HAL
archives-ouvertes.fr

Overdamped 2D Brownian motion for self-propelled and nonholonomic particles

Agostino Martinelli

To cite this version:

Agostino Martinelli. Overdamped 2D Brownian motion for self-propelled and nonholonomic particles. Journal of Statistical Mechanics : Theory and Experiment, 2014, 2014, pp.P03003. <10.1088/1742-5468/2014/03/P03003>. <hal-01096925>

HAL Id: hal-01096925
https://hal.inria.fr/hal-01096925

Submitted on 18 Dec 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Overdamped 2D-Brownian motion for self-propelled and nonholonomic particles

Agostino Martinelli

INRIA Rhone Alpes, 655 Avenue de l'Europe, Montbonnot, Grenoble

E-mail: `agostino.martinelli@inria.fr`

Abstract. This paper provides a complete analytical solution to the Brownian motion of a self-propelled particle that moves in two spatial dimensions in the overdamped limit by satisfying kinematic constraints. From a mathematical point of view, it provides the analytical solution of a Fokker Plank equation for which the detailed balance is not verified. The analytical expression of any-order moment of the probability distribution is obtained by a direct integration of the Langevin equation. Each moment is expressed as a multiple integral of the active motion performed by the particle. For special active motions, these integrals can be easily solved. In particular, they can be expressed as the real or the imaginary part of suitable analytic functions. This is explicitly shown for the first two moments.

1. Introduction

In recent years a great interest has been devoted to investigate the statistical properties of the motion of active (or self-propelled) particles. These particles differ from passive particles since they move under the action of an internal force. Typical examples of these particles are Brownian motors [5, 8, 17], biological and artificial microswimmers [4, 10, 11, 19, 21, 22, 23], macroscopic animals [1, 3, 14] and even pedestrians [24]. As in the case of passive particles, the motion can often be characterized by the Langevin equation in the overdamped limit [20]. In this case, the probability distribution satisfies the Smoluchowski equation, which is a special case of the Fokker-Plank equation [18]. On the other hand, since in most of cases of active particles the so-called detailed balance condition with respect to a given time reversal transformation is not satisfied, it is more convenient to derive the properties of the probability distribution by using the stochastic differential equation (i.e., the Langevin equation) instead of directly trying to solve the Fokker-Plank equation (we remind the reader that analytic solutions of the Fokker-Plank equation can be found when the detailed balance condition is satisfied: when this condition is not satisfied, it is in general very difficult to find analytic solutions in the non linear case and for a large number of variables [18]). Very recently, the Langevin equation has successfully been used to analytically compute the moments of the probability distribution for active particles [6, 7, 20]. Specifically, the first two moments have been derived for rodlike particles [20] and spherical particles [6]. In both these cases the orientation of the particle was confined to two dimensions. The work presented in [7] extends these works by including the three-dimensional case for both isotropic and anisotropic particles. Additionally, the first four moments were analytically derived (instead of the first two).

In this paper we consider the case of active particles that satisfy nonholonomic constraints. Specifically, our stochastic motion model (presented in section 2) differs from the one in [7] since both the active motion and the noise must satisfy the kinematic constraint. Additionally, the active motion is time-dependent and it is characterized by two independent functions of time, the former characterizes the linear speed, the latter the torque. On the other hand, we restrict the computation to the two-dimensional case even if we believe that the extension to the three-dimensional case is possible. Starting from this motion model, we compute the analytical expression of any-order moment of the probability distribution (section 3). Specifically, each moment is expressed as a multiple integral of the active motion performed by the particle. In other words, the expression is a multiple integral on the two time-dependent functions previously mentioned. Finally, when the ratio of these two functions is independent of time, these integrals can be easily solved. In particular, the active trajectory and the noise can be characterized by a suitable complex quantity and the integrals can be expressed as the real or the imaginary part of suitable analytic functions of this complex quantity. This is explicitly shown for the first two moments in section 4. Conclusions are provided in section 5.

Note that we already computed the statistics up to the second order for a similar motion model in the framework of mobile robotics [12, 13]. (in the expressions in [12] there are some typos, corrected in [13]). Here, we extend the computation in [12] by including any-order statistics. As it will be seen, this extension requires considerable analytical effort. We also show that the expressions hold for a much more general model (the one described in section 2) and not only for a simple specific odometry error model (as shown in [12]). As a result, this paper provides a complete analytical solution to a Fokker-Plank equation for which the detailed balance condition is not verified‡.

2. Stochastic motion model

We consider a particle that moves in a $2D-$environment. The configuration of the particle is characterized by its position and orientation. In Cartesian coordinates, we denote them by $\boldsymbol{r} \equiv [x,\ y]^T$ and θ, respectively. We assume that the motion of the particle satisfies the *unicycle* model [9], which is the most general nonholonomic model for $2D-$motions, where the shift of the particle can only occur along one direction ($[\cos\theta,\ \sin\theta]^T$):

$$\left[\begin{array}{ll} \dot{x} = & v\cos\theta \\ \dot{y} = & v\sin\theta \\ \dot{\theta} = & \omega \end{array}\right. \tag{1}$$

The quantities $v = v(t)$ and $\omega = \omega(t)$ are two functions of time and are the linear and angular speed, respectively. We assume that these functions are known and we call the motion that results from them the *active* motion. Now we want to introduce a stochastic model that generalizes equation (1) by accounting a Gaussian white noise. We assume that also the noise satisfies the same nonholonomic constraint. The stochastic differential equations are:

$$\left[\begin{array}{ll} dx(t) = & \cos\theta(t)\ [v(t)dt + \alpha(t)dw^r(t)] \\ dy(t) = & \sin\theta(t)\ [v(t)dt + \alpha(t)dw^r(t)] \\ d\theta(t) = & \omega(t)dt + \beta(t)dw^\theta(t) \end{array}\right. \tag{2}$$

where $[dw^r(t),\ dw^\theta(t)]^T$ is a standard Wiener process of dimension two [15]. The functions $\alpha(t)$ and $\beta(t)$ are modelled according to the following physical requirement (diffusion in the overdamped regime). We require that, when the particle moves during the infinitesimal time interval dt, the shift is a random Gaussian variable, whose variance increases linearly with the travelled distance. This is obtained by setting $\alpha(t) = \sqrt{K_r|v(t)|}$, with K_r a positive parameter that characterizes our system (interaction particle-environment). Similarly, we require that the rotation accomplished during the same time interval dt is a random Gaussian variable, whose variance increases linearly with the travelled distance. Hence, we set $\beta(t) = \sqrt{K_\theta|v(t)|}$, with K_θ another

‡ Note that, deriving any-order moment, corresponds to deriving the probability density distribution [16].

positive parameter that characterizes our system (interaction particle-environment). The equations in (2) are the Langevin equations in the overdamped limit. From now on, we assume that the particle can only move ahead, i.e., $v(t) \geq 0$. Instead of the time t, we use the curve length $ds = v(t)dt$. Note that $v(t)$ is a deterministic and known function of time. Additionally, we will denote by μ the ratio between the angular and the linear speed. Equations in (2) read:

$$\left[\begin{array}{rl} dx(s) = & \cos\theta(s)\ [ds + K_r^{\frac{1}{2}} dw^r(s)] \\ dy(s) = & \sin\theta(s)\ [ds + K_r^{\frac{1}{2}} dw^r(s)] \\ d\theta(s) = & \mu(s)ds + K_\theta^{\frac{1}{2}} dw^\theta(s) \end{array} \right. \tag{3}$$

The associated Smoluchowski equation is:

$$\frac{\partial p}{\partial s} = -\boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{D}_1} p) + \boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{D}_2} \boldsymbol{\nabla} p) = 0 \tag{4}$$

where:

- $p = p(x,\ y,\ \theta;\ s(t))$ is the probability density for the particle at the configuration $(x,\ y,\ \theta)$ and at time t;
- $\boldsymbol{\mathcal{D}_1} = [\cos\theta,\ \sin\theta,\ \mu]^T$ is the drift vector;
- $\boldsymbol{\mathcal{D}_2} = \frac{1}{2} \begin{bmatrix} K_r \cos^2\theta & K_r \cos\theta \sin\theta & 0 \\ K_r \cos\theta \sin\theta & K_r \sin^2\theta & 0 \\ 0 & 0 & K_\theta \end{bmatrix}$ is the diffusion tensor.

Our goal is to obtain the probability density $p(x,\ y,\ \theta;\ s)$. This will be done in the next two sections.

3. Computation of the probability distribution

The probability density $p(x,\ y,\ \theta;\ s)$ satisfies the Smoluchowski partial differential equation in (4), which is a special case of the Fokker-Plank equation. Since the detailed balance condition is not satisfied [18], we follow a different procedure to compute $p(x,\ y,\ \theta;\ s)$. Specifically, we use the Langevin equation in (3) to compute the moments, up to any order, of the probability distribution. First of all, we remark that the third equation in (3) is independent of the first two, is independent of θ and is linear in dw^θ. As a result, the probability density only in terms of θ, i.e., the probability $P_\theta(\theta;\ s) \equiv \int dx \int dy\ p(x,\ y,\ \theta;\ s)$, is a Gaussian distribution with mean value $\theta_0 + \int_0^s ds' \mu(s')$ and variance $K_\theta s$, where θ_0 is the initial orientation. This same result could also be obtained by integrating (4) in x and y and by using the divergence theorem. In other words, we have the following distribution for the orientation at a given value of s:

$$\theta(s) = \mathcal{N}(\overline{\theta}(s),\ K_\theta s) \tag{5}$$

where $\mathcal{N}(\cdot,\ \cdot)$ denotes the normal distribution with mean value and variance the first and the second argument; $\overline{\theta}(s) \equiv \theta_0 + \int_0^s ds'\ \mu(s')$. Note that, according to the unicycle model, a trajectory is completely characterized by its starting point and by the orientation vs the curve length, i.e., by the function $\theta(s)$. In the following, we will call the deterministic function $\overline{\theta}(s)$, the *active* trajectory.

3.1. First and second-order statistics

Let us consider the first equation in (3). We compute the expression of $x(s)$ by a direct integration. We divide the interval $(0,\ s)$ in N equal segments, $\delta s \equiv \frac{s}{N}$. We have:

$$x(s) = \lim_{N\to\infty} \sum_{j=1}^{N} (\delta s + \epsilon_j) \cos\theta_j \tag{6}$$

where ϵ_j is a random Gaussian variable satisfying $\langle \epsilon_j \rangle = 0$, $\langle \epsilon_j\ \epsilon_k \rangle = \delta_{jk} K_r \delta s$ for $j, k = 1, \cdots, N$ (δ_{jk} is the Kronecker delta) and $\theta_j \equiv \theta(j\delta s)$. On the other hand, we have:

$$\theta_j = \overline{\theta}(j\delta s) + \sum_{m=1}^{j} \delta\theta_m \equiv \overline{\theta}_j + \Delta\theta_j \tag{7}$$

where, according to our stochastic model in (3), $\delta\theta_j$ is a random Gaussian variable satisfying $\langle \delta\theta_j \rangle = 0$, $\langle \delta\theta_j\ \epsilon_k \rangle = 0$ and $\langle \delta\theta_j\ \delta\theta_k \rangle = \delta_{jk} K_\theta \delta s$ for $j, k = 1, \cdots, N$. As a result, $\Delta\theta_j \equiv \sum_{m=1}^{j} \delta\theta_m$ is also a random Gaussian variable satisfying $\langle \Delta\theta_j \rangle = 0$, $\langle \Delta\theta_j\ \Delta\theta_k \rangle = \left[\begin{array}{ll} K_\theta j \delta s & if\ j \le k \\ K_\theta k \delta s & if\ j > k \end{array} \right.$. Starting from (6) and (7) and by remarking that $\langle \cos\Delta\theta_j \rangle = e^{-\frac{k_\theta j \delta s}{2}}$ and $\langle \sin\Delta\theta_j \rangle = 0$, it is easy to obtain the mean value of $x(s)$. We have:

$$\langle x(s) \rangle = \lim_{N\to\infty} \sum_{j=1}^{N} (\delta s + \langle \epsilon_j \rangle) \langle \cos\theta_j \rangle = \lim_{N\to\infty} \sum_{j=1}^{N} \delta s\ \ \cos\overline{\theta}_j\ e^{-\frac{k_\theta j \delta s}{2}} = \tag{8}$$

$$= \int_0^s ds'\ \ \cos\overline{\theta}(s')\ e^{-\frac{K_\theta s'}{2}}$$

Similarly, it is possible to obtain the mean value of $y(s)$:

$$\langle y(s) \rangle = \int_0^s ds'\ \ \sin\overline{\theta}(s')\ e^{-\frac{K_\theta s'}{2}} \tag{9}$$

Finally, in a similar manner, but with some more computation, we obtain all the second-order moments:

$$\left\langle x(s)^2 \right\rangle = \int_0^s ds' \int_0^{s-s'} ds'' e^{-\frac{K_\theta s''}{2}} \left\{ (1 + \chi_c(s')) \cos[\overline{\theta}(s'+s'') - \overline{\theta}(s')] + \right. \tag{10}$$

$$\left. - \chi_s(s') \sin[\overline{\theta}(s'+s'') - \overline{\theta}(s')] \right\} + \frac{K_r}{2} \left[s + \int_0^s ds'\ \chi_c(s') \right]$$

$$\left\langle y(s)^2 \right\rangle = \int_0^s ds' \int_0^{s-s'} ds'' e^{-\frac{K_\theta s''}{2}} \left\{ (1 - \chi_c(s')) \cos[\overline{\theta}(s'+s'') - \overline{\theta}(s')] + \right. \tag{11}$$

$$+\chi_s(s')\sin[\overline{\theta}(s'+s'')-\overline{\theta}(s')]\Big\} + \frac{K_r}{2}\left[s - \int_0^s ds'\ \chi_c(s')\right]$$

$$\langle x(s)\ y(s)\rangle = \int_0^s ds' \int_0^{s-s'} ds'' e^{-\frac{K_\theta s''}{2}} \Big\{\chi_s(s')\cos[\overline{\theta}(s'+s'')-\overline{\theta}(s')]+ \quad (12)$$

$$+\chi_c(s')\sin[\overline{\theta}(s'+s'')-\overline{\theta}(s')]\Big\} + \frac{K_r}{2}\int_0^s ds'\ \chi_s(s')$$

$$\sigma_{x\theta}(s) \equiv \langle x(s)\ \theta(s)\rangle - \langle x(s)\rangle\,\overline{\theta}(s) = 2K_\theta \frac{\partial\,\langle y(s)\rangle}{\partial K_\theta} \quad (13)$$

$$\sigma_{y\theta}(s) \equiv \langle y(s)\ \theta(s)\rangle - \langle y(s)\rangle\,\overline{\theta}(s) = -2K_\theta \frac{\partial\,\langle x(s)\rangle}{\partial K_\theta} \quad (14)$$

where $\chi_c(s') \equiv \cos[2\overline{\theta}(s')]\ e^{-2K_\theta s'}$ and $\chi_s(s') \equiv \sin[2\overline{\theta}(s')]\ e^{-2K_\theta s'}$.

Obtaining the expression of higher-order moments will demand more tricky computation and will be dealt separately, in the next subsection. Here we conclude by considering the quantity $D(s)^2 \equiv x(s)^2 + y(s)^2$, which provides the time-evolution of the square of the distance of the particle from its initial position. From (10) and (11) we obtain a simple expression for its mean value:

$$\left\langle D(s)^2\right\rangle = K_r s + 2\int_0^s ds' \int_0^{s-s'} ds'' e^{-\frac{K_\theta s''}{2}} \cos[\overline{\theta}(s'+s'')-\overline{\theta}(s')] \quad (15)$$

3.2. Computation of any-order moment

We introduce the following two complex random quantities:

$$u(s) \equiv \lim_{N\to\infty}\sum_{j=1}^{N}(\delta s+\epsilon_j)e^{i\theta_j}; \qquad v(s) \equiv \lim_{N\to\infty}\sum_{j=1}^{N}(\delta s+\epsilon_j)e^{-i\theta_j} \quad (16)$$

From (6) and (16) it is immediate to realize that $x(s) = \frac{u(s)+v(s)}{2}$. Similarly we have $y(s) = \frac{u(s)-v(s)}{2i}$. Hence, in order to compute any-order moment that involves $x(s)$ and $y(s)$ it suffices to compute $\langle u(s)^p\ v(s)^q\rangle$ for any $p, q \in \mathcal{N}$. The computation of this quantity requires several tricky steps, which are provided in appendix Appendix A. The key is to separate all the independent random quantities in order to compute their mean values. This is obtained by arranging all the sums in a suitable manner (see appendix Appendix A for all the details). We have:

$$\langle u(s)^p\ v(s)^q\rangle = \sum_{n=0}^{\lfloor\frac{p+q}{2}\rfloor} K_r^n \sum_{l=0}^{2n}\binom{p}{2n-l}\binom{q}{l}\sum_m \binom{2n-l}{m}\binom{l}{m} m! \quad (17)$$

$$(2n-l-m-1)!!(l-m-1)!!\ s^m\ e^{i(p-q)\theta_0}\left(\frac{2n-l-m}{2}\right)!$$

$$\left(\frac{l-m}{2}\right)!(p-2n+l)!(q-l)!\sum_{\boldsymbol{c}}\int_0^s ds_1\int_{s_1}^s ds_2\cdots\int_{s_{\beta-1}}^s ds_\beta$$

$$exp\left\{\sum_{b=0}^{\beta-1}\left[i(p-q+\Phi_b)[\overline{\theta}(s_{b+1})-\overline{\theta}(s_b)] - \frac{(p-q+\Phi_b)^2(s_{b+1}-s_b)K_\theta}{2}\right]\right\}$$

where:

- $\lfloor \frac{p+q}{2} \rfloor$ is the largest integer not greater than $\frac{p+q}{2}$;
- the second sum on l (i.e., $\sum_{l=0}^{2n}$) is restricted to the values of l for which $2n - l \leq p$ and $l \leq q$ (or, equivalently, we are using the convention that $\binom{x}{y} = 0$ when $y > x$);
- the sum on m (i.e., $\sum_m$) goes from 0 to the minimum between l and $2n - l$ and it is restricted to the integers m with the same parity of l (hence, both $\frac{2n-l-m}{2}$ and $\frac{l-m}{2}$ are integers);
- the symbols "!" and "!!" denote the factorial and the double factorial, respectively [2] (note that $0! = 0!! = (-1)!! = 1$);
- $\beta = p + q - n - m$ is the dimension of the remaining multiple integral (note that a multiple integral of dimension m has already been computed and provided the term s^m);
- when $\beta = 0$ the sum on $\boldsymbol{c}$, i.e. $\sum_{\boldsymbol{c}}$, must be replaced with 1;
- when $\beta \geq 1$ the sum on $\boldsymbol{c}$ is the sum over all the vectors $\boldsymbol{c}$ of dimension β, whose entries are -2, -1, 1 and 2: specifically, each vector $\boldsymbol{c}$ has $\frac{l-m}{2}$ entries equal to 2, $q - l$ equal to 1, $p - 2n + l$ equal to -1 and $\frac{2n-l-m}{2}$ equal to -2 (note that the sum $\sum_{\boldsymbol{c}}$ consists of $\binom{\beta}{q-l}\binom{\beta-q+l}{p-2n+l}\binom{n-m}{\frac{l-m}{2}}$ addends);
- $\Phi_b \equiv \sum_{a=1}^{b} c_a$ (note that $\Phi_\beta = q - p$);
- $s_0 \equiv 0$ and $\overline{\theta}(s_0) = \theta_0$ (note that s_0 is not a variable of integration).

In order to complete the derivation of the statistics for our problem, we need to compute any-order moment that also involves the orientation θ. We provide the formula for the quantity $\left\langle u(s)^p \; v(s)^q \; \tilde{\theta}(s)^r \right\rangle$, where $\tilde{\theta}(s) \equiv \theta(s) - \overline{\theta}(s)$. The details of this computation are provided in appendix Appendix B. We have, for any $p, q, r \in \mathcal{N}$:

$$\left\langle u(s)^p \; v(s)^q \; \tilde{\theta}(s)^r \right\rangle = \sum_{n=0}^{\lfloor \frac{p+q}{2} \rfloor} K_r^n \sum_{l=0}^{2n} \binom{p}{2n-l} \binom{q}{l} \sum_m \binom{2n-l}{m} \binom{l}{m} m! \quad (18)$$

$$(2n-l-m-1)!!(l-m-1)!! \; s^m \; e^{i(p-q)\theta_0} \left(\frac{2n-l-m}{2}\right)! \left(\frac{l-m}{2}\right)!$$

$$(p-2n+l)!(q-l)! \sum_{\boldsymbol{c}} \sum_{\boldsymbol{\gamma}} \frac{r!(\gamma_{\beta+1}-1)!!K_\theta^{\frac{r}{2}}}{\prod_{b=0}^{\beta} \gamma_{b+1}!} \int_0^s ds_1 \int_{s_1}^s ds_2 \cdots \int_{s_{\beta-1}}^s ds_\beta$$

$$(s - s_\beta)^{\frac{\gamma_{\beta+1}}{2}} exp\left\{ i \sum_{b=0}^{\beta-1} (p - q + \Phi_b)[\overline{\theta}(s_{b+1}) - \overline{\theta}(s_b)] \right\} \prod_{b=0}^{\beta-1} \left\{ (s_{b+1} - s_b)^{\frac{\gamma_{b+1}}{2}} \right.$$

$$\left. e^{-\frac{(p-q+\Phi_b)^2 K_\theta (s_{b+1}-s_b)}{2}} \sum_{a=0}^{\lfloor \frac{\gamma_{b+1}}{2} \rfloor} \frac{\gamma_{b+1}! \left(i(p-q+\Phi_b)\sqrt{K_\theta(s_{b+1}-s_b)} \right)^{\gamma_{b+1}-2a}}{a!(\gamma_{b+1}-2a)!2^a} \right\}$$

where the sum over $\boldsymbol{\gamma}$, i.e. $\sum_{\boldsymbol{\gamma}}$, is the sum over all the vectors $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, \cdots, \gamma_\beta, \gamma_{\beta+1}]$, where γ_b ($b = 1, \cdots, \beta$) are positive integers and $\gamma_{\beta+1}$ is a positive integer with even parity. Additionally, they satisfy the constraint: $\sum_{b=1}^{\beta+1} \gamma_b = r$.

4. The case of circular active trajectories

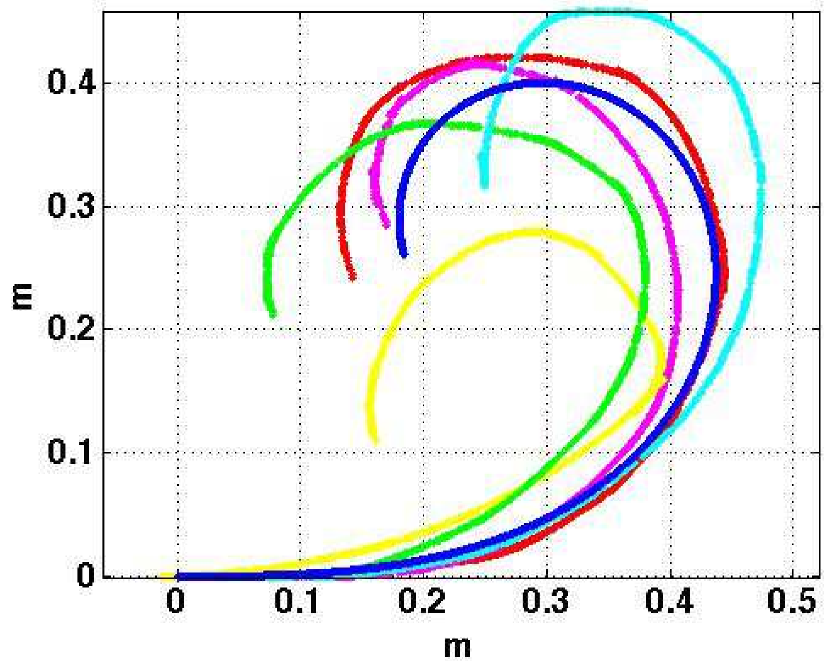

Figure 1. ...

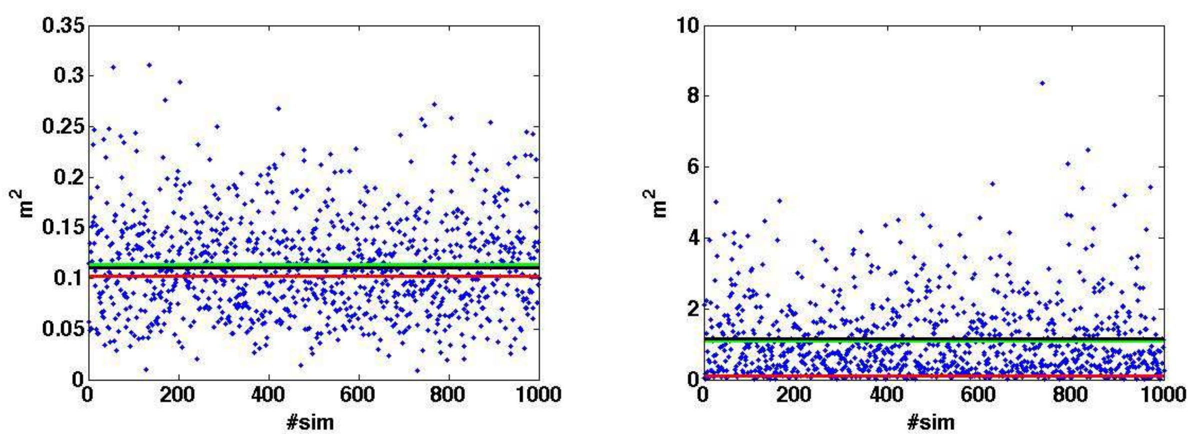

Figure 2. ...

The expressions provided in the previous section allow us to compute any order order statistics for the considered Brownian motion. These expressions contain multiple integrals on the active trajectory. In some cases, these integrals can be easily solved. This is certainly the case when the expression of $\overline{\theta}(s)$ is a linear function of the curve length s. Let us refer to this special case.

First of all, we remark that, when the orientation linearly depends on the curve length, the particle accomplishes a circular trajectory. From the third equation in (3), by only considering the deterministic part, we obtain that $\overline{\theta}(s)$ is linear in s when $\mu(s) = \mu_0$, namely, when the ratio of the angular and linear speed is constant. We have:

$$\overline{\theta}(s) = \theta_0 + \mu_0 s \tag{19}$$

and, the corresponding active trajectory, is a circumference with radius $\frac{1}{\mu_0}$ (note that when the angular speed vanishes, the radius becomes infinite and the active trajectory becomes a straight line). The computation of the integrals in (17) and (18) is immediate.

We conclude by providing the expression of the following mean values: $\langle x(s) \rangle$, $\langle y(s) \rangle$ and $\langle D(s)^2 \rangle$. Let us introduce the following complex quantity:

$$z \equiv -\frac{K_\theta}{2} + i\mu_0 \tag{20}$$

From (8) and (9) we obtain $\langle x(s) \rangle + i \langle y(s) \rangle = e^{i\theta_0} \int_0^s ds' e^{zs'} = e^{i\theta_0} \frac{e^{zs}-1}{z}$. Hence:

$$\langle x(s) \rangle = \cos\theta_0 \Re\{f(z,s)\} - \sin\theta_0 \Im\{f(z,s)\} \tag{21}$$

$$\langle y(s) \rangle = \cos\theta_0 \Im\{f(z,s)\} + \sin\theta_0 \Re\{f(z,s)\} \tag{22}$$

where $f(z,s) \equiv \frac{e^{zs}-1}{z}$ and the symbols $\Re\{\cdot\}$ and $\Im\{\cdot\}$ are the real and the imaginary part of a given complex quantity. Finally, from (15) we obtain:

$$\left\langle D(s)^2 \right\rangle = K_r s + 2\Re\left\{ \frac{f(z,s) - s}{z} \right\} \tag{23}$$

Note that this last mean value is independent of the initial orientation, as expected. Obviously, this property holds for any active trajectory and can directly be remarked in (15).

5. Conclusion

In this paper we derived the statistics, up to any order, for an overdamped $2D-$Brownian motion of an active and non-holonomic particle. The chosen kinematic constraint is modelled by the unicycle differential equation and it is a very general constraint for a $2D$ motion. According to this model, the particle can freely rotates but only one direction for the shift is allowed. The considered active motion is very general. The expressions here provided for the statistics hold for any active trajectory that satisfies the mentioned kinematic constraint. The expressions contain multiple integrals over the active trajectory. In the case when the ratio between the angular and linear speed is constant, these integrals can be easily computed. In particular, some expressions for the statistics up to the second order are provided.

We want to remark that this paper provides the analytical expressions for the statistics of a not trivial Brownian motion up to any order. To the best of our knowledge, this has never been done in the past. Deriving any-order moment corresponds to analytically derive the probability density distribution [16]. In this sense, the paper provides the analytical solution to a not trivial Fokker-Plank equation, for which the detailed balance does not hold.

Appendix A. Computation of $\langle u(s)^p\ v(s)^q \rangle$

We have:

$$\langle u(s)^p\ v(s)^q \rangle = \lim_{N\to\infty} \sum_{j_1\cdots j_p k_1\cdots k_q} \left\langle e^{i(\theta_{j_1}+\cdots+\theta_{j_p}-\theta_{k_1}-\cdots-\theta_{k_q})} \right\rangle \tag{A.1}$$

$$\left\langle (\delta s+\epsilon_{j_1})\cdots(\delta s+\epsilon_{j_p})(\delta s+\epsilon_{k_1})\cdots(\delta s+\epsilon_{k_q}) \right\rangle$$

where each index goes from 1 to N. Let us focus our attention on the quantity $\left\langle (\delta s+\epsilon_{j_1})\cdots(\delta s+\epsilon_{j_p})(\delta s+\epsilon_{k_1})\cdots(\delta s+\epsilon_{k_q}) \right\rangle$. We can write this quantity as the sum of $p+q+1$ terms, i.e., $\delta s^{p+q}C_0 + \delta s^{p+q-1}C_1 + \delta s^{p+q-2}C_2 + \cdots + \delta s C_{p+q-1} + C_{p+q}$. We trivially have $C_0 = 1$. Concerning the remaining coefficients, we remark, first of all, that only the ones with even index are different from 0. In other words, $C_{2n+1} = 0$, $n = 0, 1, \cdots, \lfloor \frac{p+q}{2} \rfloor$ (where $\lfloor x \rfloor$ is the largest integer not greater than x). Let us compute C_{2n}. This coefficient is the sum of $\binom{p+q}{2n}$ elements, each of them being the average of a product of $2n$ terms "ϵ". On the other hand, since the exponential in (A.1) is symmetric with respect to the change $j_a \leftrightarrow j_{a'}$, $a, a' = 1, \cdots, p$ and with respect to the change $k_b \leftrightarrow k_{b'}$, $b, b' = 1, \cdots, q$ but it is not symmetric with respect to the change $j_a \leftrightarrow k_b$, we need to separate the $\binom{p+q}{2n}$ elements in several groups. Specifically, let us denote by l the number of ϵ with index of type k. The number of elements belonging to this group is $\binom{p}{2n-l}\binom{q}{l}$. Note that we set $\binom{a}{b} = 0$ when $b > a$. Note also that $\sum_{l=0}^{2n} \binom{p}{2n-l}\binom{q}{l} = \binom{p+q}{2n}$. We now remark that, because of the statistical properties of ϵ, the average of the product of $2n$ terms ϵ is different from zero only when their indexes are equal two-by-two. Hence, we have to consider all the combinations of products of terms ϵ, whose indexes are equal two-by-two and that differ for at least one pair of indexes. On the other hand, for a given element in the group characterized by a given n and l, the effect in (A.1) depends on the number of pairs that are *hetero* (i.e., with an index of type j and one of type k). Let us denote by m the number of pairs that are hetero. Note that m has the same parity of l. Indeed, by definition we have l indexes of type k. Additionally, we are using m indexes of type j and m indexes of type k to make m hetero pairs. Hence, $l - m$ indexes of type k remain and, with them, we have to make $\frac{l-m}{2}$ homo pairs of type k. Similarly, we have to make $\frac{2n-l-m}{2}$ homo pairs of type j. Hence, $\frac{l-m}{2}$ must be integer.

We compute the number of combinations of the products of $2n$ terms ϵ, with l indexes of type k, where the indexes are equal two-by-two, with m pairs that are hetero and that differ for at least one pair. We obtain: $\binom{2n-l}{m}\binom{l}{m} m!(2n-l-m-1)!!(l-m-1)!!$. Following this, we can write (A.1) as follows:

$$\langle u(s)^p\ v(s)^q \rangle = \lim_{N\to\infty} \sum_{\{j\}_p\{k\}_q} \sum_{n=0}^{\lfloor \frac{p+q}{2} \rfloor} \delta s^{p+q-2n} \sum_{l=0}^{2n} \binom{p}{2n-l}\binom{q}{l} \tag{A.2}$$

$$\cdot \sum_{m=0 \;\; odd/even}^{min(l,2n-l)} \binom{2n-l}{m} \binom{l}{m} m!(2n-l-m-1)!!(l-m-1)!! \left\langle \epsilon_{j_1}^2 \right\rangle \delta_{j_1 j_2}$$

$$\left\langle \epsilon_{j_3}^2 \right\rangle \delta_{j_3 j_4} \cdots \left\langle \epsilon_{j_{2n-l-m-1}}^2 \right\rangle \delta_{j_{2n-l-m-1} j_{2n-l-m}} \left\langle \epsilon_{k_1}^2 \right\rangle \delta_{k_1 k_2} \left\langle \epsilon_{k_3}^2 \right\rangle \delta_{k_3 k_4} \cdots \left\langle \epsilon_{k_{l-m-1}}^2 \right\rangle \delta_{k_{l-m-1} k_{l-m}}$$

$$\left\langle \epsilon_{k_{l-m+1}}^2 \right\rangle \delta_{k_{l-m+1} j_{2n-l-m+1}} \cdots \left\langle \epsilon_{k_l}^2 \right\rangle \delta_{k_l j_{2n-l}} \left\langle e^{i(\theta_{j_1}+\cdots+\theta_{j_p}-\theta_{k_1}-\cdots-\theta_{k_q})} \right\rangle$$

where, for the brevity sake, we denoted by $\sum_{\{j\}_p\{k\}_q}$ the sum $\sum_{j_1\cdots j_p k_1 \cdots k_q=1}^N$. Each average $\langle \epsilon^2 \rangle$ provides $K_r \delta s$. Hence, all together, they provide $K_r^n \delta s^n$. Additionally, the number of Kronecker deltas is n. Hence, in the limit of $N \to \infty$ for each value of n we get a multiple integral of dimension $p+q-n$. Note that, when the indexes are equal h-by-h ($h > 2$), the result in the limit $N \to \infty$ vanishes since, the power of δs, is larger than the number of sums. By a direct computation in (A.2) we obtain:

$$\langle u(s)^p \; v(s)^q \rangle = \sum_{n=0}^{\lfloor \frac{p+q}{2} \rfloor} K_r^n \sum_{l=0}^{2n} \binom{p}{2n-l} \binom{q}{l} \sum_{m=0 \;\; odd/even}^{min(l,2n-l)} \binom{2n-l}{m} \binom{l}{m} \quad \text{(A.3)}$$

$$m!(2n-l-m-1)!!(l-m-1)!! \; s^m \lim_{N\to\infty} \sum_{\{j^s\}_\sigma \{j^d\}_\rho \{k^s\}_\chi \{k^d\}_\eta} \delta s^\beta$$

$$\left\langle exp \left\{ i[\theta_{j_1^s} + \cdots + \theta_{j_\sigma^s} + 2(\theta_{j_1^d} + \cdots + \theta_{j_\rho^d}) - (\theta_{k_1^s} + \cdots + \theta_{k_\chi^s}) - 2(\theta_{k_1^d} + \cdots + \theta_{k_\eta^d})] \right\} \right\rangle$$

where $\rho \equiv \frac{2n-l-m}{2}$, $\sigma \equiv p-(2n-l)$, $\eta \equiv \frac{l-m}{2}$, $\chi \equiv q-l$ and $\beta \equiv \rho+\sigma+\eta+\chi = p+q-n-m$. We must compute the average of the exponential in (A.3) and then we compute the limit $N \to \infty$. We remark that the various θ in the exponential contain random quantities (i.e., the $\delta\theta$ at different time steps). In order to proceed we have to separate all the random quantities that are independent. We start this separation by redefining the indexes in the sum $\sum_{\{j^s\}_\sigma \{j^d\}_\rho \{k^s\}_\chi \{k^d\}_\eta}$. Specifically, we consider the new indexes $\overline{j}^s \overline{j}^d \overline{k}^s \overline{k}^d$ that differ from $j^s j^d k^s k^d$ since they are ordered (in increasing order). For instance, $\overline{j}_1^s < \overline{j}_2^s < \cdots < \overline{j}_\sigma^s$. Hence, the last sum in (A.3) can be replaced with $\sum_{\{j^s\}_\sigma \{j^d\}_\rho \{k^s\}_\chi \{k^d\}_\eta} \to \rho!\sigma!\eta!\chi! \sum_{\{\overline{j}^s\}_\sigma \{\overline{j}^d\}_\rho \{\overline{k}^s\}_\chi \{\overline{k}^d\}_\eta}$. The four types of indexes are not ordered among them. Hence, the sum includes all the possible combinations that maintain the order only restricted to a single index type. For instance, a possible combination is: $\overline{k}_1^d < \overline{k}_2^d < \overline{j}_1^s < \overline{k}_1^s < \overline{j}_2^s < \overline{k}_3^d < \overline{j}_1^d < \overline{j}_2^d < \cdots$. We introduce the β ordered indexes $\overline{w}_1 < \overline{w}_2 < \cdots < \overline{w}_\beta$. Additionally, let us denote with $\Delta_a^b \equiv \sum_{c=a+1}^b \delta\theta_c$ and $\overline{\Delta}_a^b \equiv \sum_{c=a+1}^b \overline{\delta\theta}_c = \overline{\theta}(b\delta s) - \overline{\theta}(a\delta s)$, where $\overline{\theta}(s)$ is the active trajectory. Finally, let us define $\alpha \equiv 2\rho + \sigma - \chi - 2\eta$. The sum in the exponential in (A.3) contains $\alpha(\Delta_0^{\overline{w}_1} + \overline{\Delta}_0^{\overline{w}_1})$. Then, depending on the combination of the indexes $\overline{j}^s \overline{j}^d \overline{k}^s \overline{k}^d$, we have a different result for the $(\Delta_{\overline{w}_1}^{\overline{w}_2} + \overline{\Delta}_{\overline{w}_1}^{\overline{w}_2})$. Specifically, if $\overline{w}_1 = \overline{j}_1^d$ the sum in the exponential contains $(\alpha-2)(\Delta_{\overline{w}_1}^{\overline{w}_2} + \overline{\Delta}_{\overline{w}_1}^{\overline{w}_2})$. If $\overline{w}_1 = \overline{j}_1^s$ it contains $(\alpha-1)(\Delta_{\overline{w}_1}^{\overline{w}_2} + \overline{\Delta}_{\overline{w}_1}^{\overline{w}_2})$. If $\overline{w}_1 = \overline{k}_1^s$ it contains $(\alpha+1)(\Delta_{\overline{w}_1}^{\overline{w}_2} + \overline{\Delta}_{\overline{w}_1}^{\overline{w}_2})$. If $\overline{w}_1 = \overline{k}_1^d$ it contains $(\alpha+2)(\Delta_{\overline{w}_1}^{\overline{w}_2} + \overline{\Delta}_{\overline{w}_1}^{\overline{w}_2})$. We introduce the vector $\boldsymbol{c} \equiv [c_1, \cdots, c_\beta]^T$ whose entries are -2, -1, 1 and 2. Specifically, it contains η entries equal to 2, χ equal to 1, σ equal to -1 and ρ equal to -2. It is easy to realize that we have $\binom{\beta}{\chi}\binom{\beta-\chi}{\sigma}\binom{\rho+\eta}{\eta}$ vectors $\boldsymbol{c}$. Finally, we define $\Phi_b \equiv \sum_{a=1}^b c_a$. Note that $\Phi_\beta = -\alpha$. According to this, the last sum in (A.3) can be written as follows: $e^{i\alpha\theta_0}\rho!\sigma!\eta!\chi!\sum_{\boldsymbol{C}}\sum_{\{\overline{w}\}_\beta}\delta s^\beta\left\langle exp\left\{i\sum_{b=0}^{\beta-1}\left[(\alpha+\Phi_b)(\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}+\overline{\Delta}_{\overline{w}_b}^{\overline{w}_{b+1}})\right]\right\}\right\rangle$. Hence, we have:

$$\langle u(s)^p\ v(s)^q\rangle = \sum_{n=0}^{\lfloor\frac{p+q}{2}\rfloor} K_r^n \sum_{l=0}^{2n}\binom{p}{2n-l}\binom{q}{l}\sum_{\substack{m=0\\ odd/even}}^{min(l,2n-l)}\binom{2n-l}{m}\binom{l}{m} \tag{A.4}$$

$$m!(2n-l-m-1)!!(l-m-1)!!\ s^m e^{i\alpha\theta_0}\rho!\sigma!\eta!\chi!$$

$$\sum_{\boldsymbol{C}}\lim_{N\to\infty}\sum_{\{\overline{w}\}_\beta}\delta s^\beta\left\langle exp\left\{i\sum_{b=0}^{\beta-1}\left[(\alpha+\Phi_b)(\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}+\overline{\Delta}_{\overline{w}_b}^{\overline{w}_{b+1}})\right]\right\}\right\rangle$$

Now we can compute the average since we were able to separate all the independent quantities. We have:

$$\left\langle exp\left\{i\sum_{b=0}^{\beta-1}\left[(\alpha+\Phi_b)(\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}+\overline{\Delta}_{\overline{w}_b}^{\overline{w}_{b+1}})\right]\right\}\right\rangle = \tag{A.5}$$

$$= exp\left\{i\sum_{b=0}^{\beta-1}\left[(\alpha+\Phi_b)\overline{\Delta}_{\overline{w}_b}^{\overline{w}_{b+1}}\right]\right\}\prod_{b=0}^{\beta-1}\left\langle exp\left\{i\left[(\alpha+\Phi_b)\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right]\right\}\right\rangle =$$

$$= exp\left\{i\sum_{b=0}^{\beta-1}\left[(\alpha+\Phi_b)\overline{\Delta}_{\overline{w}_b}^{\overline{w}_{b+1}}\right]\right\}\prod_{b=0}^{\beta-1} exp\left\{-\frac{(\alpha+\Phi_b)^2(\overline{w}_{b+1}-\overline{w}_b)K_\theta\delta s}{2}\right\} =$$

$$= exp\left\{\sum_{b=0}^{\beta-1}\left[i(\alpha+\Phi_b)\overline{\Delta}_{\overline{w}_b}^{\overline{w}_{b+1}} - \frac{(\alpha+\Phi_b)^2(\overline{w}_{b+1}-\overline{w}_b)K_\theta\delta s}{2}\right]\right\}$$

We used the equality ($\eta = \mathcal{N}(0,1)$):

$$\left\langle e^{A\eta}\right\rangle = e^{\frac{A^2}{2}} \tag{A.6}$$

with $A = i(\alpha+\Phi_b)\sigma_{b+1}$ and $\sigma_{b+1}^2 = (\overline{w}_{b+1}-\overline{w}_b)K_\theta\delta s$. By substituting equation (A.5) in (A.4) and by taking the limit ($N\to\infty$), we finally obtain:

$$\langle u(s)^p\ v(s)^q\rangle = \sum_{n=0}^{\lfloor\frac{p+q}{2}\rfloor} K_r^n \sum_{l=0}^{2n}\binom{p}{2n-l}\binom{q}{l}\sum_{\substack{m=0\\ odd/even}}^{min(l,2n-l)}\binom{2n-l}{m}\binom{l}{m} \tag{A.7}$$

$$m!(2n-l-m-1)!!(l-m-1)!!\ s^m e^{i\alpha\theta_0}\rho!\sigma!\eta!\chi!\sum_{\boldsymbol{C}}\int_0^s ds_1\int_{s_1}^s ds_2\cdots\int_{s_{\beta-1}}^s ds_\beta$$

$$exp\left\{\sum_{b=0}^{\beta-1}\left[i(\alpha+\Phi_b)[\overline{\theta}(s_{b+1})-\overline{\theta}(s_b)] - \frac{(\alpha+\Phi_b)^2(s_{b+1}-s_b)K_\theta}{2}\right]\right\}$$

which coincides with (17).

Appendix B. Computation of $\left\langle u(s)^p \ v(s)^q \ \tilde{\theta}^r \right\rangle$

This computation follows the same initial steps carried out in appendix Appendix A. We obtain an expression equal to the one given in (A.4) but the mean value at the end must be replaced with $\left\langle exp\left\{i\sum_{b=0}^{\beta-1}\left[(\alpha+\Phi_b)(\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}+\overline{\Delta}_{\overline{w}_b}^{\overline{w}_{b+1}})\right]\right\}\cdot\tilde{\theta}^r\right\rangle$. The deterministic part can be factorized out the mean value. We need to calculate:

$$\left\langle exp\left\{i\sum_{b=0}^{\beta-1}\left[(\alpha+\Phi_b)\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right]\right\}\cdot\tilde{\theta}^r\right\rangle=\left\langle\tilde{\theta}^r\prod_{b=0}^{\beta-1}\left\{exp\left[i(\alpha+\Phi_b)\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right]\right\}\right\rangle$$

By using the multinomial theorem we can write:

$$\tilde{\theta}^r=\left(\sum_{b=0}^{\beta}\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right)^r=\sum_{\boldsymbol{\gamma}}r!\cdot\prod_{b=0}^{\beta}\frac{\left(\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right)^{\gamma_{b+1}}}{\gamma_{b+1}!}$$

where $\overline{w}_{\beta+1}\equiv N$ and the sum over $\boldsymbol{\gamma}$ is the sum over all the vectors $\boldsymbol{\gamma}=[\gamma_1,\cdots,\gamma_{\beta+1}]$, where γ_b $(b=1,\cdots,\beta+1)$ are positive integers satisfying the constraint: $\sum_{b=1}^{\beta+1}\gamma_b=r$. Hence, we obtain

$$\left\langle\tilde{\theta}^r\prod_{b=0}^{\beta-1}\left\{exp\left[i(\alpha+\Phi_b)\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right]\right\}\right\rangle= \tag{B.1}$$

$$=\sum_{\boldsymbol{\gamma}}r!\left\langle\frac{\left(\Delta_{\overline{w}_\beta}^{\overline{w}_{\beta+1}}\right)^{\gamma_{\beta+1}}}{\gamma_{\beta+1}!}\prod_{b=0}^{\beta-1}exp\left[i(\alpha+\Phi_b)\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right]\frac{\left(\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right)^{\gamma_{b+1}}}{\gamma_{b+1}!}\right\rangle=$$

$$=\sum_{\boldsymbol{\gamma}}\frac{r!}{\prod_{b=0}^{\beta}\gamma_{b+1}!}\left\langle\left(\Delta_{\overline{w}_\beta}^{\overline{w}_{\beta+1}}\right)^{\gamma_{\beta+1}}\right\rangle\prod_{b=0}^{\beta-1}\left\langle exp\left[i(\alpha+\Phi_b)\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right]\left(\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right)^{\gamma_{b+1}}\right\rangle$$

We use the following two standard results for a normal distribution:

$$\left\langle\left(\Delta_{\overline{w}_\beta}^{\overline{w}_{\beta+1}}\right)^{\gamma_{\beta+1}}\right\rangle=\left[\begin{array}{ccc}0 & \gamma_{\beta+1} & odd\\ \sigma_{\beta+1}^{\gamma_{\beta+1}}(\gamma_{\beta+1}-1)!! & \gamma_{\beta+1} & even\end{array}\right. \tag{B.2}$$

and

$$\left\langle exp\left[i(\alpha+\Phi_b)\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right]\left(\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right)^{\gamma_{b+1}}\right\rangle= \tag{B.3}$$

$$=\sigma_{b+1}^{\gamma_{b+1}}exp\left[-\frac{(\alpha+\Phi_b)^2\sigma_{b+1}^2}{2}\right]\sum_{a=0}^{\lfloor\frac{\gamma_{b+1}}{2}\rfloor}\frac{\gamma_{b+1}!\,(i(\alpha+\Phi_b)\sigma_{b+1})^{\gamma_{b+1}-2a}}{a!(\gamma_{b+1}-2a)!2^a}$$

with $\sigma_{b+1}^2\equiv K_\theta\delta s(\overline{w}_{b+1}-\overline{w}_b)$. Equation (B.3) is obtained starting from (A.6) and by differentiating γ_{b+1} times with respect to A. Hence, the expression in (B.1), becomes:

$$\left\langle\tilde{\theta}^r\prod_{b=0}^{\beta-1}\left\{exp\left[i(\alpha+\Phi_b)\Delta_{\overline{w}_b}^{\overline{w}_{b+1}}\right]\right\}\right\rangle=\sum_{\boldsymbol{\gamma}}\frac{r!\sigma_{\beta+1}^{\gamma_{\beta+1}}(\gamma_{\beta+1}-1)!!}{\prod_{b=0}^{\beta}\gamma_{b+1}!}\cdot \tag{B.4}$$

$$\cdot\prod_{b=0}^{\beta-1}\left\{\sigma_{b+1}^{\gamma_{b+1}}exp\left[-\frac{(\alpha+\Phi_b)^2\sigma_{b+1}^2}{2}\right]\sum_{a=0}^{\lfloor\frac{\gamma_{b+1}}{2}\rfloor}\frac{\gamma_{b+1}!\,(i(\alpha+\Phi_b)\sigma_{b+1})^{\gamma_{b+1}-2a}}{a!(\gamma_{b+1}-2a)!2^a}\right\}$$

where now the sum over $\boldsymbol{\gamma}$ only includes the vectors $\boldsymbol{\gamma}$ whose last entry is even. By taking the limit $N \to \infty$, this expression becomes:

$$\sum_{\boldsymbol{\gamma}} \frac{r!(\gamma_{\beta+1}-1)!!}{\prod_{b=0}^{\beta}\gamma_{b+1}!}(s-s_\beta)^{\frac{\gamma_{\beta+1}}{2}} K_\theta^{\frac{r}{2}} \prod_{b=0}^{\beta-1}\Bigg\{(s_{b+1}-s_b)^{\frac{\gamma_{b+1}}{2}} e^{-\frac{(\alpha+\Phi_b)^2 K_\theta(s_{b+1}-s_b)}{2}}$$

$$\sum_{a=0}^{\lfloor\frac{\gamma_{b+1}}{2}\rfloor} \frac{\gamma_{b+1}!\left(i(\alpha+\Phi_b)\sqrt{K_\theta(s_{b+1}-s_b)}\right)^{\gamma_{b+1}-2a}}{a!(\gamma_{b+1}-2a)!2^a}\Bigg\}$$

and we finally obtain:

$$\left\langle u(s)^p\ v(s)^q\ \tilde{\theta}(s)^r\right\rangle = \sum_{n=0}^{\lfloor\frac{p+q}{2}\rfloor} K_r^n \sum_{l=0}^{2n}\binom{p}{2n-l}\binom{q}{l}\sum_{m=0\ odd/even}^{min(l,2n-l)}\binom{2n-l}{m} \tag{B.5}$$

$$\binom{l}{m} m!(2n-l-m-1)!!(l-m-1)!!\ s^m e^{i\alpha\theta_0}\rho!\sigma!\eta!\chi!$$

$$\sum_{\boldsymbol{c}}\sum_{\boldsymbol{\gamma}} \frac{r!(\gamma_{\beta+1}-1)!!K_\theta^{\frac{r}{2}}}{\prod_{b=0}^{\beta}\gamma_{b+1}!}\int_0^s ds_1\int_{s_1}^s ds_2\cdots\int_{s_{\beta-1}}^s ds_\beta (s-s_\beta)^{\frac{\gamma_{\beta+1}}{2}}$$

$$exp\left\{i\sum_{b=0}^{\beta-1}(\alpha+\Phi_b)[\overline{\theta}(s_{b+1})-\overline{\theta}(s_b)]\right\}\prod_{b=0}^{\beta-1}\Bigg\{(s_{b+1}-s_b)^{\frac{\gamma_{b+1}}{2}}$$

$$e^{-\frac{(\alpha+\Phi_b)^2 K_\theta(s_{b+1}-s_b)}{2}}\sum_{a=0}^{\lfloor\frac{\gamma_{b+1}}{2}\rfloor} \frac{\gamma_{b+1}!\left(i(\alpha+\Phi_b)\sqrt{K_\theta(s_{b+1}-s_b)}\right)^{\gamma_{b+1}-2a}}{a!(\gamma_{b+1}-2a)!2^a}\Bigg\}$$

which coincides with (18).

References

[1] Sepideh Bazazi, Pawel Romanczuk, Sian Thomas, Lutz Schimansky-Geier, Joseph J. Hale, Gabriel A. Miller, Gregory A. Sword, Stephen J. Simpson and Iain D. Couzin, Nutritional state and collective motion: from individuals to mass migration, Proc. R. Soc. B 278, 356 (2010).

[2] Callan, David (2009), A combinatorial survey of identities for the double factorial, arXiv:0906.1317

[3] I. D. Couzin et al., Collective memory and spatial sorting in animal groups, J. Theor. Biol. 218, 1 (2002).

[4] R. Dreyfus, J. Baudry, M. L. Roper, M. Fermigier, H. A. Stone, and J. Bibette, Microscopic artificial swimmers, Nature London, 437, 862, 2005

[5] Freidlin Mark and Hu Wenqing, On Diffusion in Narrow Random Channels, J Stat Phys, 152:136–158, 2013

[6] B ten Hagen, S van Teeffelen and H Lowen, J. Phys.: Condens. Matter, 12, 725–738, 2009

[7] B ten Hagen, S van Teeffelen and H Lowen, J. Phys.: Condens. Matter, 23, 194119, 2011

[8] Hanggi Peter, Marchesoni Fabio, Artificial Brownian motors: Controlling transport on the nanoscale, Reviews of Modern Physics 81 (1): 387, 2009

[9] J. C. Latombe, Robot motion planning, Kluwer Academic Publishers, 1991

[10] E. Lauga and T. R. Powers, The hydrodynamics of swimming microorganisms, Rep. Prog. Phys. 72, 096601, 2009

[11] M. Leoni, J. Kotar, B. Bassetti, P. Cicuta, and M. C. Lagomarsino, A basic swimmer at low Reynolds number, Soft Matter 5, 472, 2009

[12] A. Martinelli, The odometry error of a mobile robot with a synchronous drive system, IEEE Transactions on Robotics and Automation, Volume 18, Issue 3, 399–405, Jun 2002.

[13] A. Martinelli, The accuracy on the parameter estimation of an odometry system of a mobile robot, ICRA '02. IEEE International Conference on Robotics and Automation, 1378–1383 vol.2, 2002.

[14] H. Niwa, Self-organizing dynamic model offish schooling, J. Theor. Biol. 171, 123 (1994).

[15] Bernt Oksendal, Stochastic Differential Equations: An Introduction with Applications, Springer, Berlin, 2003

[16] Papoulis, A. Probability, Random Variables, and Stochastic Processes, 2nd ed. New York: McGraw-Hill, 1984.

[17] Reimann Peter, Brownian motors: noisy transport far from equilibrium, Physics Reports, 361, 57–265, 2002

[18] Hannes Risken, The Fokker-Planck Equation Methods of Solution and Applications, Springer Series in Synergetics Volume 18, 1989

[19] A. Sokolov, I. S. Aranson, J. O. Kessler, and R. E. Goldstein, Concentration Dependence of the Collective Dynamics of Swimming Bacteria, Phys. Rev. Lett. 98, 158102

[20] van Teeffelen S and Lowen H, Dynamics of a Brownian circle swimmer Phys. Rev. E 78 020101(R), 2008

[21] P. Tierno, R. Golestanian, I. Pagonabarraga, and F. Sagues, Magnetically actuated colloidal microswimmers, J. Phys. Chem. B. 112, 16525 2008

[22] P Tierno, R Golestanian, I Pagonabarraga, F Sagus, Controlled swimming in confined fluids of magnetically actuated colloidal rotors, Phys Rev Lett 101 218304, 2008

[23] R. Trouilloud, T. S. Yu, A. E. Hosoi and E. Lauga, Soft swimming: Exploiting deformable interfaces for low-Reynolds number locomotion, Phys. Rev. Lett. 101, 048102, 2008

[24] Turner Jonathan, Brownian Motion Applied to Human Intersections, thesis at Texas State University-San Marcos, Dept. of Mathematics, 2012