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Abstract

Solving polynomial systems arising from applications is frequently made easier by the structure of the systems. Weighted homogeneity (or quasi-homogeneity) is one example of such a structure: given a system of weights $W = (w_1, \ldots, w_n)$, W-homogeneous polynomials are polynomials which are homogeneous w.r.t the weighted degree $\deg_W(X_1^{\alpha_1} \ldots X_n^{\alpha_n}) = \sum w_i \alpha_i$.

Gröbner bases for weighted homogeneous systems can be computed by adapting existing algorithms for homogeneous systems to the weighted homogeneous case. We show that in this case, the complexity estimate for Algorithm $\mathsf{F_5}$ $\left(\binom{n+d_{\max}-1}{d_{\max}}^{\omega}\right)$ can be divided by a factor $(\prod w_i)^{\omega}$. For zero-dimensional systems, the complexity of Algorithm FGLM nD^{ω} (where D is the number of solutions of the system) can be divided by the same factor $(\prod w_i)^{\omega}$. Under genericity assumptions, for zero-dimensional weighted homogeneous systems of W-degree $(d_1, \ldots, d_n)$, these complexity estimates are polynomial in the weighted Bézout bound $\prod_{i=1}^{n} d_i / \prod_{i=1}^{n} w_i$.

Furthermore, the maximum degree reached in a run of Algorithm $\mathsf{F_5}$ is bounded by the weighted Macaulay bound $\sum(d_i - w_i) + w_n$, and this bound is sharp if we can order the weights so that $w_n = 1$. For overdetermined semi-regular systems, estimates from the homogeneous case can be adapted to the weighted case.

We provide some experimental results based on systems arising from a cryptography problem and from polynomial inversion problems. They show that taking advantage of the weighted homogeneous structure can yield substantial speed-ups, and allows us to solve systems which were otherwise out of reach.

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1. Introduction

Algorithms for solving polynomial systems have become increasingly important over the past years, because of the many situations where such algebraic systems appear, including both theoretical problems (algorithmic geometry, polynomial inversion...), and real-life applications (cryptography, robotics...). Examples of such algorithms include eigenvalues methods for systems with a finite number of solutions, or resultant calculations for polynomial elimination (see (Dickenstein and Emiris, 2010) for a survey).

The theory of Gröbner bases is another tool which has proved useful for this purpose, and many algorithms for computing Gröbner bases have been described since their introduction. They include direct algorithms, computing the Gröbner basis of any system: to name only a few, the historical Buchberger algorithm (Buchberger, 1976), and later the Faugère F_4 (Faugère, 1999) and F_5 (Faugère, 2002) algorithms; as well as change of order algorithms, computing a Gröbner basis of an ideal from another Gröbner basis: the main examples are the FGLM algorithm (Faugère et al., 1993) for systems with a finite number of solutions, and the Gröbner walk (Collart et al., 1997) for the general case.

Systems arising from applications usually have some structure, which makes the resolution easier than for generic systems. In this paper, we consider one such structure, namely *weighted homogeneous* polynomials: a polynomial $f(X_1, \ldots, X_n)$ is weighted homogeneous with respect to a system of weights $W = (w_1, \ldots, w_n)$ (or W-homogeneous) if and only if $f(X_1^{w_1}, \ldots, X_n^{w_n})$ is homogeneous in the usual sense.

Moreover, in order to obtain precise results, we will assume that the systems satisfy some *generic* properties, which are satisfied by almost any system drawn at random. This is a usual assumption for Gröbner basis complexity estimates. More generally, we will also consider affine systems with a *weighted homogeneous structure*, that is systems whose component of maximal weighted degree will satisfy these generic properties.

The complexity estimates given in this paper can be applied to a wide range of Gröbner basis algorithms. However, we mainly focus on two algorithms: Matrix-F_5, which is a matrix variant of F_5 described in Bardet et al. (2014), allowing for complexity analyses, and FGLM.

Prior work The special case $W = (1, \ldots, 1)$ is the usual homogeneous case. In this case, all the results from this paper specialize to known results. Furthermore, some hypotheses are always satisfied, making the properties and definitions simpler. In particular, the description of the Hilbert series of a homogeneous complete intersection is adapted from Moreno-Socías (2003), and the asymptotics of the degree of regularity of a semi-regular sequence were studied in Bardet et al. (2005).

Weighted homogeneous systems have been studied before, from the angle of singularity theory and commutative algebra. In particular, some results about the Hilbert series and the Hilbert function of weighted homogeneous ideals, including the weighted Bézout bound (5), can be found in most commutative algebra textbooks.

The computational strategy for systems with a weighted structure is not new either, for example it is already implemented (partially: only for weighted homogeneous systems with a degree order) in the computer algebra system Magma (Bosma et al., 1997). Additionally, the authors of Traverso (1996) proposed another way of taking into account the weighted structure, by way of the Hilbert series of the ideal. The authors of Caboara et al.

(1996) generalized this algorithm to systems homogeneous with respect to a multigraduation. Their definition of a system of weights is more general than the one we use in the present paper.

To the best of our knowledge, nobody presented a formal description of a computational strategy for systems with a weighted homogeneous structure (not necessarily weighted homogeneous), together with complexity estimates.

Some of the results presented in this paper about regular sequences previously appeared in a shorter conference paper (Faugère et al., 2013), of which this paper is an extended version: these results are the weak form of the weighted Macaulay bound (2) and the formal description of the algorithmic strategy for weighted homogeneous systems, with the complexity estimates (1) and (4). This conference paper lacked a hypothesis (reverse chain-divisible systems of weights), and as such lacked the precise description of Hilbert series required to obtain results for semi-regular sequences. The sharp variant of the weighted Macaulay bound (3), under the assumption of simultaneous Noether position, was also added in the present paper. Finally, the benchmarks section of the current paper contains additional systems, arising in polynomial inversion problems.

The conference paper was using *quasi-homogeneous* to describe the studied structure, instead of *weighted homogeneous*. While both names exist in the literature, *weighted homogeneous* seems to be more common, and to better convey the notion that this structure is a generalization of homogenity, instead of an approximation. The same notion is sometimes also named simply *homogeneous* (in which case the weights are determined by the degree of the generators; see for example Eisenbud (1995)), or homogeneous for a *nonstandard graduation* (Dalzotto and Sbarra, 2006).

Main results By definition, weighted homogeneous polynomials can be made homogeneous by raising all variables to their weight. The resulting system can then be solved using algorithms for homogeneous systems. However, experimentally, it appears that solving such systems is much faster than generic homogeneous systems. In this paper, we show that the complexity estimates for homogeneous systems, in case the system was originally W-homogeneous, can be divided by $(\prod w_i)^\omega$, where ω is the complexity exponent of linear algebra operations ($\omega = 3$ for naive algorithms, such as the Gauss algorithm).

These complexity estimates depend on two parameters of the system: its *degree of regularity* d_{reg} and its *degree* $\deg(I)$. These parameters can be obtained from the *Hilbert series* of the ideal, which can be precisely described under generic assumptions. To be more specific, we will consider systems defined by a *regular sequence* (Def. 4) and systems which are in *simultaneous Noether position* (Def. 5).

Theorem. *Let $W = (w_1, \ldots, w_n)$ be a system of weights, and $F = (f_1, \ldots, f_m)$ a zero-dimensional W-homogeneous system of polynomials in $\mathbb{K}[X_1, \ldots, X_n]$, with respective W-degree $d_1, \ldots, d_m$. The complexity (in terms of arithmetic operations in $\mathbb{K}$) of Algorithm F_5 to compute a W-GRevLex Gröbner basis of $I := \langle F \rangle$ is bounded by*

$$C_{\mathsf{F}_5} = \mathrm{O}\left(\frac{1}{(\prod w_i)^\omega} \cdot \binom{n + d_{\mathrm{reg}} - 1}{d_{\mathrm{reg}}}^\omega\right). \tag{1}$$

If F is a regular sequence (and in particular $m = n$), then d_{reg} can be bounded by the weighted Macaulay bound*:*

$$d_{\text{reg}} \leq \sum_{i=1}^{n} (d_i - w_i) + \max\{w_j\}. \tag{2}$$

If additionally F is in simultaneous Noether position w.r.t the order $X_1 > \cdots > X_n$, then the weighted Macaulay bound can be refined:

$$d_{\text{reg}} \leq \sum_{i=1}^{n} (d_i - w_i) + w_n. \tag{3}$$

The complexity of Algorithm FGLM *to perform a change of ordering is bounded by*

$$C_{\mathsf{FGLM}} = \mathrm{O}\left(n(\deg(\mathrm{I}))^{\omega}\right). \tag{4}$$

If F forms a regular sequence, then $\deg(I)$ *is given by the* weighted Bézout bound

$$\deg(I) = \frac{\prod_{i=1}^{n} d_i}{\prod_{i=1}^{n} w_i}. \tag{5}$$

In particular, the bound (3) indicates that in order to compute a Gröbner basis faster for a generic enough system, one should order the variables by decreasing weights whenever possible.

The hypotheses of the theorem are not too restrictive. In the homogeneous case, regularity and simultaneous Noether position are generic properties. However, in the weighted homogeneous case, there are systems of weights and systems of weighted degrees for which they are not generic. In this paper, we identify large families of systems of weights and systems of weighted degrees for which they are (Prop. 5).

All sequences in simultaneous Noether position are regular. In the homogeneous case, conversely, all regular sequences are in simultaneous Noether position up to a generic linear change of coordinates. In the weighted homogeneous case, it is no longer true. Worse still, there are systems of weights for which there exists no non-trivial change of coordinates.

In order to work around this limitation, we consider *reverse chain-divisible* systems of weights, that is systems of weights such that $w_n \mid w_{n-1} \mid \ldots \mid w_1$. This property ensures that there are non-trivial change of coordinates of the form $X_i \leftarrow X_i + P_i(X_{i+1}, \ldots, X_n)$ for all i, with P_i a W-homogeneous polynomial with W-degree w_i. Under this assumption, many properties from the homogeneous case remain valid in a weighted setting, and in particular, any regular sequence is, up to a W-homogeneous change of coordinates, in simultaneous Noether position (Th. 8).

For many systems from practical applications, the weights can be chosen to be reverse chain-divisible. We give a few examples in the last section of this paper.

If $m > n$, there is no regular sequence. Instead, we will consider systems defined by a *semi-regular* sequence, that is systems for which no reduction to zero appear in a run of Algorithm F_5. This property has several equivalent definitions in the homogeneous case. While these definitions can be easily extended to the weighted case, their equivalence is not necessarily true. However, we prove that these definitions are equivalent in the special case where the weights form a reverse chain-divisible sequence.

In the homogeneous case, the property of being semi-regular is only conjectured to be generic, but this conjecture is proved in a handful of cases (Moreno-Socías (1996, Thm. 1.5). In this paper, we adapt the proof of one of these cases, namely the case $m = n + 1$ in a base field of characteristic 0.

For semi-regular systems with $m = n+1$, we obtain a bound on the degree of regularity of the system. More generally, in the homogeneous case, one can compute asymptotic estimates on the degree of regularity of a semi-regular sequence (Bardet et al., 2005; Bardet, 2004). These estimates can be adapted to the weighted homogeneous case. As an example, we give an asymptotic bound on the degree of regularity for semi-regular systems with $m = n + k$ for a given integer k:

Theorem. *Let n and k be two positive integers, and let $m = n+k$. Let w_0 and d_0 be two positive integers such that $w_0 \mid d_0$. Consider the system of n weights $W = (w_0, \ldots, w_0, 1)$. Let F be a semi-regular sequence in $\mathbb{K}[X_1, \ldots, X_n]$, made of W-homogeneous polynomials with W-degree d_0. Then the highest degree reached in the computation of a W-GRevLex Gröbner basis of $\langle F \rangle$ is asymptotically bounded by*

$$d_{\mathrm{reg}} = n\,\frac{d_0 - w_0}{2} - \alpha_k \sqrt{n\,\frac{d_0^2 - w_0^2}{6}} + \mathrm{O}\left(n^{1/4}\right).$$

where α_k is the largest root of the k'th Hermite's polynomial.

Experimentally, if we lift the assumption that the system of weights is reverse chain-divisible, the degree of regularity does not appear to rise too far beyond the bound. Future work on the topic could include characterizing the Hilbert series of W-homogeneous semi-regular sequences in full generality, in order to obtain bounds on the W-degree of regularity.

In practice, taking advantage of the weighted structure when applicable yields significant speed-ups. Some instance of a weighted structure has already been successfully exploited for an application in cryptography (Faugère et al., 2013). We also present timings obtained with several polynomial inversion problems, with speed-ups ranging from 1–2 to almost 100. In particular, we use these techniques in order to compute the relations between fundamental invariants of several groups (see (Sturmfels, 2008)). For some groups such as the Cyclic-5 group or the dihedral group D_5, computing these relations is intractable without considering the weighted structure of the system, while it takes only a few seconds or minutes when exploiting the weighted structure. All these systems are examples of applications where the weights giving the appropriate W-homogeneous structure are naturally reverse chain-divisible. These experimentations have been carried using F_5 and FGLM with the Gröbner basis library FGb (Faugère, 2010) and F_4 with the computer algebra system Magma (Bosma et al., 1997).

There are other applications where Gröbner bases are computed for polynomial systems with a weighted-homogeneous structure, for example in coding theory, both for generating codes (de Boer and Pellikaan (1999, sec. 5), (Leonard, 2009)) and for decoding through Guruswami-Sudan's algorithm (see (Guerrini and Rimoldi, 2009) for an overview).

Organisation of the paper In section 2, we define weighted graded algebras and some generic properties of weighted homogeneous systems. In section 3, we focus on regular systems and complete intersections. We describe the Hilbert series of a weighted homogeneous complete intersection and give the sharp variant of the weighted Macaulay bound. In section 4, we consider semi-regular systems. We give some equivalent definitions of this property, and we show how asymptotic estimates of the degree of regularity can be adapted from the homogeneous case to the weighted case. Additionally, we prove that Fröberg's conjecture in the case $m = n + 1$ is true in the weighted case, as in the homogeneous case, provided that the base field is large enough. In section 5, we describe strategies for computing Gröbner bases for weighted homogeneous systems, and we give complexity estimates for these strategies. Finally, in section 6, we show how weighted structures can appear in applications, and we give some benchmarks for each example.

2. Definitions and genericity statements

2.1. Definitions

Let $\mathbb{K}$ be a field. We consider the algebra $\mathbb{K}[X_1, \ldots, X_n] = \mathbb{K}[\mathbf{X}]$. This algebra can be graded with respect to a system of weights, as seen for example in (Becker and Weispfenning, 1993, sec. 10.2).

Definition 1. Let $W = (w_1, \ldots, w_n)$ be a vector of positive integers. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector of nonnegative integers. Let the integer $\deg_W(\mathbf{X}^\alpha) = \sum_{i=1}^n w_i\alpha_i$ be the *W-degree*, or *weighted degree* of the monomial $\mathbf{X}^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. We say that the vector W is a *system of weights*. We denote by $\mathbf{1}$ the system of weights defined by $(1, \ldots, 1)$, associated with the usual grading (in total degree) on $\mathbb{K}[\mathbf{X}]$.

Any grading on $\mathbb{K}[\mathbf{X}]$ comes from such a system of weights (Becker and Weispfenning, 1993, sec. 10.2). When working with a W-graduation, to clear up any ambiguity, we use the adjective *W-homogeneous* for elements or ideals, or *weighted homogeneous* if W is clear in the context. The word *homogeneous* will be reserved for $\mathbf{1}$-homogeneous items. The following property is an easy consequence of the definition.

Proposition 1. *Let $(\mathbb{K}[X_1, \ldots, X_n], W)$ be a graded polynomial algebra. Then the application*

$$\begin{aligned} \hom_W : (\mathbb{K}[X_1, \ldots, X_n], W) &\longrightarrow (\mathbb{K}[t_1, \ldots, t_n], \mathbf{1}) \\ f &\longmapsto f(t_1^{w_1}, \ldots, t_n^{w_n}) \end{aligned}$$

is an injective graded morphism, and in particular the image of a weighted homogeneous polynomial is a homogeneous polynomial.

The above morphism also provides a weighted variant of the GRevLex ordering (as found for example in (Becker and Weispfenning, 1993, 10.2)), called the W-GRevLex ordering:

$$u <_{W\text{-grevlex}} v \iff \hom_W(u) <_{\text{grevlex}} \hom_W(v).$$

Given a W-homogeneous system F, one can build the homogeneous system $\hom_W(F)$, and then apply classical algorithms (Faugère, 2002; Faugère et al., 1993) to that system to compute a GRevLex (resp. Lex) Gröbner basis of the ideal generated by $\hom_W(F)$.

Definition 2. The W-*degree of regularity* of the system F is the highest degree $d_{\mathrm{reg},W}(F)$ reached in a run of F_5 to compute a GRevLex Gröbner basis of $\hom_W(F)$. When the graduation is clear in the context, we may call it degree of regularity, and denote it d_{reg}.

Remark. Unlike what we could observe in the homogeneous case, this definition depends on the order of the variables (we shall give an example in Table 1 in section 3.2, and another, with timings, in Table 4 in section 6.1).

Definition 3. Let I be a zero-dimensional (not necessarily weighted homogeneous) ideal in $A = \mathbb{K}[X_1, \ldots, X_n]$. In that case, we define the degree D of the ideal I as the (finite) dimension of A/I, seen as a $\mathbb{K}$-vector space:

$$D = \dim_{\mathbb{K}} (A/I) .$$

Equivalently, if $\mathsf{HS}_{A/I}(T)$ is the Hilbert series (with respect to the W-graduation) of I, this series is a polynomial in T and

$$D = \mathsf{HS}_{A/I}(1).$$

Remark. This definition with the Hilbert series can be extended to ideals with positive dimension. However, in a weighted setup, varieties can end up having rational (not-necessarily integer) degrees. This is the definition used by the software Macaulay2 (Grayson and Stillman, 2014, function `degree(Module)`).

We will only consider the *affine* varieties associated with the ideals we consider. In particular, the dimension of $V(0)$ is n, and a zero-dimensional variety is defined by at least n polynomials if the base field is algebraically closed.

Definition 4 (Regular sequence)**.** Let $W = (w_1, \ldots, w_n)$ be a system of weights, let $D = (d_1, \ldots, d_m)$ be a system of W-degrees and let $F = (f_1, \ldots, f_m)$ be a sequence of W-homogeneous polynomials in $\mathbb{K}[X_1, \ldots, X_n]$, with W-degree D. The system F is called *regular* if it satisfies one of the following equivalent properties (Eisenbud, 1995):
(1) $\forall\, i \in \{1, \ldots, m\}$, f_i is not a zero-divisor in $\mathbb{K}[X_1, \ldots, X_n]/\langle f_1, \ldots, f_{i-1}\rangle$;
(2) the Hilbert series of $\langle F\rangle$ is given by

$$\mathsf{HS}_{A/I}(T) = \frac{\prod_{i=1}^{m}(1 - T^{d_i})}{\prod_{i=1}^{n}(1 - T^{w_i})}. \tag{6}$$

Definition 5 (Simultaneous Noether position)**.** Let W be a system of weights. Let $m \leq n$ and $F = (f_1, \ldots, f_m)$ be a sequence of W-homogeneous polynomials in $\mathbb{K}[X_1, \ldots, X_n]$ The system F is said to be *in Noether position w.r.t the variables* $X_1, \ldots, X_m$ if it satisfies the two following properties:

- for $i \leq m$, the canonical image of X_i in $\mathbb{K}[\mathbf{X}]/I$ is an algebraic integer over $\mathbb{K}[X_{m+1}, \ldots, X_n]$;
- $\mathbb{K}[X_{m+1}, \ldots, X_n] \cap I = 0$.

The system F is said to be *in simultaneous Noether position* (or in SNP) if for any $1 \leq i \leq m$, the system $(f_1, \ldots, f_i)$ is in Noether position w.r.t the variables $X_1, \ldots, X_i$.

The following proposition enumerates useful characterizations of the Noether position. They are mostly folklore, but we give a proof for completeness.

Proposition 2. *Let $m \leq n$, W be a system of weights and D be a system of W-degrees. Let $F = (f_1, \ldots, f_m)$ be a sequence of W-homogeneous polynomials, with W-degree D. The following statements are equivalent:*

- (NP1) *the sequence F is in Noether position w.r.t. the variables $X_1, \ldots, X_m$;*
- (NP2) *the sequence $F_{\text{ext}} := (f_1, \ldots, f_m, X_{m+1}, \ldots, X_n)$ is regular;*
- (NP3) *the sequence $F' := F(X_1, \ldots, X_m, 0, \ldots, 0)$ is in Noether position w.r.t. the variables $X_1, \ldots, X_m$;*
- (NP4) *the sequence F' is regular.*

Proof. (NP1 $\implies$ NP2). [1] Let I be the ideal generated by F. The geometric characterization of Noether position (see e.g. Milne (2012)) shows that the canonical projection onto the m first coordinates

$$\pi : V(I) \longrightarrow V(\langle X_1, \ldots, X_m \rangle)$$

is a surjective morphism with finite fibers. This implies that the variety $V(\langle F_{\text{ext}} \rangle) = \pi^{-1}(0)$ is zero-dimensional, and so the sequence is regular.

(NP2 $\implies$ NP1). Let $i \leq m$, we want to show that X_i is integral over the ring $\mathbb{K}[X_{m+1}, \ldots, X_n]$. Since F_{ext} defines a zero-dimensional ideal, there exists $n_i \in \mathbb{N}$ such that $X_i^{n_i} = \mathsf{LT}(f)$ with $f \in \langle F_{\text{ext}} \rangle$ for the GRevLex ordering with $X_1 > \cdots > X_n$. By definition of the GRevLex ordering, we can assume that f simply belongs to I. This shows that every X_i is integral over $\mathbb{K}[X_{i+1}, \ldots, X_n]/I$. We get the requested result by induction on i: first, this is clear if $i = m$. Now assume that we know that $\mathbb{K}[X_i, \ldots, X_n]/I$ is an integral extension of $\mathbb{K}[X_{m+1}, \ldots, X_n]$. From the above, we also know that X_{i-1} is integral over $\mathbb{K}[X_i, \ldots, X_n]$, and so, since the composition of integral homomorphisms is integral, we get the requested result.

Finally, we want to check the second part of the definition of Noether position. Assume that there is a non-zero polynomial in $\mathbb{K}[X_{m+1}, \ldots, X_n] \cap I$. Since the ideal is weighted homogeneous, we can assume this polynomial to be weighted homogeneous. Either this polynomial has degree 0, or it is a non-trivial syzygy between $X_{m+1}, \ldots, X_n$. So in any case, it contradicts the regularity hypothesis.

(NP2 $\implies$ NP4). For any $i \in \{1, \ldots, m\}$, write $f'_i = f_i(X_1, \ldots, X_m, 0, \ldots, 0)$. Since any permutation of a regular sequence is a regular sequence, $(X_{m+1}, \ldots, X_n, f_1, \ldots, f_m)$ is a regular sequence, that is, for any $1 \leq i \leq m$, f_i is not a zero divisor in

$$\mathbb{K}[X_1, \ldots, X_n]/\langle X_{m+1}, \ldots, X_n, f_1, \ldots, f_{i-1} \rangle$$

As a consequence, factoring in the quotient by $\langle X_{m+1}, \ldots, X_m \rangle$, f'_i is no zero-divisor in

$$\mathbb{K}[X_1, \ldots, X_m]/\langle f'_1, \ldots, f'_{i-1} \rangle.$$

[1] The proof of NP1 $\iff$ NP2 can be found in Faugère et al. (2013), we give it again here for completeness.

(NP4 $\implies$ NP2). For any i, write $f_i = f'_i + r_i$ with $f'_i \in \mathbb{K}[X_1, \ldots, X_m]$, and $r_i \in \langle X_{m+1}, \ldots, X_n \rangle$. Let $1 \leq i \leq n$. Assume that $gf_i \in \langle X_{m+1}, \ldots, X_n, f_1, \ldots, f_{i-1} \rangle$:

$$\begin{aligned} gf_i = gf'_i + gr_i &= \sum_{j=1}^{i-1} g_j f_j + \sum_{j=m+1}^{n} g_j X_j \\ &= \sum_{j=1}^{i-1} g_j f'_j + R \qquad \text{with } R \in \langle X_{m+1}, \ldots, X_n \rangle. \end{aligned}$$

As a consequence, considering only the monomials in $\mathbb{K}[X_1, \ldots, X_m]$

$$g' f'_i = \sum_{j=1}^{i-1} g_j f'_j \text{ where } g' = g(X_1, \ldots, X_m, 0, \ldots, 0).$$

Since F' is regular, $g' \in \langle f'_1, \ldots, f'_{i-1} \rangle$:

$$g = g' + r \in \langle f'_1, \ldots, f'_{i-1} \rangle + \langle X_{m+1}, \ldots, X_m \rangle = \langle f_1, \ldots, f_{i-1} \rangle + \langle X_{m+1}, \ldots, X_m \rangle.$$

And indeed, f_i is no zero-divisor in $\mathbb{K}[X_1, \ldots, X_n]/\langle X_{m+1}, \ldots, X_n, f_1, \ldots, f_{i-1} \rangle$. It means that $(X_{m+1}, \ldots, X_n, f_1, \ldots, f_m)$ is a regular sequence. By permutation, we conclude that $(f_1, \ldots, f_m, X_{m+1}, \ldots, X_n)$ is a regular sequence.

(NP4 $\iff$ NP3). The sequence $F' = (f'_1, \ldots, f'_m) \in \mathbb{K}[X_1, \ldots, X_m]^m$ is regular if and only if the sequence $(f'_1, \ldots, f'_m, X_{m+1}, \ldots, X_n)$ is regular. The equivalence between NP3 and NP4 is then a mirror of the equivalence between NP1 and NP2. □

2.2. Reverse chain-divisible systems of weights and their properties

Let W be a system of weights. Several properties from the homogeneous case turn out to be no longer true in the weighted case. For example, properties such as the Noether normalization lemma are no longer available, since in general, we cannot write any non trivial weighted homogeneous change of coordinates. However, if we add some constraints on the system of weights, some of these properties can be proved in a weighted setting. More precisely, we will consider *reverse chain-divisible* systems of weights, defined as follows.

Definition 6. We say that W is *reverse chain-divisible* if we have

$$w_n \mid w_{n-1} \mid \ldots \mid w_1$$

In this situation, the weights are coprime if and only if $w_n = 1$.

Remark. The name "chain-divisible" can be found in Alfonsín (2005), referring to a notion introduced in Alfonsín (1998).

In this setting, many results from the homogeneous case can now be adapted to the weighted homogeneous case. For example, the Noether normalization lemma states that for homogeneous polynomials with an infinite base field, all regular sequences are in Noether position up to a generic linear change of coordinates. In the weighted homogeneous case with reverse chain-divisible weights, all regular sequences are in Noether position, up to a weighted homogeneous change of coordinates, with W-degree W. More

precisely, in the weighted homogeneous case, we can prove the following version of the Noether normalization lemma (see (Eisenbud, 1995, lem. 13.2.c) for the homogeneous version of this lemma):

Lemma 7 (Noether normalization lemma, weighted case)**.** *Let $\mathbb{K}$ be an infinite field, W be a reverse chain-divisible system of weights and $f \in R = \mathbb{K}[X_1, \ldots, X_r]$ be a non-constant polynomial, W-homogeneous with W-degree d. Then there are elements $X'_1, \ldots, X'_{r-1} \in R$ such that R is a finitely generated module over $\mathbb{K}[X'_1, \ldots, X'_{r-1}, f]$. Furthermore, if the field has characteristic 0 or large enough, there exists a dense Zariski-open subset $U \subset \mathbb{K}^{r-1}$ such that for all $(a_i) \in U$, one can choose $X'_i = X_i - a_i X_r^{w_i/w_r}$.*

Proof. We follow the proof of (Eisenbud, 1995, lem. 13.2.c). For any $1 \leq i \leq r-1$, let $a_i \in \mathbb{K}$, and let $X'_i = X_i - a_i X_r^{w_i/w_r}$. We need to show that for generic a_i, under this change of variables, f is monic in X_r:

$$\begin{aligned} f(X_1, \ldots, X_r) &= f(X'_1 + a_1 X_r^{w_1/w_r}, X'_2 + a_2 X_r^{w_2/w_r}, \ldots, X_{r-1} + a_{r-1} X_r^{w_{r-1}/w_r}) \\ &= f(a_1, \ldots, a_{r-1}, 1) X_r^d + \ldots \end{aligned}$$

So the set of all a_i's such that f is monic in X_r is exactly the set of all a_i's such that $f(a_1, \ldots, a_{r-1}, 1) \neq 0$, and since f is W-homogeneous non-constant, this is a non-empty open subset of $\mathbb{K}^{r-1}$. □

Then, as in the homogeneous case (Eisenbud, 1995, th. 13.3), a consequence of this lemma is Noether's normalization theorem, which we restate in a weighted setting:

Theorem 8. *Let W be a reverse chain-divisible system of weights, and let F be a W-homogeneous zero-dimensional regular sequence in $\mathbb{K}[X_1, \ldots, X_n]$. Then, for a generic choice of W-homogeneous polynomials P_i with W-degree w_i, the change of variable*

$$X_i = X'_i + P_i(X_{i+1}, \ldots, X_n),$$

is such that $F(X_1(\mathbf{X}'), \ldots, X_n(\mathbf{X}'))$ is in simultaneous Noether position with respect to the order $X'_1 > X'_2 > \cdots > X'_n$.

Another property of reverse chain-divisible weights is the following proposition. In the homogeneous case, if $d_1 \leq d_2$ are two non-negative integers, then any monomial with degree d_2 is divisible by a monomial with degree d_1. When the system of weights is reverse chain-divisible, the following proposition states a similar result for the weighted case.

Proposition 3. *Assume that $W = (w_1, \ldots, w_n)$ is a system of weights, such that $w_1 \geq w_2 \geq \cdots \geq w_n$. The following statements are equivalent:*

(1) The system of weights W is reverse chain-divisible;

(2) Let $d_1 \leq d_2$ positive integers, $i \in \{1, \ldots, n\}$, and m_2 a monomial of W-degree d_2. Assume that w_i divides d_1, and that m_2 is not divisible by any of the variables $X_1, \ldots, X_{i-1}$. Then there exists a monomial m_1 with W-degree d_1, such that $m_1 \mid m_2$.

Proof. (1 $\implies$ 2). Fix d_1. We shall prove by induction over d_2 that for any monomial m_2 with W-degree d_2 satisfying the hypotheses of (2), there exists a monomial m_1 with W-degree d_1 dividing m_2. The case $d_2 = d_1$ is trivial.

Assume that $d_2 > d_1$, and let m_2 be a monomial of W-degree d_2. Let j be the greatest index of a variable dividing m_2, write $m_2 = X_j^\alpha m_2'$, where m_2' is a monomial in $\mathbb{K}[X_i, \dots, X_{j-1}]$, with W-degree $d_2' = d_2 - w_j\alpha$. If $d_2' \geq d_1$, the result follows by induction. If $i = j$, then $m_2 = X_i^\alpha$, and $m_1 := X_i^{d_1/w_i}$ has W-degree d_1 and divides m_2. So we can assume that $d_2' < d_1$ and that $i < j$.

Since W is a reverse chain-divisible system of weights, w_{j-1} divides w_k for any k in $\{1, \dots, j-1\}$. Hence, since $m_2 \in \mathbb{K}[X_i, \dots, X_j]$ and $m_2' \in \mathbb{K}[X_i, \dots, X_{j-1}]$, $d_2 \equiv 0 \bmod w_j$ and $d_2' \equiv 0 \bmod w_{j-1}$. By hypothesis, d_1 is divisible by w_i, and in particular it is divisible by w_{j-1}. All in all, this shows that $d_1 - d_2'$ is divisible by w_{j-1}, and so it is divisible by w_j. Let

$$m_1 = m_2' \cdot X_j^{(d_1 - d_2')/w_j}.$$

The monomial m_1 has W-degree d_1 and divides m_2.

(2 $\implies$ 1). Assume that W is a system of weights which is not reverse chain-divisible, we shall find integers $d_1 \leq d_2$ and a monomial m_2 with W-degree d_2 which is not divisible by any monomial of W-degree d_1.

Since W is not reverse chain-divisible, there exists i such that w_{i+1} does not divide w_i. In particular, $\gcd(w_i, w_{i+1}) < w_i$ and $\gcd(w_i, w_{i+1}) < w_{i+1}$. Without loss of generality, we may consider only the variables X_i, X_{i+1}. Let $d_1 = w_i w_{i+1}$, $d_2 = d_1 + \gcd(w_i, w_{i+1})$. By Paoli's lemma (see for example (Lucas, 1891, chap. 264) or the discussion after (Niven et al., 1991, th. 5.1)), there exists exactly

$$\left\lfloor \frac{d_2}{w_i w_{i+1}} \right\rfloor = \left\lfloor 1 + \frac{\gcd(w_i, w_{i+1})}{w_i w_{i+1}} \right\rfloor = 1$$

couple of non-negative integers a, b such that $aw_i + bw_{i+1} = d_2$. Let m_2 be the monomial $X_i^a X_{i+1}^b$. The W-degree d_1 is divisible by w_i, and m_2 is not divisible by $X_1, \dots, X_{i-1}$. The maximal divisors of m_2 are

$$\frac{m_2}{X_i} = X_i^{a-1} X_{i+1}^b \text{ with } W\text{-degree } d_2 - w_i = d_1 + \gcd(w_i, w_{i+1}) - w_i < d_1;$$
$$\frac{m_2}{X_{i+1}} = X_i^a X_{i+1}^{b-1} \text{ with } W\text{-degree } d_2 - w_{i+1} = d_1 + \gcd(w_i, w_{i+1}) - w_{i+1} < d_1.$$

As a consequence, m_2 is not divisible by any monomial of W-degree d_1. □

This proposition essentially states that the staircase of a W-homogeneous ideal is reasonably shaped when W is a reverse chain-divisible system of weights. For example, let W be a reverse chain-divisible system of weights, and let I be the ideal generated by all monomials of W-degree w_1 (that is, the least common multiple of the weights). Then the proposition proves that I contains all monomials of W-degree greater than w_1.

If on the other hand the system of weights is not reverse chain-divisible, this property needs not hold. For example, consider the algebra $\mathbb{K}[X_1, X_2, X_3]$ graded w.r.t. the system of weights $W = (3, 2, 1)$, the least common multiple of the weights being 6, and let I be the ideal generated by all monomials of W-degree 6. Consider the monomial $X_1 X_2^2$: it has W-degree 7, yet it is not divisible by any monomial with W-degree 6, and so it does not belong to the ideal I.

2.3. Genericity

We shall give some results about the genericity of regularity and Noether position for weighted homogeneous sequences. The fact that they define Zariski-open subsets of the sets of sequences of a given weighted degree is classical. For regular sequences, see for example (Pardue, 2010, sec. 2). The proof for sequences in (simultaneous) Noether position is a simple extension of the statement for regular sequences. However, we provide here a sketch of these proofs for completeness.

Proposition 4. *Let $m \leq n$ be two integers, $W = (w_1, \ldots, w_n)$ a system of weights, and $D = (d_1, \ldots, d_m)$ a system of W-degrees. Then*

- *the set of regular sequences,*
- *the set of sequences in Noether position with respect to the variables $X_1, \ldots, X_m$, and*
- *the set of sequences in simultaneous Noether position w.r.t. the order $X_1 > \cdots > X_m$*

are Zariski-open subsets of the affine space of W-homogeneous polynomials with W-degree D.

Proof. We shall prove that regular sequences form a Zariski-open subset of the affine space of W-homogeneous polynomials of W-degree D. The openness of Noether position will then be a corollary, since by Proposition 2, $(f_1, \ldots, f_m)$ is in Noether position w.r.t the variables $X_1, \ldots, X_m$ if and only if $(f_1, \ldots, f_m, X_{m+1}, \ldots, X_n)$ is regular. As for sequences in simultaneous Noether position, they will be given by the intersection of m open subsets, stating that the sequences $(f_1, \ldots, f_i)$, $i \in \{1, \ldots, m\}$, are in Noether position w.r.t. the variables $X_1, \ldots, X_i$.

Let $F = (f_1, \ldots, f_m)$ be a family of m generic quasi-homogeneous polynomials, that is polynomials in $\mathbb{K}[\mathbf{a}][\mathbf{X}]$, whose coefficients are algebraically independent parameters a_k. We want to prove that regular sequences are characterized by some polynomial in these coefficients a_k being non-zero, which implies that they belong to a Zariski-open set. Write $I = \langle F \rangle$. Since the Hilbert series (6) characterizes regular sequences, F is regular if and only if I contains all monomials of W-degree between $i_{\text{reg}}(I) + 1$ and $i_{\text{reg}}(I) + \max\{w_i\}$, where $i_{\text{reg}}(I)$ is given by $\sum(d_i - w_i)$. This expresses that a given set of linear equations has solutions, and so it can be coded as some determinants being non-zero, as polynomials in the coefficients a_k. □

This states that the set of regular sequences, sequences in Noether position and sequences in simultaneous Noether position are Zariski-dense subsets if and only if they are not empty. Unfortunately, depending on the weights and the weighted degrees, there may exist no regular sequence, and thus no sequences in (simultaneous) Noether position either. For example, let $W = (2, 5)$ and $D = (4, 8)$, the only W-homogeneous sequence with W-degree D in $\mathbb{K}[X, Y]$ is (up to scalar multiplication) (X^2, X^4), and it is not regular. However, this is only the case for very specific systems of W-degrees, for which there does not exist enough monomials to build non-trivial sequences.

Definition 9. Let $m \leq n$ be two integers, $W = (w_1, \ldots, w_n)$ a system of weights, and $D = (d_1, \ldots, d_m)$ a system of W-degrees. We say that D is *W-compatible* if there exists a regular W-homogeneous sequence in $\mathbb{K}[X_1, \ldots, X_n]$ with W-degree D. We say that D is *strongly W-compatible* if for any $1 \leq i \leq m$, d_i is divisible by w_i.

Using these definitions, we can identify the cases where the properties of being regular, in Noether position or in simultaneous Noether position are generic.

Proposition 5. *Let $m \leq n$ be two integers, $W = (w_1, \dots, w_n)$ a system of weights, and $D = (d_1, \dots, d_m)$ a system of W-degrees. For any $1 \leq i \leq m$, write $W_i := (w_1, \dots, w_i)$ and $D_i := (d_1, \dots, d_i)$. Write $A_{W,D}$ the affine space of W-homogeneous sequences of W-degree D. Then the following statements are true:*

(1) if D is W-compatible, then regular sequences form a Zariski-dense subset of $A_{W,D}$;
(2) if D is W_m-compatible, then sequences in Noether position with respect to the variables $X_1, \dots, X_m$ form a Zariski-dense subset of $A_{W,D}$;
(3) if D is strongly W-compatible, then D is W-compatible, W_m-compatible, and for any i, D_i is W_i-compatible;
(4) if $m = n$, D is W-compatible and W is reverse chain-divisible, then, up to some reordering of the degrees, D is strongly W-compatible.

Proof. The proofs of statements 1 and 2 follow the same technique: by Theorem 4, we know that the sets we consider are Zariski-open in $A_{W,D}$. So in order to prove the density, we only need to prove that they are non empty. Statement 1 is exactly the definition of the W-compatibility.

For statement 2, by W_m-compatibility, we know that there exists a W-homogeneous sequence $F = (f_1, \dots, f_m)$ with W-degree D in $\mathbb{K}[X_1, \dots, X_m]$, which is regular. As a consequence, the sequence $(f_1, \dots, f_m, X_{m+1}, \dots, X_n)$ is regular, and from the characterization NP4 of Noether position (prop. 2), this means that F is in Noether position with respect to the variables $X_1, \dots, X_m$.

In order to prove statement 3, we need to exhibit regular sequences of length i in $\mathbb{K}[X_1, \dots, X_i]$ for any $1 \leq i \leq m$. For $1 \leq i \leq m$, write $F_i = (X_1^{d_1/w_1}, \dots, X_i^{d_i/w_i})$, it is regular and each polynomial lies in $\mathbb{K}[X_1, \dots, X_i]$.

Finally, statement 4 is a consequence of Theorem 8. Let W be a reverse chain-divisible system of weights, and D a W-compatible system of W-degrees. Up to reordering, we can assume that the polynomials are ordered so that $d_1 \geq d_2 \geq \cdots \geq d_n$; this does not cancel the W-compatibility. Let $F = (f_1, \dots, f_n)$ be a regular sequence, W-homogeneous with W-degree D. By Theorem 8, there exist polynomials $P_i(X_{i+1}, \dots, X_n)$ which are W-homogeneous with W-degree w_i, and such that F, under the change of variables $X_i = X_i' + P_i(X_{i+1}, \dots, X_n)$, is in simultaneous Noether position with respect to the order $X_1' > X_2' > \cdots > X_n'$. From the characterization NP4 of Noether position, that means in particular that for any $i \in \{1, \dots, n\}$, $f_i(X_1(X_1', \dots, X_i'), \dots, X_n(X_1', \dots, X_i'))$ belongs to a regular sequence, and thus is not zero. And by definition of reverse chain-divisible weights, its W-degree d_i is a sum of multiples of w_i, and so it is itself a multiple of w_i. □

Remark. The statement 4 is a converse of 3 in the reverse chain-divisible case. In the non-reverse chain-divisible case, that converse is false: let $W = (3, 2)$, $D = (6, 5)$ and consider $F = (X^2 + Y^3, XY)$ in $\mathbb{K}[X, Y]$. The sequence F is in simultaneous Noether position w.r.t. the order $X > Y$, yet 5 is neither divisible by 3 nor by 2.

The weaker converse that if D is W-compatible, then D is W_m-compatible is also false: with the same weights and algebra, let $D = (5)$, the only polynomial with W-degree 5 is (up to scalar multiplication) $f = XY$. It is non-zero, so (f) is a regular sequence, but (f, Y) is not regular, so (f) is not in Noether position w.r.t X.

Remark. These examples lead to the following attempt at writing a general characterization of W-compatibility.

Let n be a positive integer, $W = (w_1, \dots, w_n)$ a system of weights, and $D = (d_1, \dots, d_n)$ a system of W-degrees. Further assume that

- for all $i \in \{1, \dots, n\}$, $\mathbb{K}[\mathbf{X}]_{d_i} \neq 0$
- the formal series
$$S_{D,W}(T) = \frac{\prod_{i=1}^{n}(1 - T^{d_i})}{\prod_{i=1}^{n}(1 - T^{w_i})}$$
is a polynomial.

Is D necessarily W-compatible?

The answer is *no*: take the system of weights $W = (3, 5, 11)$, and the system of W-degrees $D = (165, 19, 19)$. Note that 165 is the product of the weights, and 19 the sum of the weights. The series
$$S_{D,W}(T) = \frac{(1 - T^{165}) \cdot (1 - T^{19}) \cdot (1 - T^{19})}{(1 - T^3) \cdot (1 - T^5) \cdot (1 - T^{11})} = 1 + T^3 + \cdots + T^{184}$$
is a polynomial. But at W-degree 19, there are only 2 monomials, namely $X_1X_2X_3$ and $X_1^3X_2^2$, and they are not coprime, so we cannot form a regular sequence of W-degrees $(165, 19, 19)$.

3. Regular systems

3.1. Shape of the Hilbert series of a weighted homogeneous complete intersection

Let $W = (w_1, \dots, w_n)$ be a reverse chain-divisible system of weights such that $w_n = 1$, and let $D = (d_1, \dots, d_n)$ be a system of W-degrees, such that for any $i \in \{1, \dots, n\}$, d_i is divisible by all of the w_j's. Let $R = \mathbb{K}[X_1, \dots, X_n]$ be a polynomial algebra graded with respect to W.

We use the following notations, as found in (Moreno-Socías, 1996):

- $\delta_j = \sum_{i=1}^{j}(d_i - w_i)$;
- $\delta = \delta_n$, $\delta^* = \delta_{n-1}$;
- $\sigma = \min\left(\delta^*, \left\lfloor \frac{\delta}{2} \right\rfloor\right)$, $\sigma^* = \min\left(\delta_{n-2}, \left\lfloor \frac{\delta^*}{2} \right\rfloor\right)$;
- $\mu = \delta - 2\sigma$, $\mu^* = \delta^* - 2\sigma^*$.

Given a formal series $S(T) = \sum_{d=0}^{\infty} a_d T^d$, we also define
$$\begin{aligned}\Delta S(T) &= \sum_{d=0}^{\infty}(a_d - a_{d-1})T^d \quad \text{(with the convention } a_{-1} = 0\text{)} \\ &= (1 - T) \cdot S(T)\end{aligned}$$
and
$$\int S = \sum_{d=0}^{\infty}(a_0 + \cdots + a_d)T^d = \frac{S(T)}{1 - T}.$$

Lemma 10. *Under the above notations and assumptions, the following properties hold.*

$$\begin{cases} \delta^* > \left\lfloor \frac{\delta}{2} \right\rfloor & \iff & d_n - \delta^* \leq 0 \\ \delta^* = \left\lfloor \frac{\delta}{2} \right\rfloor & \iff & 1 \leq \quad d_n - \delta^* \leq 2 \\ \delta^* < \left\lfloor \frac{\delta}{2} \right\rfloor & \iff & 3 \leq \quad d_n - \delta^* \end{cases} \tag{7}$$

$$\sigma = \left\lfloor \frac{\delta}{2} \right\rfloor \implies \mu = \delta \bmod 2 \in \{0, 1\} \tag{8}$$

$$0 \leq \mu < d_n \tag{9}$$

$$d_{n-1} \leq d_n \implies \sigma^* + \mu^* \leq \sigma \tag{10}$$

Proof. The proof of statements (7) and (9) can be found in (Moreno-Socías, 1996, Lemma 2.1). This proof depends only on the value of w_n, and since we assume it to be 1, it is also valid in our setting. It also proves (8) as a side-result.

For the statement (10), we proceed by case disjunction on the values of σ.

- If $\sigma = \delta^*$:

$$\sigma^* + \mu^* = \delta^* - \sigma^* \leq \delta^* = \sigma.$$

- If $\sigma = \lfloor \delta/2 \rfloor$, then $\sigma = \lfloor (\delta^* + d_n - 1)/2 \rfloor$ which implies $2\sigma = \delta^* + d_n - 1 - \mu$ and $\mu = \delta \bmod 2 \in \{0, 1\}$ (from statement (8)). Now consider the possible values of σ^*:
  - if $\sigma^* = \lfloor \delta^*/2 \rfloor$, then $\mu^* = \delta^* \bmod 2$, and thus $2\sigma = 2\sigma^* + \mu^* + d_n - 1 - \mu$. It implies that $d_n - 1 - \mu + \mu^*$ is even, we shall prove that it is greater than or equal to 0.

    From statement (9), $d_n - 1 - \mu \geq 0$, so if $\mu^* = 0$, we are done. If $\mu^* = 1$, by parity $d_n - 1 - \mu$ is odd, and thus $d_n - 1 - \mu \geq 1 = \mu^*$.

    It implies that:

$$2\sigma = 2\sigma^* + \mu^* + d_n - 1 - \mu \geq 2\sigma^* + 2\mu^*;$$

  - otherwise, $\sigma^* = \delta^{**}$, and in that case

$$\sigma^* + \mu^* = \delta^* - \sigma^* = \delta^* - \delta^{**} = d_{n-1} - w_{n-1}$$

    which implies that:

$$d_n - 1 \geq \sigma^* + \mu^* \text{ (since } w_{n-1} \geq w_n \text{ and } d_{n-1} \leq d_n)$$

    and

$$\delta^* = \delta^{**} + d_{n-1} - w_{n-1} \geq \sigma^* + \mu^*.$$

    So we have:

$$\begin{aligned} 2\sigma &= \delta^* + d_n - 1 - \mu \\ &\geq \sigma^* + \mu^* + \sigma^* + \mu^* - \mu. \end{aligned}$$

    Recall that $\mu \in \{0, 1\}$, so by parity, $2\sigma \geq 2\sigma^* + 2\mu^*$, hence $\sigma \geq \sigma^* + \mu^*$. □

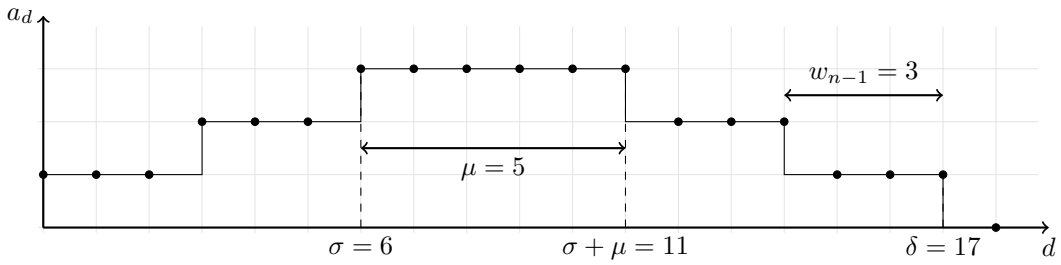

Figure 1. Shape of the Hilbert series of a W-homogeneous complete intersection for $W = (3, 3, 1)$ and $D = (9, 6, 3)$

The following theorem is a description of the shape of the Hilbert series of a zero-dimensional complete intersection. It states that it is a self-reciprocal (or palindromic) polynomial, that is a polynomial with symmetrical coefficients, and that these coefficients increase at small degrees, then station, then decrease again. Furthermore, between every strict increase, they reach a step, which has width w_{n-1}. For an example, see figure 1, where the width of the steps is 3, and the width of the central plateau is 5.

This is a generalization of a known result in the homogeneous case, which has been proved for example in (Moreno-Socías, 1996, prop. 2.2) (we will follow that proof for the weighted case). In the homogeneous case, there is no such step in the growth of the coefficients, and they are strictly increasing, then stationary, then strictly decreasing.

Theorem 11. *Let $W = (w_1, \dots, w_n)$ be a reverse chain-divisible system of weights, and $D = (d_1, \dots, d_n)$ a system of degrees such that for any $i \in \{1, \dots, n\}$, d_i is divisible by w_1. Consider the formal series*

$$S_{W,D}(T) = \frac{\prod_{i=1}^{n}(1 - T^{d_i})}{\prod_{i=1}^{n}(1 - T^{w_i})} = \sum_{d=0}^{\delta} a_d T^d$$

The series $S_{W,D}$ is a self-reciprocal polynomial in T (i.e. for any $d \leq \delta$, $a_d = a_{\delta - d}$) and its coefficients satisfy the inequalities:

$$\begin{aligned} \forall d \in \{0, \dots, \sigma - 1\}, &\quad a_d \leq a_{d+1} \\ \forall d \in \{\sigma, \dots, \sigma + \mu - 1\}, &\quad a_d = a_{d+1} \\ \forall d \in \{\sigma + \mu, \dots, \delta\}, &\quad a_d \geq a_{d+1} \end{aligned}$$

Furthermore, if $d < \sigma$ (resp. $d > \sigma + \mu$), the coefficients increase (resp. decrease) with steps, and these steps have width w_{n-1}:

$$\forall d \in \{0, \dots, \sigma - 1\},\ a_d - a_{d-1} \begin{cases} > 0 & \text{if } w_{n-1} \text{ divides } d \\ = 0 & \text{otherwise.} \end{cases}$$

Proof. We adapt the proof from (Moreno-Socías, 1996, Prop. 2.2) for the homogeneous case to the weighted case. Up to permutation of the d_i's, we can assume that for any i, $d_i \geq d_{i-1}$. We proceed by induction on n. The result for the case $n = 1$ is a consequence of the homogeneous case, since $w_n = 1$.

Let $n > 1$. Let $\overline{W}^* = (w_1/w_{n-1}, \ldots, w_{n-1}/w_{n-1})$ and $\overline{D}^* = (d_1/w_{n-1}, \ldots, d_{n-1}/w_{n-1})$, and consider the series

$$\overline{S}^* := S_{W^*,D^*} = \frac{\prod_{i=1}^{n-1}(1-T^{d_i/w_{n-1}})}{\prod_{i=1}^{n-1}(1-T^{w_i/w_{n-1}})} = \sum_{d=0}^{\delta} \overline{a}_d^* T^d.$$

The Hilbert series S can be computed from $\overline{S}^*$ with

$$S(T) = \frac{1-T^{d_n}}{1-T}\overline{S}^*(T^{w_{n-1}}) = (1-T^{d_n}) \cdot \int \overline{S}^*(T^{w_{n-1}}),$$

and so for any d, we have:

$$a_d = a^*_{d-d_n+1} + \cdots + a^*_d$$
$$a'_d := a_d - a_{d-1} = a^*_d - a^*_{d-d_n}$$

where

$$a^*_d = \begin{cases} \overline{a}^*_{\overline{d}} & \text{if } d = \overline{d} w_{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

This proves that the polynomial is self-reciprocal:

$$\begin{aligned} a_{\delta-d} &= a^*_{\delta-d-d_n+1} + \cdots + a^*_{\delta-d} \\ &= a^*_{d-d_n+1} + \cdots + a^*_d \text{ since, by induction hypothesis, } \overline{S}^* \text{ is self-reciprocal} \\ &= a_d \end{aligned}$$

To prove the properties regarding the sign of $a'_d = a_d - a_{d-1}$, we shall consider two cases, according to the value of d_n.

- If $d_n \geq \delta^* + 1$, then from statement (7) in Lemma 10, and the definition of σ and μ, $\sigma = \delta^*$ and $\sigma + \mu = d_n - 1$. Let $0 \leq d \leq \sigma$, then $d \leq \delta^* < d_n$, and thus:

$$a'_d = a^*_d = \begin{cases} \overline{a}^*_{d/w_{n-1}} > 0 & \text{if } w_{n-1} \text{ divides } d; \\ 0 & \text{otherwise.} \end{cases}$$

  Let $d \in \{\sigma+1, \ldots, \sigma+\mu\}$, that implies that $\delta^* < d \leq d_n - 1$, and thus:

$$a'_d = a^*_d = 0 \text{ (since } \delta^* \text{ is the degree of } S^*).$$

- If $d_n \leq \delta^*$, then from statement (7) again, $\sigma = \lfloor \delta/2 \rfloor$ and $\mu = \delta \bmod 2$. Let $d \leq \sigma$, we want to prove that $a_d - a_{d-1}$ is greater or equal to zero, depending on whether d is divisible by w_{n-1}. We shall consider two ranges of values for d:
  - if $d \leq \sigma^* + \mu^*$, then $d - d_n \leq \sigma^* + \mu^* - d_n < \sigma^*$ (since $\mu^* < d_n$). Recall that $a'_d = a^*_d - a^*_{d-d_n}$. By hypothesis, d_n is divisible by w_{n-1}, and so, either both d and $d - d_n$ are divisible by w_{n-1}, or both are not. Thus,

$$a'_d \begin{cases} > 0 & \text{if both } d \text{ and } d-d_n \text{ are divisible by } w_{n-1} \\ = 0 & \text{if neither } d \text{ nor } d-d_n \text{ is divisible by } w_{n-1}; \end{cases}$$

  - if $\sigma^* + \mu^* < d \leq \sigma$, then $2d \leq 2\sigma \leq \delta$; by definition, $\delta = \delta^* + d_n - 1$, so $d - d_n < \delta^* - d$; furthermore, $\delta^* - d < \delta^* - (\sigma^* + \mu^*) = \sigma^*$, so in the end:

$$d - d_n < \delta^* - d < \sigma^*.$$

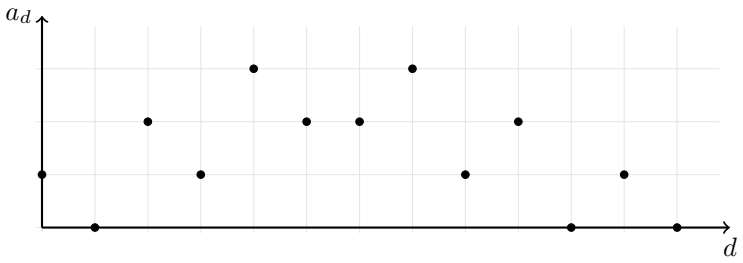

Figure 2. Hilbert series of a weighted homogeneous complete intersection with $W = (3, 2, 2)$ and $D = (6, 6, 6)$

Since by construction, δ^* is divisible by w_{n-1}, the same reasoning as before yields that

$$a'_d = a^*_d - a^*_{d-d_n} = a^*_{\delta^*-d} - a^*_{d-d_n}$$

and

$$a'_d \begin{cases} > 0 & \text{if both } \delta^* - d \text{ and } d - d_n \text{ are divisible by } w_{n-1}; \\ = 0 & \text{if neither } \delta^* - d \text{ nor } d - d_n \text{ is divisible by } w_{n-1}. \end{cases}$$

Still assuming that d_n, let now $d \in \{\sigma + 1, \ldots, \sigma + \mu\}$, we want to prove that $a_d - a_{d-1} = 0$. If $\mu = 0$ there is nothing to prove, so assume that $\mu = 1$ and $d = \sigma + 1$. But then $\sigma + 1 - d_n = \delta - \sigma - d_n = \delta^* - \sigma$, and so by symmetry, $a'_d = a^*_{\sigma+1} - a^*_{\sigma+1-d_n} = 0$. □

Remark. The hypothesis that the weights are reverse chain-divisible is necessary. As a counter-example, let $W = (3, 2, 2)$ and $D = (6, 6, 6)$. Then the Hilbert series of a complete intersection of W-degree D is illustrated in Figure 2. It is self-reciprocal, but the coefficients do not vary as predicted by Theorem 11.

The hypothesis that each of the W-degrees should be divisible by w_1 is also necessary. As a counter-example, let $W = (4, 2, 1)$ and $D = (8, 8, 2)$. Then the Hilbert series of a complete intersection of W-degree D is illustrated in Figure 3: the width of the steps is greater than w_{n-1}. Furthermore, following the proof, the parameters for this series should be defined by $\sigma = \lfloor \delta/2 \rfloor$ and $\mu = \delta \bmod 2$, where $\delta = 11$, so that $\sigma = 5$ and $\mu = 1$. However, we cannot reorder the degrees such that $d_3 \geq d_2 \geq d_1$, and we cannot deduce from statement (10) in Lemma 10 that $\sigma^* + \mu^* \leq \sigma$: indeed, we have $\sigma = 4$ but $\sigma^* + \mu^* = 6$.

However, the fact that the Hilbert series is self-reciprocal for complete intersections is true even for general system of weights, and is a consequence of the Gorenstein property of complete intersections (see (Eisenbud, 1995, Chap. 21); this property is also central to the proof of Theorem 9).

3.2. Degree of regularity of a weighted homogeneous complete intersection

The degree of regularity of a zero-dimensional homogeneous regular system is bounded by Macaulay's bound

$$d_{\text{reg}} \leq \sum_{i=1}^{n} (d_i - 1) + 1,$$

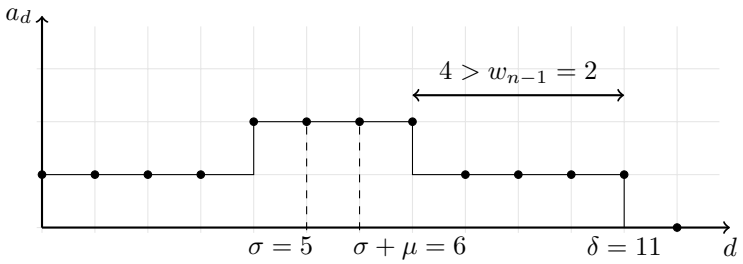

Figure 3. Hilbert series of a weighted homogeneous complete intersection with $W = (4, 2, 1)$ and $D = (8, 8, 2)$

and, in practice, that bound is reached for generic systems. The proof of this result uses the degree of the Hilbert series of the system. However, in the weighted case, the best result we can obtain from the degree of the Hilbert series is (Faugère et al., 2013):

$$d_{\mathrm{reg},W} \leq \sum_{i=1}^{n}(d_i - w_i) + \max\{w_j\}, \tag{11}$$

and this bound is not sharp in general. In particular, it appears that this degree of regularity depends on the order we set on the variables.

The following theorem is an improvement over the previous bound, under the additional assumption that the system is in simultaneous Noether position. Recall that this property is generic, and that for reverse chain-divisible systems of weights, it is always true for regular sequences, up to a weighted homogeneous change of coordinates.

Theorem 12. *Let $W = (w_1, \ldots, w_n)$ be a (not necessarily reverse chain-divisible) system of weights and $D = (d_1, \ldots, d_n)$ be a strongly W-compatible system of W-degrees. Further assume that for any $j \in \{2, \ldots, n\}$, $d_j \geq w_{j-1}$. Let $F = (f_1, \ldots, f_n)$ be a system of W-homogeneous polynomials, with W-degree D, and assume that F is in simultaneous Noether position for the variable ordering $X_1 > X_2 > \cdots > X_n$. Then the W-degree of regularity of F is bounded by*

$$d_{\mathrm{reg},W}(F) \leq \sum_{i=1}^{n}(d_i - w_i) + w_n. \tag{12}$$

Proof. We prove this by induction on n. If $n = 1$, we simply have one W-homogeneous polynomial to consider, and so $d_{\mathrm{reg},W} = d_1$.

So assume that $n > 1$. We consider the system F^* defined by:

$$F^* = \big(f_1(X_1, \ldots, X_{n-1}, 0), \ldots, f_{n-1}(X_{n-1}, \ldots, X_{n-1}, 0)\big).$$

This system is W^*-homogeneous, for $W^* := (w_1, \ldots, w_{n-1})$. From the characterization NP3 of Noether position, the sequence F^* is in simultaneous Noether position. As a consequence, the induction hypothesis applies to F^*, and the W^*-degree of regularity of F^* is bounded by

$$d_{\mathrm{reg},W^*}(F^*) \leq \sum_{i=1}^{n-1}(d_i - w_i) + w_{n-1}.$$

Denote by δ the degree of the Hilbert series of F, that is $\delta = \sum_{i=1}^{n}(d_i - w_i)$. We want to prove that $d_{\text{reg}} \leq \delta + w_n$, i.e. that the Gröbner basis of F need not contain any polynomial with W-degree greater than $\delta + w_n$. Equivalently, let μ be a monomial with W-degree $d > \delta + w_n$, we will prove that μ is strictly divisible by a monomial in the initial ideal generated by F.

Write $\mu = X_n^{\alpha} \cdot \mu'$, with $\mu' \in \mathbb{K}[X_1, \ldots, X_{n-1}]$, and proceed by induction on α:

- if $\alpha = 0$, then $\mu \in \mathbb{K}[X_1, \ldots, X_{n-1}]$. By assumption, $d_n \geq w_{n-1}$, hence:

$$\delta + w_n = \delta^* + w_{n-1} - w_{n-1} + d_n - w_n + w_n \geq d^*_{\text{reg}} + d_n - w_{n-1} \geq d^*_{\text{reg}}; \quad (13)$$

  so μ has W-degree greater than d^*_{reg}, and by induction hypothesis, μ is strictly divisible by a monomial in the initial ideal generated by F^*;
- if $\alpha > 0$, then consider $\mu'' = X_n^{\alpha-1}\mu'$, it is a strict divisor of μ. Furthermore, since $\deg(\mu) > \delta + w_n$, $\deg(\mu'') = \deg(\mu) - w_n > \delta$. Recall that δ is by definition the degree of the Hilbert series of the ideal generated by F, so μ lies in that ideal. □

Remark. The hypothesis stating that for any i, $d_i \geq w_{i-1}$ is necessary. For example, let $W = (2, 1)$, $D = (2, 1)$ and the system $F = (X, Y)$ in $\mathbb{K}[X, Y]$, it is W-homogeneous with W-degree D and in simultaneous Noether position. This system has Hilbert series 1 (the quotient vector span is generated by $\{1\}$), which has degree $\delta = 0$. But the Gröbner basis of the system is given by F itself, and contains X, with W-degree 2.

More generally, without that hypothesis, we obtain the following bound for $d_{\text{reg},W}(F)$:

$$d_{\text{reg},W}(F) \leq \max\left\{\sum_{i=1}^{k}(d_i - w_i) + w_k : k \in \{1, \ldots, n\}\right\},$$

and the proof is the same as that of Theorem 12, with the weaker induction hypothesis that $d_{\text{reg},W}(F) \leq \max\left(\delta + w_n, d_{\text{reg},W^*}(F^*)\right)$, which does not need inequality (13).

Remark. We give examples of the behavior of both bounds in Table 1: we give the degree of regularity of a generic W-homogeneous system of W-degree D, and show how this degree of regularity varies if we change the order of the weights W.

Remark. Theorem 12 gives an indication as to how to choose the order of the variables. Generically, in order to compute a W-GRevLex Gröbner basis of the system, the complexity estimates will be better if we set the variables in decreasing weight order.

While the new bound (12) is not sharp in full generality, it is sharp whenever $w_n = 1$. We conjecture that the sharp formula is the following.

Conjecture 13. Let $W = (w_1, \ldots, w_n)$ be a system of weights, and $D = (d_1, \ldots, d_n)$ a strongly W-compatible system of W-degrees. Let $F \in \mathbb{K}[X_1, \ldots, X_n]$ be a generic system of W-homogeneous polynomials. Let $\delta = \sum_{i=1}^{n}(d_i - w_i)$ be the degree of the Hilbert series of $\langle F \rangle$, and let d_0 be defined as

$$d_0 = \begin{cases} \delta + 1 & \text{if there exists } i \text{ such that } w_i = 1 \\ \delta - g & \text{otherwise ,} \end{cases}$$

where g is the Frobenius number of W (that is, the greatest W-degree at which the set of monomials is empty). In other words, d_0 is the degree of the first "unexpected"

W	D	d_{reg}	Bound (11)	Bound (12)
$(3,2,1)$	$(6,6,6)$	13	15	13
$(3,1,2)$	$(6,6,6)$	14	15	14
$(1,2,3)$	$(6,6,6)$	15	15	15

Table 1. Macaulay's bound on the degree of regularity of generic weighted homogeneous systems

zero coefficient in the Hilbert series (by definition of the degree in the first case, and by self-reciprocality of the Hilbert series in the second case).

Then the degree of regularity of F is the first multiple of w_n greater than d_0:

$$d_{\text{reg}} = w_n \left\lceil \frac{d_0}{w_n} \right\rceil . \tag{14}$$

4. Semi-regular systems

The study of systems with m equations and n unknowns, when $m \leq n$, is reduced to the (generic) case of regular sequences, sequences in Noether position or sequences in simultaneous Noether position.

However, it is frequent in some applications that polynomial systems arise with more equations than unknowns. Experimentally, this usually makes the resolution faster. In the homogeneous case, this has been studied through the notion of *semi-regularity*. This property extends the regularity to the overdetermined case. Fröberg's conjecture (Fröberg, 1985) states that this property is generic, and as of today, it is only known in a handful of cases (see (Moreno-Socías, 1991, Thm. 1.5) for a survey).

In this section, we give a definition of semi-regularity in the weighted case, and some consequences on the Hilbert series and the degree of regularity of the system. Additionally, we show that Fröberg's conjecture is true if $m = n + 1$, as in the homogeneous case.

4.1. Definitions and notations

Let n and m be two integers, $m \geq n$, $W = (w_1, \ldots, w_n)$ a system of weights, and $D = (d_1, \ldots, d_n)$ a system of W-degrees. Let $F = (f_1, \ldots, f_m)$ be a system of W-homogeneous polynomials with W-degree D. For any $i \in \{i, \ldots, n\}$, write $F_i = (f_1, \ldots, f_i)$.

Definition 14 (Semi-regularity)**.** We say that F is *semi-regular* if, for any $i \in \{1, \ldots, m\}$ and for any $d \in \mathbb{N}$, the linear map given by the multiplication by f_i:

$$s_{i,d} : \left(\mathbb{K}[X_1, \ldots, X_n]/\langle F_{i-1}\rangle\right)_d \xrightarrow{\cdot f_i} \left(\mathbb{K}[X_1, \ldots, X_n]/\langle F_{i-1}\rangle\right)_{d+d_i}$$

is full-rank (either injective or surjective).

Furthermore, let

$$S_{D,W}(T) = \frac{\prod_{i=1}^{m}(1 - T^{d_i})}{\prod_{i=1}^{n}(1 - T^{w_i})} = \sum_{d=0}^{\infty} a_d T^d.$$

We say that F *has a semi-regular Hilbert series* if the Hilbert series of F is equal to $\lfloor S_{D,W}(T) \rfloor$, that is the series truncated at the first coefficient less than or equal to zero.

The motivation behind these definitions is given by the following classical result in the homogeneous case (see for example (Pardue, 2010, prop. 1)):

Proposition 6. *If $W = (1, \dots, 1)$, the following conditions are equivalent:*
(1) the system F is semi-regular;
(2) for any $1 \leq i \leq n$, the system F_i has a semi-regular Hilbert series.

For weighted homogeneous systems, the converse implication ($2 \implies 1$) is still true:

Proposition 7. *Let F be a W-homogeneous system such that, for any $1 \leq i \leq n$, the system F_i has a semi-regular Hilbert series. Then F is semi-regular.*

Proof. We prove this by induction on the number m of polynomials. The initial case is $m = n$, and it is a direct consequence of the characterization of a regular sequence.

Assume $m > n$. Write $R^* = \mathbb{K}[X_1, \dots, X_n]/\langle f_1, \dots, f_{m-1}\rangle$, and for any $d \in \mathbb{N}$, consider the multiplication map

$$s_{m,d} = R^*_d \xrightarrow{\cdot f_m} R^*_{d+d_m}$$

Let $K_{m,d} = \ker(s_{m,d})$. Write $S(T)$ the Hilbert series of F, a_d its coefficient at degree d, δ its degree, $H(T) = \left(\prod_{i=1}^{m}(1-T^{d_i})\right) / \left(\prod_{i=1}^{n}(1-T^{w_i})\right)$, b_d its coefficient at degree d, and $S^*(T)$, a^*_d, δ^*, $H^*(T)$ and b^*_d their counterparts with $m-1$ polynomials. We know, from the exact sequence

$$0 \longrightarrow K_{m,d} \longrightarrow R^*_d \xrightarrow{s_{m,d}} R^*_{d+d_m} \longrightarrow R_{d+d_m} \longrightarrow 0$$

that the following identity holds

$$a_{d+d_m} = a^*_{d+d_m} - a^*_d + \dim(K_{m,d}).$$

We want to prove that either $a_{d+d_m} = 0$ or $\dim(K_{m,d}) = 0$. Assume that $a_{d+d_m} > 0$, that means that $d + d_m \leq \delta$ and $a_{d+d_m} = b_{d+d_m}$, so:

$$\begin{aligned} a_{d+d_m} &= a_{d+d_m} - a^*_d + \dim(K_{m,d}) \\ &= b_{d+d_m} \\ &= b^*_{d+d_m} - b^*_d \text{ by definition of } H(T) \\ &= a^*_{d+d_m} - a^*_d \text{ since } \delta^* \geq \delta. \end{aligned}$$

Thus we have $K_{m,d} = 0$. □

4.2. Hilbert series of a semi-regular sequence

In this section, we prove that for reverse chain-divisible systems of weights, semi-regular sequences have a semi-regular Hilbert series. First, we characterize the shape of semi-regular Hilbert series, by extending Theorem 11 to the overdetermined case.

Theorem 15. *Let $m \geq n \geq 0$ be two integers. Let $W = (w_1, \dots, w_n)$ be a reverse chain-divisible system of weights, and let $D = (d_1, \dots, d_m)$ be a system of W-degrees such that $d_1, \dots, d_n$ are all divisible by w_1. Write*

$$S_{D,W}(T) = \frac{\prod_{i=1}^{m}(1-T^{d_i})}{\prod_{i=1}^{n}(1-T^{w_i})} = \sum_{d=0}^{\infty} a_d T^d.$$

Then there exist W-degrees σ, δ such that

$$\forall\, d \in \{1, \dots, \sigma\},\ a_d \geq a_{d-1} \tag{σ1}$$
$$a_\sigma > a_{\sigma-1} \tag{σ2}$$
$$\forall\, d \in \{\sigma+1, \dots, \delta\},\ a_d \leq a_{d-1} \tag{σ3}$$
$$a_\delta > 0,\ a_{\delta+1} \leq 0. \tag{δ1}$$

Furthermore, if $m > n$, let $D^* = (d_1, \dots, d_{m-1})$ and define δ^* as above for the series $S_{D^*,W}$. Then the following statements hold:

$$\begin{cases} \forall\, d \in \{\delta+1, \dots, \delta^*\},\ a_d \leq 0 & \text{if } n = 0 \\ \forall\, d \in \{\delta+1, \dots, \delta^* + d_m\},\ a_d \leq 0 & \text{if } n > 0. \end{cases} \tag{δ2}$$

If $n > 0$, let $W^* = (w_1, \dots, w_{n-1})$, and let δ' be the degree of $\lfloor S_{D,W^*}(T) \rfloor$. If $n = 0$, let $\delta' = 0$. Then the following equality holds

$$\sigma = \delta'. \tag{σ4}$$

Proof. We prove the theorem by induction on n, and for any given n, by induction over m. The base cases are:

- $n = 0$, $m \geq 0$: then $S_{D,W}(T) = 1 - a_k T^k + O(T^{k+1})$ with $a_k > 0$, and we can conclude, taking $\delta = 0$ and $\sigma = 0$.
- $n = m > 0$: then this is a consequence of Theorem 11 (shape of the Hilbert series of a complete intersection).

Assume that $m > n > 0$. Let $D^* = (d_1, \dots, d_{m-1})$, $W^* = (w_1, \dots, w_{n-1})$, and write:

$$S(T) := S_{D,W}(T) = \sum_{d=0}^{\infty} a_d T^d;$$
$$S^*(T) := S_{D^*,W}(T) = \sum_{d=0}^{\infty} a^*_d T^d.$$

The derivatives of these series are

$$\Delta S(T) = S_{D,W^*}(T) = \sum_{d=0}^{\infty} a'_d T^d;$$
$$\Delta S^*(T) = S_{D^*,W^*}(T) = \sum_{d=0}^{\infty} a'^*_d T^d.$$

Furthermore, let $w = w_{n-1}$, $\overline{W^*} = (w_1/w, \dots, w_{n-1}/w)$ and $\overline{D^*} = (d_1/w, \dots, d_{n-1}/w)$, and consider the series

$$\overline{\Delta S}(T) = S_{\overline{D},\overline{W^*}}(T) = \sum_{d=0}^{\infty} \overline{a'} d T^d;$$
$$\overline{\Delta S^*}(T) = S_{\overline{D^*},\overline{W^*}}(T) \sum_{d=0}^{\infty} \overline{a'^*} d T^d.$$

In particular,

$$\Delta S(T) = \overline{\Delta S}(T^w) \text{ and } \Delta S^*(T) = \overline{\Delta S^*}(T^w).$$

All the series S^*, $\overline{\Delta S}$ and $\overline{\Delta S^*}$ satisfy the induction hypothesis. The W-degrees for which the coefficients of S^* satisfy properties ($\sigma 1$)-($\sigma 4$) and ($\delta 1$)-($\delta 2$) are denoted by σ^* and δ^*. We write $\overline{\sigma'}$, $\overline{\delta'}$, $\overline{\sigma'^*}$, $\overline{\delta'^*}$ the respective values of the W-degrees for which these properties apply to $\overline{\Delta S}$ and $\overline{\Delta S^*}$.

From $S(T) = (1 - T^{d_m})S^*(T)$, we deduce the recurrence relation

$$a_d = a^*_d - a^*_{d-d_m}.$$

Since S^* satisfies the induction hypothesis, we know that there exists a degree δ such that

$$\begin{cases} \forall\, d \in \{0, \dots, \delta\} & a^*_d > a^*_{d-d_m} \\ \forall\, d \in \{\delta + 1, \dots, \delta^* + d_m\} & a^*_d \leq a^*_{d-d_m}. \end{cases}$$

This proves statements ($\delta 1$) and ($\delta 2$). As a side result, since $a^*_\delta > a^*_{\delta - d_m}$, we also deduce that

$$\delta - d_m < \sigma^*. \tag{15}$$

Let $\sigma = \delta'$, we prove that it satisfies equations ($\sigma 1$), ($\sigma 2$) and ($\sigma 3$). We need to evaluate the sign of $a_d - a_{d-1}$, depending on d. The generating series of $a_d - a_{d-1}$ is:

$$(1 - T)S(T) = (1 - T) \cdot \frac{\prod_{i=1}^{m}(1 - T^{d_i})}{\prod_{i=1}^{n}(1 - T^{w_i})} = \Delta S(T) \text{ since } w_n = 1.$$

In other words, $a_d \geq a_{d-1}$ if and only if $a'_d \geq 0$, which proves statements ($\sigma 1$) and ($\sigma 2$), by definition of δ':

$$\begin{gathered} \forall\, d \in \{0, \dots, \sigma\},\ a_d - a_{d-1} = a'_d \geq 0 \\ a_\sigma - a_{\sigma-1} = a'_\sigma = a'_{\delta'} > 0. \end{gathered}$$

To finish the proof, we need to prove that for any $d \in \{\delta' + 1, \dots, \delta\}$, $a'_d \leq 0$.

From the induction hypothesis (statement ($\sigma 4$)) applied to S^*, we know that $\delta'^* = \sigma^*$. Moreover, statement ($\delta 2$) from the induction hypothesis applied to $\overline{\Delta S}$ yields that:

$$\forall\, \overline{d} \in \{\overline{\delta'} + 1, \dots, \overline{\delta'^*}\},\ \overline{a'_d} \leq 0.$$

As a consequence, since $\sigma^* = \delta'^* = w\overline{\delta'^*}$:

$$\forall\, d \in \{\delta' + 1, \dots, \sigma^*\},\ a'_d = \begin{cases} \overline{a'_{\overline{d}}} \leq 0 & \text{if } d = w\overline{d}; \\ 0 & \text{otherwise.} \end{cases} \tag{i}$$

Now assume that $\sigma^* < d \leq \delta$. We can write a'_d as

$$\begin{aligned} a'_d = a_d - a_{d-1} &= a^*_d - a^*_{d-d_m} - a^*_{d-1} + a^*_{d-d_m-1} \\ &= (a^*_d - a^*_{d-1}) - (a^*_{d-d_m} - a^*_{d-d_m-1}) = a'^*_d - a'^*_{d-d_m}. \end{aligned}$$

Since $a_d \leq a^*_d$ for any d, we necessarily have $\delta \leq \delta^*$, hence $\sigma^* < d \leq \delta^*$. So by induction hypothesis (statement ($\sigma 3$)), we know that $a^*_d - a^*_{d-1} \leq 0$. Additionally, equation (15) and induction hypothesis (statement ($\sigma 1$)) together yield that $a^*_{d-d_m} - a^*_{d-d_m-1} \geq 0$, so we conclude that

$$\forall\, d \in \{\sigma^* + 1, \dots, \delta\},\ a'_d \leq 0. \tag{ii}$$

And so, sticking the ranges of statements (i) and (ii) together, we prove statement ($\sigma 3$) which completes the proof. □

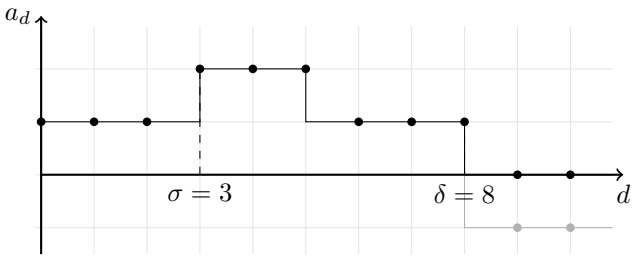

Figure 4. Shape of the Hilbert series of a semi-regular W-homogeneous sequence with $W = (3, 3, 1)$ and $D = (12, 9, 6, 6, 3)$

Using this description of semi-regular Hilbert series, we now prove that for reverse chain-divisible systems of weights, semi-regular sequences have a semi-regular Hilbert series. As an illustration, Figure 4 shows the coefficient of a semi-regular Hilbert series. The black dots correspond to the actual coefficients, and the gray dots are the coefficients which were truncated away.

Theorem 16. *Let $m \geq n \geq 0$ be two integers, $W = (w_1, \ldots, w_n)$ be a reverse chain-divisible system of weights and $D = (d_1, \ldots, d_m)$ be a system of W-degrees such that $d_1, \ldots, d_n$ are all divisible by w_1. Let $F = (f_1, \ldots, f_m)$ be a system of W-homogeneous polynomials, with respective W-degree D. If F is a semi-regular sequence, then F has a semi-regular Hilbert series.*

Proof. We proceed by induction on m. If $m = n$, then the result is a consequence of the characterization of regular sequences.

Assume that $m > n$. We consider the series $S(T) = S_{D,W}(T)$ with generic coefficient a_d, $S^*(T) = S_{D^*,W}(T)$ with generic coefficient a^*_d, $H(T)$ the Hilbert series of F with generic coefficient b_d, and $H^*(T)$ the Hilbert series of $F^* := (f_1, \ldots, f_{m-1})$ with generic coefficient b^*_d. By induction hypothesis, $H^*(T) = \lfloor S^*(T) \rfloor$. And since F is semi-regular, we have the exact sequence

$$0 \longrightarrow K_{m,d} \longrightarrow R^*_d \xrightarrow{s_{m,d}} R^*_{d+d_m} \longrightarrow R_{d+d_m} \longrightarrow 0$$

where $R = \mathbb{K}[X_1, \ldots, X_n]/\langle F \rangle$ and $R^* = \mathbb{K}[X_1, \ldots, X_n]/\langle F^* \rangle$. As a consequence, for any $d \geq 0$, the coefficient b_d satisfies the recurrence relation:

$$b_{d+d_m} = b^*_{d+d_m} - b^*_d + \dim(K_{m,d})$$

where either $K_{m,d} = 0$ or $b_{d+d_m} = 0$. Since $s_{m,d}$ is defined from a space of dimension b^*_d to a space of dimension $b^*_{d+d_m}$, this can be rephrased as

$$b_d = \max\left(b^*_d - b^*_{d-d_m}, 0\right).$$

From Theorem 15 applied to $S(T)$, there exists δ such that

$$\forall d \in \{0, \ldots, \delta\},\ a_d = a^*_d - a^*_{d-d_m} > 0.$$

Furthermore, the induction hypothesis shows that there exists a degree δ^* such that

$$\begin{aligned} \forall d \in \{0, \dots, \delta^*\},\ a^*_d = b^*_d > 0 \\ \forall d > \delta^*, b^*_d = 0, \end{aligned}$$

and that δ^* is defined as in Theorem 15. In particular, it implies that $\delta^* \geq \delta$.

We shall prove that the Hilbert series H of F is equal to S, truncated at degree δ. Let $d \geq 0$:

- if $0 \leq d \leq \delta \leq \delta^*$:

$$\begin{aligned} b_d &= b^*_d - b^*_{d_m} \text{ since } d \leq \delta \\ &= a^*_d - a^*_{d_m} \text{ since } d \leq \delta^* \\ &= a_d \end{aligned}$$

- if $\delta < d$:

$$\begin{aligned} b_d &= \max\left(b^*_d - b^*_{d-d_m}, 0\right) \\ &= 0 \text{ since } b^*_d = 0 \text{ and } b^*_{d-d_m} \geq 0 \end{aligned}$$

And since $a_{\delta+1} \leq 0$, this proves that

$$H(T) = \lfloor S(T) \rfloor. \qquad \square$$

Another consequence of Theorem 15 is an explicit value for the degree δ of the Hilbert series of an ideal defined by a semi-regular sequence with $m = n + 1$ polynomials in n variables. In the homogeneous case, it is known that this degree is bounded by

$$\delta = \min\left(\sum_{i=1}^{n} d_i - n, \left\lfloor \frac{\sum_{i=1}^{n+1} d_i - n}{2} \right\rfloor\right).$$

Proposition 8. *Let n be a positive integer, and $m = n + 1$. Let $W = (w_1, \dots, w_n)$ be a system of weights, and $F = (f_1, \dots, f_m)$ a system of W-homogeneous polynomials, and assume that the hypotheses of Theorem 16 are satisfied. For all $i \in \{1, \dots, m\}$, let $d_i := \deg_W(f_i)$. Then the degree δ of the Hilbert series of $\langle F \rangle$ is given by:*

$$\delta = \min\left(\sum_{i=1}^{n} d_i - \sum_{i=1}^{n} w_i, \left\lfloor \frac{\sum_{i=1}^{n+1} d_i - \sum_{i=1}^{n} w_i}{2} \right\rfloor\right).$$

Proof. Consider the system of weights $W^+ = (w_1, \dots, w_n, 1)$, and the series S_{D,W^+} as defined in Theorem 15. It satisfies the hypotheses of Theorem 11, which implies that its coefficients are increasing up to degree

$$\sigma^+ = \min\left(\sum_{i=1}^{n} d_i - \sum_{i=1}^{n} w_i, \left\lfloor \frac{\sum_{i=1}^{n+1} d_i - \sum_{i=1}^{n} w_i}{2} \right\rfloor\right).$$

Theorem 15 (statement ($\sigma 4$)) states that the degree δ of the Hilbert series of $\langle F \rangle$ satisfies

$$\delta = \sigma^+,$$

hence the result. $\square$

4.3. Asymptotic analysis of the degree of regularity of weighted homogeneous semi-regular sequences

In this section, we show how the results from Bardet et al. (2005) and (Bardet, 2004, Chap. 4) about the degree of regularity of semi-regular homogeneous sequences can be adapted to the weighted case.

Theorem 17. *Let k, and n be non-negative integers and let $m := n + k$. Let w_0 and d_0 be non-negative integers such that $w_0 \mid d_0$. Consider the system of n weights $W = (w_0, \ldots, w_0, 1)$ and the system of m W-degrees $D = (d_0, \ldots, d_0)$. Let $F = (f_1, \ldots, f_m) \subset A = \mathbb{K}[X_1, \ldots, X_n]$ be a semi-regular sequence of weighted homogeneous polynomials with W-degree D. Then the asymptotic developement of the W-degree of regularity d_{reg} of F as n tends to infinity is given by*

$$d_{\text{reg}} = n\left(\frac{d_0 - w_0}{2}\right) - \alpha_k \sqrt{n\left(\frac{d_0^2 - w_0^2}{6}\right)} + \mathrm{O}\left(n^{1/4}\right).$$

Remark. In the non-weighted case, this asymptotic developement is

$$d_{\text{reg}} = n\left(\frac{d_0 - 1}{2}\right) - \alpha_k \sqrt{n\left(\frac{d_0^2 - 1}{6}\right)} + \mathrm{O}\left(n^{1/4}\right).$$

Overall, the bound is improved by $\mathrm{O}\left(nw_0\right) = \mathrm{O}\left(\sum w_i\right)$.

Proof. Let $I := \langle F \rangle$, the Hilbert series of A/I is given by

$$\mathsf{HS}_{A/I}(T) = \left\lfloor \frac{\left(1 - T^{d_0}\right)^m}{\left(1 - T^{w_0}\right)^{n-1}(1 - T)} \right\rfloor.$$

Write

$$H(T) = \frac{\left(1 - T^{d_0}\right)^m}{\left(1 - T^{w_0}\right)^{n-1}(1 - T)} = \sum_{d=0}^{\delta} a_d T^d;$$
$$H^*(T) = \frac{\left(1 - T^{d_0/w_0}\right)^{m-1}}{\left(1 - T\right)^{n-1}} = \sum_{d=0}^{\delta} a_d^* T^d,$$

these series are related through

$$H(T) = H^*(T^{w_0}) \cdot \frac{1 - T^{d_0}}{1 - T} = H^*(T^{w_0}) \cdot \left(1 + T + \cdots + T^{d_0 - 1}\right).$$

For the coefficients, it means that, for any d in $\mathbb{N}$:

$$a_d = a^*_{\lfloor d/w_0 \rfloor} + \cdots + a^*_{\lceil (d - d_0 + 1)/w_0 \rceil}$$

The series H^*, if truncated before its first non-positive coefficient, is the Hilbert series of a semi-regular $\mathbb{1}$-homogeneous sequence of $m - 1$ polynomials in $n - 1$ variables, with degree d_0/w_0. Let δ^* be the degree of this truncated series, so that $\delta^* + 1$ is an upper bound for the degree of regularity of such a sequence.

Statement ($\delta 2$) of Theorem 15 states that:

$$\forall\, d \in \{\delta^* + 1, \ldots, \delta^* + d_0/w_0\},\ a^*_d \leq 0.$$

Let $\delta_0 := w_0\delta^* + d_0$, we have

$$\delta^* < \frac{\delta_0 - d_0 + 1}{w_0} \leq \delta^* + 1$$

and

$$\frac{\delta_0}{w_0} = \delta^* + \frac{d_0}{w_0},$$

and as a consequence

$$a_{\delta^0} = a^*_{\lfloor \delta_0/w_0 \rfloor} + \cdots + a^*_{\lceil (\delta_0 - d_0 + 1)/w_0 \rceil} \leq 0.$$

In other words, the degree of regularity d_{reg} of F is bounded by

$$w_0\delta^* < d_{\mathrm{reg}} \leq \delta_0 = w_0\delta^* + d_0.$$

The degree δ^* is the degree of the Hilbert series of a homogeneous semi-regular sequence, and as such, it follows the asymptotic estimates proved in (Bardet, 2004, Chap. 4). For example, in our setting where k is an integer and $m = n+k$, the asymptotic developement of δ^* when n tends to infinity is given by

$$\delta^* + 1 = n\,\frac{d_0/w_0 - 1}{2} - \alpha_k \sqrt{n\,\frac{(d_0/w_0)^2 - 1}{6}} + \mathrm{O}\left(n^{1/4}\right)$$

where α_k is the largest root of the k'th Hermite's polynomial. [2]

As a consequence,

$$\begin{aligned} d_{\mathrm{reg}} &= w_0\delta^* + \mathrm{O}\,(1) \\ &= w_0 \left(n\,\frac{d_0/w_0 - 1}{2} - \alpha_k \sqrt{n\,\frac{(d_0/w_0)^2 - 1}{6}} + \mathrm{O}\left(n^{1/4}\right) \right) + \mathrm{O}\,(1) \\ &= n\,\frac{d_0 - w_0}{2} - \alpha_k \sqrt{n\,\frac{d_0^2 - w_0^2}{6}} + \mathrm{O}\left(n^{1/4}\right). \qquad \square \end{aligned}$$

4.4. Fröberg's conjecture

Fröberg's conjecture states that homogeneous semi-regular sequences are generic among sequences of fixed degree. The fact that semi-regularity is a Zariski-open condition is a known fact (the proof is the same as for regularity), so the conjecture states that for any system of degrees, there exists a semi-regular homogeneous sequence with these degrees.

This conjecture extends naturally to the weighted case. In this case, semi-regularity is still a Zariski-open condition.

We extend here one known result from the homogeneous case (see for example Reid et al. (1991)), stating that Fröberg's conjecture is true in characteristic 0 if $m = n + 1$. We follow the proof given in Reid et al. (1991).

[2] In (Bardet, 2004, Chap. 4), the remainder $\mathrm{O}\left(n^{1/4}\right)$ was written as $\mathrm{o}\left(\sqrt{n}\right)$. However, it appears that in the proof, this $\mathrm{o}\left(\sqrt{n}\right)$ is a rewriting of $\sqrt{n} \cdot \mathrm{O}\left(\sqrt{\Delta z}\right)$, where $\Delta z = \mathrm{O}\left(1/\sqrt{n}\right)$.

Proposition 9. *Let $m = n+1$, $W = (w_1, \ldots, w_n)$ a reverse chain-divisible system of weights, $D = (d_1, \ldots, d_n)$ a strongly W-compatible system of degrees, and d_{n+1} an integer divisible by w_1. Write $f_{n+1} = (X_1 + X_2^{w_1/w_2} + \cdots + X_n^{w_1})^{d_{n+1}/w_1}$, then the sequence $F = (X_1^{d_1/w_1}, \ldots, X_n^{d_n/w_n}, f_{n+1})$ is semi-regular.*

Lemma 18. *Let f be a polynomial such that*

$$f \cdot f_{n+1} = 0 \text{ in } A = \mathbb{K}[X_1, \ldots, X_n]/(X_1^{d_1/w_1}, \ldots, X_n^{d_n/w_n}).$$

Let $\delta = \sum_{i=1}^{n}(d_i - w_i)$, then we have

$$\deg_W(f) \geq \frac{\delta - d_{n+1} + 1}{2}.$$

Proof. If the W-degree of f is 0, that means that $(X_1 + X_2^{w_1/w_2} + \cdots + X_n^{w_1})^{d_{n+1}/w_1} = 0$ in A. Assume that $\deg_W(f) < (\delta - d_{n+1} + 1)/2$, that means that $\delta - d_{n+1} + 1 \geq 1$, so $\delta \geq d_{n+1}$. Consider the expansion of f_{n+1}, all coefficients are nonzero since the base field has characteristic 0. Its support is the set of monomials of degree d_{n+1}. Since $d_{n+1} \leq \delta$, $\dim\left(\mathbb{K}[\mathbf{X}]/\langle f_1, \ldots, f_n\rangle\right)_{d_{n+1}} > 0$, which means that there exists at least one monomial with W-degree d_{n+1} which does not lie in the initial ideal of $\langle f_1, \ldots, f_n\rangle$. As a consequence, f is non-zero in the quotient.

Now assume that $\deg_W(f) > 0$. Write $B = \mathbb{K}[X_2, \ldots, X_n]/(X_2^{d_2/w_2}, \ldots, X_n^{d_n/w_n})$, $X = X_1$, $R = B[X]$, $d = d_1/w_1$, such that $A = R/X^d$. Let $S = (X + X_2^{w_1/w_2} + \cdots + X_n^{w_1})$, and let F be a weighted homogeneous polynomial in R with image f in A. By assumption, there exists $G \in R$ such that $S^{d_{n+1}/w_1} \cdot F = G \cdot X^d$. Derive this equality along X to obtain:

$$mS'S^{d_{n+1}/w_1 - 1}F + S^{(d_{n+1})/w_1}F = dG'X^{d-1} + G'X^d$$

or, modulo X^{d-1}

$$\begin{aligned} S^{d_{n+1}-1}(mF + SF') \equiv 0 \bmod X^{d-1} &\implies S^{d_{n+1}/w_1}(mF + SF') \equiv 0 \bmod X^{d-1} \\ &\implies S^{d_{n+1}/w_1 + 1}F' \equiv mFS^{d_{n+1}/w_1} \equiv 0 \bmod X^{d-1} \end{aligned}$$

Since $X = X_1$ has weight w_1, F' is W-homogeneous with W-degree $\deg_W(f) - w_1$, and we can use the induction hypothesis on $F' \bmod X \in A$ and $\deg(F) = d_{n+1} + w_1$ to deduce:

$$\begin{aligned} \deg_W(f) = \deg_W(F) &= \deg_W(F') + 1 \\ &\geq \frac{(\delta - 1) - (d_{n+1} + 1) + 1}{2} + 1 \\ &\geq \frac{\delta - d_{n+1} + 1}{2}. \end{aligned}$$

□

Proof of the proposition. The proof given in (Reid et al., 1991, before prop. 7) still holds in the weighted case. □

5. Taking into account a weighted homogeneous structure when computing Gröbner bases

5.1. Weighted homogeneous systems

Let n, m be two integers, let $W = (w_1, \ldots, w_n)$ be a system of weights, and let $F = (f_1, \ldots, f_m)$ in $\mathbb{K}[X_1, \ldots, X_n]$ be a system of weighted homogeneous polynomials.

In order to solve the system F, we need to compute a Gröbner basis for some monomial order, usually an elimination order or the lexicographical order. The usual strategy for that purpose is to perform the computation in two steps, first computing a Gröbner basis for some "easy" order, using a fast direct algorithm (Buchberger, $\mathsf{F_4}$ or $\mathsf{F_5}$), and then computing a Gröbner basis for the wanted order with either a direct algorithm or a change of order algorithm (Gröbner walk in positive dimension, FGLM in zero dimension).

The first step of the computation involves choosing a monomial order making the computations easier. In the homogeneous case, the usual choice is the GRevLex order, together with a strategy for selecting critical pairs for reduction by lowest degree first. In order to take advantage from the weighted homogeneous structure of the system F, we may choose the W-GRevLex order instead, with a selection strategy by lowest W-degree first.

For algorithms proceeding purely with critical pairs, such as Buchberger, $\mathsf{F_4}$ or $\mathsf{F_5}$, but unlike Matrix-$\mathsf{F_5}$ for example, this computation can be performed without changing the algorithm or its implementation, by transforming the system beforehand:

Proposition 10. *Let $F = (f_1, \ldots, f_m)$ be a family of polynomials in $\mathbb{K}[X_1, \ldots, X_n]$, assumed to be weighted homogeneous for a system of weights $W = (w_1, \ldots, w_n)$. Let $<_1$ be a monomial order, G be the reduced Gröbner basis of $\hom_W(F)$ for this order, and $<_2$ be the pullback of $<_1$ through $\hom_W$. Then*

(1) all elements of G are in the image of $\hom_W$;

(2) the family $G' := \hom_W^{-1}(G)$ is a reduced Gröbner basis of the system F for the order $<_2$.

Proof. The morphism $\hom_W$ preserves S-polynomials, in the sense that

$$S\text{-pol}(\hom_W(f), \hom_W(g)) = \hom_W(S\text{-pol}(f, g)).$$

Recall that we can compute a reduced Gröbner basis by running the Buchberger algorithm, which involves only multiplications, additions, tests of divisibility and computation of S-polynomials. Since all these operations are compatible with $\hom_W$, if we run the Buchberger algorithm on both F and $\hom_W(F)$ simultaneously, they will follow exactly the same computations up to application of $\hom_W$. The consequences on the final reduced Gröbner basis follow. □

Actually, the fact that $\hom_W$ preserves S-polynomials proves that running any critical pairs algorithm on $\hom_W(F)$ for the GRevLex order involves exactly the same reductions as running the same algorithm on F for the W-GRevLex order.

We will only give estimates for the complexity of the F_5 algorithm, as it is usually faster than Buchberger and F_4. The complexity of this algorithm is usually studied through its variant Matrix-F_5. This complexity is given by

$$C_{\mathsf{F}_5} = \mathrm{O}\left(M_{W,d_{\mathrm{reg}}}^{\omega}\right)$$

where $M_{W,d}$ is the size of the matrix we need to build at W-degree d, d_{reg} is the degree of regularity and ω is the exponent in the complexity of matrix multiplication.

For a W-homogeneous system, the size of the matrix at W-degree d is given by the number of monomials at W-degree d. This number of monomials is known as the *Sylvester denumerant* $d(d; w_1, \ldots, w_n)$. There is no formula for this denumerant, but its asymptotics are known (see for example Alfonsín (2005, sec. 4.2)):

$$M_{W,d} \simeq \frac{1}{\prod w_i} M_{\mathbb{1},d} = \frac{1}{\prod w_i} \binom{n+d-1}{d}.$$

As for the degree of regularity of the system, depending on the hypotheses satisfied by the system F (regularity, Noether position or semi-regularity), we can use the corresponding estimates.

All in all, the complexity of computing an "easy" Gröbner basis for a weighted homogeneous system is divided by $(\prod w_i)^{\omega}$ when compared to an homogeneous system with the same degree. The degree of regularity is also reduced, yielding an important practical gain for the F_5 algorithm:

$$C_{\mathsf{F}_5} = \mathrm{O}\left(\frac{1}{(\prod w_i)}^{\omega} \cdot \binom{n+d_{\mathrm{reg}}-1}{d_{\mathrm{reg}}}^{\omega}\right). \tag{16}$$

The gain from the reduced number of monomials applies to other algorithms as well, provided they are run on $\hom_W(F)$ if they are only using critical pairs, or use the W-GRevLex order otherwise.

Solving zero-dimensional weighted homogeneous systems is rarely needed. The reason is that generically, such a system only admits the trivial solution $(0, \ldots, 0)$. For most applications, a W-GRevLex Gröbner basis is enough, without the need for a change of ordering.

For positive dimension, depending on the situation, it may be interesting to perform a two-steps computation, or to simply use one of the direct algorithms with the desired order. In the former case, the usual algorithm used for the change of order is the Gröbner walk. This algorithm is much more complex and to the best of our knowledge, does not have good complexity estimates. However, it involves computing successive Gröbner bases, using algorithm F_4 or F_5 as a blackbox. As such, assigning weights to a polynomial system will yield similar improvements for the computing time.

5.2. Affine systems

Affine systems can be solved with the same methods as homogeneous or weighted homogeneous systems, by homogenizing the system with an homogenization variable H. However, reducing affine systems can lead to *degree falls*, that is reductions leading to affine polynomials of lesser W-degree, or equivalently, to weighted homogeneous polynomials divisible by H. If the algorithm carries on the computation on the homogenized system, then it will be led to examine polynomials divisible by large powers of H. This

effect can be mitigated by detecting these reductions and reinjecting these polynomials at the relevant W-degree, but overall, degree falls usually make the computation slower.

Such a degree fall is a reduction to zero of the highest W-degree components of a pair of polynomials. However, if the highest W-degree components form a regular sequence (or a sequence in Noether position, or a sequence in simultaneous Noether position), all results from the W-homogeneous case apply. For semi-regular sequences, the F_5 Criterion can only eliminate degree falls up to the last W-degree δ at which all of the multiplication applications $s_{i,d}$ $(n < i \leq m)$ are injective. At this degree, a degree fall is unavoidable, and the algorithm is left to proceed with the lower W-degree components of the system, for which no regularity assumption was made. However, the degree of these subsequent reductions will not go above δ, and complexity estimates can be obtained by considering the full Macaulay matrix at W-degree δ.

Assuming the affine system is zero-dimensional, we may ultimately want to compute its solutions. This is done by writing triangular generators of the ideal. Using Gröbner bases, generically, it requires computing a Gröbner basis of the ideal for the lexicographical order, which can be done with a change of order from the W-GRevLex order. The usual algorithm for that purpose is the FGLM algorithm. Its complexity is given by

$$C_{\mathsf{FGLM}} = \mathrm{O}\left(n \deg^{\omega}\right)$$

where deg is the degree of the system.

Let $F = (f_1, \ldots, f_n) \subset \mathbb{K}[X_1, \ldots, X_n]$ be a zero-dimensional affine system. For any system of weights $W = (w_1, \ldots, w_n)$, one may W-homogenize the system F, that is compute a system $F^h = (f_1^h, \ldots, f_n^h) \subset \mathbb{K}[X_1, \ldots, X_n, H]$ such that for any $i \in \{1, \ldots, n\}$,

$$f_i(X_1, \ldots, X_n) = f_i^h(X_1, \ldots, X_n, 1),$$

and such that F^h is W^h-homogeneous, with $W^h = (w_1, \ldots, w_n, 1)$.

If F is regular, then its homogenized F^h is also regular. Assume that the system of weights W is chosen so that F is regular in the affine sense, i.e. its highest W-degree components form a regular sequence. Since the system of these highest W-degree components is exactly $F^h(H := 0)$, by the characterization NP4, F^h is in Noether position with respect to the variables $X_1, \ldots, X_n$. As a consequence, the degree of $\langle F^h \rangle$ is

$$\deg = \frac{\prod_{i=1}^{n} d_i}{\prod_{i=1}^{n} w_i}$$

and the complexity bounds for the change of ordering are also improved by a factor $\left(\prod_{i=1}^{n} w_i\right)^{\omega}$:

$$C_{\mathsf{FGLM}} = \mathrm{O}\left(n\left(\frac{\prod d_i}{\prod w_i}\right)^{\omega}\right). \tag{17}$$

6. Applications

In this section, we present some applications where taking into account the weighted structure of the system yields speed-ups. For each system, we compare two strategies: the "standard" strategy consists of computing a Gröbner basis without considering the weighted structure; the "weighted" strategy is the strategy we described at section 5. For all these examples, we use a more compact notation for degrees and weights, so that for example, $(2^3, 1)$ is equivalent to $(2, 2, 2, 1)$.

6.1. *Generic systems*

First, we present some timings obtained with generic systems, in both the complete intersection ($m = n$), the positive-dimensional ($m < n$) and the over-determined ($m > n$) cases. In both cases, we fix a system of weights $W = (w_1, \ldots, w_n)$ and a system of W-degrees $D = (d_1, \ldots, d_m)$, and we pick at random m polynomials $(f_i)_{i=1\ldots m}$, such that for any $i \in \{1, \ldots, m\}$, f_i has dense support in the set of monomials with W-degree less than or equal to d_i.

For complete intersection systems, we compute a lexicographical Gröbner basis, using a two-steps strategy in Magma, with algorithm F_4 as a first step (first block of lines in Table 2a) and algorithm FGLM for the change of ordering (Table 2b).

For over-determined systems, we compute a Gröbner basis for the GRevLex ordering, using algorithm F_4 from Magma (second block of lines in Table 2a).

For positive-dimensional systems, we compute a basis for an elimination order, using a two-steps strategy with FGb [3]: first we compute a GRevLex basis with algorithm F_4 (third block of lines in Table 2a), and then we compute a basis for the wanted elimination order, again with F_4 (Table 2c). In this table, the second column describes what variables we eliminate: for example, 3 means that we eliminate the first 3 variables, while $1 \to 3$ means that we first eliminate the first variable, then the next 2 variables, again resulting in a basis eliminating the first 3 variables.

For algorithm F_4 with the GRevLex ordering, the behavior we observe is coherent with the previous complexity studies: we observe some speed-ups when taking into account the weighted structure of the system, and these speed-ups seem to increase with the weights. However, the speed-ups cannot be expected to match rigorously the ones predicted by the complexity bounds, because the systems are usually not regular for the standard strategy. Experiments also confirm that it is more effective to order the variables with highest weight first.

For the lexicographical ordering with FGLM, we also observe some speed-ups when applying the weights (we will observe this behavior again in Section 6.2). These differences are not explained by the theoretical complexity bounds, since both ideals have the same degree in each case. However, it appears that the slower FGLM runs are those where the FGLM matrix is denser, and that this difference in density seems to match quantitatively the speed-ups we observe.

Finally, for elimination bases, the results are similar to what we observed with the GRevLex ordering: when possible, one should take into account the weights, and order the variables such that the smallest weights are also the smallest variables. However, when eliminating variables, the largest variables need to be the ones that should be eliminated. If the variables need to be ordered such that those with the smallest weights are first, in most cases, taking into account the weighted structure is still profitable. However, if the smallest weight is on the largest variable and there is only one such variable, this

[3] The Gröbner basis algorithms from Magma seem to behave strangely with elimination orders, as seen in the detailed logs, and it coincides with significant slowdowns. This behavior was not observed on other implementations of the same algorithms: F_4 from FGb and Buchberger from Singular (Decker et al., 2012). For example, for the system in the first line of table 2c, without the weights, with Magma's F_4 algorithm, the first degree fall comes at step 4, and the algorithm needs more than 66 steps to compute the basis. With FGb's implementation of F_4 in Maple, the first degree fall appears at step 13, and the algorithm finishes at step 32.

Table 2. Benchmarks with Magma for generic systems

Parameters	Without weights (s)	With weights (s)	Speed-up
$n = 8$, $W = (2^6, 1^2)$, $D = (4^8)$	8.0	2.5	3.2
$n = 9$, $W = (2^7, 1^2)$, $D = (4^9)$	101.2	12.5	8.1
$n = 7$, $W = (2^5, 1^2)$, $D = (8^{15})$	31.6	7.5	4.2
$n = 7$, $W = (2^5, 1^2)$, $D = (8^{14})$	29.0	9.4	3.1
$n = 7$, $W = (2^5, 1^2)$, $D = (8^{13})$	40.0	12.0	3.3
$n = 5$, $m = 4$, $W = (2^4, 1)$, $D = (8^4)$	2.6	0.2	13.0
$n = 5$, $m = 4$, $W = (1, 2^4)$, $D = (8^4)$	2.5	0.3	8.3
$n = 5$, $m = 4$, $W = (1^3, 2^2)$, $D = (4^4)$	23.6	0.0	2360.0
$n = 5$, $m = 4$, $W = (2^2, 1^3)$, $D = (4^4)$	407.5	0.0	40 750.0

(a) Benchmarks for the F_4 algorithm for the GRevLex ordering

Parameters	Degree	Without weights (s)	With weights (s)	Speed-up
$n = 8$, $W = (2^6, 1^2)$, $D = (4^8)$	1024.0	500.4	495.0	1.0
$n = 9$, $W = (2^7, 1^2)$, $D = (4^9)$	2048.0	11 995.8	7462.1	1.6

(b) Benchmarks for the FGLM algorithm (lexicographical ordering)

is no longer true (see for example the second line in Table 2c). Experiments suggest that these systems naturally possess a good weighted structure for the weights $(1, \ldots, 1)$: their construction ensures that every such polynomial of total degree d will have a large homogeneous component at degree $d/2$, and the higher degree components will be small, and divisible by large powers of X_1. On the other hand, with weights $(1, 2, \ldots, 2)$, the same polynomial will have a large W-homogeneous component at W-degree d, overall leading to reductions at higher degree (an example is given in Table 3).

We conclude this section with timings illustrating the consequences of the estimates of the degree of regularity of a system, depending on the order of the variables (Section 3.2). For this purpose, we generate a generic system of W-degree (60^4) with weights $(20, 5, 5, 1)$. Then we compute a W-GRevLex Gröbner basis for the orders $X_1 > \cdots > X_4$ (smallest weights last) and for the reverse order $X_n < \cdots < X_1$. We give the degree of regularity, the value predicted by the previous bound (11), by the new bound (12) and by the conjectured bound (14), as well as the timings. This experiment was run using algorithm F_5 from the FGb library, the results are in Table 4.

6.2. Discrete logarithm problem

Taking advantage of a weighted homogeneous structure has allowed the authors of the article (Faugère et al., 2013) to obtain significant speed-ups for solving a system arising from the DLP on Edwards elliptic curves (Gaudry (2009)). They observed that

Parameters	Elim. vars.	Without weights (s)	With weights (s)	Speed-up
$n = 5$, $m = 4$, $W = (2^4, 1)$, $D = (8^4)$	1	120.3	12.0	10.0
$n = 5$, $m = 4$, $W = (1, 2^4)$, $D = (8^4)$	1	27.6	30.4	0.9
$n = 5$, $m = 4$, $W = (1^3, 2^2)$, $D = (4^4)$	2	146.3	6.9	21.2
$n = 5$, $m = 4$, $W = (1^3, 2^2)$, $D = (4^4)$	$1 \longrightarrow 2$	162.0	3.3	49.1
$n = 5$, $m = 4$, $W = (2^2, 1^3)$, $D = (4^4)$	1	>750	0.1	>7500
$n = 5$, $m = 4$, $W = (2^2, 1^3)$, $D = (4^4)$	$1 \longrightarrow 2$	NA	0.1	NA
$n = 5$, $m = 4$, $W = (2^2, 1^3)$, $D = (4^4)$	$1 \longrightarrow 2 \longrightarrow 3$	NA	7.9	NA

(c) Benchmarks for the F_4 algorithm for elimination

Table 3. Size of the W-homogeneous components for a generic polynomial with W_0-degree 4 for $W_0 = (1, 2, 2, 2)$

W-degree	$W = (1, 2, 2, 2)$	$W = (1, 1, 2, 2)$	$W = (1, 1, 1, 1)$
0	1	1	1
1	1	2	4
2	4	5	10
3	4	6	4
4	10	6	1

Table 4. Impact of the order of the variables on the degree of regularity and the computation times (generic weighted homogeneous system with W-degree (60^4) w.r.t. $W = (20, 5, 5, 1)$)

Order	d_{reg}	Macaulay's bound (11)	Bound (12)	Conjectured bound (14)	F_5 time
$X_1 > X_2 > X_3 > X_4$	210	229	210	210	101.9
$X_4 > X_3 > X_2 > X_1$	220	229	229	220	255.5

the system of equations they had to solve has symmetries, and rewrote it in terms of the invariants of the symmetry group. For a system in n equations, these invariants are

$$\begin{aligned} E_1 &= e_1(X_1^2, \ldots, X_n^2) \\ E_2 &= e_2(X_1^2, \ldots, X_n^2) \\ &\vdots \\ E_{n-1} &= e_{n-1}(X_1^2, \ldots, X_n^2) \\ E_n &= e_n(X_1, \ldots, X_n). \end{aligned}$$

Table 5. Benchmarks with FGb and Magma for DLP systems

System	deg(I)	F_5 w (s)	F_5 std (s)	Speed-up for F_5	FGLM w (s)	FGLM std (s)	Speed-up for FGLM
DLP Edwards: $n = 4$, $W = (2^3, 1)$, $D = (8^4)$	512	0.1	0.1	1.0	0.1	0.1	1.0
DLP Edwards: $n = 5$, $W = (2^4, 1)$, $D = (16^5)$	65 536	935.4	6461.2	6.9	2164.4	6935.6	3.2

(a) Benchmarks with FGb

System	deg(I)	F_4 w (s)	F_4 std (s)	Speed-up for F_4	FGLM w (s)	FGLM std (s)	Speed-up for FGLM
DLP Edwards: $n = 4$, $W = (2^3, 1)$, $D = (8^4)$	512	1	1	1.0	1	27	27
DLP Edwards: $n = 5$, $W = (2^4, 1)$, $D = (16^5)$	65 536	6044	56 105	9.3	∞	∞	NA

(b) Benchmarks with Magma

The system they obtained is sparser, but does not have a good homogeneous structure. In particular, the highest total degree components of the system do not form a regular sequence, and in practice, a Gröbner basis computation will follow many degree falls.

However, the system had a weighted homogeneous structure for the weights $(2, \ldots, 2, 1)$ (only E_n has weight 1), with respective W-degree $(2^n, \ldots, 2^n)$. The highest W-degree components forming a sequence in simultaneous Noether position with respect to the order $E_1 > E_2 > \cdots > E_n$, one could compute a Gröbner basis without any W-degree fall, with complexity bounded by the estimates (16) and (17).

6.3. Polynomial inversion

The polynomial inversion problem consists of finding polynomial relations between polynomials. More precisely, given a system of polynomial equations

$$\begin{cases} f_1(X_1, \ldots, X_n) = 0 \\ f_2(X_1, \ldots, X_n) = 0 \\ \qquad\qquad\qquad \vdots \\ f_m(X_1, \ldots, X_n) = 0\,, \end{cases}$$

we want to compute all the relations of the form

$$g_i(f_1, \ldots, f_r) = 0.$$

One can compute these relations with Gröbner bases by computing an elimination ideal: consider the ideal generated by the polynomials

$$\begin{aligned} T_1 &- f_1(X_1, \ldots, X_n) \\ T_2 &- f_1(X_1, \ldots, X_n) \\ &\vdots \\ T_m &- f_m(X_1, \ldots, X_n) \end{aligned}$$

in $R := \mathbb{K}[X_1, \ldots, X_n, T_1, \ldots, T_m]$. Order R with an elimination order for the variables $X_1, \ldots, X_n$, that is an order such that

$$m_X(X_1, \ldots, X_n) m_T(T_1, \ldots, T_m) <_{\text{elim}} m'_X(X_1, \ldots, X_n) m'_T(T_1, \ldots, T_m)$$

$$\iff \begin{cases} m_X <_X m'_X \\ \text{or} \\ m_X = m'_X \text{ and } m_T <_T m'_T \end{cases}$$

for some monomial orders $<_X$ and $<_T$. The usual choice is a block-GRevLex order.

This problem can benefit from being given a weighted structure (see (Traverso, 1996, sec. 6.1)). For any $i \in \{1, \ldots, m\}$, let d_i be the degree of f_i. By setting the weight of T_i to be d_i, the monomial T_i becomes part of the highest W-degree component of $T_i - f_i(X_1, \ldots, X_n)$, giving this equation a weighted homogeneous structure.

More precisely:

Proposition 11. *Let $f_1, \ldots, f_m$ be a system of polynomials with respective degree $d_1, \ldots, d_m$ in $\mathbb{K}[X_1, \ldots, X_n]$. Consider the algebra $R := \mathbb{K}[X_1, \ldots, X_n, T_1, \ldots, T_m]$, graded with the weights $W = (1, \ldots, 1, d_1, \ldots, d_m)$, and consider the system $F = (T_1 - f_1\mathbf{X}, \ldots, T_m - f_m(\mathbf{X}))$ in R. Then the system F^h formed with the highest W-degree components of F is in Noether position with respect to the variables $T_1, \ldots, T_m$, and in particular it forms a regular sequence.*

Proof. By the choice of the weights, the system F^h is defined by

$$F^h = (T_1 - f_1^h(\mathbf{X}), \ldots, T_m - f_m^h(\mathbf{X})),$$

where for any $i \in \{1, \ldots, m\}$, f_i^h is the highest degree component of f_i. As a consequence, by the characterization NP4 of the Noether position, the system F^h is indeed in Noether position with respect to the variables $T_1, \ldots, T_m$. □

In Tables 6, we present timings for a few systems with this kind of problem:

- group invariants (Sturmfels (2008)): given a group, compute its fundamental invariants, and then the relations between these invariants. Since these examples can lead to very long computations, in some cases, we only compute the relations between the k first invariants;
- monomials: given m monomials of degree d in $\mathbb{K}[X_1, \ldots, X_n]$, compute the relations between them;
- matrix minors: given a $p \times q$ matrix of linear forms in n indeterminates, compute all its minors of rank r as polynomials in the $X_{i,j}$'s, and compute the relations between them.

Table 6. Benchmarks with Magma on some polynomial inversion systems

System	Without weights (s)	With weights (s)	Speed-up
Cyclic invariants, $n = 4$	4.2	0.0	140.0
Cyclic invariants, $n = 5$, $k = 12$	2612.6	54.7	47.8
Cyclic invariants, $n = 5$	$>$ 75 000 [a]	392.7	NA
Cyclic invariants, $n = 6$, $k = 14$	32 987.6	2787.7	11.8
Cyclic invariants, $n = 6$, $k = 15$	$>$280 000 [a]	14 535.4	NA
Dihedral invariants, $n = 5$	$>$ 70 000 [a]	6.3	NA
Generic monomials, $d = 2$, $n = 24$, $m = 48$	216.1	0.2	1350.6
Generic monomials, $d = 2$, $n = 25$, $m = 50$	14 034.7	0.1	116 955.8
Generic monomials, $d = 2$, $n = 26$, $m = 52$	14 630.6	0.2	66 502.7
Generic monomials, $d = 2$, $n = 27$, $m = 54$	8887.6	0.2	55 547.5
Generic monomials, $d = 3$, $n = 11$, $m = 22$	370.9	0.1	6181.7
Generic monomials, $d = 3$, $n = 12$, $m = 24$	4485.0	0.2	26 382.4
Matrix minors, $n = 5$, 7×7, $r = 3$	125.7	93.3	1.3
Matrix minors, $n = 6$, 7×7, $r = 3$	1941.0	1029.1	1.9
Matrix minors, $n = 6$, 8×8, $r = 3$	4115.8	2295.8	1.8
Matrix minors, $n = 4$, 6×6, $r = 5$	612.6	159.2	3.8
Matrix minors, $n = 4$, 7×7, $r = 6$	8043.3	2126.9	3.8
Matrix minors, $n = 4$, 7×10, $r = 7$	69 386.1	43 910.1	1.6

a. Memory usage was over 120 GB

(a) First step (F_4 for the GRevLex order)

In each case, we compute an elimination basis using a two-steps strategy: first we compute a GRevLex basis (Table 6a), then we compute the elimination basis (Table 6b). In Table 6c, we show some timings for the computation of the elimination basis directly from the input system. All these experiments were run using algorithm F_4 from Magma.

System	Without weights (s)	With weights (s)	Speed-up
Cyclic invariants, $n = 4$	7.0	0.1	70.0
Cyclic invariants, $n = 5$, $k = 12$	1683.2	70.7	23.8
Cyclic invariants, $n = 5$	NA	382.5	NA
Cyclic invariants, $n = 6$, $k = 14$	9236.4	1456.0	6.3
Cyclic invariants, $n = 6$, $k = 15$	NA	7179.7	NA
Dihedral invariants, $n = 5$	NA	20.3	NA
Generic monomials, $d = 2$, $n = 24$, $m = 48$	250.3	117.4	2.1
Generic monomials, $d = 2$, $n = 25$, $m = 50$	13 471.2	15 932.9	0.8
Generic monomials, $d = 2$, $n = 26$, $m = 52$	17 599.5	8054.2	2.2
Generic monomials, $d = 2$, $n = 27$, $m = 54$	9681.0	3605.6	2.7
Generic monomials, $d = 3$, $n = 11$, $m = 22$	624.5	199.9	3.1
Generic monomials, $d = 3$, $n = 12$, $m = 24$	9751.6	3060.1	3.2
Matrix minors, $n = 5$, 7×7, $r = 3$	52.6	66.6	0.8
Matrix minors, $n = 6$, 7×7, $r = 3$	556.5	779.1	0.7
Matrix minors, $n = 6$, 8×8, $r = 3$	1257.9	1714.0	0.7
Matrix minors, $n = 4$, 6×6, $r = 5$	262.7	328.1	0.8
Matrix minors, $n = 4$, 7×7, $r = 6$	2872.2	4299.8	0.7
Matrix minors, $n = 4$, 7×10, $r = 7$	4728.4	5485.8	0.9

(b) Second step (F_4 for an elimination order)

System	Without weights (s)	With weights (s)	Speed-up
Cyclic invariants, $n = 4$	4.0	0.3	13.3
Cyclic invariants, $n = 5$, $k = 12$	2705.8	73.4	36.9
Cyclic invariants, $n = 5$	$>$ 90 000 [b]	370.0	$>$243
Cyclic invariants, $n = 6$, $k = 14$	35 922.4	2256.2	15.9
Cyclic invariants, $n = 6$, $k = 15$	$>$300 000 [b]	7426.7	$>$40
Dihedral invariants, $n = 5$	$>$ 40 000 [b]	18.5	$>$2162
Generic monomials, $d = 2$, $n = 24$, $m = 48$	216.5	110.9	2.0
Generic monomials, $d = 2$, $n = 25$, $m = 50$	31 135.2	16 352.2	1.9
Generic monomials, $d = 2$, $n = 26$, $m = 52$	14 919.2	8142.8	1.8
Generic monomials, $d = 2$, $n = 27$, $m = 54$	5645.8	4619.0	1.2
Generic monomials, $d = 3$, $n = 11$, $m = 22$	370.1	193.1	1.9
Generic monomials, $d = 3$, $n = 12$, $m = 24$	4527.2	2904.6	1.6
Matrix minors, $n = 7$, 7×7, $r = 3$	41 220.0	26 340.0	1.6
Matrix minors, $n = 7$, 8×8, $r = 3$	48 000.0	18 060.0	2.7
Matrix minors, $n = 8$, 8×8, $r = 3$	711 690.0	390 235.0	1.8
Matrix minors, $n = 4$, 6×6, $r = 5$	613.9	325.4	1.9
Matrix minors, $n = 4$, 7×7, $r = 6$	8059.4	3955.5	2.0
Matrix minors, $n = 4$, 7×10, $r = 7$	71 067.8	32 721.5	2.2

b. Memory usage was over 120 GB.

(c) Direct strategy

References

Alfonsín, J. L. R., 1998. On variations of the subset sum problem. Discrete Applied Mathematics 81 (1–3), 1 – 7.
URL http://www.sciencedirect.com/science/article/pii/S0166218X96001059

Alfonsín, J. L. R., 2005. The Diophantine Frobenius Problem. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, Oxford.

Bardet, M., Dec. 2004. Étude des systèmes algébriques surdéterminés. applications aux codes correcteurs et à la cryptographie. Phd thesis, Université Pierre et Marie Curie - Paris VI.
URL http://tel.archives-ouvertes.fr/tel-00449609

Bardet, M., Faugère, J.-C., Salvy, B., Sep. 2014. On the Complexity of the F5 Gröbner basis Algorithm. Journal of Symbolic Computation, 1–24.
URL http://hal.inria.fr/hal-00915522

Bardet, M., Faugère, J.-C., Salvy, B., Yang, B.-Y., 2005. Asymptotic behaviour of the degree of regularity of semi-regular polynomial systems. In: MEGA'05, 2005. Eighth International Symposium on Effective Methods in Algebraic Geometry.

Becker, T., Weispfenning, V., 1993. Gröbner bases. Vol. 141 of Graduate Texts in Mathematics. Springer-Verlag, New York, a computational approach to commutative algebra, In cooperation with Heinz Kredel.
URL http://dx.doi.org/10.1007/978-1-4612-0913-3

Bosma, W., Cannon, J., Playoust, C., 1997. The Magma algebra system. I. The user language. Journal of Symbolic Computation 24 (3-4), 235–265, computational algebra and number theory (London, 1993).
URL http://dx.doi.org/10.1006/jsco.1996.0125

Buchberger, B., 1976. A theoretical basis for the reduction of polynomials to canonical forms. ACM SIGSAM Bulletin 10 (3), 19–29.

Caboara, M., de Dominicis, G., Robbiano, L., 1996. Multigraded hilbert functions and buchberger algorithm. In: Engeler, E., Caviness, B. F., Lakshman, Y. N. (Eds.), Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation, ISSAC '96, Zurich, Switzerland, July 24-26, 1996. ACM, pp. 72–78.
URL http://doi.acm.org/10.1145/236869.236901

Collart, S., Kalkbrenner, M., Mall, D., 1997. Converting bases with the Gröbner walk. Journal of Symbolic Computation 24 (3-4), 465–469.
URL dx.doi.org/10.1006/jsco.1996.0145

Dalzotto, G., Sbarra, E., 2006. Computations in weighted polynomial rings. Analele Stiintifice ale Universitatii Ovidius Constanta 14(2), 31–44.
URL http://www.anstuocmath.ro/mathematics/pdf12/31_44_GDalzotto_ESbarra.pdf

de Boer, M., Pellikaan, R., 1999. Gröbner bases for codes. No. 4 in Algorithms and Computation in Mathematics. Springer, pp. 237–259.
URL http://www.win.tue.nl/~{}ruudp/paper/34.pdf

Decker, W., Greuel, G.-M., Pfister, G., Schönemann, H., 2012. SINGULAR 3-1-6 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de.

Dickenstein, A., Emiris, I. Z., 2010. Solving Polynomial Equations: Foundations, Algorithms, and Applications, 1st Edition. Springer Publishing Company, Incorporated.

Eisenbud, D., 1995. Commutative algebra. Vol. 150 of Graduate Texts in Mathematics. Springer-Verlag, New York, with a view toward algebraic geometry.
URL http://dx.doi.org/10.1007/978-1-4612-5350-1

Faugère, J.-C., 1999. A new efficient algorithm for computing Gröbner bases (F_4). Journal of Pure and Applied Algebra 139 (1-3), 61–88, effective methods in algebraic geometry (Saint-Malo, 1998).
URL http://dx.doi.org/10.1016/S0022-4049(99)00005-5

Faugère, J.-C., 2002. A new efficient algorithm for computing Gröbner bases without reduction to zero (F_5). In: Proceedings of the 2002 International Symposium on Symbolic and Algebraic Computation. ACM, New York, pp. 75–83 (electronic).
URL http://dx.doi.org/10.1145/780506.780516

Faugère, J.-C., Gianni, P., Lazard, D., Mora, T., 1993. Efficient computation of zero-dimensional Gröbner bases by change of ordering. Journal of Symbolic Computation 16 (4), 329–344.
URL http://dx.doi.org/10.1006/jsco.1993.1051

Faugère, J.-C., Safey El Din, M., Verron, T., 2013. On the complexity of computing Gröbner bases for quasi-homogeneous systems. In: Proceedings of the 2013 International Symposium on Symbolic and Algebraic Computation. ISSAC '13. ACM, New York, NY, USA.

Faugère, J.-C., Sep. 2010. FGb: A Library for Computing Gröbner Bases. In: Fukuda, K., Hoeven, J., Joswig, M., Takayama, N. (Eds.), Mathematical Software - ICMS 2010. Vol. 6327 of Lecture Notes in Computer Science. Springer Berlin / Heidelberg, Berlin, Heidelberg, pp. 84–87.
URL http://www-polsys.lip6.fr/~jcf/Papers/ICMS.pdf

Faugère, J.-C., Gaudry, P., Huot, L., Renault, G., 2013. Using symmetries in the index calculus for elliptic curves discrete logarithm. Journal of Cryptology, 1–41.
URL http://dx.doi.org/10.1007/s00145-013-9158-5

Fröberg, R., 1985. An inequality for Hilbert series of graded algebras. Mathematica Scandinavica 56, 117–144.
URL http://eudml.org/doc/166929

Gaudry, P., 2009. Index calculus for abelian varieties of small dimension and the elliptic curve discrete logarithm problem. Journal of Symbolic Computation 44 (12), 1690 – 1702, gröbner Bases in Cryptography, Coding Theory, and Algebraic Combinatorics.
URL http://www.sciencedirect.com/science/article/pii/S074771710800182X

Grayson, D. R., Stillman, M. E., 2014. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

Guerrini, E., Rimoldi, A., 2009. Fglm-like decoding: from fitzpatrick's approach to recent developments. In: Sala, M., Sakata, S., Mora, T., Traverso, C., Perret, L. (Eds.), Gröbner Bases, Coding, and Cryptography. Springer Berlin Heidelberg, pp. 197–218.
URL http://dx.doi.org/10.1007/978-3-540-93806-4_12

Leonard, D. A., 2009. A weighted module view of integral closures of affine domains of type i. Advances in Mathematics of Communications 3 (1), 1–11.
URL http://aimsciences.org/journals/displayArticlesnew.jsp?paperID=3953

Lucas, É., 1891. Théorie des nombres. Vol. 1 of Théorie des nombres. Gauthier-Villars et fils.

Milne, J. S., 2012. Algebraic geometry (v5.22). Available at www.jmilne.org/math/.

Moreno-Socías, G., 1991. Autour de la fonction de Hilbert-Samuel (escaliers d'idéaux polynomiaux). Ph.D. thesis.
URL http://cat.inist.fr/?aModele=afficheN&cpsidt=151245

Moreno-Socías, G., 1996. Revlex standard bases of generic complete intersections. Technical report.

Moreno-Socías, G., 2003. Degrevlex Gröbner bases of generic complete intersections. Journal of Pure and Applied Algebra 180, 263–283.
URL http://www.sciencedirect.com/science/article/pii/S0022404902002979

Niven, I. M., Zuckerman, H. S., Montgomery, H. L., 1991. An introduction to the theory of numbers. Wiley.

Pardue, K., 2010. Generic sequences of polynomials. Journal of Algebra 324 (4), 579–590.
URL http://dx.doi.org/10.1016/j.jalgebra.2010.04.018

Reid, L., Roberts, L. G., Roitman, M., 1991. On complete intersections and their Hilbert functions. Canadian Mathematical Bulletin 34 (4), 525–535.
URL http://dx.doi.org/10.4153/CMB-1991-083-9

Sturmfels, B., 2008. Algorithms in Invariant Theory (Texts and Monographs in Symbolic Computation), 2nd Edition. Springer Publishing Company, Incorporated.

Traverso, C., 1996. Hilbert Functions and the Buchberger Algorithm. Journal of Symbolic Computation 22 (4), 355 – 376.
URL http://www.sciencedirect.com/science/article/pii/S0747717196900565