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**Abstract**

Theorems on Implicit Lyapunov Functions (ILF) for finite-time and fixed-time stability analysis of nonlinear systems are presented. Based on these resutls, new nonlinear control laws are designed for robust stabilization of a chain of integrators. High order sliding mode (HOSM) algorithms are obtained as particular cases. Some aspects of digital implementations of the presented algorithms are studied, it is shown that they possess a chattering reduction ability. Theoretical results are supported by numerical simulations.

*Key words:* Lyapunov Methods, Robust Control, Sliding Mode Control

## 1 Introduction and related works

Many practical applications require severe time response constraints (for security reasons, or simply to improve productivity). That is why, finite-time stability and stabilization problems have been intensively studied, see [2], [3], [4], [5], [6]. Time constraint may also appear in observation problems when a finite-time convergence of the state estimate to the real values is required: [7], [8], [9], [10], [11].

Let us stress that finite-time stability property is frequently associated with HOSM controls, since these algorithms should ensure finite-time convergence to a sliding manifold [12], [13], [14], [15]. Typically, the associated controllers have mechanical and electromechanical applications [16], [17], [18], [19].

The theoretical background of HOSM control systems is very well developed [12], [20], [21]. However, applications of the existing HOSM control algorithms are complicated, since there are a few constructive algorithms for tuning the HOSM control parameters. Most of them are restricted to the second order sliding mode systems [21], [15], [22], [23].

*Fixed-time stability*, that demands *boundedness of the settling-time function* for a globally finite-time stable system, was studied in [23], [22], [24]. This property was originally discovered in the context of homogeneous systems [25]. Fixed-time stability looks promising if a controller (observer) has to be designed in order to provide some required control (observation) precision in *a given time* and *independently of initial conditions*.

The main tool for analysis of finite-time and fixed-time stability is the Lyapunov function method (see, for example, [4], [5], [24]), which is lacking for constructive design in the nonlinear case. This paper deals with ILF method [26], which relies on Lyapunov functions defined, implicitly, as solutions to an algebraic equation. Stability analysis does not require solving this equation, since the implicit function theorem [27] helps to check all stability conditions directly from the implicit formulation. The similar approach was presented in [28] for control design and called the controllability function method (see, also [29]).

This paper addresses the problem of a control design for the robust finite-time and fixed-time stabilization of a chain of integrators. The ILF method is used to design the control laws together with Lyapunov functions for closed-loop systems. This method allows us to analyze robustness of the closed-loop system and to design *a high order sliding mode control algorithm*, which rejects bounded matched exogenous disturbances. Finite-time and fixed-time stability conditions were obtained in the form of Linear Matrix Inequalities (LMI). They provide *simple constructive schemes for*

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*tuning the control parameters* in order to predefine the required convergence time and/or to guarantee stability and robustness with respect to disturbances of a given magnitude.

Through the paper the following notation will be used:

- $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$;
- $\frac{dV}{dt}\big|_{(.)}$ is the time derivative of a function $V$ along the solution of a differential equation numbered as (.);
- $\|\cdot\|$ is the Euclidian norm in $\mathbb{R}^n$;
- $\mathrm{diag}\{\lambda_i\}_{i=1}^n$ is the diagonal matrix with the elements $\lambda_i$;
- a continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is monotone increasing and $\sigma(h) \to 0^+$ as $h \to 0^+$;
- for a symmetric matrix $P = P^T$ the minimal and maximal eigenvalues are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$,
- $\mathrm{int}(\Omega)$ is the interior of the set $\Omega \subseteq \mathbb{R}^n$.

## 2 Problem statement

The paper deals with finite-time and fixed-time stabilization problems for the disturbed chain of integrators. Note that a control design scheme developed for such systems usually admits extension to feedback linearizable nonlinear multi-input multi-output systems [30]. The problem statement presented below is also typical for high-order sliding mode control design [12].

Consider a linear single input system of the following form

$$\dot{x}(t) = Ax(t) + bu(t) + d(t, x(t)), \quad t > 0, \tag{1}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ 1 \end{pmatrix},$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, and the function $d : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ describes the system uncertainties and disturbances. The whole state vector $x$ is assumed to be measured. Let the function $d$ be measurable locally bounded uniformly in time, i.e. $\sup\limits_{t\in\mathbb{R}_+, x\in\mathbb{R}^n : \|x\|<\delta} \|d(t,x)\| < \infty$ for any $\delta > 0$. Both the control function $u$ and the function $d$ are admitted to be discontinuous with respect to $x$. For example, the function $d$ may describe unknown dry friction of a mechanical model. The analysis of such systems requires a special mathematical framework. In this paper we use Filippov theory [31].

The goal of the paper is to develop control laws such that the origin of the closed-loop system (1) will be globally asymptotically stable and all its trajectories will reach the origin in a finite time or in the fixed time $T_{\max} \in \mathbb{R}_+$. In addition, the control algorithms to be developed should have effective schemes for tuning the control parameters and assigning of the settling time.

The control design is based of ILF approach to finite-time and fixed-time stability analysis, which is developed in the next section.

## 3 Preliminaries

### *3.1 Finite-time and fixed-time stability*

Consider the system defined by

$$\dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}_+, \quad x(0) = x_0, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector field locally bounded uniformly in time. If $f$ is a locally measurable function that is discontinuous with respect to the state variable $x$ then a solution of the Cauchy problem (2) is understood in the sense of Filippov [31], namely, as an absolutely continuous function satisfying the differential inclusion

$$\dot{x}(t) \in K[f](t, x(t))$$

for almost all $t \in [0, t^*)$, where $t^* \in \mathbb{R}_+$ or $t^* = +\infty$. The set-valued function $K[f] : \mathbb{R}_+ \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined for any fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ as follows

$$K[f](t,x) = \bigcap_{\varepsilon>0} \bigcap_{N:m(N)=0} \mathrm{co}\, f(t, B(x,\varepsilon)\backslash N),$$

where $\mathrm{co}(M)$ defines the convex closure of the set $M \subset \mathbb{R}^n$, $B(x,\varepsilon)$ is the ball with the center at $x \in \mathbb{R}^n$ and the radius $\varepsilon \in \mathbb{R}_+$, the equality $m(N) = 0$ means that the Lebesgue measure of the set $N \subset \mathbb{R}^n$ is zero.

Let the origin be an equilibrium of (2), i.e. $0 \in K[f](t, 0)$. The system (2) may have non-unique solutions and may admit both weak and strong stability (see, for example, [31]). This paper deals only with the *strong stability* properties, which ask for stable behavior of all solutions of the system (2). The next definition of uniform finite-time stability is just a compact representation of Definition 2.5 from [13].

**Definition 1** *The origin of system* (2) *is said to be globally* ***uniformly finite-time stable*** *if it is globally uniformly asymptotically stable (see, for example, [13]) and there exists a* ***locally bounded*** *function* $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$, *such that* $x(t, x_0) = 0$ *for all* $t \geq T(x_0)$, *where* $x(\cdot, x_0)$ *is an arbitrary solution of the Cauchy problem* (2). *The function* $T$ *is called the* ***settling-time function***.

Asymptotic stability of the time-independent (autonomous) system always implies its uniform asymptotic stability (see, for example, [32]). For finite-time stable systems this is not true in general case (see, for example, [24]), since Definition 1 additionally asks a uniformity of the settling time with respect to initial conditions.

The origin of system $\dot{x}(t) = -|x(t)|^{0.5}\,\mathrm{sign}[x(t)], x \in \mathbb{R}$ is globally uniformly finite-time stable, since its settling-time function $T$ is locally bounded: $T(x_0) = 2\sqrt{|x_0|}$.

**Definition 2 ([23])** *The origin of system* (2) *is said to be globally* ***fixed-time stable*** *if it is globally uniformly finite-time stable and the settling-time function $T$ is* ***globally bounded****, i.e.* $\exists T_{\max} \in \mathbb{R}_+$ *such that* $T(x_0) \leq T_{\max}$, $\forall x_0 \in \mathbb{R}^n$.

The presented definition just asks more: strong uniformity of finite-time stability with respect to initial condition. The origin of system $\dot{x}(t) = -\left(|x|^{0.5}(t)+|x|^{1.5}(t)\right)\operatorname{sign}(x(t)), x \in \mathbb{R}$, is globally fixed-time stable, since its settling time function $T(x_0) = 2\arctan(\sqrt{|x_0|})$ is bounded by $\pi \approx 3.14$ .

The uniformity of finite-time and fixed-time stability with respect to system disturbances can also be analyzed. For instance, finite-time stability, which is uniform (in some sense) with respect to both initial conditions and system disturbances, was called equiuniform finite-time stability [13].

### *3.2 Implicit Lyapunov Function Method*

This subsection introduces some stability theorems further used for control design. They refine the known results about global uniform asymptotic, finite-time and fixed-time stability of differential inclusions to the case of implicit definition of Lyapunov function. The next theorem extends Theorem 2 from [26].

**Theorem 3** *If there exists a continuous function*

$$Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$$
$$(V, x) \to Q(V, x)$$

*satisfying the conditions*
*C1) $Q$ is continuously differentiable outside the origin;*
*C2) for any $x \in \mathbb{R}^n\backslash\{0\}$ there exists $V \in \mathbb{R}_+$ such that* $Q(V, x) = 0$*;*
*C3) let* $\Omega = \{(V, x) \in \mathbb{R}_+ \times \mathbb{R}^n : Q(V, x) = 0\}$ *and*
$$\lim_{\substack{x\to 0\\(V,x)\in\Omega}} V = 0^+, \quad \lim_{\substack{V\to 0^+\\(V,x)\in\Omega}} \|x\| = 0, \quad \lim_{\substack{\|x\|\to\infty\\(V,x)\in\Omega}} V = +\infty;$$
*C4)* $\frac{\partial Q(V,x)}{\partial V} < 0$ *for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n\backslash\{0\}$;*
*C5)* $\sup\limits_{t\in\mathbb{R}_+, y\in K[f](t,x)} \frac{\partial Q(V,x)}{\partial x} y < 0$ *for all $(V, x) \in \Omega$;*
*then the origin of* (2) *is globally uniformly asymptotically stable.*

**Proof.** The conditions *C1*), *C2*), *C4*) and the implicit function theorem [27] imply that the equation $Q(V, x) = 0$ implicitly defines a unique function $V : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}_+$ such that $Q(V(x), x) = 0$ for all $x \in \mathbb{R}^n\backslash\{0\}$. The function $V$ is continuously differentiable outside the origin and $\frac{\partial V}{\partial x} = -\left[\frac{\partial Q(V,x)}{\partial V}\right]^{-1} \frac{\partial Q(V,x)}{\partial x}$ for $Q(V, x) = 0$, $x \neq 0$. Due to the condition *C3*) the function $V$ can be continuously prolonged at the origin by setting $V(0) = 0$. In addition, it is radially unbounded and positive definite. Denote $Z(V, x) = \left[\frac{\partial Q}{\partial V}\right]^{-1} \sup\limits_{t\in\mathbb{R}_+, y\in K[f](t,x)} \frac{\partial Q(V,x)}{\partial x} y$ and $W(x) = Z(V(x), x)$. The conditions *C4* and *C5* imply $W(x) > 0$ for $x \neq 0$. Let $x(t, x_0)$ be a solution of the system (2) with initial condition $x(0, x_0) = x_0$ then the function $V(x(t, x_0))$ is differentiable for almost all $t$ such that $x(t, x_0) \neq 0$ and $\frac{d}{dt}V(x(t, x_0)) \leq -W(x(t, x_0))$. Finally, we finish the proof using Theorem 4.1 from [33] and, for example, Lemmas 4, 6 from [24]. ■

Evidently, the conditions of Theorem 3 mainly repeat (in the implicit form) the requirements of the classical theorem on global asymptotic stability (see, for example, [33]). Indeed, Condition *C1)* asks for smoothness of the Lyapunov function. Condition *C2)* and the first two limits from Condition *C3)* provide its positive definiteness. The last limit from Condition *C3)* implies radial unboundedness of the Lyapunov function. Condition *C5)* guarantees the negative definiteness of the total derivative of the Lyapunov function calculated along trajectories of the system (2). The only specific condition is *C4)*, which is imposed by implicit function theorem (see, for example, [27]). This condition is required in order to guarantee that the Lyapunov function is (uniquely) well-defined by the equation $Q(V, x) = 0$.

**Theorem 4** *If there exists a continuous function* $Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *that satisfies the conditions C1)-C4) of Theorem 3 and the condition*
*C5bis) there exist $c > 0$ and $0 < \mu \leq 1$ such that*
$$\sup_{t\in\mathbb{R}_+, y\in K[f](t,x)} \frac{\partial Q(V,x)}{\partial x} y \leq cV^{1-\mu} \frac{\partial Q(V,x)}{\partial V} \text{ for } (V, x) \in \Omega,$$
*then the origin of the system* (2) *is globally uniformly finite-time stable and* $T(x_0) \leq \frac{V_0^\mu}{c\mu}$, *where* $Q(V_0, x_0) = 0$.

**Proof.** Theorem 3 implies global uniform asymptotic stability of the origin of (2). The uniform finite-time stablity of the origin follows from the differential inequality
$$\frac{dV(x(t,x_0))}{dt} \leq -cV^{1-\mu}(x(t, x_0)),$$
which, due to the condition *C5bis*), holds for almost all $t$ such that $x(t, x_0) \neq 0$. For a detailed analysis of the obtained differential inequality see, for example, [4], [5], [24]. ■

**Theorem 5** *Let there exist two functions $Q_1$ and $Q_2$ satisfying the conditions C1)-C4) of Theorem 3 and the conditions*
*C6)* $Q_1(1, x) = Q_2(1, x)$ *for all $x \in \mathbb{R}^n\backslash\{0\}$;*
*C7) there exist $c_1 > 0$ and $0 < \mu < 1$ such that the inequality*
$$\sup_{t\in\mathbb{R}_+, y\in K[f](t,x)} \frac{\partial Q_1(V,x)}{\partial x} y \leq c_1 V^{1-\mu} \frac{\partial Q_1(V,x)}{\partial V},$$
*holds for all $V \in (0, 1]$ and $x \in \mathbb{R}^n\backslash\{0\}$ satisfying the equation $Q_1(V, x) = 0$;*
*C8) there exist $c_2 > 0$ and $\nu > 0$ such that the inequality*
$$\sup_{t\in\mathbb{R}_+, y\in K[f](t,x)} \frac{\partial Q_2(V,x)}{\partial x} y \leq c_2 V^{1+\nu} \frac{\partial Q_2(V,x)}{\partial V},$$
*holds for all $V \geq 1$ and $x \in \mathbb{R}^n\backslash\{0\}$ satisfying the equation $Q_2(V, x) = 0$, then the system* (2) *is globally fixed-time stable with the settling-time estimate* $T(x_0) \leq \frac{1}{c_1\mu} + \frac{1}{c_2\nu}$.

**Proof.** Let the two functions $V_1$ and $V_2$ be defined by the equations $Q_1(V, x) = 0$ and $Q_2(V, x) = 0$ (see, the proof of Theorem 3). Consider the sets $\Sigma_1 = \{x \in \mathbb{R}^n : V_1(x) > 1\}$,

$\Sigma_2 = \{x \in \mathbb{R}^n : V_2(x) > 1\}$ and prove that $\Sigma_1 = \Sigma_2$. Suppose the contrary, i.e. $\exists z \in \mathbb{R}^n$ such that $z \in \Sigma_1$ and $z \notin \Sigma_2$. On the one hand, $Q_1(V_1, z) = 0$ implies $V_1 > 1$ and $Q_1(1, z) > Q_1(V_1, z) = 0$ due to Condition C4). On the other hand, $Q_2(V_2, z) = 0$ implies $V_2 \leq 1$ and $Q_2(1, z) \leq Q_2(V_2, z) = 0$. The contradiction follows from Condition *C6)*.

Therefore, due to *C6*) and *C4*) the function $V : \mathbb{R}^n \to \mathbb{R}$ defined by the equality

$$V(x) = \begin{cases} V_1(x) & \text{for} \quad V_1(x) < 1, \\ V_2(x) & \text{for} \quad V_2(x) > 1, \\ 1 & \text{for} \quad V_1(x) = V_2(x) = 1, \end{cases}$$

is positive definite, continuous in $\mathbb{R}^n$ and continuously differentiable for $x \notin \{0\} \cup \{x \in \mathbb{R}^n : V(x) = 1\}$. The function $V$ is Lipschitz continuous outside the origin and has the following Clarke's gradient [34]:

$$\nabla_C V(x) = \xi \nabla V_1(x) + (1-\xi)\nabla V_2(x), \quad x \in \mathbb{R}^n,$$

where $\xi = 1$ for $0 < V_1(x) < 1$, $\xi = 0$ for $V_2(x) > 1$, $\xi = [0, 1]$ for $V_1(x) = V_2(x) = 1$ and $\nabla V_i$ is the gradient of the function $V_i$, $i = 1, 2$. Hence, due to conditions *C7*) and *C8*), the inequality

$$\frac{dV(x(t,x_0))}{dt} \leq \begin{cases} -c_1 V^{1-\mu}(x(t,x_0)) & \text{for} \quad V(x(t,x_0)) < 1, \\ -c_2 V^{1+\nu}(x(t,x_0)) & \text{for} \quad V(x(t,x_0)) > 1, \\ -\min\{c_1, c_2\} & \text{for} \quad V(x(t,x_0)) = 1, \end{cases}$$

holds for almost all $t$ such that $x(t, x_0) \neq 0$, where $x(t, x_0)$ is a solution of the system (2) with the initial condition $x(0) = x_0$. This implies the fixed-time stability of the origin of the system (2) with the estimate of settling-time function given above. Please see [23] or [24] for more details. ■

## 4 Control design using Implicit Lyapunov Function Method

### 4.1 Finite-Time Stabilization

Introduce the implicit Lyapunov function candidate

$$Q(V, x) := x^T D_r(V^{-1}) P D_r(V^{-1}) x - 1, \tag{3}$$

where $V \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}, P > 0$, $D_r(\lambda) := \mathrm{diag}\{\lambda^{r_i}\}$ is the matrix with $r = (r_1, ..., r_n)^T \in \mathbb{R}^n_+$ and

$$r_i = 1 + (n-i)\mu, \quad 0 < \mu \leq 1.$$

Denote $H_\mu := \mathrm{diag}\{-r_i\}_{i=1}^n$.

The function $Q$ is an implicit analog of the quadratic Lyapunov function. Indeed, for $\mu = 0$ the equality $Q(V, x) = 0$ gives $V(x) = \sqrt{x^T P x}$. For $\mu = 1$ it is coincides with the implicit Lyapunov function considered in [26].

**Theorem 6 (Finite-time Stabilization)** *Let 1) $\mu \in (0, 1]$, $\alpha, \beta, \gamma \in \mathbb{R}_+$ such that $\alpha > \beta$ and the system of matrix inequalities*

$$\begin{cases} AX + XA^T + by + y^T b^T + \alpha X + \beta I_n \leq 0, \\ -\gamma X \leq XH_\mu + H_\mu X < 0, \quad X > 0 \end{cases} \tag{4}$$

*is feasible for some $X = X^T \in \mathbb{R}^{n\times n}$ and $y \in \mathbb{R}^{1\times n}$;*

*2) the control $u$ has the form*

$$u(V, x) = V^{1-\mu} k D_r(V^{-1}) x, \tag{5}$$

*where $k := yX^{-1}$, $V \in \mathbb{R}_+$ is such that $Q(V, x) = 0$ and $Q$ is defined by* (3) *with $P = X^{-1}$;*

*3) the function $d(t, x)$ satisfy the inequality*

$$\beta^2 V^{-2\mu} \geq \sup_{t\in\mathbb{R}_+} d^T(t, x) D_r^2(V^{-1}) d(t, x), \tag{6}$$

*for all $V \in \mathbb{R}_+, x \in \mathbb{R}^n$ such that $Q(V, x) = 0$;*

*then the closed-loop system* (1) *is globally uniformly finite-time stable and $T(x) \leq \frac{\gamma V^\mu}{\mu(\alpha-\beta)}$, where $Q(V, x) = 0$.*

**Proof.** It can be shown that the following inequalities $\frac{\lambda_{\min}(P)\|x\|^2}{\max\{V^{2+2(n-1)\mu}, V^2\}} \leq Q(V, x) + 1 \leq \frac{\lambda_{\max}(P)\|x\|^2}{\min\{V^{2+2(n-1)\mu}, V^2\}}$ hold for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$. The function (3) satisfies conditions *C1*)-*C3*) of Theorem 3. The condition *C4*) of Theorem 3 also holds, since $0 > \frac{\partial Q}{\partial V} = V^{-1} x^T D_r(V^{-1})(H_\mu P + P H_\mu) D_r(V^{-1}) x \geq -\gamma V^{-1}$. Taking into account that $D_r(V^{-1}) A D_r^{-1}(V^{-1}) = V^{-\mu} A$ and $D_r(V^{-1}) b u(x) = V^{-\mu} b k D_r(V^{-1}) x$ we obtain $\frac{\partial Q}{\partial x}(Ax + bu(x) + d(t, x)) =$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^T W \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \beta^{-1} V^\mu z_2^T z_2 - \alpha V^{-\mu} z_1^T P z_1,$$

where $z_1 = D_r(V^{-1})x$, $z_2 = D_r(V^{-1})d$, the matrix

$$W := \begin{pmatrix} \frac{1}{V^\mu}\left(P(A+bk) + (A+bk)^T P + \alpha P\right) & P \\ P & -\frac{V^\mu}{\beta} I_n \end{pmatrix}$$

is negative semidefinite due to (4) and the Schur complement. Since $z_1^T P z_1 = 1$ and $z_2^T z_2 \leq \beta^2 V^{-2\mu}$ by (6) then the condition *C5bis*) of Theorem 4 holds for $c = (\alpha - \beta)/\gamma$. ■

Let us make remarks about the presented control scheme:

- The practical implementation of the control (5) requires to find the solution $V(x)$ of the equation $Q(V, x) = 0$, which can be solved numerically and on-line using the current value of the state vector. A simple numerical scheme that can be utilized for this purpose is presented in Section 5.
- If $n = 2$ and $\mu = 1$ the function $V(x)$ can be found analytically. Indeed, the equation $Q(V, x) = 0$ becomes
$$V^4 - p_{22}x_2^2 V^2 - 2p_{12}x_1x_2V - p_{11}x_1^2 = 0,$$
where $\{p_{ij}\}$ are elements of the matrix $P > 0$ and

$(x_1, x_2)^T = x \in \mathbb{R}^2$. The roots of the obtained quartic (with respect to $V$) equation can be found using, for example, Ferrari formulas. Due to Theorem 6 the equation has a unique positive root for any $(x_1, x_2)^T \in \mathbb{R}^2$.

Denote $z(x_1, x_2) = \frac{2p_{22}x_2^2 - C_1(x_1,x_2) - C_2(x_1,x_2)}{3}$, where $C_i(x_1, x_2) = \sqrt[3]{\frac{(-1)^i\sqrt{\Delta_1^2(x_1,x_2) - 4\Delta_0^3(x_1,x_2)} + \Delta_1(x_1,x_2)}{2}}$, $\Delta_0(x_1, x_2) = p_{22}^2 x_2^4 - 12p_{11}x_1^2$ and $\Delta_1(x_1, x_2) = p_{22}^3 x_2^6 - 180p_{12}^2 x_1^2 x_2^2$. The roots of the quartic equation coincide with roots of two quadratic equations [35]:

$$V^2 + (-1)^i\sqrt{z(x_1, x_2)}V + r_i(x_1, x_2) = 0, \quad i = 1, 2,$$

where $r_i(x_1, x_2) = \frac{p_{22}x_2^2 + z(x_1,x_2)}{2} + (-1)^i \frac{p_{12}x_1x_2}{\sqrt{z(x_1,x_2)}}$. The explicit representations of roots are rather cumbersome. They are omitted in order to safe the space.

- The implicit restriction (6) to the system disturbances and uncertainties takes an explicit form when $\mu = 1$ and the matching condition [14] holds, i.e. $d_i(t, x) \equiv 0$ for $i = 1, ..., n-1$ and $d = (d_1, ..., d_n)^T$. In this case the condition (6) becomes $|d_n(t, x)| \leq \beta$ and the term $\beta I_n$ in (4) can be replaced with $\beta E$, where the matrix $E = \{e_{ij}\} \in \mathbb{R}^{n\times n}$ has only one nonzero element: $e_{n,n} = 1$. So, the finite-time control (5) designed for $\mu = 1$ rejects the bounded disturbances realized **HOSM** algorithm. The HOSM version of (5) has a discontinuity only at zero similarly to the quasi-continuous HOSM algorithm [20].
- For $\mu = 1$ the control (5) is globally bounded. Indeed, the equality $x^T D_r(V^{-1})PD_r(V^{-1})x = 1$ implies $\|D_r(V^{-1})x\|^2 \leq \frac{1}{\lambda_{\min}(P)}$ and
$$\|u(x)\| \leq \|k\| \cdot \|D_r(V^{-1})x\| \leq \frac{\|k\|}{\sqrt{\lambda_{\min}(P)}}.$$
Hence the condition $\|u(x)\| \leq u_0$, $u_0 \in \mathbb{R}_+$ can be formulated as additional LMI:
$$\begin{pmatrix} X & y^T \\ y & u_0^2 \end{pmatrix} \geq 0. \tag{7}$$
Indeed, the inequality $u^2 = x^T D_r(V^{-1})k^T k D_r(V^{-1})x \leq u_0^2 = u_0^2 x^T D_r(V^{-1})PD_r(V^{-1})x$ holds for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$ such that $Q(V, x) = 0$. Hence $k^T k \leq u_0^2 P$ and Schur complement gives (7) for $X = P^{-1}$, $y = kX$.
- The advantage of the control design scheme presented in Theorem 6 is related to simplicity of tuning the control parameters, which is based on LMI technique. The parameters $\gamma$ and $\alpha$ are introduced in (4) in order to tune the settling time of the closed-loop system.

**Corollary 7** *Let the conditions of Theorem 6 hold and the control law is modified as $u(x) = u(V_0, x)$, where $V_0 \in \mathbb{R}_+$ is an arbitrary fixed number, then the ellipsoid*

$$\Pi(V_0) = \left\{ x \in \mathbb{R}^n : x^T \left(D_r(V_0)XD_r(V_0)\right)^{-1} x \leq 1 \right\} \tag{8}$$

*is strictly positively invariant set of the closed-loop system* (1), *i.e.* $x(t_0) \in \Pi(V_0) \Rightarrow x(t) \in \operatorname{int}\Pi(V_0), t > t_0$, *where* $x(\cdot)$ *is any trajectory of the closed-loop system* (1) *and* $t_0 \geq 0$ *is an arbitrary instant of time.*

**Proof.** Rewrite the matrix inequality (4) in the form: $PA + A^T P + Pbk + k^T b^T P + \alpha P + \beta P^2 \leq 0$. Hence we derive $D_r(V_0^{-1})(PA + A^T P + Pbk + k^T b^T P + \alpha P + \beta P^2)D_r(V_0^{-1}) \leq 0$. Denoting $P_0 = D_r(V_0^{-1})PD_r(V_0^{-1}) > 0$ and taking into account $D_r^{-1}(V_0^{-1})AD_r(V_0^{-1}) = V_0^{\mu}A$, $D_r^{-1}(V_0^{-1})b = V_0 b$ we obtain the following matrix inequality:

$$P_0A + A^T P_0 + P_0bk_0 + k_0^T b^T P_0 + \frac{\alpha P_0 + \beta P_0 D_r^2(V_0) P_0}{V_0^{\mu}} \leq 0,$$

where $k_0 = V_0^{1-\mu}kD_r(V_0^{-1})$, $\alpha > \beta > 0$ and $P_0 > 0$. This means that the matrix $A + bk_0$ is Hurwitz, i.e. $u(x) = u(V_0, x) = k_0 x$ is a stabilizing linear feedback control for the system (1) with the Lyapunov function $\tilde{V}(x) = x^T P_0 x$. Note that using Schur Complement the obtained matrix inequality can be transformed into the form $W_0 \leq 0$, where

$$W_0 = \begin{pmatrix} P_0(A+bk_0) + (A+bk_0)^T P_0 + \frac{\alpha}{V_0^{\mu}}P_0 & P_0 \\ P_0 & -\frac{V_0^{\mu}D^2(V_0^{-1})}{\beta} \end{pmatrix}.$$

Taking into account the inequality (6) for $V = V_0$ (or equivalently $\tilde{V} = 1$) we derive

$$\begin{aligned} \frac{d}{dt}\tilde{V}\Big|_{(1)} &= x^T(P_0A + A^T P_0 + P_0bk_0 + k_0^T b^T P_0)x + 2d^T P_0 x \\ &= \begin{pmatrix} x \\ d \end{pmatrix}^T W_0 \begin{pmatrix} x \\ d \end{pmatrix} + \frac{V_0^{\mu}}{\beta}d^T D_r^2(V_0^{-1})d - \frac{\alpha\tilde{V}}{V_0^{\mu}} \\ &\leq \frac{\beta - \alpha}{V_0^{\mu}} < 0 \quad \text{for} \quad \tilde{V}(x) = 1. \end{aligned}$$

The obtained inequality implies that the ellipsoid (8) is strictly positively invariant. ■

The next corollary proves stability of the sampled-time realization for the ILF-based control algorithm (5).

**Corollary 8** *Let $\{t_i\}_{i=0}^{\infty}$ be a strictly increasing sequence of arbitrary time instants, $0 = t_0 < t_1 < t_2 < ...$ such that $\lim_{i\to\infty} t_i = +\infty$. Let all conditions of Theorem 6 hold and the control $u$ is applied as follows: $u(t) = u(V_i, x(t))$ for $t \in [t_i, t_{i+1})$, where $V_i > 0 : Q(V_i, x(t_i)) = 0$. Then the origin of the system* (1) *is globally asymptotically stable.*

**Proof.** Let $V(x)$ be a positive definite function implicitly defined by the equation $Q(V, x) = 0$ and $x(t)$ be a trajectory of the closed-loop system (1) with the sampled control application described above. Let us prove that the sequence $\{V(x(t_i))\}_{i=1}^{\infty}$ is monotone decreasing and tends to 0. This, obviously, implies convergence of $x(t)$ to the origin.

Consider the time interval $[t_i, t_{i+1})$ and the function $\tilde{V}_i(x) = x^T P_i x$, where $P_i = D_r(V^{-1}(x(t_i)))PD_r(V^{-1}(x(t_i)) >$

0. The switching control $u(x) = u_i(x)$ on this interval takes the form $u_i(x) = u(V(x(t_i)), x) = k_i x$, where $k_i := V^{1-\mu}(x(t_i))kD_r(V^{-1}(x(t_i)))$. Repeating the proof of Corollary 7 we show $\tilde{V}_i(x(t_i)) > \tilde{V}_i(x(t))$ for all $t \in (t_i, t_{i+1})$. Since $V(x)$ is such that $Q(V(x), x) = 0$ then for all $t \in (t_i, t_{i+1})$

$$Q(V(x(t_i)), x(t)) =$$
$$x^T(t)D_r(V^{-1}(x(t_i))PD_r(V^{-1}(x(t_i))x(t) - 1 =$$
$$\tilde{V}_i(x(t)) - 1 < \tilde{V}_i(x(t_i)) - 1 = Q(V(t_i), x(t_i)) =$$
$$0 = Q(V(x(t)), x(t)).$$

For any given $x \in \mathbb{R}^n \backslash \{0\}$ the function $Q(V, x)$ is monotone decreasing for all $V \in \mathbb{R}^+$ (see Condition *C4*) of Theorem 3 and the proof of Theorem 6). Then the obtained inequality implies $V(x(t)) < V(x(t_i)), \forall t \in (t_i, t_{i+1}]$. Moreover $V(x(t)) \leq V(x(0))$ for all $t > 0$, i.e. *the origin of the closed-loop system* (1) *is Lyapunov stable*. The proven properties of closed-loop system also imply that $\|x(t)\|$ and $\|d(t, x(t))\|$ are bounded by some constants for all $t > 0$.

Since the function $V(x)$ is positive definite then the monotone decreasing sequence $\{V(x(t_i))\}_{i=1}^{\infty}$ converge to some limit. Let us show now that this limit is zero. Suppose the contrary, i.e. $\lim\limits_{i\to\infty} V(x(t_i)) = V_* > 0$ or equivalently
$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) : 0 \leq V(x(t_i)) - V_* < \varepsilon, \ \forall i \geq N.$
Let us represent the sampled control law $u$ in the form

$$u(V(x(t_i)), x) = u(V_*, x) + \Delta_i x,$$
$$\Delta_i = V^{1-\mu}(x(t_i))kD_r(V^{-1}(x(t_i))) - V_*^{1-\mu}kD_r(V_*^{-1}).$$

Since the control function $u = u(V, x)$ is continuous in $\mathbb{R}_+ \times \mathbb{R}^n$ and then there exists $\gamma \in \mathcal{K}$ (possibly depended on $\mu, k, V_*, n$) such that $\|\Delta_i\| \leq \gamma(\varepsilon)$ for all $i \geq N(\varepsilon)$. This means that for $t > t_{N(\varepsilon)}$ the closed-loop system (1) can be presented in the form

$$\dot{x} = (A + bk_*)x + b\Delta_i x + d(t, x) \tag{9}$$

where $k_* = V_*^{1-\mu}kD_r(V_*^{-1})$. Consider the Lyapunov function candidate $\tilde{V}(x) = x^T P_* x$, where $P_* = D_r(V_*^{-1})PD_r(V_*^{-1})$. Under the assumptions made above we have $\tilde{V}(x(t)) \geq 1$ for all $t > 0$ and repeating the proof of Corollary 7 we derive

$$\left.\frac{d\tilde{V}(x(t))}{dt}\right|_{(1)} \leq \frac{V_*^{\mu}d^T D_r^2(V_*^{-1})d}{\beta} - \frac{\alpha\tilde{V}(x(t))}{V_*^{\mu}} + 2x^T(t)P_* b\Delta_i x(t)$$
$$\leq \frac{-\alpha+\beta}{V_*^{\mu}} + 2x^T(t)P_* b\Delta_i x(t)$$
$$+\frac{V_*^{\mu}(d^T D_r^2(V_*^{-1})d - \beta^2 V_*^{-2\mu})}{\beta}.$$

Since $\|x(t)\|$ and $\|d(t, x(t))\|$ are bounded by some constants for all $t > 0$ then for sufficiently small $\varepsilon > 0$ we have $\left.\frac{d\tilde{V}(x(t))}{dt}\right|_{(1)} \leq \frac{-\alpha+\beta}{2V_*^{\mu}} < 0$ for all $t > t_{N(\varepsilon)}$. This contradicts the condition $V(x(t)) \geq V^*$ for all $t > 0$. Consequently, $\lim_{i\to\infty} V(x(t_i)) = 0$. Hence, we derive the asymptotic stability of the closed-loop system (1). ■

The proven corollary shows that the sampled-time realization of the developed "implicit" control scheme preserves the asymptotic stability to the origin of the closed-loop system (1) *independently on the sampling period*. Between two switching instants the unperturbed system is *linear*, so analysis of its discrete-time version can be studied using discretization schemes of linear systems (see, e.g. [36]).

### *4.2 Fixed-time Stabilization*

Consider now two functions:

$$Q_1(V, x) = x^T D_{r_1}(V^{-1})PD_{r_1}(V^{-1})x - 1,$$
$$Q_2(V, x) = x^T D_{r_2}(V^{-1})PD_{r_2}(V^{-1})x - 1,$$

where $P \in \mathbb{R}^{n\times n}$, $P = P^T > 0$,
$r_1 = (1 + (n-1)\mu, \ 1 + (n-2)\mu, \ ..., \ 1 + \mu, \ 1)^T \in \mathbb{R}^n$,
$r_2 = (1, \ 1 + \nu, \ ..., \ 1 + (n-2)\nu, \ 1 + (n-1)\nu)^T \in \mathbb{R}^n$,
$0 < \mu \leq 1$ and $\nu \in \mathbb{R}_+$. Denote $H_\nu = \text{diag}\{-(r_2)_i\}_{i=1}^n$.

**Theorem 9 (Fixed-time stabilization)** *If 1) the system of matrix inequalities*

$$\begin{cases} AX + XA^T + by + y^T b^T + \alpha X + \beta I_n \leq 0, \\ -\gamma_1 X \leq XH_\mu + H_\mu X < 0, \\ -\gamma_2 X \leq XH_\nu + H_\nu X < 0, \\ X > 0, \end{cases} \tag{10}$$

*is feasible for some* $X = X^T \in \mathbb{R}^{n\times n}$, $y \in \mathbb{R}^{1\times n}$ *and numbers* $\mu \in (0, 1]$, $\nu, \alpha, \beta, \gamma_1, \gamma_2 \in \mathbb{R}_+$ *such that* $\alpha > \beta$*;*

*2) the control* $u$ *has the form*

$$u(V, x) = \begin{cases} V^{1-\mu}kD_{r_1}(V^{-1})x & for \ \ x^T Px < 1, \\ V^{1+n\nu}kD_{r_2}(V^{-1})x & for \ \ x^T Px \geq 1, \end{cases} \tag{11}$$

*where* $k = yX^{-1}$, $P = X^{-1}$ *and* $V$ *defined by*

$$V = V(x) \ such \ that \begin{cases} Q_1(V, x) = 0 & for \ \ x^T Px < 1, \\ Q_2(V, x) = 0 & for \ \ x^T Px \geq 1; \end{cases}$$

*3) the disturbance function* $d$ *satisfy*

$$\beta^2 V^{-2\mu} \geq \sup_{t\in\mathbb{R}_+} d^T(t, x)D_{r_1}^2(V^{-1})d(t, x) \ \ if \ x^T Px < 1,$$
$$\beta^2 V^{2\nu} \geq \sup_{t\in\mathbb{R}_+} d^T(t, x)D_{r_2}^2(V^{-1})d(t, x) \ \ if \ x^T Px \geq 1;$$

*then the closed-loop system* (1) *is fixed-time stable with the settling-time estimate:* $T(x) \leq (\gamma_1/\mu + \gamma_2/\nu)(\alpha - \beta)^{-1}$.

**Proof.** The functions $Q_1(V, x)$ and $Q_2(V, x)$ satisfy the conditions *C1*)-*C4*) of Theorem 3. Since $Q_1(1, x) = Q_2(1, x)$, then the condition *C6*) of Theorem 5 also holds.

Repeating the proof of Theorem 6 it can be shown that $\frac{\partial Q_1}{\partial x}(Ax+bu(x)+d(t,x)) \leq \frac{\alpha-\beta}{\gamma_1}V^{1-\mu}\frac{\partial Q_1}{\partial V}$ for $x^TPx \leq 1$, $\frac{\partial Q_2}{\partial x}(Ax+bu(x)+d(t,x)) \leq \frac{\alpha-\beta}{\gamma_2}V^{1+\nu}\frac{\partial Q_2}{\partial V}$ for $x^TPx \geq 1$. Applying Theorem 5 we finish the proof. ■

If $\mu = 1$, $|d_n(t,x)| \leq \beta$ and the matching condition holds then Theorem 9 provides a HOSM control with fixed-time reaching phase.

Let us call the controls (5) and (11) by *the finite-time ILF control* and *the fixed-time ILF control*, respectively. The ILF control with $\mu = 1$ we call *the HOSM ILF control*.

It is worth to stress that Corollaries 7 and 8 stay true for the case of fixed-time control (11) application. We do not prove the finite-time and fixed-time stability properties for the sampled-time realization of the ILF controls. However, the motion of the sampled system will be close to the original one if the sampling period is sufficiently small.

## 5 Aspects of practical implementation

Corollaries 7 and 8 give some remarks on possible implementation of the developed control scheme. A detailed study of sampled-time and discrete-time versions of the presented control algorithms goes beyond the scope of this paper providing the subject for a future research. In this section we provide just some general ideas to be used for ILF control implementation.

The control scheme (5) can be realized in digital control devices, which allow us to solve the equation $Q(V,x) = 0$ numerically and on-line for any given $x \in \mathbb{R}^n\backslash\{0\}$ . Rather simple numerical procedures can be utilized for this purpose. The function $Q(V,x)$ satisfy the properties *C1*)-*C4*) of Theorem 3. So, for each fixed $\overline{x} \in \mathbb{R}^n\backslash\{0\}$ the function $\overline{Q}(V) = Q(V,\overline{x})$ is monotone decreasing and has the unique zero on the interval $(0,+\infty)$. In this case we may use, for example, the bisection method in order to solve the scalar equation $\overline{Q}(V) = 0$.

Let the control $u(V,x)$ be given by (5) and the parameter $V$ may change its value at some time instants $t_0 = 0$, $t_i > 0$, $i = 1,2,....$ Namely, let the control signal $u(t)$ be defined as $u(t) = u(V_i, x(t))$ for $t \in [t_i, t_{i+1})$ and $V_i \in \mathbb{R}_+$. Recall that $u(V_i,x)$ is a linear stabilizing feedback for any $V_i \in \mathbb{R}_+$ (see, the proof of Corollary 7). Denote $x_i = x(t_i)$. If $V_i = V(x_i)$ then stability of the sampling control scheme follows from Corollary 8.

The simplest algorithm for finding the switching control parameter $V_i$ is given below.

**Algorithm 10 (Implementation of the ILF control)**

INITIALIZATION: $a = V_{\min}$; $V_0 > V_{\min}$; $b = V_0$;
STEP :
If $x_i^T D_r(b^{-1})PD_r(b^{-1})x_i > 1$ then $a = b$; $b = 2b$;
  elseif $x_i^T D_r(a^{-1})PD_r(a^{-1})x_i < 1$ then
    $b = a$; $a = \max\{\frac{a}{2}, V_{\min}\}$;
  else $c = \frac{a+b}{2}$;
    If $x_i^T D_r(c^{-1})PD_r(c^{-1})x_i < 1$ then $b = c$;
      else $a = \max\{V_{\min}, c\}$;
    endif;
endif;
$V_i = b$,

*where $V_{\min} \in \mathbb{R}_+$ is a minimal admissible value of $V_i$.*

INITIALIZATION defines a linear feedback $u(V_0,x)$ on the first sampling interval $[t_0,t_1)$ by means of selection of the value $V_0 \in \mathbb{R}_+$. STEP of the algorithm is applied at each sampling time instances $t_i$ in order to define the value $V_i$ (and the linear feedback $u(V_i,x)$) for the next interval $[t_i,t_{i+1})$. Let $x_i \in \mathbb{R}^n$ is a current state vector. If at the sampling instant $t_i$ STEP of the algorithm is repeated many times for the same $x_i$ (for example, there exists a loop containing STEP) then Algorithm 10 realizes: 1) a localization of the unique positive root of the equation $Q(V,x_i) = 0$, i.e. $V(x_i) \in [a,b]$; 2) improvement of the obtained localization by means of the bisection method, i.e. $|b-a| \to 0$. Such an application of Algorithm 10 allows us to calculate $V(x_i)$ with a high precision, however, it requests a high computational capability of a control device.

Therefore, it is more reasonable to realize STEP of Algorithm 10 just once or a few (2 or 3) times at each sampling instant. Some additional considerations are needed in order to show that the algorithm will work properly in this case.

Denote $\Pi(V_i)$=$\{x \in \mathbb{R}^n : x^T D_r(V_i^{-1})PD_r(V_i^{-1})x \leq 1\}$. If the set of initial conditions is known then the value $V_0 \in \mathbb{R}_+$ can be always selected such that $x_0 \in \Pi(V_0)$. Otherwise, applying Algorithm 10 for sufficiently small sampling period there always exists a finite number $i^* \in \{0,1,2,...\}$ such that $x_{i^*} \in \Pi(V_{i^*})$. Corollary 7 implies that the ellipsoid $\Pi(V_i)$ is *strictly* positively invariant for the closed-loop system (1) with the feedback law $u(x) = u(V_i,x)$ of the form (5). Monotonicity condition $\frac{\partial Q(V,x)}{\partial V} < 0$ implies that $\Pi(V') \subset \Pi(V'')$ for $V' < V''$. In order to guarantee stability of the sampled-time realization of the developed control, on each STEP of Algorithm 10 the *upper estimate* (i.e. $V_i = b$) is selected in order to design the linear feedback $u(V_i,x)$ for the next interval $[t_i,t_{i+1})$. Such selection ensures $V_{i+1} \in (0,V_i]$ if $x_i \in \Pi(V_i)$ and the sequence $\{V_i\}_{i=i^*}^{\infty}$ generated by Algorithm 10 is non-increasing. The bisection procedure will operate until $x(t) \notin \Pi(V_{\min})$. So, it can be shown that $V_i \to V_{\min}$ as $i \to \infty$.

The parameter $V_{\min}$ defines the lower possible value of $V$ and the "minimal" attractive invariant set $\Pi(V_{min})$ for the closed-loop system. This parameter cannot be selected arbitrary small due to finite numerical precision of digital devices. Fixed-time control application can be realized with small changes of Algorithm 10.

**Remark 11** *An additional advantage of the developed control scheme is related to the possible reduction of the chattering effect for HOSM ILF control application. Indeed, the HOSM control of the form* (5) *with* $\mu = 1$ *has the unique discontinuity point* $x = 0$. *According Algorithm 10, near discontinuity point the feedback law is defined as follows:* $u(x) = kD(V_{\min})x$, $\forall x \in \Pi(V_{\min})$, *where* $V_{\min} \in \mathbb{R}_+$, *i.e. we always have a linear continuous control inside the ellipsoid* $\Pi(V_{\min})$. *Such a modification of the control law obviously follows the classical idea of the chattering reduction developed for the first order sliding mode algorithms [14]. Namely, for practical realization of the sliding mode control, the discontinuous feedback law can be replaced with a high-gain linear feedback if the system state is close to the switching manifold [14].*

## 6 Numerical examples

### *6.1 HOSM ILF control*

Consider the system (1) for $n = 4$, $d_1(t, x) = d_2(t, x) = d_3(t, x) = 0$ and $|d_4(t, x)| \leq \beta = 0.2$. We take $d_4(t, x) = 0.1 \sin(t) + 0.1 \cos(x_4)$ and $x(0) = (2, 0, 0, 0)^T$.

The numerical solution of ODE for the closed-loop system has been obtained using the explicit Euler method with a fixed step size $h \in \mathbb{R}_+$. In order to show the effectiveness of the developed control scheme with respect to the chattering reduction, we select quite a large step size $h = 0.1$.

The Fig. 1(a) shows the simulation results for the closed-loop system with the HOSM ILF control (5) ($\mu = 1$) that is restricted by $|u(x)| \leq 1$ (see, (7)) and applied by the scheme presented in Algorithm 10 for $V_{\min} = 0.1$ and $h = 0.1$.

The parameters of the HOSM ILF control (5) were selected by means of solving the LMI system (4), (7) for $\mu = 1$ and $\alpha = 0.5$, $\beta = 0.2$, $\gamma = 10$, $u_0 = 1$:

$$P = \begin{pmatrix} 0.0058 & 0.0318 & 0.0644 & 0.0517 \\ 0.0318 & 0.1983 & 0.4351 & 0.3798 \\ 0.0644 & 0.4351 & 1.0798 & 1.0393 \\ 0.0517 & 0.3798 & 1.0393 & 1.3857 \end{pmatrix},$$
$$k = (-0.0330, -0.2522, -0.7823, -1.1386).$$

In order to compare the obtained results with some existing HOSM controller, the simulations also have been done for the fourth order sliding mode (4-SM) nested controller [12] of the form

$$u = -\operatorname{sign}\left[x_4 + 0.5\left(|x_1|^3 + |x_2|^4 + |x_3|^6\right)^{\frac{1}{12}} \operatorname{sign}(\phi)\right],$$
$$\phi = x_3 + \tfrac{1}{4}(|x_1|^3 + |x_2|^4)^{\frac{1}{6}} \operatorname{sign}\left[x_2 + 0.15|x_1|^{\frac{3}{4}} \operatorname{sign}[x_1]\right].$$

The effective procedures for parameters adjustment of high order sliding mode control algorithms are not developed yet for $n \geq 3$. So, the parameters of this controller have been selected manually. The simulations of the 4-SM nested control algorithm were initially done for the small sampling period: $h = 10^{-3}$. They showed good performances of this control: fast convergence rate and rejection of matched bounded disturbances. The chattering effect appears in the system, when the sampling period becomes larger. The simulation results with $h = 0.1$ for the 4-SM nested control are depicted on the Fig. 1(b). The Fig. 1(c) shows the control inputs generated by two considered algorithms.

Note that the comparison of the sliding mode control laws for the sufficiently large sampling period $h = 0.1$ is motivated by practical reasons. For instance, the HOSM control is frequently demonstrated on the control problems for mobile robotics (see, for instance, [12]). However, the computational restrictions of autonomous mobile systems does not allow us to spend to much computational power for the generation of control actions. In practice, the reasonable sampling period can be $0.05 - 0.1$ second [37].

### *6.2 Fixed-time ILF control*

Define the fixed-time control $u$ in the form (11) with $n = 3$, $\mu = 0.5$, $\nu = 0.1$, where the matrix

$$P = \begin{pmatrix} 5.0575 & 4.5868 & 1.4558 \\ 4.5868 & 7.4884 & 2.2003 \\ 1.4558 & 2.2003 & 1.7959 \end{pmatrix}$$

and the vector $k = (-2.9319, -5.6235, -2.6998)$ are obtained from the LMI (10) with $\beta = 0$, $\alpha = 1$, $\gamma_1 = 4.5$, $\gamma_2 = 4.5$. In this case the settling time estimate provided by Theorem 9 gives $T(x_0) \leq 54$.

The numerical simulation has been done using the Euler method with a fixed step size $h = 0.01$. The control has been applied using Algorithm 10 with $V_{\min} = 0.001$. The simulation results are presented on Fig. 2 for $x_0 = (1, 0, 0)^T$ and $x_0 = (9, 0, 0)^T$. They show a "week" dependence of the convergence time on the initial conditions. Definitely, it requires a rather high magnitude of the control.

## 7 Conclusion

The paper develops the Implicit Lyapunov Function method for finite-time and fixed-time stability analysis. Using this theoretical framework new algorithms of finite-time and fixed-time stabilization for a chain of integrators and high order sliding mode control design are developed. The obtained control schemes have the following **advantages**:

- The control design algorithms are constructive. The schemes for tuning the control parameters have LMI representations.
- For $\mu = 1$ the algorithms provide high order sliding mode control rejecting matched bounded disturbances.

- The digital implementation of the developed control scheme admits the effective chattering reduction by means of tuning the parameter $V_{\min}$.
- The fixed-time ILF controllers allow us to prescribe the convergence time independently of the initial condition.

and **disadvantages**:

- The algorithms are applicable only in digital controllers.
- The practical realization of the developed control schemes asks for additional computational power of the digital control device, which is required for on-line computation of the ILF value at the current state.

The developed Implicit Lyapunov Function method for finite-time and fixed-time stability analysis is promising to tackle many other problems such as fast observation and estimation, development of fast adaptation algorithms or fast consensus protocols, etc.

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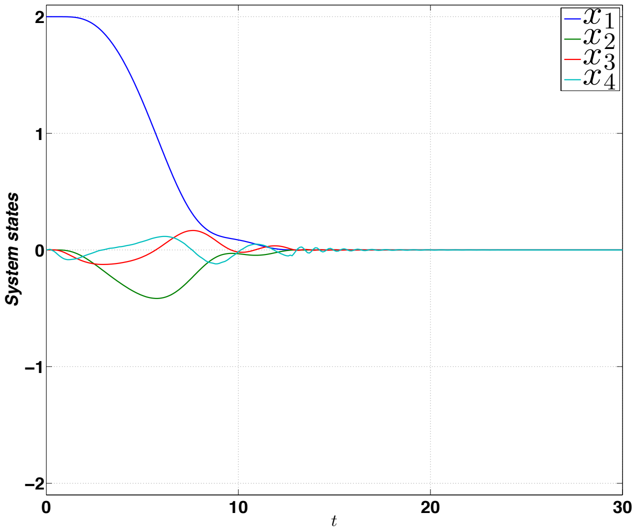
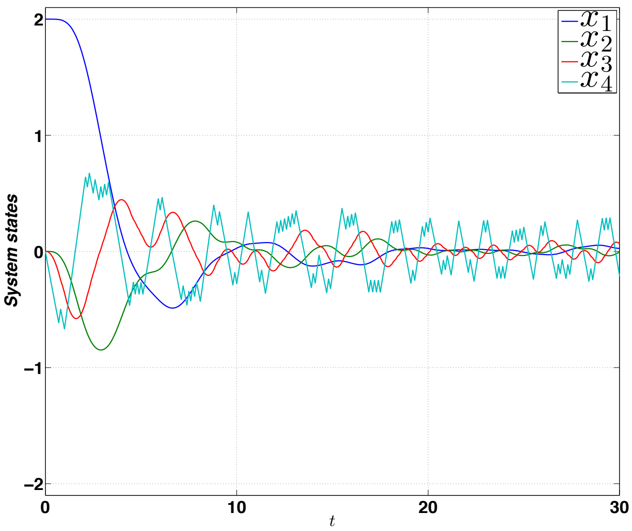
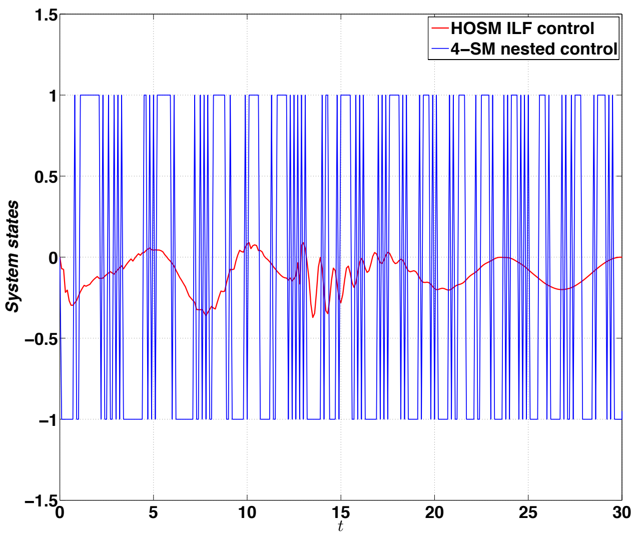



(a) The HOSM ILF control.

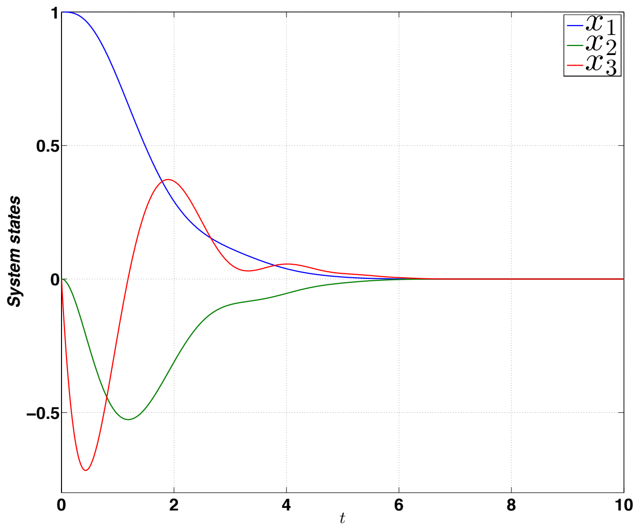



(b) The 4-SM nested control [12].

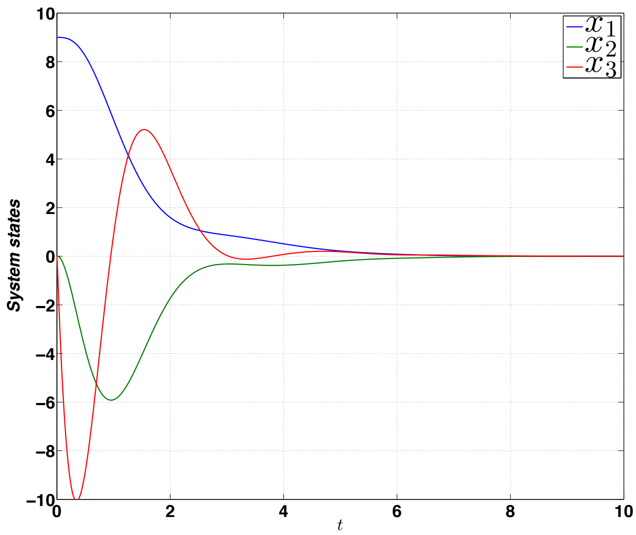



(c) Comparison of control inputs.

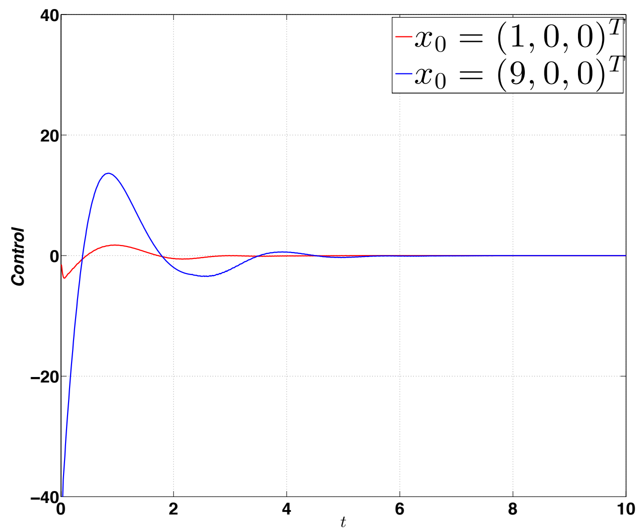
Fig. 1. The simulation results for the HOSM ILF control ($V_{\min} = 0.1$) and the 4-SM nested control.




(a) $x_0 = (1, 0, 0)^T$




(b) $x_0 = (9, 0, 0)^T$




(c) Control inputs.

Fig. 2. The simulation results for the fixed-time ILF control.