} = \mathrm{Vor}_\omega(\sigma')$ where σ' is the projection of σ onto $T_p\mathcal{M}$. Also we can prove, using P1, that δ^2-power protection of a simplex $\sigma \in \mathrm{Del}_\omega(L, T\mathcal{M})$ incident to p on $T_p\mathcal{M}$ implies δ^2-power protection of $\sigma' \in \mathrm{Del}_\psi(L')$. Using this correspondance, we can show that all m-simplices incident to p in $\mathrm{Del}_\psi(L')$ are δ^2-power protected since all the m-simplices incident to p in $\mathrm{Del}_\omega(L, T\mathcal{M})$ are δ^2-power protected on $T_p\mathcal{M}$. We can now use Lemma 8. Using the bound on λ, we can show that $\mathrm{Vor}_\psi(p) = \mathrm{Vor}_\omega(p) \cap T_p\mathcal{M}$ is bounded, see [BG14, Lem. 4.4]. From Lemma 8, we then get that all j-simplices in $\mathrm{Del}_\psi(L')$ incident to p' are $\frac{\delta^2}{m+1}$-power protected. This result, together with the correspondence we have established with the power protection of simplices incident to p in $\mathrm{Del}_\psi(L')$ with the power protection of simplices incident to p in $\mathrm{Del}_\omega(L, T\mathcal{M})$ on $T_p\mathcal{M}$, we deduce that all j-simplices incident to p in $\mathrm{Del}_\omega(L, T\mathcal{M})$ are $\frac{\delta^2}{m+1}$-power protected on $T_p\mathcal{M}$.

Let σ be $\frac{\delta^2}{m+1}$-power protected at $c \in \mathrm{Vor}_\omega(\sigma) \cap T_p\mathcal{M}$, where $p \in \sigma$. We can show that there exists $c' \in \mathcal{M}$, such that $\|c - c'\|$ is small compared to $\frac{\delta^2}{m+1}$ and the line passing through c and c' is orthogonal to $\mathrm{aff}(\sigma)$. Using simple triangle inequalities, we can prove that σ is $\Omega(\frac{\delta^2}{m})$-power protected at c'.

As W is an ε-sample of $\mathcal{M}$, we can find a $w \in W$ such that $\|w - c'\| < \varepsilon$. Using the facts that ε is much smaller than $\delta^2 = \delta_0^2\lambda^2$ and σ is $\Omega(\frac{\delta^2}{m})$-power protected at c', we get w to be a ω-witness of σ. Since σ is an arbitrary simplex of $\mathrm{Del}_\omega(L, T\mathcal{M})$, we have proved that $\mathrm{Del}_\omega(L, T\mathcal{M}) \subseteq \mathrm{Wit}_\omega(L, W)$.

5 Reconstruction algorithm

Let $\mathcal{M}$ be a smooth submanifold with known dimension m, let $W \subset \mathcal{M}$ be an ε-sample of $\mathcal{M}$, and let $L \subset W$ be a λ-net of W for some known λ. We will also assume that $\varepsilon < \lambda$, which implies that L is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$. We will discuss the reasonability of these assumptions in Section 5.3.

The main important part of the algorithm is to find a stable weight assignment $\omega : L \to [0, \infty)$. We will prove that this is possible if Γ_0, δ_0, and the absolute constant $\alpha_0 < \frac{1}{2}$ satisfy Inequality 5 (Lemma 17).

Once we have calculated a stable weight assignment ω, we can just output the witness complex $\mathrm{Wit}_\omega(L, W)$, which is a faithful reconstruction of $\mathcal{M}$ by Theorem 15.

5.1 Outline of the algorithm

We initialize all weights by setting $\omega_0(p) = 0$ for all $p \in L$. We then process each point $p_i \in L$, $i \in \{1, \ldots, n\}$. At step i, we compute a new weight assignment ω_i satisfying the following properties:

C1. $\tilde{\omega}_i \leq \tilde{\alpha}_0$, and $\forall\ p \in L \setminus \{p_i\}$, $\omega_i(p) = \omega_{i-1}(p)$.

C2. ω_i is locally stable at p.

Once we have assigned weights to all the points of L in the above manner, the algorithm outputs $\mathrm{Wit}_\omega(L, W)$ where $\omega = \omega_n$ is the final weight assignment $\omega_n : L \to [0, \infty)$.

The crux of our approach is that weight assignments will be done without computing the cocone complex or any other sort of Voronoi/Delaunay subdivision. Rather, we just look at local neighborhoods

$$N(p_i) \stackrel{\text{def}}{=} \left\{x \in L : \#(\overline{B}(p_i, d(p_i, x)) \cap L) \leq N_1\right\}$$

where N_1 is defined in Lemma 18. The main idea is the following. We define the *candidate simplices* of p_i as the Γ_0-slivers σ of dimension $\leq m + 1$, with vertices in $N(p_i)$, $p_i \in \sigma$, and of diameter $\Delta(\sigma) \leq 16\lambda$. For such a candidate simplex σ, we compute a *forbidden interval* $I_{\omega_{i-1}}(\sigma, p_i)$ (to be defined in Section 5.2). We then select a weight for p_i that is outside all the forbidden intervals of the candidate simplices of p_i.

5.2 Analysis

5.2.1 Correctness of the algorithm

As shown in the next section, forbidden intervals and elementary weight perturbations are closely related (see Lemma 16) and we will prove in Lemma 17 that, if Inequality 5

Algorithm 1 Pseudocode of the algorithm

Input: L, W, Γ_0, δ_0 and m
// let $L = \{p_1, \dots, p_n\}$
// parameters Γ_0, δ_0 and m satisfy Eq. (5)
Initi: $\omega_0 : L \to [0, \infty)$ with $\omega_0(p) = 0$, $\forall\, p \in L$;
Compute: $L(p)$, $N(p)$ for all $p \in L$
// $L(p) \stackrel{\text{def}}{=} \min_{x \in L \setminus \{p\}} d(x, p)$
for $i = 1$ to n **do**
  Compute: *candidate simplices* $S(p_i)$;
  $I \leftarrow \bigcup_{\sigma \in S(p_i)} I_{\omega_{i-1}}(\sigma, p_i)$;
  $\omega_i(q) \leftarrow \omega_{i-1}(q)$ for all $q \in L \setminus \{p_i\}$;
  $x \leftarrow$ a point from $[0, \tilde{\alpha}_0^2 L(p_i)^2] \setminus I$;
  $\omega_i(p_i) \leftarrow \sqrt{x}$;
end for
Output: $\mathrm{Wit}_{\omega_n}(L, W)$;

is satisfied, we can find a locally stable weight assignment ω_i at each iteration of the algorithm. Moreover, we will prove that if all ω_i are locally stable, then we will end up with a stable weight assignment $\omega = \omega_n$ for which Theorem 15 applies. In this respect our algorithm is in the same vein as the seminal work of Cheng *et al.* [CDE+00]. See also [CDR05, BGO09, BG14].

For a given weight assignment $\omega : L \to [0, \infty)$, and a simplex σ with vertices in L, we define

$$F_\omega(p, \sigma) \stackrel{\text{def}}{=} D(p, \sigma)^2 + d(p, N_\omega(\sigma_p))^2 - R_\omega(\sigma_p)^2, \tag{2}$$

the terms $N_\omega(\sigma_p)$ and $R_\omega(\sigma_p)$ are defined in page 35. Note that $F_\omega(p, \sigma)$ depends on the weights of the vertices of σ_p and not on the weight of p. This crucial fact will be used in the analysis of the algorithm.

If σ is a candidate simplex of p, the *forbidden interval* of σ with respect to p is

$$I_\omega(\sigma, p) \stackrel{\text{def}}{=} \left[F_\omega(p, \sigma) - \frac{\eta}{2},\, F_\omega(p, \sigma) + \frac{\eta}{2}\right], \tag{3}$$

where

$$\eta \stackrel{\text{def}}{=} 2^{14} \left(\Gamma_0 + \frac{\delta_0^2}{\Gamma_0^m}\right) \lambda^2. \tag{4}$$

The following result relates forbidden intervals and stable weight assignments. The proof is included in Appendix B.

Lemma 16 *Let $L \subset \mathcal{M}$ be a $(\lambda, 2\lambda)$-net of $\mathcal{M}$ with $\lambda < \frac{1}{18}(1 - \sin\theta_0)^2 rch(\mathcal{M})$, $\omega : L \to [0, \infty)$ be a weight assignment with $\tilde{\omega} \le \tilde{\alpha}_0$. Let, in addition, p be a point of L, and σ a candidate simplex of p. If there exists an ewp ω_1 of ω satisfying $\tilde{\omega}_1 \le \alpha_0$ and $\sigma \in \mathrm{K}_{\omega_1}(L)$, then $\omega(p)^2 \in I_\omega(p, \sigma)$.*

The following lemma shows that *good weights*, i.e., weights which do not lie in any forbidden intervals, exist, which ensures that the algorithm will terminate. For a point p

in L, we write

$$L(p) \stackrel{\mathrm{def}}{=} \min_{q \in L \setminus p} \|p - q\|.$$

Lemma 17 (Existence of good weights) *Assume that* $\lambda \leq \frac{rch(\mathcal{M})}{512}$*, and* Γ_0*,* δ_0 *and* $\tilde{\alpha}_0$ $(= \sqrt{\alpha_0^2 - \delta_0^2})$ *satisfy*

$$\Gamma_0 + \frac{\delta_0^2}{\Gamma_0^m} < \frac{\tilde{\alpha}_0^2}{2^{14} N} \tag{5}$$

where $N = 2^{O(m^2)}$ *and will be defined explicitly in the proof. Then, at the* i^{th} *step, one can find a weight* $\omega_i(p_i) \in [0, \tilde{\alpha}_0 L(p_i)]$ *outside the forbidden intervals of the candidate simplices of* $S(p_i)$*. Moreover,* ω_i *satisfies properties* **C1** *and* **C2***.*

Using simple packing arguments and [GW04, Lem. 6], we get the following bound (similar arguments were used, for example, in [GW04, Lem. 9] and [BG14, Lem. 4.12]).

Lemma 18 *If* $\lambda \leq \frac{rch(\mathcal{M})}{512}$*, then for any* $p \in L$*,* $\#(B(p, 16\lambda) \cap L) \leq 66^m \stackrel{\mathrm{def}}{=} N_1$*.*

Proof of Lemma 17 Write $S(p_i)$ for the set of candidate simplices of p_i. We have

$$\#S(p_i) \leq N \stackrel{\mathrm{def}}{=} \sum_{j=2}^{m+1} N_1^j.$$

For all $\omega : L \to [0, \infty)$ with $\tilde{\omega} \leq \omega_0$, we get from Lemmas 22 (2) and 18 that the set of Γ_0-slivers of dimension $\leq m+1$ in $\mathrm{K}_\omega(L)$ that are incident to p_i is a subset of $S(p_i)$.

Since

$$\begin{aligned} \mu\left(\bigcup_{\sigma \in S(p_i)} I_{\omega_{i-1}}(\sigma, p_i) \right) &\leq \sum_{\sigma \in S(p_i)} \mu(I_{\omega_{i-1}}(\sigma, p_i)) \\ &\leq N\eta \;<\; \tilde{\alpha}_0^2 \lambda^2 \;\leq\; \tilde{\alpha}_0^2 L(p_i)^2, \end{aligned}$$

we can select $\omega(p_i) \in [0, \tilde{\alpha}_0 L(p_i)]$ such that $\omega(p_i)^2$ is outside the forbidden intervals of the candidate simplices of p_i, i.e.,

$$\omega(p_i)^2 \notin \bigcup_{\sigma \in S(p_i)} I_{\omega_{i-1}}(\sigma, p_i).$$

By Lemma 16, the weight assignment ω_i we obtain is a locally stable weight assignment for p_i. $\square$

The following lemma shows that getting a locally stable weight assignment ω_i at each iteration of the algorithm gives a globally stable weight assignment ω_n at the end of the algorithm.

Lemma 19 *The weight assignment* $\omega_n : L \to [0, \infty)$ *is stable.*

Proof It is easy to see that $\tilde{\omega}_n \le \tilde{\alpha}_0$, since for all $p \in L$, the weights were chosen from the interval $[0, \tilde{\alpha}_0 L(p)]$.

We will prove the stability of ω_n by contradiction. Let $\xi : L \to [0, \infty)$ be an ewp of ω_n that modifies the weight of $q \in L$, and assume that there exists a Γ_0-sliver $\sigma = [p_{i_0}, \dots, p_{i_k}] \in \mathrm{K}_\xi(L)$. Note that $\tilde{\xi} \le \alpha_0$, and that for any $p \in \mathring{\sigma}$, $\sigma \in S(p)$ (from the definition of $S(p)$ and Lemma 18). Without loss of generality assume that

$$i_0 < \cdots < i_k.$$

We will have to consider the following two cases:

Case 1. q is not a vertex of σ. This implies that $\sigma \in \mathrm{K}_{\omega_n}(L)$ since $\xi(x) = \omega_n(x)$ for all $x \in L \setminus \{q\}$, and $\xi(q) \ge \omega_n(q)$. Using the same arguments, we can show that $\sigma \in \mathrm{K}_{\omega_{i_k}}(L)$. From Lemma 17 and the fact that ω_{i_k} is an ewp of itself, we have reached a contradiction as ω_{i_k} is a locally stable weight assignment for p_{i_k}.

Case 2. q is a vertex of σ. Using the same arguments as in Case 1 we can show that $\sigma \in \mathrm{K}_{\xi_1}(L)$ where $\xi_1 : L \to [0, \infty)$ is a weight assignment satisfying: $\xi_1(q) = \xi(q)$ and $\xi_1(x) = \omega_{i_k}(x)$ for all $x \in L \setminus \{q\}$. Observe that ξ_1 is an ewp of ω_{i_k}. As in Case 1, we have reached a contradiction since ω_{i_k} is a locally stable weight assignment for p_{i_k}. □

5.2.2 Complexity of the algorithm

The following theorem easily follows from the algorithm and the previous analysis.

Theorem 20 *Time and space complexity of the algorithm is*

$$O\left(d\#L(\#W + 2^{O(m^2)}\#L)\right).$$

5.3 Regarding the assumptions

We have assumed that we know the dimension of the manifold m, and the value of λ (having an upper bound would have been good enough) where L is a λ-net of W.

We will address the second question first. Given a point sample W, and beginning with an arbitrary point from W, it is simple to show that a furthest point sampling from W will generate a λ-net of W, for some $\lambda > 0$, and it is possible to keep track of the value of λ. For an analysis of this procedure, refer to [BGO09, Lem. 5.1].

Let $\mathsf{P} \subset \mathcal{M}$ be an (ν, ϵ)-net of $\mathcal{M}$. If $\frac{\nu}{\epsilon} = O(1)$ and if we know an upper bound on this quantity and if $\epsilon \le \epsilon_0$, where ϵ_0 depends only on the reach and the dimension of $\mathcal{M}$, then we can learn the local dimension of the manifold at each sample point with time and space complexity $2^{O(m)}(\#\mathsf{P})^2$ and $2^{O(m)}\#\mathsf{P}$ respectively, see [CWW08, CC09, GW04]. Note that, in these papers, the dimension estimation is done locally around each sample point and therefore is exactly in the spirit of this paper.

6 Conclusion: only distances required

The algorithm we have outlined can be simply adapted to work in the setting where the input is just a *distance matrix* corresponding to a dense point sample on the submanifold $\mathcal{M}$. Rather than giving explicit coordinates of the points, we will be given a distance matrix $M = (a_{ij})$ where $a_{ij} = \|p_i - p_j\|$ and $p_i,\, p_j \in W$.

λ-net L of W. The distance matrix can be used to generate a λ-net L of W by repeatedly inserting a farthest point. Moreover, the interpoint distance matrix can be used to estimate the dimension m of the manifold $\mathcal{M}$ as well.

In our reconstruction algorithm, we have to compute for all $p \in L$, lists of local neighbors $N(p)$, candidate simplices $S(p)$ and forbidden intervals $I_\omega(\sigma, p)$, and finally the witness complex.

Computing $N(p)$ and $S(p)$. Computing $N(p)$ is simple. For computing $S(p)$, we need to compute the altitude of simplices. This reduces to computing volume of simplices,

$$D(p, \sigma^j) = \frac{j\,\mathrm{vol}(\sigma^j)}{\mathrm{vol}(\sigma^j_p)},$$

which can be done from the knowledge of the lengths of its edges. Observe that for a simplex $\sigma = [p_0,\, \ldots,\, p_k]$

$$\mathrm{vol}(\sigma) = \frac{1}{k!}\sqrt{|\,\det M(\sigma)\,|}$$

where $M(\sigma) \stackrel{\text{def}}{=} (b_{ij})_{1 \le i,\, j \le k}$ with

$$b_{ij} = \langle p_i - p_0,\, p_j - p_0\rangle^1 = \frac{\|p_i - p_0\|^2 + \|p_j - p_0\|^2 - \|p_i - p_j\|^2}{2}.$$

Computing forbidden intervals $I_\omega(\sigma, p)$. Assume σ is a k-simplex. Recall that computing $I_\omega(\sigma, p)$ will boil down to computing $D(p, \sigma)$, $d(p, N_\omega(\sigma_p))$ and $R_\omega(\sigma_p)$, see Eq.s (2), (3) and (4). We have already discussed how to compute $D(p, \sigma)$, but observe that $d(p, N_\omega(\sigma_p))$ and $R_\omega(\sigma_p)$ can be computed if we can find a distance preserving embedding of σ. Since we know the pairwise distance between vertices of the simplex, a distance preserving embedding of σ can be computed in $O(k^3)$, where σ is a k-simplex. See [Mat02, Mat13].

Computing witness complex. By its very definition, the witness complex can be built from an interpoint distance matrix. So, we can easily adapt our algorithm, without increasing its complexity, to the setting of interpoint distance matrices, which was not possible with the other reconstruction algorithms that explicitly needs coordinates of the points [CDR05, BGO09, BG14].

[1]Given two vectors u and v, $\langle u,\, v\rangle$ denotes the Euclidean inner product of the vectors u and v.

Acknowledgements

This research has been partially supported by the 7th Framework Programme for Research of the European Commission, under FET-Open grant number 255827 (CGL Computational Geometry Learning). Partial support has also been provided by the Advanced Grant of the European Research Council GUDHI (Geometric Understanding in Higher Dimensions).

Arijit Ghosh is supported by the Indo-German Max Planck Center for Computer Science (IMPECS). Part of this work was done when Arijit Ghosh was a visiting scientist at ACM Unit, Indian Statistical Institute, Kolkata, India.

A Proof of Theorem 15

This section is devoted to the proof of Theorem 15. In this section we will need the definition of weighted normal space $N_\omega(\sigma)$ defined in Section D.

When we talk about properties P1, P2, P3, P4 and P5 in this section, we are actually referring to properties indroduced in Section 4.

We will use the following structural result from [BGO09].

Lemma 21 *Let $\theta \in [0, \frac{\pi}{2})$, and $\mathsf{P} \subset \mathcal{M}$ be an ϵ-sample of $\mathcal{M}$ with $\epsilon < \frac{1}{9}(1-\sin\theta)^2 rch(\mathcal{M})$. For any weight assignment $\omega : L \to [0,\infty)$ with $\tilde{\omega} < \frac{1}{2}$, for any $p \in \mathsf{P}$ and $x \in \mathrm{Vor}_\omega(p) \cap \mathrm{K}^\theta(p)$, we have*

$$\|p - x\| \leq \frac{3\epsilon}{1-\sin\theta} .$$

The following result is a direct consequence of Lemma 21.

Lemma 22 *Let L be an ϵ-sample of $\mathcal{M}$ with $\epsilon < \frac{1}{9}(1-\sin\theta_0)^2 rch(\mathcal{M})$, and let $\omega : L \to [0,\infty)$ be a weight assignment with $\tilde{\omega} < \frac{1}{2}$. Let $\sigma \in \mathrm{K}_\omega(L)$.*

1. *Let v be a vertex of σ with $\mathrm{Vor}_\omega(v) \cap \mathrm{K}^{\theta_0}(v) \neq \emptyset$. For all vertices p of σ and $x \in \mathrm{Vor}_\omega(v) \cap \mathrm{K}^{\theta_0}(v)$, we have $\|x - p\| < 4\epsilon$, This implies for all the vertices p of σ, $\|p - C_\omega(\sigma)\| < 4\epsilon$.*

2. *$\Delta(\sigma) < 8\epsilon$.*

3. **(Property P1)** *Assume $\dim\sigma = k \leq m$ and $\Upsilon(\sigma) \geq \Gamma_0^k$. Additionally, if $\epsilon < \frac{rch(\mathcal{M})}{8}$ then for all vertices p of σ we have*

$$\sin\angle(\mathrm{aff}(\sigma), T_p\mathcal{M}) \leq \frac{8\epsilon}{\Gamma_0^m rch(\mathcal{M})}.$$

4. **(Property P2)** *If $\mathrm{K}_\omega(L)$ does not contain any Γ_0-slivers of dimension $\leq m+1$ and*

$$\epsilon < \frac{\Gamma_0^{2m+1} rch(\mathcal{M})}{12},$$

then dimension of maximal simplices in $\mathrm{K}_\omega(L)$ is at most m.

5. **(Property P3)** *Assume hypothesis in part (4) of this lemma. Additionally, if L is $\frac{\epsilon}{2}$-sparse and*
$$\epsilon \leq \frac{3\sin\theta_0 \Gamma_0^m}{2^{10}} rch(\mathcal{M}),$$
then for all m-simplex $\sigma^m \in \mathrm{K}_\omega(L)$ and $\forall\ p \in \sigma$, we have $\mathrm{Vor}_\omega(\sigma^m) \cap T_p\mathcal{M} \neq \emptyset$.

Proof Part 1 and 2 directly follows from Lemma 21 and using triangle inequality.

Part 3 follows directly from Corollary 5 and part 2 of the lemma.

Using part 3 and exactly the proof idea used in the proof of [BG14, Lem. 4.9], we can show that all $m+1$-simplices in $\mathrm{K}_\omega(L)$ are either Γ_0-bad or have thickness $\frac{12\epsilon}{\Gamma_0^m rch(\mathcal{M})}$. Using the bound on ϵ, we can complete the proof of part 4.

Using the facts that L is $\frac{\epsilon}{2}$-sparse and $\tilde{\omega} < \frac{1}{2}$, we can show $\forall\ p \in \sigma$, $d(p, N_\omega(\sigma)) \geq \frac{3\epsilon}{16}$. Let $\sigma^m \in \mathrm{Del}_\omega(L, T\mathcal{M})$ and p be a vertex of σ^m such that $\mathrm{Vor}_\omega(\sigma) \cap T_p\mathcal{M} = \emptyset$. Using the bound on ϵ and part 3 of this lemma, we can show, for all $x \in \sigma^m$, that
$$\sin(\operatorname{aff}\sigma^m, T_x\mathcal{M}) \leq \rho \stackrel{\text{def}}{=} \frac{8\epsilon}{\Gamma_0^m rch(\mathcal{M})} < \frac{1}{2}$$
and
$$\#(N_\omega(\sigma) \cap T_x\mathcal{M}) = 1.$$
Let $c = N_\omega(\sigma) \cap T_p\mathcal{M}$ and $c' = N_\omega(\sigma) \cap T_q\mathcal{M}$ where $q \in \sigma^m \setminus p$. Using part 1, and the bound on angle between tangent spaces at the vertices and the simplex, we have $\|c - c'\| \leq 8\rho\epsilon$. Using the bound on ϵ, the facts that $\|c - c'\| \in N_\omega(\sigma)$, $\|c - c'\| \leq 8\rho\epsilon$ and $d(p, N_\omega(\sigma)) \geq \frac{3\epsilon}{16}$, we get $[c, c'] \in \mathrm{K}^{\theta_0}(p)$. So, $\mathrm{Vor}(\sigma^m) \cap T_p\mathcal{M} = \emptyset$ implies there exists a $\sigma^{m+1} \in \mathrm{K}_\omega(L)$ with $\sigma^m < \sigma^{m+1}$. We have reached a contradiction via part 4. □

The following corollary about witness complex is from [dS08, Cor. 7.6].

Corollary 23 *For any subsets W, $L \subseteq \mathbb{R}^d$ with L finite, for any $\omega : L \to [0, \infty)$, we have $\mathrm{Wit}_\omega(L, W) \subseteq \mathrm{Del}_\omega(L)$. Moreover, for any simplex σ of $\mathrm{Wit}_\omega(L, W)$, the weighted Voronoi face of σ intersects the convex hull of the ω-witnesses (among the points of W) of σ and of its subsimplices.*

Following result is a direct consequence of Lemma 6 (2).

Lemma 24 *Let $W \subseteq \mathcal{M}$ be an ε-sample of $\mathcal{M}$, $L \subseteq W$ be a λ-net of W with $\lambda + \varepsilon < \frac{rch(\mathcal{M})}{4}$, and $\omega : L \to [0, \infty)$ be a weight assignment with $\tilde{\omega} < \frac{1}{2}$.*

1. *For all $pq \in \mathrm{Wit}_\omega(L, W)$ are at most $(4 + 10\tilde{\omega})(\lambda + \varepsilon)$.*

2. *Let $\sigma \in \mathrm{Wit}_\omega(L, W)$. The distance between any vertex v of σ any witness w of $\tau \leq \sigma$ is at most $(5 + 12\tilde{\omega})(\lambda + \varepsilon)$.*

Lemma 25 (Property P4) *Let $W \subseteq \mathcal{M}$ be a ε-sample of $\mathcal{M}$, $L \subset W$ be a (λ, λ)-sample of W with $\varepsilon \leq \lambda$, and $\omega : L \to [0,\ \infty)$ be a weight assignment with $\tilde{\omega} < \frac{1}{2}$ and $\mathrm{K}_\omega^{\theta_0}(L)$ does not contain any Γ_0-sliver of dimension $\leq m + 1$. If*
$$\lambda < \min\left\{\frac{3\sin\theta_0}{2^{11}(1+m)}, \frac{\Gamma_0^{2m+1}}{24}\right\} rch(\mathcal{M})$$

then

$$\mathrm{Wit}_\omega(L, W) \subseteq \mathrm{Del}_\omega(L, T\mathcal{M}).$$

Proof Note that L is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$.

To reach a contradiction, let σ^k be a k-simplex in $\mathrm{Wit}_\omega(L, W)$ and p be a vertex of σ such that $\mathrm{Vor}_\omega(\sigma^k) \cap T_p\mathcal{M} = \emptyset$.

Let $w \in W$ be a ω-witness of a subface of σ^k. From Lemma 24, and the facts that $\tilde{\omega} < \frac{1}{2}$ and $\varepsilon \leq \lambda$, we have

$$\|p - w\| \leq (5 + 12\tilde{\omega})(\lambda + \varepsilon) < 22\lambda.$$

From Lemma 2, we have

$$d(w, T_p\mathcal{M}) \leq \frac{2 \times 11^2 \lambda^2}{rch(\mathcal{M})}.$$

From Corollary 23, there exist $c_k \in \mathrm{Vor}_\omega(\sigma)$ that lies in the convex hull of the ω-witness of σ (in W) and its subfaces. This implies,

$$\mu \stackrel{\text{def}}{=} d(c_k, T_p\mathcal{M}) \leq \frac{2 \times 11^2 \lambda^2}{rch(\mathcal{M})}.$$

Note that since L is λ-sparse, $\|p - c_k\| \leq \frac{3\lambda}{8}$.

Note that λ is sufficiently small such that

$$\frac{\mu}{\frac{3\lambda}{8} - 2m\mu} \leq \sin\theta_0 \tag{6}$$

as

$$\sin\theta \stackrel{\text{def}}{=} \frac{16\lambda}{\Gamma_0^m rch(\mathcal{M})} < \frac{\sqrt{3}}{2}. \tag{7}$$

We will now generate sequence of simplices

$$\sigma^k < \sigma^{k+1} < \cdots < \sigma^m < \sigma^{m+1}$$

and points

$$c_k, \, c_{k+1}, \, \ldots, \, c_m, \, c_{m+1}$$

by walking on $\mathrm{Vor}_\omega(\sigma^k)$ satisfying the following properties:

Prop-1. For all σ^{k+i}, there exists $c_{k+i} \in \mathrm{Vor}_\omega(\sigma^{k+i})$ such that

$$d(c_{k+i}, T_p\mathcal{M}) \leq \mu$$

and

$$\|p - c_{k+i}\| \geq \|p - c_k\| - 2i\mu.$$

From Eq. (6), this implies $\sigma^{k+i} \in \mathrm{K}_\omega^{\theta_0}(L)$.

Prop-2. For all σ^{k+i}, we have

$$\mathrm{Vor}_\omega(\sigma^{k+i}) \cap T_p\mathcal{M} = \emptyset.$$

Note that once we have shown that such sequence of simplices exists, then we would have reached a contradiction from Lemma 22 (4).

We will now show how to generate the above sequence of simplices.

Base case. From Eq. (6), it is easy to see that σ^k and c_k satisfy Prop-1 and Prop-2.

Inductive step. Wlog lets assume that we have generated till σ^{k+i}, satisfying properties **Prop-1** and **Prop-2**, and we also assume $k+i \leq m$. Since $\sigma^{k+i} \in \mathrm{K}_\omega(L)$, we can show, using Lemma 22 (3), that

$$\sin\angle(N_p\mathcal{M}, N_\omega(\sigma^{k+i})) \leq \sin\theta.$$

From **Prop-1**, we have $\|p - c_{k+i}\| \geq t - 2i\mu$ and $d(c_{k+i}, T_p\mathcal{M}) \leq \mu$. Therefore, from Eq. 7, there exists $\tilde{c}_{k+i} \in T_p\mathcal{M} \cap N_\omega(\sigma^{k+i})$ such that

$$\|c_{k+i} - \tilde{c}_{k+i}\| \leq \frac{\mu}{\cos\theta} \leq 2\mu.$$

As $\mathrm{Vor}_\omega(\sigma^{k+i}) \cap T_p\mathcal{M} = \emptyset$ hence there exists $c_{k+i+1} \in [c_{k+i},\, \tilde{c}_{k+i})$ such that $c_{k+i+1} \in \mathrm{Vor}_\omega(\sigma^{k+i+1})$ with $\sigma^{k+i} < \sigma^{k+i+1}$. Note that, as in the base case, we can show that

$$d(c_{k+i+1}, T_p\mathcal{M}) \leq \mu$$

and

$$\|p - c_{k+i+1}\| \leq \|p - c_k\| - 2(i+1)\mu\,.$$

□

Property **P5** is a direct consequence of the following lemma[2] from [BG14].

Lemma 26 *Let $L \subset \mathcal{M}$ be an $(\lambda, 2\lambda)$-net of $\mathcal{M}$, and $\omega : \mathsf{P} \to [0, \infty)$ be a weight assignment satisfying the following properties:*

1. $\tilde{\omega} \leq \alpha_0$.
2. *Dimension of maximal simplices in* $\mathrm{Del}_\omega(\mathsf{P}, T\mathcal{M})$ *is equal to* m
3. *All the simplices in* $\mathrm{Del}_\omega(\mathsf{P}, T\mathcal{M})$ *are* Γ_0*-good.*
4. *For all* $\sigma = [p_0, \ldots, p_k] \in \mathrm{Del}_\omega(\mathsf{P}, T\mathcal{M})$ *and* $\forall\, i \in \{0, \ldots, k\}$, $\mathrm{Vor}_\omega(p_i) \cap T_{p_i}\mathcal{M} \neq \emptyset$.

There exists $\epsilon_0 > 0$ that depends only on α_0, Γ_0 and m such that for $\lambda \leq \epsilon_0$, $\mathrm{Del}_\omega(\mathsf{P}, T\mathcal{M})$ is homeomorphic to and a close geometric approximation of $\mathcal{M}$.

Now, to complete the proof of Theorem 15 we only need to prove Property **P4**, and rest of the setion is devoted to the proof of this property.

The following lemma connects power protection of m-dimensional simplices in $\mathrm{Del}_\omega(T\mathcal{M})$ with stability of ω.

[2] Note that this lemma is a special case of the result proved in [BG14].

Lemma 27 *Let $L \subset \mathcal{M}$ be a $(\lambda, 2\lambda)$-net of $\mathcal{M}$ with*

$$\lambda < \min\left\{\frac{3\sin\theta_0\Gamma_0^m}{2^{10}}, \frac{\Gamma_0^{2m+1}}{24}\right\} rch(\mathcal{M}),$$

and let $\omega : L \to [0, \infty)$, with $\tilde{\omega} \leq \tilde{\alpha}_0$, be a stable weight assignment. Then all the m-simplices $\sigma \in \mathrm{Del}_\omega(L, T\mathcal{M})$ are δ^2-power protected on $T_p\mathcal{M}$ for all $p \in \mathring{\sigma}$, where $\delta = \delta_0\lambda$.

Proof For all m-simplices σ in $\mathrm{Del}_\omega(L, T\mathcal{M})$, we have $\mathrm{Vor}_\omega(\sigma) \cap T_p\mathcal{M}$ for all $p \in \mathring{\sigma}$, Lemma 22 part 5, and from part 3 of the same lemma and the bound on λ, we have

$$\sin\angle(\mathrm{aff}\,\sigma, T_p\mathcal{M}) \leq \frac{16\lambda}{\Gamma_0^m rch(\mathcal{M})} < 1. \tag{8}$$

This implies

$$\#(N_\omega(\sigma) \cap T_p\mathcal{M}) = 1. \tag{9}$$

To reach a contradiction, lets assume that σ to be not δ^2-power protected on $T_p\mathcal{M}$. Let $c = \mathrm{Vor}_\omega(\sigma) \cap T_p\mathcal{M}$ (from Eq. (9)) and $q \in L \setminus \sigma$ such that for all $x \in \mathring{\sigma}$

$$\|q - c\|^2 - \omega(q)^2 - \delta^2 \leq \|x - c\|^2 - \omega(x)^2.$$

Let $\beta^2 = \|q - c\|^2 - \|p - c\|^2 - (\omega(q)^2 - \omega(p)^2)$ where $p \in \mathring{\sigma}$. Note that $\beta \leq \delta$.

Let $\xi : L \to [0, \infty)$

$$\xi(x) = \begin{cases} \omega(x) & \text{if } x \neq q \\ \sqrt{\omega(q)^2 + \beta^2} & \text{if } x = q \end{cases}$$

Since L is λ-sparse, $\delta = \delta_0\lambda$ and $\tilde{\alpha}_0^2 + \delta_0^2 \leq \alpha_0^2$, we have $\tilde{\xi} \leq \alpha_0$. It is easy to see ξ is an ewp of ω. As $\mathrm{Del}_\xi(L, T\mathcal{M}) \subseteq \mathrm{K}_\xi^{\theta_0}(L)$, we have reached a contradiction from part 4 of Lemma 22 and the fact that ω is a stable weight assignment. □

We will need the following result is due to Boissonnat et al. [BGO09, Lem 2.2].

Lemma 28 *Let $L \subset \mathbb{R}^d$ be a point set, $\omega : L \to [0, \infty)$ be a weight distribution, and $H \subseteq \mathbb{R}^d$ be a k-dimensional flat. Also, let L' denotes the projection of the point set L onto H, and p' denotes the projection of $p \in L$ onto H. For all $p \in L$, we have*

$$\mathrm{Vor}_\omega(p) \cap H = \mathrm{Vor}_\xi(p')$$

where $\xi : L' \to [0, \infty)$ with

$$\xi(q')^2 = \omega(q)^2 - \|q - q'\|^2 + \max_{x \in \mathsf{P}} \|x - x'\|^2$$

and $\mathrm{Vor}_\xi(p')$ denotes the Voronoi diagram of p' in H and not in $\mathbb{R}^d$.

From Lemma 28, we have get the following corollary.

Corollary 29 *Let $L \subset \mathbb{R}^d$ be a finite set, $\omega : L \to [0, \infty)$, and let $H \subseteq \mathbb{R}^d$ be k-flat. For a point $p \in L$, if $\mathrm{Vor}_\omega(p) \cap H$ is bounded then the dimension of maximal simplices incident to p in $\mathrm{Del}_\omega(L, H) \stackrel{\mathrm{def}}{=} \{\sigma : \mathrm{Vor}_\omega(\sigma) \cap H \neq \emptyset\}$ is greater than k.*

Following lemma connects power protection of m-simplices on the tangent space to that on the manifold. The proof of the lemma is given in the Appendix 30.

Lemma 30 *Let $L \subset \mathcal{M}$ be a $(\lambda, 2\lambda)$-net of $\mathcal{M}$, and $\delta = \delta_0 \lambda$ with $\delta_0 < 1$. Let the weight assignment $\omega : L \to [0, \infty)$, $\tilde{\omega} \le \alpha_0$, satisfy the following properties:*

1. $\mathrm{Del}_\omega(\mathsf{P}, \mathcal{M})$ *does not contain any Γ_0-sliver, and*
2. $\forall\, \sigma^m \in \mathrm{Del}_\omega(\mathsf{P}, T\mathcal{M})$, *$\sigma^m$ is δ^2-power protected on $T_p\mathcal{M}$ for all $p \in \sigma^m$.*

If

$$\lambda \le \frac{\Gamma_0^m rch(\mathcal{M})}{2^{11}},$$

then all $\sigma \in \mathrm{Del}_\omega(L, \mathcal{M})$ are δ_1^2-power protected on $\mathcal{M}$ where

$$\delta_1^2 = \frac{\delta^2}{m+1} - \frac{B\lambda^3}{rch(\mathcal{M})}$$

and $B \stackrel{\text{def}}{=} 2^{15}$.

Proof Let p be a point in L, and L' denotes the projection of the point sample L onto $T_p\mathcal{M}$. For a point $x \in L$, x' is the projection of x onto $T_p\mathcal{M}$ and vise versa, and similarly, let $\sigma = [p_0, \ldots, p_k]$ be a simplex with p_i's in L then σ' denotes the simplex $[p'_0, \ldots, p'_k]$ and vise versa. Note that $p' = p$.

The weight assignment $\xi_p : L' \to [0, \infty)$ is defined in the following way:

$$\xi(x')^2 = \omega(x)^2 - \|x - x'\|^2 + \max_{y \in L} \|y - y'\|^2 .$$

For $\sigma' \subseteq L'$, $\mathrm{Vor}_\xi(\sigma')$ denotes the Voronoi cell in $T_p\mathcal{M}$ and not in $\mathbb{R}^d$.

From Lemmas 28 and 22 (1) we have:

Prop. (a) For $\sigma \subseteq L$, $\mathrm{Vor}_\omega(\sigma) \cap T_p\mathcal{M} = \mathrm{Vor}_\xi(\sigma')$.

Prop. (b) $\mathrm{Vor}_\omega(p) \cap T_p\mathcal{M} = \mathrm{Vor}_\xi(p) \subset B(p, 8\lambda) \cap T_p\mathcal{M}$.

From Prop. (a) and the definition of tangential complex, if $\sigma' \in \mathrm{st}(p; \mathrm{Del}_{\xi_p}(L'))$ then $\sigma \in \mathrm{Del}_\omega(L, T\mathcal{M})$. Since all the m-simplices of $\mathrm{Del}_\omega(L, T\mathcal{M})$ are δ^2-power protected on the tangent space of the vertices (Hyp. 4), therefore, from the definition of $\xi :\to [0, \infty)$, all the m-simplices σ' incident to p in $\mathrm{Del}_\xi(L')$ are also δ^2-power protected on $T_p\mathcal{M}$, i.e., there exists $x \in \mathrm{Vor}_{\xi_p}(\sigma')$ such that for all $q' \in \mathring{\sigma}'$ and $r' \in L' \setminus \mathring{\sigma}'$

$$\|r' - x\|^2 - \xi(r')^2 > \|q' - x\|^2 - \xi(q')^2 + \delta^2 .$$

Following properties are a direct consequence of Prop. (b), and Lemmas 9 (1) and 8

Prop. (c) Dimension of maximal simplices incident to p in $\mathrm{Del}_\xi(L')$ is equal to m.

Prop. (d) Let σ' be a m-simplex incident to p in $\mathrm{Del}_\xi(L')$. Then $\mathrm{Vor}_\xi(\sigma') = c_\xi(\sigma')$.

Prop. (e) Let $\sigma' \in \mathrm{Del}_\xi(L')$ be a j-simplex incident, with $p \in \sigma'$, then σ' is $\frac{\delta^2}{m-j+1}$-power protected.

Note that Prop. (c) and the definition of tangential complex implies the following

Prop. (f) Dimension of maximal simplices in $\mathrm{Del}_\omega(L, T\mathcal{M})$ is equal to m.

We will now prove the power protection of simplices in $\mathrm{Del}_\omega(L, T\mathcal{M})$ on the manifold $\mathcal{M}$. Let $\sigma \in \mathrm{Del}_\omega(L, T\mathcal{M})$ be a k-simplex, with $k \le m$, incident to p. From Prop. (e), $\exists\, c' \in \mathrm{Vor}_\xi(\sigma')$ such that $\forall\, x' \in \mathring{\sigma}'$ and $\forall\, y' \in L' \setminus \mathring{\sigma}'$

$$\|y' - c'\|^2 - \xi(y')^2 > \|x' - c'\|^2 - \xi(x')^2 + \frac{\delta^2}{m+1}.$$

Which, from the definition of ξ and Prop. (a), implies $\forall\, x \in \mathring{\sigma}$ and $\forall\, y \in L \setminus \mathring{\sigma}$

$$\|y - c'\|^2 - \omega(y)^2 > \|x - c'\|^2 - \omega(x)^2 + \frac{\delta^2}{m+1},$$

and $c' \in \mathrm{Vor}_\omega(\sigma)$.

Let $\hat{c}$ be the point closest to c' on $\mathcal{M}$ and c denotes the point closest to c' in $\mathcal{M} \cap N_\omega(\sigma)$.

Using the facts that $\|p - c'\| \le 8\lambda$ (from Lemma 22 (1)) and

$$\|c' - \hat{c}\| \le \frac{2^7\lambda^2}{rch(\mathcal{M})} \le \frac{\lambda}{16} < \frac{rch(\mathcal{M})}{25} \tag{10}$$

from part 2(b) of Lemma 2 and $\lambda \le \frac{\Gamma_0^m rch(\mathcal{M})}{2^{11}} \le \frac{rch(\mathcal{M})}{2^{11}}$, we get

$$\|p - \hat{c}\| \le \|p - c'\| + \|c' - \hat{c}\| \le \left(8 + \frac{1}{16}\right)\lambda < \frac{rch(\mathcal{M})}{4}. \tag{11}$$

Therefore, using $\sin\angle(\mathrm{aff}(\sigma), T_p\mathcal{M}) \le \frac{16\lambda}{\Gamma_0^m rch(\mathcal{M})}$ (from Lemma 22 (3)) and $\sin\angle(T_p\mathcal{M}, T_{\hat{c}}\mathcal{M}) < \frac{6\|p-\hat{c}\|}{rch(\mathcal{M})}$ (from part 2(b) of Lemma 2 and $\|p - \hat{c}\| < \frac{rch(\mathcal{M})}{4}$), we have

$$\begin{aligned}
\sin\angle(\mathrm{aff}\,\sigma, T_{\hat{c}}\mathcal{M}) &\le \sin\angle(\mathrm{aff}\,\sigma, T_p\mathcal{M}) + \sin\angle(T_p\mathcal{M}, T_{\hat{c}}\mathcal{M}) \\
&\le \frac{16\lambda}{\Gamma_0^m rch(\mathcal{M})} + \frac{6\|p - \hat{c}\|}{rch(\mathcal{M})} \\
&\le \frac{16\lambda}{\Gamma_0^m rch(\mathcal{M})} + \frac{387\lambda}{8\, rch(\mathcal{M})} && \text{as } \|p - \hat{c}\| \le \tfrac{129\lambda}{16} \\
&\le \frac{1}{4} && \text{as } \lambda \le \tfrac{\Gamma_0^m rch(\mathcal{M})}{2^{11}}
\end{aligned}$$

Using the above bound on $\sin\angle(\mathrm{aff}\,\sigma, T_{\hat{c}}\mathcal{M})$, the fact that $\|c' - \hat{c}\| \le \frac{2^7\lambda^2}{rch(\mathcal{M})} < \frac{rch(\mathcal{M})}{25}$ (Eq. (10)) and Lemma 40, we get

$$\|c' - c\| \le 4\|c' - \hat{c}\| \le \frac{2^9\lambda^2}{rch(\mathcal{M})} \stackrel{\text{def}}{=} \frac{C\lambda^2}{rch(\mathcal{M})}.$$

Let $q \in L \setminus \mathring{\sigma}$ and $p \in \mathring{\sigma}$. We will consider the following two cases:

Case-1. $\|q-c\|^2 > \|p-c\|^2 + 2(2\lambda)^2$. Using the facts that $\omega(q) \le 4\alpha_0\lambda$ (from part 2(a) of Lemma 6) and $\alpha_0 < \frac{1}{2}$, we have

$$\begin{aligned}
\|q-c\|^2 - \omega(q)^2 - (\|p-c\|^2 - \omega(p)^2) &> 8\lambda^2 - \omega(q)^2 + \omega(p)^2 \\
&> 8\lambda^2 - \omega(q)^2 \\
&> 4\lambda^2
\end{aligned}$$

Case-2. $\|q-c\|^2 \le \|p-c\|^2 + 8\lambda^2$. This implies $\|q-c\| < \|p-c\| + 3\lambda$.

Using the facts that $\|p-c'\| \le 8\lambda$ (from Lemma 22 (1)), $\|c-c'\| \le \frac{C\lambda^2}{rch(\mathcal{M})} \le 16\lambda$,

$$\begin{aligned}
\|p-c\| &\le \|p-c'\| + \|c-c'\|\| \le \left(8+\frac{1}{4}\right)\lambda, \text{ and} \\
\|q-c'\| &\le \|q-c\| + \|c-c'\| \le \|p-c\| + 19\lambda \le \left(8+\frac{13}{4}\right)\lambda,
\end{aligned}$$

we get

$$\begin{aligned}
\|q-c\|^2 - \omega(q)^2 &\ge (\|q-c'\| - \|c-c'\|)^2 - \omega(q)^2 \\
&\ge \|q-c'\|^2 - \omega(q)^2 - 2\|c-c'\|\|q-c'\| \\
&> \|p-c'\|^2 - \omega(p)^2 + \frac{\delta^2}{m+1} - 2\|c-c'\|\|q-c'\| \\
&\ge (\|p-c\| - \|c-c'\|)^2 - \omega(p)^2 + \frac{\delta^2}{m+1} - 2\|c-c'\|\|q-c'\| \\
&\ge \|p-c\|^2 - \omega(p)^2 + \frac{\delta^2}{m+1} - 2\|c-c'\|(\|q-c'\| + \|p-c\|) \\
&> \|p-c\|^2 - \omega(p)^2 + \frac{\delta^2}{m+1} - \frac{B\lambda^3}{rch(\mathcal{M})}
\end{aligned}$$

where $B = 2^{15}$.

From **Case-1** and **2**, we get

$$\|q-c\|^2 - \omega(q)^2 > \|p-c\|^2 - \omega(p)^2 + \frac{\delta^2}{m+1} - \frac{B\lambda^3}{rch(\mathcal{M})}.$$

$\square$

Lemma 31 (Property P4) *Let $W \subseteq \mathcal{M}$ be an ε-sample of $\mathcal{M}$, $L \subseteq W$ be a (λ,λ)-sample of W with $\varepsilon \le \lambda$, and $\delta = \delta_0\lambda$. Also, let $\omega : L \to [0,\infty)$ be a weight assignment satisfying conditions (1) to (4) of Lemma 30. If $\delta = \delta_0\lambda$,*

$$\lambda < \min\left\{\frac{\Gamma_0^m}{2^{11}}, \frac{\delta_0^2}{B(m+1)}\right\} rch(\mathcal{M})$$

and

$$\varepsilon < \frac{\lambda}{24}\left(\frac{\delta_0^2}{m+1} - \frac{B\lambda}{rch(\mathcal{M})}\right),$$

then

$$\mathrm{Del}_\omega(L, T\mathcal{M}) \subseteq \mathrm{Wit}_\omega(L, W).$$

Proof Note that, as $\varepsilon \leq \lambda$, L is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$.

Let $\sigma^k \in \mathrm{Del}_\omega(L, \mathcal{M})$. From Lemma 30, there exists $c \in \mathrm{Vor}_\omega(\sigma^k) \cap \mathcal{M}$ such that σ^k is δ_1^2-protected at c, where $\delta_1^2 = \frac{\delta^2}{m+1} - \frac{B\lambda^3}{rch(\mathcal{M})}$. From Lemma 6 (2) as $c \in \mathrm{Vor}_\omega(\sigma^k) \cap \mathcal{M}$, we have for all $p \in \sigma^k$, $\|p - c\| \leq 4\lambda$.

Let $w \in W$ be such that $\|c - w\| \leq \varepsilon$. For all $q \in L \setminus \sigma^k$ and $p \in \sigma^k$ we have

$$\begin{aligned}
\|p - w\|^2 - \omega(p)^2 &\leq (\|p - c\| + \|c - w\|)^2 - \omega(p)^2 \\
&= \|p - c\|^2 - \omega(p)^2 + \|c - w\| \left(\|c - w\| + 2\|p - c\|\right) \\
&\leq \|p - c\|^2 - \omega(p)^2 + 9\varepsilon\lambda \\
&< \|q - c\|^2 - \omega(q)^2 - (\delta_1^2 - 9\varepsilon\lambda) \\
&\leq \|q - w\|^2 - \omega(q)^2 + \beta - (\delta_1^2 - 9\varepsilon\lambda) \qquad (12)
\end{aligned}$$

Where $\beta = \|w - c\| \left(\|w - c\| + 2\|q - w\|\right)$.

We have to consider the following two case:

1. If $\|q - w\|^2 > \|p - w\|^2 + 4\lambda^2$. Using the fact that $\omega(q) < 2\lambda$, from Lemma 6 (1), we get This implies

$$\|q - w\|^2 - \omega(q)^2 > \|p - w\|^2 + 4\lambda^2 - \omega(q)^2 > \|p - w\|^2 \geq \|p - w\|^2 - \omega(p)^2$$

2. If $\|q - w\|^2 \leq \|p - w\|^2 + 4\lambda^2$. This implies

$$\|q - w\| \leq \|p - w\| + 2\lambda \leq \|p - c\| + \|c - x\| + 2\lambda \leq 7\lambda.$$

Now, using Eq. (12) and the facts that $\|q - w\| = 7\lambda$ and $\|c - w\| \leq \varepsilon \leq \lambda$, we get

$$\begin{aligned}
\|p - w\|^2 - \omega(p)^2 &\leq \|q - w\|^2 - \omega(q)^2 + \beta - (\delta_1^2 - 9\varepsilon\lambda) \\
&\leq \|q - w\|^2 - \omega(q)^2 - (\delta_1^2 - 24\varepsilon\lambda) \qquad \text{as } \beta \leq 15\varepsilon\lambda \\
&< \|q - w\|^2 - \omega(q)^2
\end{aligned}$$

The last inequality follows from the fact that $\lambda < \frac{\delta_0^2 rch(\mathcal{M})}{B\,m}$ and $\varepsilon < \frac{\lambda}{24}\left(\frac{\delta_0^2}{m} - \frac{B\lambda}{rch(\mathcal{M})}\right)$.

This implis w is a witness of σ^k.

As this is true for all $\sigma^k \in \mathrm{Del}_\omega(L, T\mathcal{M})$, we get $\mathrm{Del}_\omega(L, T\mathcal{M}) \subseteq \mathrm{Wit}_\omega(L, W)$. □

B Proof of Lemma 16

B.1 Outline of the proof

We will use a variant of Pumping equation, Lemma 33, from [CDE+00] and bound on the height of slivers, Lemma 34, from [BDG13b]. Let $\omega : L \to [0, \infty)$ be a weight assignment with $\tilde{\omega} \leq \tilde{\alpha}_0$, and $\sigma \subset L$ be a Γ_0-sliver incident to the point $p \in L$. As in Lemma 16, ω_1 is an ewp of ω such that $\sigma \in \mathrm{K}_\omega(L)$. To prove Lemma 16, we distinguish the following two cases depending on the point whose weight is changed when replacing ω by ω_1:

Case 1. The point whose weight is changed is p. Lemma 35 takes care of this case and states that

$$\omega(p)^2 \in J_\omega(p,\sigma) = \left[F_\omega(p,\sigma) - \frac{\eta_1}{2} - \delta_0^2\lambda^2,\ F_\omega(p,\sigma) + \frac{\eta_1}{2}\right],$$

for some $\eta_1 \leq \eta - 2\delta_0^2\lambda^2$.

Case 2. The point whose weight is changed is *not* p. Lemma 16 takes care of this case and states that

$$\omega(p)^2 \in I_\omega(\sigma,p) = \left[F_\omega(p,\sigma) - \frac{\eta}{2},\ F_\omega(p,\sigma) + \frac{\eta}{2}\right].$$

Since $J_\omega(\sigma,p) \subset I_\omega(\sigma,p)$, Lemma 16 is proved.

The proof of **Case 1** is in the same vein as the proofs of [CDR05, Lem. 10] and [BG14, Lem. 4.14].

The main technical ingredient in completing the proof of **Case 2** is in showing that

$$|F_\omega(p,\sigma) - F_{\omega_1}(p,\sigma)| = O\left(\frac{\delta_0^2\lambda^2}{\Gamma_0^m}\right)$$

One way to proving this is by proving $\left|R_\omega(\sigma_p)^2 - R_{\omega_1}(p,\sigma)^2\right|$ and $\left|d(p, N_\omega(\sigma_p))^2 - d(p, N_{\omega_1}(p,\sigma))^2\right|$ is $O\left(\frac{\delta_0^2\lambda^2}{\Gamma_0^m}\right)$, and this will be done in Lemma 38 using Lemma 36 and Corollary 37,

B.2 Details of the proof

For the rest of this section we will assume the following hypothesis

Hypothesis 32 *$L \subset \mathcal{M}$ is a $(\lambda, 2\lambda)$-net of $\mathcal{M}$ with $\lambda < \frac{1}{18}(1-\sin\theta_0)^2\, rch(\mathcal{M})$.*

For a simplex σ and a vertex $p \in \sigma$, *excentricity* $H_\omega(p,\sigma)$ of σ with respect to p is the signed distance of $C_\omega(\sigma)$ from aff σ_p, i.e., $H_\omega(p,\sigma)$ is positive if $C_\omega(\sigma)$ and p lie on the same side of aff σ_p and negative if they lie on different sides of aff σ_p.

The following lemma is a variant of the pumping equation from [CDE+00, BG14, CDR05].

Lemma 33 (Pumping equation) *We will assume that the weight of p is varying and the weight of the other vertices of σ are fixed. Then*

$$2D(p,\sigma)\, H_\omega(p,\sigma) = F_\omega(p,\sigma) - \omega(p)^2.$$

The above "pumping equation" will be used to bound the length of the forbidden intervals.

The following result is from [BDG13b].

Lemma 34 (Sliver altitude bound) *If a $(k+1)$-simplex τ is a Γ_0-sliver, then for any vertex p of σ we have*

$$D(p,\sigma) < \frac{2\Gamma_0\Delta(\sigma)^2}{L(\sigma)}.$$

A variant of the following result can be found in [CDR05, Lem. 10] and [BG14, Lem. 4.14]. We have included the proof for completeness.

Lemma 35 (Case 1) *Let $\omega : L \to [0, \infty]$ be a weight assignment with $\tilde{\omega} \le \tilde{\alpha}_0$, and $\sigma \subset L$ be a Γ_0-sliver incident to the point $p \in L$. Let ω_1 be a ewp of ω satisfying the following conditions*

$$\omega(q) = \omega_1(q),\ \forall\ q \in \sigma \setminus p,\ \text{and } \sigma \in \mathrm{K}_{\omega_1}(L).$$

If λ is sufficiently small then

$$\omega(p)^2 \in \left[F_\omega(p, \sigma) - \frac{\eta_1}{2} - \delta_0^2\lambda^2,\ F_\omega(p, \sigma) - \frac{\eta_1}{2}\right]$$

where $\eta_1 \stackrel{\text{def}}{=} 2^{14}\Gamma_0\lambda^2$.

Proof Since $|H_{\omega_1}(p, \sigma)| \le \|C_{\omega_1}(\sigma) - p\|$, we have from Lemma 22 (1)

$$|H_{\omega_1}(p, \sigma)| \le \|C_{\omega_1}(\sigma) - p\| < 8\lambda.$$

Since L is λ-sparse, we have from Lemma 22 (2)

$$\lambda \le L(\sigma) \le \Delta(\sigma) < 16\lambda$$

From Lemma 34, we have

$$D(p, \sigma) < \frac{2\Gamma_0\Delta(\sigma)^2}{L(\sigma)} < 2^9\Gamma_0\lambda.$$

Therefore, using Lemma 33, we have

$$\begin{aligned} F_{\omega_1}(p, \sigma) - 2D(p, \sigma)|H_{\omega_1}(p, \sigma)| \le \quad \omega_1(p)^2 \quad &\le F_{\omega_1}(p, \sigma) + 2D(p, \sigma)|H_{\omega_1}(p, \sigma)| \\ F_{\omega_1}(p, \sigma) - 2^{13}\Gamma_0\lambda^2 \le \quad \omega_1(p)^2 \quad &\le F_{\omega_1}(p, \sigma) + 2^{13}\Gamma_0\lambda^2 \end{aligned}$$

The result now follows from the facts that

- $F_{\omega_1}(p, \sigma) = F_\omega(p, \sigma)$ as, from the definition, $F_{\omega_1}(p, \sigma)$ (and $F_\omega(p, \sigma)$) depends only on the weights of the vertices in σ_p and for all $q \in \sigma \setminus p$, $\omega(q) = \omega_1(q)$.
- $\omega_1(p)^2 \in [\omega(p)^2, \omega(p)^2 + \delta_0^2\lambda^2]$.

□

The following lemmas show the stability of weighted centers of well shaped simplices under small perturbations of weight assignments. The proof is in the same vein as the proof of [BDG13c, Lem. 4.1], and will use singular values of matrices associated with the simplices. See, the section on notations at the end of paper.

Lemma 36 *Let σ be a simplex with $L(\sigma) \ge \lambda$ and $\Upsilon(\sigma) > 0$, and $\xi_i : \sigma \to [0, \infty)$, with $i \in \{1, 2\}$, be weights assignments, with $\tilde{\xi}_i \le \alpha_0$, satisfy the following properties: $\exists\ p \in \sigma$ such that*

1. $\forall\, q \in \sigma \setminus p$, $\xi_1(q) = \xi_2(q)$, *and*

2. $|\xi_1(p)^2 - \xi_2(p)^2| \leq \delta_0^2 \lambda^2$.

Then

$$\|C_{\xi_1}(\sigma) - C_{\xi_2}(\sigma)\| \leq \frac{\delta_0^2 \lambda}{2\Upsilon(\sigma)},$$

and for $r \notin \sigma$, *we have*

$$\left|d(r, N_{\xi_1}(\sigma)) - d(r, N_{\xi_2}(\sigma))\right| \leq \frac{\delta_0^2 \lambda}{2\Upsilon(\sigma)}$$

Proof Let $\sigma = [p_0 \ \ldots \ p_k]$, and wlog let $p \neq p_0$ and $\sigma \subset \mathbb{R}^k$. The ortho-radius of σ satisfy the following system of k-linear equations:

$$(p_j - p_0)^{\mathrm{T}} C_{\xi_i}(\sigma) = \frac{1}{2}(\|p_j\|^2 - \xi_i(p_j)^2 - \|p_0\|^2 + \xi_i(p_0)^2)$$

Rewriting the above system of equation we get

$$\begin{aligned}(p_j - p_0)^{\mathrm{T}}(C_{\xi_2}(\sigma) - C_{\xi_1}(\sigma)) &= \frac{1}{2}(\xi_1(p_j)^2 - \xi_2(p_j)^2 + \xi_2(p_0)^2 - \xi_1(p_0)^2) \\ &= \frac{1}{2}(\xi_1(p_j)^2 - \xi_2(p_j)^2) \quad \text{as } p \neq p_0\end{aligned}$$

Letting P be a $k \times k$ matrix whose j^{th} column is $(p_j - p_0)$, we have

$$P^T(C_{\xi_2}(\sigma) - C_{\xi_1}(\sigma)) = \frac{x_\xi}{2}$$

where $x_\xi = (\xi_1(p_1)^2 - \xi_2(p_1)^2, \ldots, \xi_1(p_k)^2 - \xi_2(p_k)^2)^{\mathrm{T}}$. Therefore

$$\begin{aligned}\|C_{\xi_2}(\sigma) - C_{\xi_1}(\sigma)\| &= \frac{1}{2}\|P^{-\mathrm{T}} x_\xi\| \\ &\leq \frac{1}{2}\|P^{-1}\|\,\|x_\xi\| \\ &\leq s_1(P^{-1}) \times \frac{\delta_0^2\lambda^2}{2} && \text{as } s_1(P^{-1}) = \|P^{-1}\| \text{ and } \|x_\xi\| \leq \delta_0^2\lambda^2 \\ &= s_k(P)^{-1} \times \frac{\delta_0^2\lambda^2}{2} && \text{as } s_1(P^{-1}) = s_k(P)^{-1}, \text{ Lem. 45 page 35} \\ &\leq \frac{\delta_0^2\lambda^2}{2\sqrt{k}\Upsilon(\sigma)\Delta(\sigma)} && \text{as } s_k(P) \geq \sqrt{k}\Upsilon(\sigma)\Delta(\sigma), \text{ Lem. 46 page 35} \\ &\leq \frac{\delta_0^2\lambda}{2\Upsilon(\sigma)} && \text{as } \tfrac{\Delta(\sigma)}{\lambda} \geq 1\end{aligned}$$

The bound on $\left|d(r, N_{\xi_1}(\sigma)) - d(r, N_{\xi_2}(\sigma))\right|$ follows directly from the part 1 of the lemma and the fact that

$$\left|d(r, N_{\xi_1}(\sigma)) - d(r, N_{\xi_2}(\sigma))\right| \leq \|C_{\xi_1}(\sigma) - C_{\xi_2}(\sigma)\|.$$

□

Corollary 37 *Let $\omega : L \to [0, \infty]$ be a weight assignment with $\tilde{\omega} \leq \tilde{\alpha}_0$, and $\sigma \subset L$ be a Γ_0-sliver with $p \in \sigma$ and $\#\sigma \leq m+2$. In addition, we assume*

$$\frac{\delta_0^2}{\Gamma_0^m} \leq 2.$$

If ω_1 be an ewp of ω satisfying the following: $\exists\, q \in \sigma_p$ such that $\forall\, x \in L \setminus \{q\}$, $\omega(x) = \omega_1(x)$ and $\sigma \in \mathrm{K}_{\omega_1}(L, \mathcal{M})$. Then

$$\left|d(p, N_\omega(\sigma_p))^2 - d(p, N_{\omega_1}(\sigma_p))^2\right|, \ \left|R_\omega(\sigma_p)^2 - R_{\omega_1}(\sigma_p)^2\right| \leq \frac{49\delta_0^2\lambda^2}{2\Gamma_0^m}.$$

Proof Using the fact that L is an $(\lambda, 2\lambda)$-net of $\mathcal{M}$, and from Lemmas 22 (1) and (2) we have

$$\begin{aligned} d(p, N_{\omega_1}(\sigma_p)) &\leq \|C_{\omega_1}(\sigma_p) - q\| + \|p - q\| \\ &\leq 24\lambda \end{aligned}$$

From Lemma 36 we have

$$\begin{aligned} d(p, N_\omega(\sigma_p)) + d(p, N_{\omega_1}(\sigma_p)) &\leq 2d(p, N_{\omega_1}(\sigma_p)) + \frac{\delta_0^2\lambda}{2\Upsilon(\sigma_p)} \\ &\leq 2d(p, N_{\omega_1}(\sigma_p)) + \frac{\delta_0^2\lambda}{2\Gamma_0^m} \qquad\qquad \text{as } \sigma \text{ is a } \Gamma_0\text{-sliver} \\ &\leq 49\lambda \end{aligned}$$

From Lemma 36 and the fact that $d(p, N_\omega(\sigma_p)) + d(p, N_{\omega_1}(\sigma_p)) \leq 49\lambda$, we have

$$\left|d(p, N_\omega(\sigma_p))^2 - d(p, N_{\omega_1}(\sigma_p))^2\right| \leq \frac{49\delta_0^2\lambda^2}{2\Gamma_0^m}$$

As $k \geq 1$, there exists $r \in \sigma_p \setminus q$. This implies $\omega(r) = \omega_1(r)$.
Using the facts that $\omega_1(r) = \omega(r)$, $\|C_{\omega_1}(\sigma_p) - r\| \leq 8\lambda$ (from Lemma 22 (1)) and

$$\|C_\omega(\sigma_p) - C_{\omega_1}(\sigma_p)\| \leq \frac{\delta_0^2\lambda}{2\Gamma_0^m}$$

(from Lemma 36 and the fact that $\Upsilon(\sigma_p) \geq \Gamma_0^k \geq \Gamma_0^m$), we get

$$\begin{aligned} R_\omega(\sigma_p)^2 &= \|C_\omega(\sigma_p) - r\|^2 - \omega(r)^2 \\ &\leq (\|C_{\omega_1}(\sigma_p) - r\| + \|C_\omega(\sigma_p) - C_{\omega_1}(\sigma_p)\|)^2 - \omega_1(r)^2 \\ &\leq R_{\omega_1}(\sigma_p)^2 + \Big(2\|C_{\omega_1}(\sigma_p) - r\| \\ &\qquad\qquad + \|C_\omega(\sigma_p) - C_{\omega_1}(\sigma_p)\|\Big) \|C_\omega(\sigma_p) - C_{\omega_1}(\sigma_p)\| \\ &\leq R_{\omega_1}(\sigma_p)^2 + \left(16\lambda + \frac{\delta_0^2\lambda}{2\Gamma_0^m}\right) \frac{\delta_0^2\lambda}{2\Gamma_0^m} \\ &\leq R_{\omega_1}(\sigma_p)^2 + \frac{17\delta_0^2\lambda^2}{2\Gamma_0^m} \end{aligned}$$

and

$$
\begin{aligned}
R_\omega(\sigma_p)^2 &= \|C_\omega(\sigma_p) - r\|^2 - \omega(r)^2 \\
&\geq (\|C_{\omega_1}(\sigma_p) - r\| - \|C_\omega(\sigma_p) - C_{\omega_1}(\sigma_p)\|)^2 - \omega_1(r)^2 \\
&\geq R_{\omega_1}(\sigma_p)^2 - \Big(2\|C_{\omega_1}(\sigma_p) - r\| \\
&\qquad\qquad - \|C_\omega(\sigma_p) - C_{\omega_1}(\sigma_p)\|\Big) \|C_\omega(\sigma_p) - C_{\omega_1}(\sigma_p)\| \\
&\geq R_{\omega_1}(\sigma_p)^2 - 2\|C_{\omega_1}(\sigma_p) - r\| \, \|C_\omega(\sigma_p) - C_{\omega_1}(\sigma_p)\| \\
&\geq R_{\omega_1}(\sigma_p)^2 - \frac{8\delta_0^2}{\Gamma_0^m}\lambda^2
\end{aligned}
$$

□

Lemma 38 (Case 2) *Assuming the same conditions on ω, ω_1, p, q and σ as in Corollary 37, we get*

$$\omega(p)^2 \in \left[F_\omega(p,\sigma) - \frac{\eta}{2},\, F_\omega(p,\sigma) + \frac{\eta}{2}\right].$$

Proof As in the proof of Lemma 35, we can show that $|2D(p,\sigma)H_{\omega_1}(p,\sigma)| \leq 2^{13}\Gamma_0\lambda$.
From Corollary 37, we have

$$\left|d(p, N_\omega(\sigma_p))^2 - d(p, N_{\omega_1}(\sigma_p))^2\right|, \; \left|R_\omega(\sigma_p)^2 - R_{\omega_1}(\sigma_p)^2\right| \leq \frac{49\delta_0^2}{2\Gamma_0^m}\lambda^2.$$

This implies, from the definition of $F_{\omega_1}(p,\sigma)$,

$$\left|F_{\omega_1}(p,\sigma) - F_\omega(p,\sigma)\right| \leq \frac{49\delta_0^2}{\Gamma_0^m}\lambda^2.$$

From Lemma 33, and the above bounds we get

$$\omega_1(p)^2 \in \left[F_\omega(p,\sigma) - \frac{\eta}{2},\, F_\omega(p,\sigma) + \frac{\eta}{2}\right].$$

The result now follows from the fact that $\omega(p) = \omega_1(p)$. □

Remark 39 *Note that $\eta \geq \eta_1 + 2\delta_0^2\lambda^2$.*

Combining Lemmas 35 and 38, completes the proof of Lemma 16.

C Almost normal flats intersecting submanifolds

The following technical lemma, which asserts that, for $j \leq m = \dim \mathcal{M}$, if a $(d-j)$-flat, $\mathcal{N}$, passes through a point $\tilde{c}$ that is close to $\mathcal{M}$, and the normal space at the point on $\mathcal{M}$ closest to $\tilde{c}$ makes a small angle with $\mathcal{N}$, then $\mathcal{N}$ must intersect $\mathcal{M}$ in that vicinity. The technical difficulty stems from the fact that the codimension may be greater than one.

Lemma 40 *Let $\tilde{c} \in \mathbb{R}^d$ be such that it has a unique closest point $\hat{c}$ on $\mathcal{M}$ and $\|\tilde{c} - \hat{c}\| \leq \mu \leq \frac{rch(\mathcal{M})}{25}$. Let $j \leq m = \dim \mathcal{M}$, and let $\mathcal{N}$ be a $(d-j)$-dimensional affine flat passing through $\tilde{c}$ such that $\angle(N_{\hat{c}}\mathcal{M}, \mathcal{N}) \leq \alpha$ with $\sin \alpha \leq \frac{1}{4}$. Then there exists an $x \in \mathcal{N} \cap \mathcal{M}$ such that $\|\tilde{c} - x\| \leq 4\mu$.*

The idea of the proof is to consider the m-dimensional affine space $\widetilde{T_{\hat{c}}}\mathcal{M}$ that passes through $\hat{c}$ and is orthogonal to $\mathcal{N}$. We show that the orthogonal projection onto $\widetilde{T_{\hat{c}}}\mathcal{M}$ induces, in some neighbourhood V of $\hat{c}$, a diffeomorphism between $\mathcal{M} \cap V$, and $\widetilde{T_{\hat{c}}}\mathcal{M} \cap V$ (Lemma 43). We use $T_{\hat{c}}\mathcal{M}$ as an intermediary in this calculation (Lemma 42). Then, since $\mathcal{N}$ intersects $T_{\hat{c}}\mathcal{M}$ near $\hat{c}$ (Lemma 41), we can argue that it must also intersect $\mathcal{M}$ because the established diffeomorphisms make a correspondence between points along segments parallel to $\mathcal{N}$.

The final bounds are established in Lemma 44, from which Lemma 40 follows by a direct calculation, together with the following observations: If $\dim \mathcal{N} = \dim N_{\hat{c}}\mathcal{M}$, then $\angle(N_{\hat{c}}\mathcal{M}, \mathcal{N}) = \angle(\mathcal{N}, N_{\hat{c}}\mathcal{M})$, and if $\dim \mathcal{N} \geq \dim N_{\hat{c}}\mathcal{M}$, then there is an affine subspace $\widetilde{\mathcal{N}} \subset \mathcal{N}$, such that $\dim \widetilde{\mathcal{N}} = \dim N_{\hat{c}}\mathcal{M}$, and $\angle(N_{\hat{c}}\mathcal{M}, \widetilde{\mathcal{N}}) = \angle(N_{\hat{c}}\mathcal{M}, \mathcal{N})$. Indeed, we may take $\widetilde{\mathcal{N}}$ to be the orthogonal projection of $N_{\hat{c}}\mathcal{M}$ into $\mathcal{N}$.

We now bound distances to the intersection of $\mathcal{N}$ and $T_{\hat{c}}\mathcal{M}$.

Lemma 41 *Let $\tilde{c}$, $\hat{c}$ be points in $\mathbb{R}^d$ such that the projection of $\tilde{c}$ onto $\mathcal{M}$ is $\hat{c}$ and $\|\tilde{c} - \hat{c}\| \leq \mu$. Let $\mathcal{N}$ be a $d-m$ dimensional affine flat passing through $\tilde{c}$ such that $\angle(\mathcal{N}, N_{\hat{c}}\mathcal{M}) \leq \alpha$. For all $x \in \mathcal{N} \cap T_{\hat{c}}\mathcal{M}$, we have*

1. $\|\tilde{c} - x\| \leq \frac{\mu}{\cos \alpha}$
2. $\|\hat{c} - x\| \leq \left(1 + \frac{1}{\cos \alpha}\right) \mu$

Proof For a point $x \in \mathcal{N} \cap T_{\hat{c}}\mathcal{M}$, let u_x denote the unit vector from $\tilde{c}$ to x, and let $v_x \in N_{\hat{c}}\mathcal{M}$ be the unit vector that makes the smallest angle with u_x. Let H denote the hyperplane passing through $\hat{c}$ and orthogonal to v_x. Since $\|\tilde{c} - \hat{c}\| \leq \mu$, $\mathrm{dist}(\tilde{c}, H) \leq \mu$. Therefore,

$$\|\tilde{c} - x\| \leq \frac{\mathrm{dist}(\tilde{c}, H)}{\cos \alpha}$$

and

$$\|\hat{c} - x\| \leq \|\hat{c} - \tilde{c}\| + \|\tilde{c} - x\| \leq \left(1 + \frac{1}{\cos \alpha}\right) \mu$$

□

The following lemma is a direct consequence of the definition of the angle between two affine spaces.

Lemma 42 *Let p be a point in $\mathcal{M}$ and let $\widetilde{T_p}\mathcal{M}$ denote a k-dimensional flat passing through p with $\angle(T_p\mathcal{M}, \widetilde{T_p}\mathcal{M}) \leq \alpha < \frac{\pi}{2}$. If f_p^α denote the orthogonal projection of $T_p\mathcal{M}$ onto $\widetilde{T_p}\mathcal{M}$, then*

1. *The map f_p^α is bijective.*

2. *For $r > 0$, $f_p^\alpha(B_p(r)) \supseteq \widetilde{B}_p(r\cos\alpha)$ where $B_p(r) = B(p,r) \cap T_p\mathcal{M}$ and $\widetilde{B}_p(r) = B(p,r) \cap \widetilde{T}_p\mathcal{M}$.*

Lemma 43 *Let p be a point in $\mathcal{M}$, and let $\widetilde{T}_p\mathcal{M}$ be a k-dimensional affine flat passing through p with $\angle(T_p\mathcal{M}, \widetilde{T}_p\mathcal{M}) \leq \alpha$. There exists an $r(\alpha)$ satisfying :*

$$\frac{7\,r(\alpha)}{rch(\mathcal{M})} + \sin\alpha < 1 \quad \textit{and} \quad r(\alpha) \leq \frac{rch(\mathcal{M})}{10}$$

such that the orthogonal projection map, g_p^α, of $B_\mathcal{M}(p, r(\alpha)) = B(p, r(\alpha)) \cap \mathcal{M}$ into $\widetilde{T}_p\mathcal{M}$ satisfy the following conditions:

1. *g_p^α is a diffeomorphism.*
2. *$g_p^\alpha(B_\mathcal{M}(p, r(\alpha))) \supseteq \widetilde{B}_p(r(\alpha)\cos\alpha_1)$ where $\sin\alpha_1 = \frac{r(\alpha)}{2rch(\mathcal{M})} + \sin\alpha$.*
3. *Let $x \in g_p^\alpha(B_\mathcal{M}(p, r(\alpha)))$, then $\|x - (g_p^\alpha)^{-1}(x)\| \leq \|p - x\| \tan\alpha_1$.*

Proof 1. Let $\pi_{\widetilde{T}_p\mathcal{M}}$ denote the orthogonal projection of $\mathbb{R}^d$ onto $\widetilde{T}_p\mathcal{M}$. The derivative of this map, $D\pi_{\widetilde{T}_p\mathcal{M}}$, has a kernel of dimension $(d-m)$ that is parallel to the orthogonal complement of $\widetilde{T}_p\mathcal{M}$ in $\mathbb{R}^d$.

We will first show that Dg_p^α is nonsingular for all $x \in B_\mathcal{M}(p, r(\alpha))$. From Lemma 2 (3) and the fact that $\angle(T_p\mathcal{M}, \widetilde{T}_p\mathcal{M}) \leq \alpha$, we have

$$\begin{aligned} \sin\angle(\widetilde{T}_p\mathcal{M}, T_x\mathcal{M}) &\leq \sin\angle(T_x\mathcal{M}, T_p\mathcal{M}) + \sin\angle(T_p\mathcal{M}, \widetilde{T}_p\mathcal{M}) \\ &\leq \frac{6r(\alpha)}{rch(\mathcal{M})} + \sin\alpha < 1 \end{aligned}$$

Since g_p^α is the restriction of $\pi_{\widetilde{T}_p\mathcal{M}}$ to $B_\mathcal{M}(p, r(\alpha))$, the above inequality implies that Dg_p^α is non-singular. Therefore, g_p^α is a local diffeomorphism.

Let x, $y \in B_\mathcal{M}(p, r(\alpha))$. From Lemma 2 part (1) and (3), we have

$$\begin{aligned} \sin\angle([x,y], \widetilde{T}_p\mathcal{M}) &\leq \sin\angle([x,y], T_x\mathcal{M}) + \sin\angle(T_x\mathcal{M}, T_p\mathcal{M}) + \sin\angle(\widetilde{T}_p, T_p\mathcal{M}) \\ &\leq \frac{\|x-y\|}{2rch(\mathcal{M})} + \frac{6\|p-x\|}{rch(\mathcal{M})} + \sin\alpha \\ &\leq \frac{7r(\alpha)}{rch(\mathcal{M})} + \sin\alpha < 1 \end{aligned}$$

This implies $g_p^\alpha(x) \neq g_p^\alpha(y)$.

Since g_p^α is nonsingular and injective on $B_\mathcal{M}(p, r(\alpha))$, it is a diffeomorphism onto its image.

2. Notice that, for $x \in B_\mathcal{M}(p, r(\alpha))$, the angle α_1 is a bound on the angle between $[p,x]$ and $\tilde{T}_p\mathcal{M}$. The inclusion $g_p^\alpha(B_\mathcal{M}(p, r(\alpha))) \supseteq \widetilde{B}_p(r(\alpha)\cos\alpha_1)$ follows since $[x, g_P^\alpha(x)]$ is orthogonal to $\tilde{T}_p\mathcal{M}$.

3. Follows similarly. $\square$

Lemma 44 *Let $\tilde{c}$, $\hat{c}$ be points in $\mathbb{R}^d$ such that the projection of $\tilde{c}$ onto $\mathcal{M}$ is $\hat{c}$ and $\|\tilde{c}-\hat{c}\| \leq \mu$. Let $\mathcal{N}$ be a $d-k$ dimensional affine flat passing through $\tilde{c}$ such that $\angle(\mathcal{N}, N_{\hat{c}}\mathcal{M}) \leq \alpha$. If*

$$\mu \leq \frac{r(\alpha)\cos\alpha\cos\alpha_1}{1+\cos\alpha}$$

Then there exists an $x \in \mathcal{N} \cap \mathcal{M}$ such that

$$\|\tilde{c} - x\| \leq \left(\frac{1}{\cos\alpha} + \left(1 + \frac{1}{\cos\alpha}\right)(\sin\alpha + \sin\alpha_1)\right)\mu\,.$$

Proof Let $\tilde{T}_{\hat{c}}\mathcal{M}$ denote the orthogonal complement of N in $\mathbb{R}^d$ passing through $\hat{c}$. Note that $\angle(T_{\hat{c}}\mathcal{M}, \tilde{T}_{\hat{c}}\mathcal{M}) = \angle(\mathcal{N}, N_{\hat{c}}\mathcal{M})$.

Let $\hat{x} \in \mathcal{N} \cap T_{\hat{c}}\mathcal{M}$ and $\tilde{x} = f^{\alpha}_{\hat{c}}(\hat{x})$. Then from Lemma 41, we have

$$\|\tilde{x} - \hat{c}\| \leq \|\hat{x} - \hat{c}\| \leq \left(1 + \frac{1}{\cos\alpha}\right)\mu$$

and

$$\|\hat{x} - \tilde{x}\| \leq \|\hat{x} - \hat{c}\|\sin\alpha \leq \left(1 + \frac{1}{\cos\alpha}\right)\sin\alpha\,\mu\,.$$

Using the fact that $\mu \leq \frac{r(\alpha)\cos\alpha\cos\alpha_1}{1+\cos\alpha}$, we have

$$\|\tilde{x} - \hat{c}\| \leq \left(1 + \frac{1}{\cos\alpha}\right)\mu \leq r(\alpha)\cos\alpha_1.$$

Therefore, from Lemma 41, there exists an $x \in B_{\mathcal{M}}(p, r(\alpha))$ such that $g^{\alpha}_{p}(x) = \tilde{x}$ and

$$\|\tilde{x} - x\| \leq \|\tilde{x} - \hat{c}\|\tan\alpha_1 \leq \left(1 + \frac{1}{\cos\alpha}\right)\tan\alpha_1\,\mu.$$

Therefore

$$\begin{aligned}
\|\tilde{c} - x\| &\leq \|\tilde{c} - \hat{x}\| + \|\hat{x} - \tilde{x}\| + \|\tilde{x} - x\| \\
&\leq \frac{\mu}{\cos\alpha} + \left(1 + \frac{1}{\cos\alpha}\right)(\sin\alpha + \tan\alpha_1)\,\mu \\
&= \left(\frac{1}{\cos\alpha} + \left(1 + \frac{1}{\cos\alpha}\right)(\sin\alpha + \tan\alpha_1)\right)\mu.
\end{aligned}$$

Note that the line segment $[\tilde{c},\, x] \in \mathcal{N}$. □

This completes the proof of Lemma 40.

D Missing notations

D.1 Weighted Voronoi diagrams

For a simplex $\sigma = [p_0, \ldots, p_k]$ with vertices in L and $\omega : L \to [0, \infty)$ be a weight assignment, we define *ω-weighted normal space*, or just weighted normal space, $N_\omega(\sigma)$ of σ as

$$N_\omega(\sigma) = \left\{ x \in \mathbb{R}^d : \; d(x, p_i^\omega) = d(x, p_j^\omega), \, \forall \, p_i, \, p_j \in \sigma \right\}.$$

We define *ω-weighted* (or just weighted) center of σ as

$$C_\omega(\sigma) = \operatorname{argmin}_{x \in N_\omega(\sigma)} d(x, p_0^\omega)$$

and *ω-weighted* (or just weighted) ortho-radius of σ as

$$R_\omega(\sigma)^2 = d(C_\omega(\sigma), p_0^\omega).$$

Note that weighted radius can be imaginary, and $N_\omega(\sigma)$ is an orthogonal compliment of $\mathrm{aff}(\sigma)$ and intersecting $\mathrm{aff}(\sigma)$ at $C_\omega(\sigma)$. We call $S(c, r)$ an *ω-ortho sphere*, or just *ortho sphere*, of σ if, for all $p_i \in \mathring{\sigma}$, we have $r^2 = d(c, p_i^\omega)$.

For a point $p \in L$ we define the *weighted Voronoi cell* $\mathrm{Vor}_\omega(p)$ of p as

$$\mathrm{Vor}_\omega(p) = \{x \in \mathbb{R}^d : \forall \, q \in L \setminus p, \; d(x, p^\omega) \leq d(x, q^\omega)\}.$$

For a subset $\sigma = \{p_0, \ldots, p_k\}$ of L or a simplex $\sigma = [p_0, \ldots, p_k]$ with vertices in L, the *weighted Voronoi face* $\mathrm{Vor}_\omega(\sigma)$ of σ is defines as

$$\mathrm{Vor}_\omega(\sigma) = \bigcap_{i=0}^{k} \mathrm{Vor}_\omega(p_i).$$

The weighted Voronoi cells give a decomposition of $\mathbb{R}^d$, denoted $\mathrm{Vor}_\omega(L)$, called the *weighted Voronoi diagram* of L corresponding to the weight assignment ω. Let $c \in \mathrm{Vor}_\omega(\sigma)$ and $r^2 = d(c, p_i^\omega)$ where $p_i \in \mathring{\sigma}$. We will call $S(c, r)$ *ω-ortho Delaunay sphere*, or just *ortho Delaunay sphere*, of σ.

D.2 Singular values of matrices

Let $s_i(A)$ denotes the i^{th} singular value of the matrix A. The singular values are non-negative and ordered by decreasing order of magnitude. The largest singular value $s_i(A)$ is equal to the norm $\|A\|$ of the matrix, i.e.,

$$s_1(A) = \|A\| = \sup_{\|x\|=1} \|Ax\|.$$

It is easy to see that

Lemma 45 *If A is an invertible $j \times j$ matrix, then $s_1(A^{-1}) = s_j(A)^{-1}$.*

Boissonnat et al. [BDG13c] connected the geometric properties of a simplex to the largest and smallest singular values of the associated matrix:

Lemma 46 (Thickness and singular value [BDG13c]) *Let $\sigma = [p_0, \ldots, p_j]$ be a non-degenerate j-simplex in $\mathbb{R}^m$, with $j > 0$, and let P be the $m \times j$ matrix whose i^{th} column is $p_i - p_0$. Then*

1. $s_1(P) \leq \sqrt{j}\Delta(\sigma)$, *and*
2. $s_j(P) \geq \sqrt{j}\Upsilon(\sigma)\Delta(\sigma)$.

References

[ACDL02] N. Amenta, S. Choi, T. K. Dey, and N. Leekha. A simple algorithm for homeomorphic surface reconstruction. *Int. J. Comp. Geom. & App.*, 12(2):125–141, 2002.

[AEM07] D. Attali, H. Edelsbrunner, and Y. Mileyko. Weak witnesses for Delaunay triangulations of submanifolds. In *Symp. Solid & Phy. Modeling*, pages 143–150, 2007.

[BDG13a] J.-D. Boissonnat, R. Dyer, and A. Ghosh. Constructing intrinsic Delaunay triangulations of submanifolds. Technical Report RR-8273, INRIA, 2013.

[BDG13b] J.-D. Boissonnat, R. Dyer, and A. Ghosh. Constructing stable Delaunay triangulations. Technical Report RR-8275, INRIA, 2013.

[BDG13c] J.-D. Boissonnat, R. Dyer, and A. Ghosh. The stability of Delaunay triangulations. *Int. J. Comput. Geometry Appl.*, 23(4-5):303–334, 2013.

[BG14] J.-D. Boissonnat and A. Ghosh. Manifold reconstruction using tangential Delaunay complexes. *Discrete & Computational Geometry*, 51(1):221–267, 2014.

[BGO09] J.-D. Boissonnat, L. J. Guibas, and S. Y. Oudot. Manifold reconstruction in arbitrary dimensions using witness complexes. *Discrete & Comp. Geom.*, 42:37–70, 2009.

[BY98] J.-D. Boissonnat and M. Yvinec. *Algorithmic geometry*. Cambridge University Press, 1998.

[CC09] S.-W. Cheng and M.-K. Chiu. Dimension detection via slivers. In *SODA*, pages 1001–1010, 2009.

[CDE+00] S.-W. Cheng, T. K. Dey, H. Edelsbrunner, M. A. Facello, and S. H Teng. Sliver exudation. *Journal of the ACM*, 47(5):883–904, 2000.

[CDR05] S.-W. Cheng, T. K. Dey, and E. A. Ramos. Manifold reconstruction from point samples. In *SODA*, pages 1018–1027, 2005.

[CdS04] G. Carlsson and V. de Silva. Topological estimation using witness complexes. In *Proc. Symps. Point-Based Graphics*, pages 157–166, 2004.

[CIDSZ06] G. Carlsson, T. Ishkhanov, V. De Silva, and A Zomorodian. On the local behavior of spaces of natural images. *Internat. J. Comput. Vision*, 2007, 2006.

[CO08] F. Chazal and S. Y. Oudot. Towards persistence-based reconstruction in euclidean spaces. In *SoCG*, pages 232–241, 2008.

[CWW08] S.-W. Cheng, Y. Wang, and Z. Wu. Provable Dimension Detection using Principal Component Analysis. *Int. J. Comp. Geom. & App.*, 18(05):415–440, 2008.

[dS08] V. de Silva. A weak characterisation of the Delaunay triangulation. *Geometriae Dedicata*, 135:39–64, 2008.

[Fed59] H. Federer. Curvature Measures. *Trans. Amer. Math. Soc.*, 93(3):418–491, 1959.

[GO08] L. J. Guibas and S. Y. Oudot. Reconstruction using witness complexes. *Discrete & Comp. Geom.*, 40(3):325–356, 2008.

[GW04] J. Giesen and U. Wagner. Shape dimension and intrinsic metric from samples of manifolds. *Discrete & Comp. Geom.*, 32:245–267, 2004.

[Mat02] J. Matousek. *Lectures on Discrete Geometry*. Graduate Texts in Mathematics. Springer, 1 edition, 2002.

[Mat13] J. Matousek. *Lecture Notes on Metric Embeddings*. 2013.