

## A point on fixpoints in posets

Frédéric Blanqui

► **To cite this version:**

| Frédéric Blanqui. A point on fixpoints in posets. 2014. hal-01097809

**HAL Id: hal-01097809**

**<https://hal.inria.fr/hal-01097809>**

Preprint submitted on 22 Dec 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A point on fixpoints in posets

Frédéric Blanqui (INRIA)

24 January 2014

Let  $(X, \leq)$  be a *non-empty strictly inductive poset*, that is, a non-empty partially ordered set such that every *non-empty chain*  $Y$  has a least upper bound  $\text{lub}(Y) \in X$ , a chain being a subset of  $X$  totally ordered by  $\leq$ .

We are interested in sufficient conditions such that, given an element  $a_0 \in X$  and a function  $f : X \rightarrow X$ , there is some ordinal  $k$  such that  $a_{k+1} = a_k$ , where  $a_k$  is the transfinite sequence of iterates of  $f$  starting from  $a_0$  (implying that  $a_k$  is a fixpoint of  $f$ ):

- $a_{k+1} = f(a_k)$
- $a_l = \text{lub}\{a_k \mid k < l\}$  if  $l$  is a limit ordinal, *i.e.*  $l = \text{lub}(l)$

This note summarizes known results about this problem and provides a slight generalization of some of them.

## 1 Definitions

Let AC denote the axiom of choice<sup>1</sup> [Zer04].

Let  $\mathbb{O}$  be the class of von Neumann ordinals [vN23].

Let  $A = \{x \in X \mid \exists k \in \mathbb{O}, x = a_k\}$  be the set of all transfinite iterates of  $f$  from  $a_0$ .

Let  $N$  be the smallest subset of  $X$  containing  $a_0$  and closed by  $f$  ( $Z \subseteq X$  is closed by  $f$  if, for all  $z \in Z$ ,  $f(z) \in Z$ ) and non-empty lub's ( $Z \subseteq X$  is closed by non-empty lub's if, for all  $P \subseteq Z$ , if  $P \neq \emptyset$ , then  $\text{lub}(P)$  exists and belongs to  $Z$ ).

Given  $Z \subseteq X$ , let  $\text{glb}(Z)$  be, if it exists, the greatest lower bound of  $Z$ .

Let  $\text{PreFP}(f) = \{x \in X \mid x \leq f(x)\}$  be the set of *pre-fixpoints* of  $f$ ,  $\text{PostFP}(f) = \{x \in X \mid x \geq f(x)\}$  be the set of *post-fixpoints* of  $f$ , and  $\text{FP}(f) = \text{PreFP}(f) \cap \text{PostFP}(f)$  be the set of fixpoints of  $f$ .

$f$  is (resp. *strictly monotone* (or *isotone* or *increasing*) (opposites: *anti-monotone*, *antitone* or *decreasing*) if  $f(x) \leq f(y)$  whenever  $x \leq y$  (resp.  $f(x) < f(y)$  whenever  $x < y$ ).

---

<sup>1</sup>Implicitly used by Cantor and others, but first explicitly mentioned (and rejected) by Peano in [Pea90].

$f$  is (resp. *strictly*) *extensive* (or *inflationary* or *progressive*) (opposites: *reductive*, *deflationary* or *regressive*) if, for all  $x$ ,  $f(x) \geq x$  (resp.  $f(x) > x$ ).

Note that: if  $X$  has a least element  $\perp$ , then it is a pre-fixpoint of  $f$ ; if  $f$  is extensive, then *every* element of  $X$  is a pre-fixpoint of  $f$ ; if  $f$  is monotone, then  $\text{PreFP}(f)$  and  $\text{PostFP}(f)$  are closed by  $f$  (pre-fixpoints are mapped to pre-fixpoints, and post-fixpoints to post-fixpoints).

## 2 Searching for the origins

We found useful references in [LNS82, Cam78, RR63, RR85, GKW12, Fel62]. The oldest result that we could find is:

In 1909, Hessenberg proved that, if  $X$  is a set of sets,  $\leq$  is inclusion, and  $f$  is extensive, then  $f$  has a fixpoint [Hes09]. This is explained in [Fel62] (as mentioned in [Fel00]<sup>2</sup> and [GKW12] p. 158). Unfortunately, these two papers are in German and I cannot get a more precise idea of their contents.

In 1922, Kuratowski proved that, if  $X$  is a set of sets,  $\leq$  is inclusion, and  $f$  is extensive, then  $f$  has a fixpoint, namely  $\text{lub}(N)$  [Kur22]. He also proved that  $N$  is equal to  $A$  and that, if  $f$  is also monotone, then  $\text{lub}(N)$  is the smallest fixpoint of  $f$  that is greater than or equal to  $a_0$ .

In 1927, Knaster and Tarski proved that, if  $X$  is a set of sets,  $\leq$  is inclusion and  $f$  is monotone, then  $f$  has a fixpoint [KT28].

In 1939, Tarski extended this result to arbitrary complete lattices (every subset, empty or not, has a lub and a glb) [Tar55]. In fact, he proved that  $\text{FP}(f)$  itself is a non-empty complete lattice. In 1951, Davis proved the converse, that is, a lattice is complete if every monotone map has a fixpoint [Dav55]. By the way, note that, in 1941, Frink proved that any complete lattice is compact in the interval topology [Fri42].

On the other hand, Tarski did not study whether the least fixpoint of  $f$ , proved to be  $\text{glb}(\text{PostFP}(f))$ , can be reached by transfinite iteration. However, he mentioned p. 305 that, if  $X$  is an  $\omega$ -complete lattice (every countable subset has a lub),  $f$  is  $\omega$ -continuous ( $f(\text{lub}Y) = \text{lub}f(Y)$  for every countable  $Y \subseteq X$ )

---

<sup>2</sup>[Fel00]: "Returning to Hessenberg, his paper Kettentheorie und Wohlordnung. Crelle 135 (1909) 81-133

can hardly be underestimated in its importance. Not that it was understood by his contemporaries. But Hessenberg, analyzing Zermelos second proof of the well ordering theorem, studied the general ways to construct well ordered subsets of ordered sets - with the one restriction that order always was inclusion and ordered sets were subfamilies of power sets. In the course of this, Hessenberg stated and proved the fixpoint theorem which thirty years later was rediscovered - for ordered sets now - by Nicolas Bourbaki. The amazing thing is that Hessenberg's proof is precisely the same as that given by Bourbaki ! (only that at one small point a simpler argument can be used due to the circumstance that Hessenberg's order is inclusion). For details, I refer to my article in Archiv d.Math. 13 (1962) 160-165 and to my book Naive Mengen und Abstrakte Zahlen from 1979 , p.200 ff ."

and  $a_0$  is the least element  $\perp$  of  $X$ , then  $a_{\omega+1} = a_\omega$ , a result sometimes attributed to Kleene because it is indeed used in his proof of the *first* recursion theorem with  $X$  being the set of partial functions in [Kle52] p. 348.

At the end of the 30's, Bourbaki<sup>3</sup> proved that, if  $X$  is a strictly inductive poset and  $f$  is extensive, then  $f$  has a fixpoint [Bou39, Bou49]. This is a generalization of Hessenberg and Kurakowski's results. This result was rediscovered or proved (because Bourbaki published a proof in 1949 only) at the end of the 40's or beginning of the 50's by many other people: [Kne50, Sze50, Wit50b, Wit50a, Vau52, Ina52].

Note that fixpoint theorems assuming that  $f$  is extensive easily follow from Zorn's maximal principle (equivalent to AC) saying that any non-empty inductive (= chain-complete) set of sets ordered by inclusion (every  $\subseteq$ -chain, including the empty one, has a lub) has a maximal element (for inclusion) [Zor35], or Tukey's maximal principle (equivalent to AC too) generalizing Zorn's one to arbitrary inductive posets (hence saying that any non-empty inductive poset has a maximal element) [Tuk40]. Indeed,  $A$  being a non-empty inductive poset (see Lemma 3 below), it has a maximal element  $a_k$ . Since  $a_k \leq a_{k+1}$  and  $a_k$  is maximal,  $a_{k+1} = a_k$ . But the point of the previous authors was to prove Zermelo's result that AC implies the well order theorem (every set can be well ordered) [Zer04] without using ordinal theory (like Zermelo in the second proof of his theorem [Zer08]).

At the end of the 50's, by using Hartogs theorem (for any set  $A$ , there is an ordinal  $k$  that cannot be injected into  $A$ ) [Har15], Rubin and Rubin proved that, if  $X$  is a strictly inductive set of sets,  $\leq$  is inclusion and  $f$  is extensive, then  $a_{k+1} = a_k$  for some  $k$  [RR63] (p. 18).

In 1957, Ward extended Tarski, Frink and Davis results to complete semi-lattices (every non-empty subset has a lub but not necessarily a glb) [War57]. Hence, a semi-lattice  $X$  is complete iff, for every  $x \in X$ ,  $x \downarrow = \{y \in X \mid y \leq x\}$  is compact in the interval topology;  $\text{FP}(f)$  is a complete semi-lattice if  $X$  is a complete semi-lattice and  $f : X \rightarrow X$  is monotone; a semi-lattice is compact in the interval topology iff every monotone  $f : X \rightarrow X$  has a fixed point.

In 1959, by refining Bourbaki's result, Abian and Brown extended Tarski's result to strictly inductive posets having a pre-fixpoint of  $f$  [AB61].

In 1962, using Hartogs theorem, Devidé proved that, in a complete lattice, if  $f(x) = a_0 \vee g(x)$  with  $g$  monotone, then there is  $k \in \mathbb{O}$  such that  $a_{k+1} = a_k$  [Dev64] (note that  $f$  does not need to be extensive, although it is so on  $A$ ).

Up to now, we have seen that all conditions for  $f$  to have a fixpoint are

---

<sup>3</sup>A result due to Chevalley after [Cam78].

requiring either  $f$  to be monotone or  $f$  to be extensive. So, one may wonder what relations are known between these two classes of functions, and whether one cannot devise a condition generalizing both.

As for a relation between monotony and extensivity, we have:

**Lemma 1 ([Bou53] p. 35)** On a well ordered poset  $(X, \leq)$  (every non-empty subset has a least element), any strictly monotone function  $f : X \rightarrow X$  is extensive.

**Proof.** Assume that  $f$  is not extensive. Then, the set  $E = \{x \in X \mid f(x) < x\}$  is not empty. Let  $e$  be its least element. By definition,  $f(e) < e$ . By strict monotony,  $f(f(e)) < f(e)$ . Hence,  $f(e) \in E$  and,  $e \leq f(e)$  by definition of  $e$ . Contradiction. ■

As for a condition subsuming both notions, we have:

In 1969, Salinas extended the previous fixpoint theorems by requiring (P1)  $a_0 \in \text{PreFP}(f)$ , *i.e.*  $a_0 \leq f(a_0)$ , and:

$$(P2) \quad f(x) \leq f(y) \text{ if } x \leq f(x) \leq y \text{ [Sal69].}$$

He provided two proofs, one using AC and Hartogs theorem, and another one not using AC but some notion of chain<sup>4</sup>. He also proved (using AC) that, if  $X$  is not strictly inductive but the set of upper bounds of every non-empty chain of  $X$  has a minimal element,  $f$  satisfies (P1) and (P2), and  $\text{glb}\{x, f(x)\}$  exists for all  $x$ , then  $f$  has a fixpoint.

In 1973, by using Bourbaki's theorem, Markowsky extended Tarski's results to inductive (= chain-complete) posets (every chain has a lub), that is,  $\text{FP}(f)$  is chain-complete if  $X$  is chain-complete, and proved the converse, that is,  $X$  is chain-complete if every monotone (or glb-preserving) map  $f : X \rightarrow X$  has a *least* fixpoint [Mar76].

In 1974, Pasini extended Salinas' result by proving that  $\text{FP}(f)$  has a maximal element [Pas74].

In 1977, Cousot and Cousot studied the properties of  $a = (a_k)_{k \in \mathbb{N}}$  when  $f$  is monotone and  $X$  a complete lattice (remarking however that their results extend to posets every chain of which has a lub and a glb) [CC79]. In particular, they proved that, in a complete lattice,  $A$  is bounded by every post-fixpoint bigger than or equal to  $a_0$ , hence that  $a$  can only converge to the *least* fixpoint of  $f$  bigger than or equal to  $a_0$ , and indeed converges to this fixpoint if  $f$  is monotone and  $a_0 \leq f(a_0)$ . They also extended Tarski's result by showing that  $\text{PreFP}(f)$  and  $\text{PostFP}(f)$  are non-empty complete lattices too.

---

<sup>4</sup>The notion of chain wrt a function  $f$  has been first introduced by Dedekind for defining the notion of infinite set [Ded88]. It has been used by Zermelo in his second proof that AC implies the well order theorem [Zer08], and studied in details by Hessenberg [Hes09].

### 3 Synthesis

In conclusion, the most direct argument not using AC why there must be some  $k \in \mathbb{O}$  such that  $a_{k+1} = a_k$  is the one of Rubin [RR63] based on Hartogs theorem [Har15]. We hereafter split this result in two parts by showing first that, by Hartogs theorem,  $f$  has a fixpoint if  $a$  is monotone, and second that,  $a$  is monotone if  $f$  satisfies Salinas conditions (P2).

**Lemma 2** If  $a$  is monotone, then there is an ordinal  $k$  such that  $a_{k+1} = a_k$ .

**Proof.** By Hartogs theorem, there is an ordinal  $k$  that cannot be injected into  $A$  (the smallest such one is a cardinal). Therefore,  $a|_k$ , the restriction of  $a$  to  $k$ , is not an injection, that is, there are  $l_1 < l_2 < k$  such that  $a_{l_1} = a_{l_2}$ . Since  $a$  is monotone, we have  $a_{l_1+1} = a_{l_1}$ . ■

Now, one can easily prove that  $a$  is monotone whenever  $a_0$  is a pre-fixpoint of  $f$  and  $f$  satisfies Salinas condition (P2) above:

**Lemma 3**  $a$  is monotone if  $a_0 \leq f(a_0)$  and  $f(x) \leq f(y)$  whenever  $x \leq f(x) \leq y$ , for all  $x$  and  $y$  in  $A$ .

**Proof.** We prove that  $a_k \leq a_l$  whenever  $k < l$  by induction on  $l$  (1).

- If  $l$  is a limit ordinal, then this is immediate.
- Otherwise,  $l = m + 1$  and  $k \leq m$ . If  $k < m$  then, by induction hypothesis (1), we have  $a_k \leq a_m$ . We now prove that, for all  $i \leq m$ ,  $a_i \leq a_{i+1}$ , by induction on  $i$  (2).
  - $i = 0$ .  $a_0 \leq a_1 = f(a_0)$  by assumption.
  - $i = j + 1$ . By induction hypothesis (2),  $a_j \leq a_{j+1} = a_i$ . Therefore, by (P2),  $a_{j+1} = a_i \leq a_{i+1}$ .
  - $i = \text{lub}(i)$ . Let  $j < i$ . By induction hypothesis (2),  $a_j \leq a_{j+1}$ . Since  $j + 1 < i \leq m$ , by induction hypothesis (1),  $a_{j+1} \leq a_i$ . Hence, by (P2),  $a_{j+1} \leq a_{i+1}$ . Therefore,  $a_j \leq a_{i+1}$  and  $a_i = \text{lub}\{a_j \mid j < i\} \leq a_{i+1}$ . ■

Using a nice result of Abian and Brown [AB61] for *any* poset  $(X, \leq)$  and *any* function  $f$ , we can go a little bit further and instead consider the condition:

(P2')  $f(x) \leq f(y)$  if  $x < f(x) \leq y$  and there is no  $z$  such that  $x < z < f(x)$

**Definition 1** ([AB61]) A set  $C \subseteq X$  is an  $a_0$ -chain if:

- $C$  is well ordered;
- $C$  has  $a_0$  as least element;
- $C$  is closed by non-empty lub's;
- if  $z \in C - \{\text{lub}(C)\}$ , then:

- $f(z) \in C$ ,
- $z < f(z)$ ,
- there is no  $y \in C$  such that  $z < y < f(z)$ .

Let  $W$  be the set of elements  $x \in X$  such that  $x$  is the lub of an  $a_0$ -chain.

In [Sal69], Salinas considered a similar (equal?) set called the set of admissible subsets of  $X$ .

**Theorem 1 ([AB61])** For any poset  $(X, \leq)$ ,  $a_0 \in X$  and function  $f : X \rightarrow X$ :

- $W$  is well ordered;
- $W$  has  $a_0$  as least element;
- if  $W$  has a lub  $\xi$ , then  $W$  is an  $a_0$ -chain with  $\xi$  as greatest element and  $\xi \not\leq f(\xi)$ .

Abian and Brown proved also that, for every  $x \in W$ , there is only one  $a_0$ -chain  $C$  such that  $x = \text{lub}(C)$ , namely  $\{y \in W \mid y \leq x\}$ .

Note also that  $W$  is not closed by  $f$  in general. However, they proved that, if  $x \in W$  and  $x \leq f(x)$ , then  $f(x) \in W$ .

**Theorem 2** In a non-empty strictly inductive poset  $(X, \leq)$ , if  $a_0 \in \text{PreFP}(f)$  and  $f : X \rightarrow X$  is monotone on  $W$ , then  $\text{lub}(W)$  is a fixpoint of  $f$ .

**Proof.** We simply follow the proof of Abian and Brown and check that, indeed, the monotony of  $f$  is used only on elements of  $W$ .

Since  $X$  is strictly inductive,  $\xi = \text{lub}(W)$  exists. By the previous theorem,  $W$  is an  $a_0$ -chain and  $\xi \not\leq f(\xi)$ . Since  $W$  has  $a_0$  as least element, we have  $a_0 \leq \xi$ . Since  $\xi \not\leq f(\xi)$ , it suffices to check that  $\xi \leq f(\xi)$ . If  $a_0 = \xi$ , then this is immediate since, by assumption,  $a_0 \leq f(a_0)$ . Assume now that  $a_0 < \xi$ . Then, since  $W$  is an  $a_0$ -chain, we have  $\xi \in W$  and  $V = W - \{\xi\}$  not empty, thus  $\theta = \text{lub}(V)$  exists and  $\theta \leq \xi$ . There are two cases:

- $\theta = \xi$ . Let  $x \in V$ . Then,  $x < \xi$ ,  $x < f(x) \in W$  and, by monotony of  $f$  on  $W$ ,  $f(x) \leq f(\xi)$ . Hence, for all  $x \in V$ ,  $x < f(\xi)$ . Therefore,  $\xi \leq f(\xi)$ .
- $\theta < \xi = f(\theta)$ . Then, by monotony of  $f$  on  $W$ ,  $\xi \leq f(\xi)$ . ■

Now, one can easily check that:

**Lemma 4**  $f$  is monotone on  $W$  if  $f$  satisfies (P2').

We now provide precise statements for Abian and Brown's claim that  $W = N$ . (Salinas also proved in [Sal69] that his set of *admissible* subsets of  $X$  is  $N$ .)

**Lemma 5**  $W \subseteq N$ .

**Proof.** It suffices to prove that  $W$  is included in every set  $Z$  containing  $a_0$  and closed by  $f$  and non-empty lub's. We proceed by well-founded induction on  $<$ . Let  $x \in W$ . If  $x = a_0$ , then we are done. Assume now that  $a_0 < x$ . After Lemma 4 in [AB61],  $x = \text{lub}(C)$  with  $C$  the  $a_0$ -chain  $\{y \in W \mid y \leq x\}$ . We then proceed as in Theorem 2. Since  $C$  is an  $a_0$ -chain and  $a_0 < x$ , we have  $D = C - \{x\}$  not empty, thus  $\theta = \text{lub}(D)$  exists and  $\theta \leq x$ . For all  $y < x$ , we have  $y \in Z$  by induction hypothesis. Therefore,  $\theta \in Z$  since  $Z$  is closed by non-empty lub's. If  $\theta = x$ , then we are done. Otherwise,  $x = f(\theta) \in Z$  since  $Z$  is closed by  $f$ . ■

**Lemma 6**  $N \subseteq W$  if  $X$  is strictly inductive,  $a_0 \in \text{PreFP}(f)$  and  $f$  is monotone on  $W$ .

**Proof.** It suffices to show that  $W$  contains  $a_0$  and is closed by  $f$  and non-empty lub's. By Theorem 1, we have  $a_0 \in W$  and  $W$  an  $a_0$ -chain. Hence,  $W$  is closed by non-empty lub's. Let  $\xi = \text{lub}(W)$  and  $x \in W$ . If  $x = \xi$ , then  $f(x) = x \in W$  since  $\xi \in \text{FP}(f)$  by Theorem 2. Otherwise,  $x < \xi$  and  $f(x) \in W$  since  $W$  is an  $a_0$ -chain. ■

For the sake of completeness, we also make precise Kuratowski's relations between  $A$  and  $N$ , when  $X$  is strictly inductive (for  $A$  to be well defined).

**Lemma 7** ([Kur22])  $A \subseteq N$ .

**Proof.** It suffices to prove that  $A$  is included in every set  $Z \subseteq X$  containing  $a$  and closed by  $f$  and non-empty lub's, by transfinite induction. Let  $a_k \in A$ . If  $k = 0$ , then  $a_k \in Z$  by assumption. If  $k = j + 1$ , then  $a_k = f(a_j) \in Z$  since, by induction hypothesis,  $a_j \in Z$  and  $Z$  is closed by  $f$ . Finally, if  $k = \text{lub}(k)$ , then  $a_k = \text{lub}\{a_j \mid j < k\} \in Z$  since  $Z$  is closed by non-empty lub's and, for all  $j < k$ ,  $a_j \in Z$  by induction hypothesis. ■

**Lemma 8**  $N \subseteq A$  if  $a$  is monotone.

**Proof.** Since  $A$  contains  $a_0$  and is closed by  $f$ , it suffices to prove that  $A$  is closed by non-empty lub's. Let  $Z$  be a non-empty subset of  $A$ . Then, there is a set  $K$  of ordinals such that  $Z = \{x \mid \exists k \in K, x = a_k\}$ . Since  $a$  is monotone,  $\text{lub}(Z)$  exists and equals  $a_k$  where  $k = \text{lub}(K)$  (every set of ordinals has a lub). ■

Hence, we can conclude:

**Theorem 3** If  $X$  is strictly inductive,  $a_0 \in \text{PreFP}(f)$  and  $f$  satisfies (P2'), then  $N = W = A$ . Therefore,  $N$ ,  $W$  and  $A$  are  $a_0$ -chains and there is  $k$  such that  $a_{k+1} = a_k = \text{lub}(N)$  is the *least* fixpoint of  $f$  bigger than or equal to  $a_0$ .

**Proof.** Since  $f$  satisfies (P2'),  $f$  is monotone on  $W$ . Hence,  $N = W$ . Since  $A \subseteq N$ ,  $f$  is monotone on  $A$ . Hence,  $a$  is monotone. Therefore,  $N = A$ . ■



Note that, if  $\xi = \text{lub}(W)$ , then  $f$  is both strictly monotone and strictly extensive on  $W - \{\xi\}$ .

We finish with some ultimate remarks.

In [Dev64], with  $X$  a complete lattice, Devidé takes  $f(x) = a_0 \vee g(x)$  with  $g$  monotone. In this case, one can easily check that, if  $a_0 \leq g(a_0)$ , then  $f$  and  $g$  have the same set of fixpoints bigger than or equal to  $a_0$ . Moreover, the transfinite iterates of  $f$  and  $g$  are equal.

Now, consider  $f(x) = x \vee g(x)$  with  $g$  monotone. Then,  $f$  is both monotone and extensive, and  $f$  and  $g$  are equal on  $\text{PreFP}(g)$ . Moreover,  $\text{FP}(g) \subseteq \text{FP}(f) = \text{PostFP}(g)$  and, if  $X$  has at least two distinct elements, then  $\text{FP}(f) \not\subseteq \text{FP}(g)$ : if  $g$  is the constant function equal to the least element  $\perp$  of  $X$ , then  $\text{FP}(g) = \{\perp\}$  and  $\text{FP}(f) = X \neq \{\perp\}$  since  $f$  is the identity and  $X$  has at least two elements. However, the *least* fixpoint of  $f$  is also the *least* fixpoint of  $g$ . Moreover, if  $a_0 \leq g(a_0)$ , then the transfinite iterates of  $f$  and  $g$  are equal.

## References

- [AB61] S. Abian and A. B. Brown. A theorem on partially ordered sets with applications to fixed point theorems. *Canadian Journal of Mathematics*, 13:78–83, 1961.
- [Bou39] N. Bourbaki. *Eléments de mathématique, 1ère partie: les structures fondamentales de l'analyse, Livre I: théorie des ensembles, fascicule de résultats*. Number 846 in *Actualités scientifiques et industrielles*. Hermann et Cie, 1939.
- [Bou49] N. Bourbaki. Sur le théorème de Zorn. *Archiv der Mathematik*, 2(6):434–437, 1949.
- [Bou53] N. Bourbaki. Livre I. Théorie des ensembles. Chapitre III (état 6). Ensembles ordonnés, cardinaux, nombres entiers. Technical Report 189 NBR 092, Association des Collaborateurs de Nicolas Bourbaki, 1953.
- [Cam78] P. J. Campbell. The origin of "Zorn's lemma". *Historia Mathematica*, 5:77–89, 1978.
- [CC79] P. Cousot and R. Cousot. Constructive versions of Tarski's fixed point theorems. *Pacific Journal of Mathematics*, 82(1):43–57, 1979.
- [Dav55] A. C. Davis. A characterization of complete lattices. *Pacific Journal of Mathematics*, 5:311–319, 1955.
- [Ded88] R. Dedekind. *Was sind und was sollen die Zahlen?* Vieweg, Braunschweig, 1888. English translation in [PRS95].
- [Dev64] V. Devidé. On monotonous mappings of complete lattices. *Fundamenta Mathematicae*, LIII:147–154, 1964.

- [Fel62] W. Felscher. Doppelte hülleninduktion und ein satz von Hessenberg und Bourbaki. *Archiv der Mathematik*, 13(1):160–165, 1962.
- [Fel00] W. Felscher. Re: [HM] early set theory texts, 2000.
- [Fri42] O. Frink. Topology in lattices. *Trans. Amer. Math. Soc.*, 51:569–582, 1942.
- [GKW12] D. M. Gabbay, A. Kanamori, and J. Woods, editors. *Sets and Extensions in the Twentieth Century*, volume 6 of *Handbook of the History of Logic*. North-Holland, 2012.
- [Har15] F. Hartogs. über das problem der wohlordnung. *Mathematische Annalen*, 76:438–443, 1915.
- [Hes09] G. Hessenberg. Kettentheorie and wohlordnung. *Journal für die reine und angewandte Mathematik*, 135:81–133, 1909.
- [Ina52] T. Inagaki. Sur deux théorèmes concernant un ensemble partiellement ordonné. *Mathematical Journal of Okayama University*, 1(1):167–176, 1952.
- [Kle52] S. C. Kleene. *Introduction to metamathematics*. North-Holland, 1952.
- [Kne50] H. Kneser. Eine direkte ableitung des zornschen lemmas aus dem auswahlaxiom. *Mathematische Zeitschrift*, 53:110–113, 1950.
- [KT28] B. Knaster and A. Tarski. Un théorème sur les fonctions d’ensembles. *Annales de la Société Polonaise de Mathématiques*, 6:133–134, 1928.
- [Kur22] C. Kuratowski. Une méthode d’élimination des nombres transfinis des raisonnements mathématiques. *Fundamenta Mathematicae*, 3(1):76–108, 1922.
- [LNS82] J.-L. Lassez, V. L. Nguyen, and E. A. Sonenberg. Fixed point theorems and semantics: a folk tale. *Information Processing Letters*, 14(3):112–116, 1982.
- [Mar76] G. Markowsky. Chain-complete posets and directed sets with applications. *Algebra Universalis*, 6:53–68, 1976.
- [Pas74] A. Pasini. Some fixed point theorems of the mappings of partially ordered sets. Technical Report Rendiconti del Seminario Matematico 51, Università di Padova, 1974.
- [Pea90] G. Peano. Démonstration de l’intégrabilité des équations différentielles ordinaires. *Mathematische Annalen*, 37:182–229, 1890.
- [PRS95] H. Pogorzelski, W. Ryan, and W. Snyder, editors. *What Are Numbers and What Should They Be?* Research Institute for Mathematics, Orono, Maine, 1995. English translation of [Ded88].

- [RR63] H. Rubin and J. E. Rubin. *Equivalents of the Axiom of Choice*. North-Holland, 1963.
- [RR85] H. Rubin and J. E. Rubin. *Equivalents of the Axiom of Choice, II*. Number 116 in Studies in Logic and the Foundations of Mathematics. North-Holland, 1985.
- [Sal69] B. R. Salinas. Puntos fijos en conjuntos ordenados. Technical report, Publicaciones del Seminario Matematico Garcia de Galdeano, Consejo Superior de Investigaciones Cientificas, Facultad de Ciencias de Zaragoza, 1969. English summary on <http://www.ams.org/mathscinet-getitem?mr=250943>.
- [Sze50] T. Szele. On Zorn's lemma. *Publicationes Mathematicae*, pages 254–257, 1950.
- [Tar55] A. Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5:285–309, 1955.
- [Tuk40] J. W. Tukey. *Convergence and uniformity in topology*, volume 2 of *Annals of Mathematics Studies*. Princeton University Press, 1940.
- [Vau52] H. E. Vaughan. Well-ordered subsets and maximal members of ordered sets. *Pacific Journal of Mathematics*, 2(3):271–429, 1952.
- [vH77] J. v. Heijenoort, editor. *From Frege to Gödel, a source book in mathematical logic, 1879-1931*. Harvard University Press, 1977.
- [vN23] J. von Neumann. Zur einföhrung der transfiniten zahlen. *Acta Szeged*, 1:199–208, 1923. Reprinted in [vN61].
- [vN61] J. von Neumann. *Collected Works, vol. I: Logic, Theory of Sets and Quantum Mechanics*. Pergamon Press, 1961.
- [War57] L. E. Ward Jr. Completeness in semi-lattices. *Canadian Journal of Mathematics*, 9:578–582, 1957.
- [Wit50a] E. Witt. Beweisstudien zum satz von M. Zorn. *Mathematische Nachrichten*, 4:434–438, 1950.
- [Wit50b] E. Witt. Sobre el teorema de Zorn. *Revista matemática hispanoamericana*, 10(2):82–85, 1950.
- [Zer04] E. Zermelo. Beweis, das jede menge wohlgeordnet werden kann. *Mathematische Annalen*, 59(4):514–516, 1904. English translation in [vH77].
- [Zer08] E. Zermelo. Untersuchungen über die grundlagen der mengenlehre I. *Mathematische Annalen*, 65(2):261–281, 1908. English translation in [vH77].
- [Zor35] M. Zorn. A remark on method in transfinite algebra. *Bulletin of American Mathematical Society*, 41:667–670, 1935.