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# OPTIMALITY CONDITIONS AND REGULARITY RESULTS FOR TIME OPTIMAL CONTROL PROBLEMS WITH DIFFERENTIAL INCLUSIONS

PIERMARCO CANNARSA, ANTONIO MARIGONDA, AND KHAI T. NGUYEN

ABSTRACT. We study the time optimal control problem with a general target  $\mathcal{S}$  for a class of differential inclusions that satisfy mild smoothness and controllability assumptions. In particular, we do not require Petrov's condition at the boundary of  $\mathcal{S}$ . Consequently, the minimum time function  $T(\cdot)$  fails to be locally Lipschitz—never mind semiconcave—near  $\mathcal{S}$ . Instead of such a regularity, we use an exterior sphere condition for the hypograph of  $T(\cdot)$  to develop the analysis. In this way, we obtain dual arc inclusions which we apply to show the constancy of the Hamiltonian along optimal trajectories and other optimality conditions in Hamiltonian form. We also prove an upper bound for the Hausdorff measure of the set of all nonlipschitz points of  $T(\cdot)$  which implies that the minimum time function is of special bounded variation.

## 1. INTRODUCTION

In this paper we study the time optimal control problem for the differential inclusion

$$(1) \quad \begin{cases} \dot{x}(t) \in F(x(t)), & \text{for a.e. } t > 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

with a given closed target set  $\mathcal{S} \subseteq \mathbb{R}^n$ . The dynamics is described by a set-valued Lipschitz continuous function  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ , whose values are assumed to be compact, convex and nonempty.

For each trajectory of (1) starting from  $x_0$ , i.e., for each absolutely continuous function  $y^{x_0}(\cdot) \in AC([0, +\infty[; \mathbb{R}^n)$  satisfying  $\dot{y}^{x_0}(t) \in F(y^{x_0}(t))$  for a.e.  $t > 0$  and  $y^{x_0}(0) = x_0$ , we set

$$\theta(y^{x_0}(\cdot)) := \inf\{t \geq 0 : y^{x_0}(t) \in \mathcal{S}\},$$

with the convention  $\inf \emptyset = +\infty$ . The *minimum time function* is defined by

$$T(x_0) := \inf\{\theta(y^{x_0}(\cdot)) : y^{x_0}(\cdot) \text{ is a trajectory of (1) with } y^{x_0}(0) = x_0\}.$$

When finite,  $T(x_0)$  represents the minimum time needed to steer the point  $x_0$  to the target  $\mathcal{S}$  following the trajectories of (1). The study of the regularity of  $T(\cdot)$  is a central problem in optimal control theory, and the literature offers a huge choice of papers on such a topic. In particular, starting from the seminal paper [6], the

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regularity property of  $T(\cdot)$  called *semiconcavity* was extensively studied and used to deduce new optimality conditions. Roughly speaking, semiconcavity amounts to the local Lipschitz continuity of  $T(\cdot)$  plus a uniform exterior sphere condition for the hypograph of  $T(\cdot)$ . Using these geometric properties, one can derive further regularity results for  $T(\cdot)$ , such as  $BV$  estimates for  $T(\cdot)$  and  $\nabla T(\cdot)$ , the existence of Taylor's expansion of order two around a. e. point, and bounds for the singular sets of  $T$ . We refer to [7] for an introduction to semiconcave functions.

Semiconcavity (and semiconvexity) results, however, rely essentially on a strong controllability assumption—the so-called Petrov condition—which is actually equivalent to the local Lipschitz continuity of  $T$ . When this assumption is removed,  $T(\cdot)$  may fail to be semiconcave, but it still retains the external sphere property of the hypograph (or epigraph). In [12], for parameterized linear multifunctions, the epigraph of  $T(\cdot)$  was proved to satisfy a strong external sphere condition, which is called *positive reach property* in the sense of Federer (also known, in literature, as  $\varphi$ -convexity or *proximal smoothness*). The regularity properties of the class of functions whose epigraph has positive reach turned out to be comparable with the properties of semiconcave/semiconvex functions, we refer to [10] for details.

The above geometric approach was also generalized to the nonlinear case: in [14], the positive reach property of the hypograph of  $T(\cdot)$  was obtained replacing Petrov's condition by the assumption that the normal cone to the hypograph is pointed (i.e., contains no lines). Without such an assumption, in [18], the hypograph of  $T(\cdot)$  was proved to satisfy a weaker external sphere condition. In the same paper, it was shown that such a condition yields essentially the same regularity properties as in the case when the hypograph has positive reach.

The study of the singular set for  $T(\cdot)$ , beyond the semiconcave case, was performed in [11] for the positive reach case, in [19] for the weak external sphere property, and in [17] for a further generalized case. In all these papers the structure of the singular set was analyzed, providing upper bounds for the Hausdorff measure of such a set. Lower estimates were obtained in [1] and [16].

So far, all the results in the above papers were proved for a state equation given in the form of a smooth  $C^{1,1}$ -parameterized multifunction. The smoothness of the parameterization was a crucial issue to gain semiconcavity estimates, and was extensively used in all its generalizations as well. Note that, although a Lipschitz multifunction always possesses a Lipschitz parameterization (see, for instance, [3]), it is still an open problem to find conditions for such a parameterization to be smooth.

This motivates a separate study of the minimum time problem for differential inclusions. The first result in this direction is due to [8], where the semiconcavity of the value function of the Mayer problem for system (1) was proved. Instead of searching for a smooth parametrization of  $F(\cdot)$ , the main idea of [8] is to use the smoothness of the Hamiltonian associated with system (1), defined by

$$(2) \quad H(x, p) = \sup_{v \in F(x)} \langle p, v \rangle, \quad (x, p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

This approach was extended in [9] to the study of the minimum time function, still in the semiconcavity framework. A further step to extend this kind of analysis beyond semiconcavity, was performed in [5], which can be seen as the counterpart of

[14] and [18] for differential inclusions, proving the external sphere property enjoyed by hypo  $T$  under milder controllability assumptions.

The main tool used in our analysis is the *dual arc inclusion* (see [9]), i.e., the natural nonsmooth counterpart of the Hamiltonian system associated with the problem, that is,

$$(3) \quad \begin{cases} \dot{x}(t) = \nabla_p H(x(t), p(t)), \\ -\dot{p}(t) \in \partial_x H(x(t), p(t)). \end{cases}$$

If the trajectory  $x(\cdot)$  starting from  $x_0$  at  $t = 0$  is optimal, then the above system admits a solution when coupled with a terminal condition for  $p(T(x_0))$  that turns out to be a proximal inner normal to the target set  $\mathcal{S}$  at  $\bar{x} = x(T(x_0))$ .

In this paper, we prove two results for system (3): in the first one, we state that the superdifferentiability of the minimum time function  $T(\cdot)$  along an optimal trajectory is completely determined by the value of the Hamiltonian at the endpoint  $(x(T(x_0)), p(T(x_0)))$ . More precisely, if we are able to find a nontrivial proximal inner normal to the target  $\mathcal{S}$  at the endpoint  $\bar{x} = x(T(x_0))$  such that the value of the Hamiltonian computed at this inner normal at  $\bar{x}$  is nonzero, then the  $p$ -part of the solution of (3) yields a continuous selection of  $\partial^P T$  along the trajectory  $x(\cdot)$ . In particular, we have that  $\partial^P T(x(t)) \neq \emptyset$  for all  $t \in [0, T(x_0)]$ .

Due to the presence of horizontal normals to hypo  $T$ , the nonsmoothness of  $T(\cdot)$  propagates similarly along the flow of an optimal trajectory: indeed we show that, if the value of the Hamiltonian computed at this inner normal at  $\bar{x}$  is zero, then the same  $p$ -part of the solution yields a continuous selection of  $\partial^\infty T$  along the optimal trajectory. So, once again,  $\partial^\infty T(x(t)) \neq \emptyset$  for all  $t \in [0, T(x_0)]$ .

The second result of this paper, which is a consequence of the above propagation of superdifferentiability, is concerned with the constancy of the Hamiltonian along optimal trajectories, which somewhat resembles the classical case. In order to prove this result, we use the fact that the value of the Hamiltonian, at the terminal point of an optimal trajectory, computed at a suitable inner normal, yields the existence of a supergradient of  $T$ , or of an horizontal supergradient of  $T$ , at *all* points of the trajectory. This fact forces the value of the Hamiltonian, along the solution of the dual arc inclusion, to be identically 1 in the former cases, and 0 in the latter.

A partial converse of the above result, i.e., a sufficient condition for optimality, is also established in the case of superdifferentiability. We show that, if we have a solution of the generalized characteristic system

$$\begin{cases} \dot{x}(t) = \nabla_p H(x(t), p(t)), \\ -\dot{p}(t) \in \partial^P T(x(t)), \end{cases}$$

along which the Hamiltonian is constantly equal to 1, then the  $x$ -part of the solution is actually an optimal trajectory.

The last part of the paper is devoted to the study of *SBV* regularity of the minimum time function, which guarantees that the singular part of the distributional gradient of  $T(\cdot)$  has no Cantor component—a property that has several applications to the calculus of variations (see, e.g., [4]). Actually, the result we prove is slightly stronger than just establishing the *SBV* regularity of  $T(\cdot)$ . Indeed, Theorem 4.6 ensures that, if the subset of  $\partial\mathcal{S}$  at which  $H$  vanishes for a nontrivial choice of

inner normals is *small enough*, then there are *few* optimal trajectories along which the normal cone to  $\text{hypo}T$  has an horizontal part. So, at *most* points,  $\partial^P T(x)$  is bounded and  $T(\cdot)$  turns out to exhibit a Lipschitz behaviour.

The paper is structured as follows: in Section 2 we fix the notation, and recall definitions and preliminaries from nonsmooth analysis, especially concerning differential inclusions. In Section 3, we discuss our standing assumptions and recall some useful results from [8], while Section 4 is devoted to the main results of the paper and the analysis of some of their consequences.

## 2. PRELIMINARIES AND NOTATION

Our ground space will be the Euclidean space  $\mathbb{R}^n$ .

**Definition 2.1.** Let  $\Omega$ ,  $K$ , and  $S$  be, respectively, an open, closed, and any subset of  $\mathbb{R}^n$ , let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  be points of  $\mathbb{R}^n$ , let  $r > 0$ , and let  $k \geq 0$  be an integer. We denote by:

$\langle x, y \rangle := \sum_{i=1}^n x_i y_i$	the <i>scalar product</i> in $\mathbb{R}^n$ ;
$\ x\  := \sqrt{\langle x, x \rangle}$	the <i>Euclidean norm</i> in $\mathbb{R}^n$ ;
$\partial S, \text{int}(S), \overline{S}$	the <i>topological boundary</i> , <i>interior</i> and <i>closure</i> of $S$ ;
$\text{diam}(S) := \sup\{\ z_1 - z_2\  : z_1, z_2 \in S\}$	the <i>diameter</i> of $S$ ;
$\mathcal{P}(S) := \{B \subseteq \mathbb{R}^n : B \subseteq S\}$	the <i>power set</i> of $S$ ;
$\mathbb{B}^n := \{w \in \mathbb{R}^n : \ w\  < 1\}$	the <i>unit open ball</i> (centered at the origin);
$\mathbb{S}^{n-1} := \{w \in \mathbb{R}^n : \ w\  = 1\} = \partial \mathbb{B}^n$	the <i>unit sphere</i> (centered at the origin);
$B(y, r) := \{z \in \mathbb{R}^n : \ z - y\  < r\} = y + r\mathbb{B}^n$	the <i>open ball</i> centered at $y$ of radius $r$ ;
$d_K(y) := \text{dist}(y, K) = \min\{\ z - y\  : z \in K\}$	the <i>distance</i> of $y$ from $K$ ;
$\pi_K(y) := \{z \in K : \ z - y\  = d_K(y)\}$	the <i>set of projections</i> of $y$ onto $K$ ;
$S^c := \mathbb{R}^n \setminus S$	the complement of $S$ .

Also,  $C^k(\Omega)$  stands for the space of all  $f : \Omega \rightarrow \mathbb{R}$  with continuous derivatives up to order  $k$ ,  $C_b^k(\Omega)$  collects the functions of  $C^k(\Omega)$  with bounded derivatives of order  $k$ , and  $C_c^k(\Omega)$  the ones with compact support.

If  $\pi_K(y) = \{\xi\}$ , i.e. it is a singleton, we will identify the set  $\pi_K(y)$  with its unique element and write  $\pi_K(y) = \xi$ . The *characteristic function*  $\chi_S : \mathbb{R}^d \rightarrow \{0, 1\}$  of  $S$  is defined as  $\chi_S(x) = 1$  if  $x \in S$  and  $\chi_S(x) = 0$  if  $x \notin S$ .

**Definition 2.2.** Let  $X$  be a vector space. A set  $C \subseteq X$  is *convex* if for every  $x_1, x_2 \in C$ ,  $\lambda \in [0, 1]$ , we have that  $\lambda x_1 + (1 - \lambda)x_2 \in C$ . If  $S \subseteq X$  is a set, the smallest (with respect to inclusion) convex set which contains  $S$  is called the *convex hull* of  $S$  and it is defined as

$$\text{co } S := \bigcap_{C \in \mathcal{F}(S)} C,$$

where  $\mathcal{F}(S) := \{C \subseteq X : C \supseteq S \text{ and } C \text{ is convex}\}$ . We have that  $S = \text{co } S$  iff  $S$  is convex.

**Definition 2.3.** Let  $X$  be a vector space and  $f : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$  be a function. We recall the definitions of

$$\begin{aligned} \text{dom } f &:= \{x \in X : f(x) \in \mathbb{R}\}, \text{ the domain of } f; \\ \text{epi } f &:= \{(x, \beta) \in X \times \mathbb{R} : x \in \text{dom } f, \beta \geq f(x)\}, \text{ the epigraph of } f; \\ \text{hypo } f &:= \{(x, \alpha) \in X \times \mathbb{R} : x \in \text{dom } f, \alpha \leq f(x)\}, \text{ the hypograph of } f. \end{aligned}$$

If  $X$  is a topological vector space, we say that  $f$  is *lower semicontinuous* (shortly: l.s.c.) if  $\text{epi } f$  is closed in  $X \times \mathbb{R}$  with respect to the product topology on  $X \times \mathbb{R}$ , i.e.

$$\liminf_{y \rightarrow x} f(y) \geq f(x).$$

A function  $g$  is called *upper semicontinuous* (shortly u.s.c.) if  $-g$  is l.s.c.

**Definition 2.4** (Lipschitz functions). Given the Banach spaces  $X, Y$ , and two open sets  $U \subseteq X, V \subseteq Y$ , a function  $f : U \rightarrow V$  is said to be a *Lipschitz continuous function* ( $f \in \text{Lip}(U)$ ) if there exists  $C > 0$ , called a *Lipschitz constant*, such that for every  $x_1, x_2 \in U$

$$\|f(x_1) - f(x_2)\|_Y \leq C\|x_1 - x_2\|_X.$$

A function  $f : U \rightarrow V$  is called *locally Lipschitz continuous* ( $f \in \text{Lip}_{\text{loc}}(U)$ ) if it is Lipschitz continuous on every compact subset of  $U$ . Given  $x \in X$ , we say that  $f : X \rightarrow Y$  is Lipschitz continuous at  $x$  if there exists a neighborhood  $U$  of  $x$  in  $X$  such that  $f : U \rightarrow Y$  is Lipschitz continuous.

We recall the following classical result on regularity of Lipschitz functions holding for spaces of finite dimensions:

**Theorem 2.5** (Rademacher's Theorem). *Let  $X$  be a Banach space of finite dimension, let  $U \subseteq X$  be open, and let  $f : U \rightarrow \mathbb{R}$  be a locally Lipschitz function. Then  $f$  is differentiable almost everywhere with respect to Lebesgue measure.*

**Definition 2.6** (BV functions). Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ ,  $u \in L^1(\Omega)$ . We say that  $u$  is a *function of bounded variation* in  $\Omega$  and write  $u \in BV(\Omega)$  if there exist Radon measures  $\mu_1, \dots, \mu_n$  on  $\Omega$  such that

$$\int_{\Omega} u(x) \partial_{x_i} \varphi(x) dx = - \int_{\Omega} \varphi(x) d\mu_i(x), \quad \text{for all } \varphi \in C_c^1(\Omega).$$

The vector-valued measure  $Du := (\mu_1, \dots, \mu_n)$  is called the distributional (full) gradient of  $u$ . We say that  $u \in BV_{\text{loc}}(\Omega)$  if  $u \in BV(\Omega')$  for every bounded open subset  $\Omega'$  of  $\Omega$ .

We now recall a few notions related to functions of bounded variation, referring the reader to [4] for more details. If  $u \in BV$ , then  $Du$  can be decomposed into an *absolutely continuous part* w.r.t. the Lebesgue measure, hereafter denoted by  $\nabla u dx$  or  $\nabla u$ , and a *singular part*,  $D^s u$ . One can further introduce the *jump part* of  $D^s u$ ,  $Ju$ , taking the restriction of  $D^s u$  to the set of all points of  $\Omega$  at which  $u$  has no approximate limit. Then, the difference  $D^s u - Ju$  is the so-call *Cantor part* of  $D^s u$ , labeled  $D^c u$ . Finally, we say that  $u \in BV_{\text{loc}}(\Omega)$  is a special function of locally bounded variation, and we write  $u \in SBV_{\text{loc}}(\Omega)$ , if the Cantor part  $D^c u$  vanishes.

**Proposition 2.7** (Proposition 4.2 in [4]). *Let  $u \in BV(\Omega)$ . Then  $u \in SBV(\Omega)$  if and only if  $D^s u$  is concentrated on a Borel set of  $\sigma$ -finite  $\mathcal{H}^{n-1}$  measure.*

Thus,  $u \in SBV(\Omega)$  if it vanishes outside a countably  $\mathcal{H}^{n-1}$ -rectifiable set.

**Definition 2.8** (Proximal normals). Let  $S$  be a closed subset of  $\mathbb{R}^n$ . A vector  $v$  is called a *proximal normal* to  $S$  at  $x \in S$  if there exists  $\sigma = \sigma(v, x) \geq 0$  such that

$$(4) \quad \langle v, y - x \rangle \leq \sigma \|y - x\|^2, \text{ for every } y \in S.$$

The set of all proximal normals to  $S$  at  $x$  will be denoted by  $N_S^P(x)$ .

Given  $\rho > 0$ , we say that  $v \in N_S^P(x)$  is *realized by a ball with radius  $\rho$*  if one can take  $\sigma = \frac{\|v\|}{2\rho}$  in (4).

Observe that  $x \in N_Q^P(x)$  if and only  $d_Q(x + \lambda v) = \lambda \|v\|$  for some  $\lambda > 0$ . Moreover, if  $v \in N_S^P(x)$  is realized by a ball with radius  $\rho$ , then  $B(x + \rho v, \rho) \cap S = \emptyset$ .

*Remark 2.9.* We recall that, when  $S$  is convex, we can take  $\sigma = 0$  in the above definition. Hence, the proximal normal cone at  $x$  reduces to the *normal cone in the sense of convex analysis*, namely the set of vectors  $v \in \mathbb{R}^n$  such that  $\langle v, y - x \rangle \leq 0$  for all  $y \in S$ .

**Definition 2.10** (External sphere condition). Let  $S \subset \mathbb{R}^n$  be closed and let  $\theta : \partial S \rightarrow ]0, \infty[$  be continuous. We say that  $S$  satisfies the  *$\theta$ -external sphere condition* if for every  $x \in \partial S$ , there exists a nonzero vector  $v \in N_Q^P(x)$  which is realized by a ball of radius  $\theta(x)$ .

We say that a closed subset  $Q$  of  $\mathbb{R}^n$  satisfies the  *$\theta$ -internal sphere condition* if and only if  $\overline{\mathbb{R}^n \setminus Q}$  satisfies the  $\theta$ -external sphere condition.

**Definition 2.11** (Proximal supergradients). Let  $f : \mathbb{R}^n \rightarrow [-\infty, +\infty]$  be an upper semicontinuous function. For every  $x \in \text{dom}(f)$ , we denote by

$$(5) \quad \partial^P f(x) \doteq \{ \xi \in \mathbb{R}^n : (-\xi, 1) \in N_{\text{hypo}(f)}^P(x, f(x)) \},$$

the set of *proximal supergradients* of  $f$  at  $x$ , and by

$$(6) \quad \partial^\infty f(x) \doteq \{ \xi \in \mathbb{R}^n : (-\xi, 0) \in N_{\text{hypo}(f)}^P(x, f(x)) \}.$$

the set of *horizontal proximal supergradients* of  $f$  at  $x$ .

We recall that, if  $f$  is Lipschitz continuous at  $x$ , then  $\partial^\infty f(x)$  is empty. The converse in general is false.

**Definition 2.12** (Support function). Let  $C \subseteq \mathbb{R}^n$  be nonempty. The *support function*  $\sigma_C : \mathbb{R}^n \rightarrow ]0, +\infty]$  is defined as  $\sigma_C(p) = \sup_{v \in C} \langle p, v \rangle$ .

It can be proved that  $\sigma_C(p) = \sigma_{\overline{\text{co}C}}(p)$ .

Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ , let  $x_0 \in \Omega$ , and let  $f : \Omega \rightarrow \mathbb{R}$  be Lipschitz continuous at  $x_0$ .

**Definition 2.13** (Clarke's generalized gradient). We define *Clarke's generalized gradient of  $f$  at  $x_0$*  by setting

$$\partial f(x) = \text{co} \left\{ v \in \mathbb{R}^n : \exists \{y_i\}_{i \in \mathbb{N}} \subset \Omega \text{ s.t. } f \text{ is differentiable at } y_i, \begin{matrix} y_i \rightarrow x_0 \\ \nabla f(y_i) \rightarrow v \end{matrix} \right\}.$$

We now recall the definition and some properties of semiconcave functions, referring the reader to [7] for further properties and characterizations. Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and let  $c \in \mathbb{R}$ .

**Definition 2.14** (Semiconcave functions). We say that  $f : \Omega \rightarrow \mathbb{R}$  is *semiconcave* in  $\Omega$  with *semiconcavity constant*  $c_0 > 0$  if  $f$  is continuous and, for every  $x_1, x_2 \in \Omega$  such that the line segment  $\{\lambda x_1 + (1 - \lambda)x_2 : \lambda \in [0, 1]\}$  is contained in  $\Omega$ , we have

$$\frac{f(x_1) + f(x_2)}{2} - f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{c_0}{4} \|x_1 - x_2\|^2.$$

In this case we will write  $f \in SC(\Omega)$ . We say that  $f$  is *locally semiconcave* in  $\Omega$ , and write  $f \in LSC(\Omega)$ , if  $f \in SC(A)$  for every  $A \subset \Omega$ ,  $A$  open and bounded. We say that  $g : \Omega \rightarrow \mathbb{R}$  is semiconvex iff  $-g$  is semiconcave.

**Proposition 2.15** (Properties of semiconcave functions). *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ , let  $c \in \mathbb{R}$ , and let  $f : \Omega \rightarrow \mathbb{R}$  be continuous. Then the following are equivalent:*

- (1)  $f \in SC(\Omega)$  with semiconcavity constant  $c$ ;
- (2)  $f(y) - f(x) \leq \langle \xi, y - x \rangle + c\|y - x\|^2$  for all  $\xi \in \partial^P f(x)$  and  $y \in \Omega$ .

**Definition 2.16** (Multifunctions). Let  $\Omega$  be a subset of  $\mathbb{R}^n$ . A map  $F : \Omega \rightarrow \mathcal{P}(\mathbb{R}^m)$  will be called a *multifunction* or *set-valued function*, and we will write  $F : \Omega \rightrightarrows \mathbb{R}^m$ .

We say that a multifunction  $F : \Omega \rightrightarrows \mathbb{R}^m$  is:

- *measurable*, if for all closed  $C \subset \mathbb{R}^m$  we have that  $\{x \in \Omega : F(x) \cap C \neq \emptyset\}$  is Lebesgue measurable;
- *upper semicontinuous* at  $x_0 \in \Omega$ , if for any open set  $M$  containing  $F(x_0)$  there exists an open set  $A \subset \Omega$  with  $x_0 \in A$  such that

$$F(A) := \bigcup_{a \in A} F(a) \subseteq M;$$

- *lower semicontinuous* at  $x_0 \in \Omega$ , if for any  $y_0 \in F(x_0)$  and any open set  $M$  containing  $y_0$  there exists an open set  $A \subset \Omega$  with  $x_0 \in A$  such that  $F(x) \cap M \neq \emptyset$ ;
- *continuous* at  $x_0 \in \Omega$ , if  $F$  is both lower and upper semicontinuous at  $x_0$ ;
- *Lipschitz continuous*, if there exists  $K > 0$ , called a Lipschitz constant of  $F$ , such that for any  $x_1, x_2 \in \Omega$  we have  $F(x_2) \subseteq F(x_1) + K\|x_1 - x_2\|\mathbb{B}^m$ .

**Definition 2.17** (Differential inclusions). Let  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a measurable multifunction. A *solution* or *trajectory* of the *differential inclusion*

$$(7) \quad \begin{cases} \dot{x}(t) \in F(x(t)), \\ x(t_0) = x_0 \in \mathbb{R}^n, \end{cases}$$

is an absolutely continuous map  $y^{x_0} : I \rightarrow \mathbb{R}^n$  such that  $I$  is an open interval containing  $t_0$ ,  $y^{x_0}(t_0) = x_0$ , and  $\dot{y}^{x_0}(t) \in F(t, y^{x_0}(t))$  for a.e.  $t \in I$ .

We refer the reader to the monograph [2] for further properties and results on multifunctions and differential inclusions.

**Definition 2.18** (Minimum time function). Let  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a measurable multifunction, and let  $\mathcal{S}$  be a given closed subset of  $\mathbb{R}^n$ . We define the *minimum time function*  $T : \mathbb{R}^n \rightarrow ]0, +\infty]$  by setting:

$$(8) \quad T(x) = \inf \{ \tau > 0 : \exists y^x(\cdot) \text{ satisfying } \dot{x}(t) \in F(x(t)), x(0) = x, y^x(\tau) \in \mathcal{S} \}$$

with the usual convention  $\inf \emptyset = +\infty$ .

The *reachable* (or *controllable*) set is defined as  $\mathcal{R} = \text{dom}(T)$ .



## 3. STANDING HYPOTHESIS AND FIRST CONSEQUENCES

From now on, we will consider  $t_0 = 0$  in the differential inclusion (7), and the associated minimum time function  $T$  defined in (8).

**Definition 3.1** (Hamiltonian). The *Hamiltonian*  $H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  associated with  $F$  is given by

$$(9) \quad H(x, p) = \sup_{v \in F(x)} \langle p, v \rangle.$$

In other terms,  $H(x, p)$  is the support function to the set  $F(x)$ , evaluated at  $p$ .

**Definition 3.2** (Hypothesis (F)). We say that the hypothesis (F) is satisfied by system (7) if the following holds true:

- (F1)  $F(x)$  is nonempty, convex, and compact for each  $x \in \mathbb{R}^n$ ;
- (F2)  $F$  is Lipschitz continuous.

**Definition 3.3** (Hypothesis (H)). We say that the hypothesis (H) is satisfied by system (7) if the following holds true:

- (H1) There exists a constant  $c_0 \geq 0$  such that, for every  $p \in \mathbb{R}^n$ ,  $x \mapsto H(x, p)$  is semiconvex with semiconvexity constant  $c_0|p|$ .
- (H2) For all  $p \neq 0$ , the gradient  $\nabla_p H(\cdot, p)$  exists and is globally Lipschitz continuous, i.e. there exists  $K_1 \geq 0$  such that

$$(10) \quad \|\nabla_p H(x, p) - \nabla_p H(y, p)\| \leq K_1 \|y - x\|, \quad \text{for all } x, y \in \mathbb{R}^n, p \in \mathbb{R}^n \setminus \{0\}.$$

*Remark 3.4.* If hypothesis (F) hold, and  $K$  is a Lipschitz constant for  $F$ , then for all  $v \in F(x_2)$  there exists  $\eta \in \mathbb{B}^n$ ,  $\rho \in [0, 1]$ ,  $v' \in F(x_1)$  such that we have  $v = v' + K\rho\eta\|x_1 - x_2\|$ . Moreover,  $H(x, p) \in \mathbb{R}$  for every  $x, p \in \mathbb{R}^n$ . In particular,

$$\begin{aligned} \langle p, v \rangle &= \langle p, v' \rangle + K\rho\|x_1 - x_2\| \langle \eta, p \rangle \\ &\leq \sup_{w \in F(x_1)} \langle p, w \rangle + K\|p\| \cdot \|x_1 - x_2\| = H(x_1, p) + K\|p\| \cdot \|x_1 - x_2\|. \end{aligned}$$

By passing to the supremum on  $v \in F(x_2)$ , we have

$$H(x_2, p) - H(x_1, p) \leq K\|p\| \cdot \|x_1 - x_2\|,$$

whence, reversing the role of  $x_1$  and  $x_2$ , we end up with

$$(11) \quad |H(x_1, p) - H(x_2, p)| \leq K\|p\| \cdot \|x_1 - x_2\|,$$

for all  $x_1, x_2, p \in \mathbb{R}^n$ . Take now  $x_1 = x$ ,  $x_2 = 0$ , and let  $v \in F(x)$  be such that  $v \neq 0$  and  $\|v\| = \max_{w \in F(x)} \|w\|$ . Set  $p = v/\|v\|$ . Then we have

$$\|v\| = \max_{w \in F(x)} \|w\| = H(x, p) \leq K\|x\| + H(0, p) \leq K\|x\| + \max_{w \in F(0)} \|w\|.$$

So, there exists  $K_2 > 0$  such that if  $F(x) \neq \{0\}$  it holds:

$$(12) \quad \max_{w \in F(x)} \|w\| \leq K_2(1 + \|x\|).$$

Since the above inequality still holds even if  $F(x) = \{0\}$ , it holds for every  $x \in \mathbb{R}^n$ .

*Remark 3.5.* Global Lipschitz continuity in both (F2) and (H2) was assumed just to simplify computations. Indeed, our results still hold if  $F$  is locally Lipschitz with respect to the Hausdorff distance, and  $\nabla_p H(\cdot, p)$  is locally Lipschitz in  $x$ , uniformly so over  $p$  in  $\mathbb{R} \setminus \{0\}$ . In that case, however, we need to assume (12) as an extra condition.

We collect now, in the following, some consequences of assumptions (F) and (H):

**Proposition 3.6.** *Suppose that  $F$  and  $H$  satisfy hypotheses (F) and (H).*

(i) *For every  $p \neq 0$  and  $x \in \mathbb{R}^n$ , it holds*

$$\partial H(x, p) \subseteq \partial_x H(x, p) \times \partial_p H(x, p),$$

*where  $\partial_x$  and  $\partial_p$  denote the Clarke's generalized gradient with respect to  $x$  variables and  $p$  variables, respectively.*

(ii) *For every  $p \neq 0$ , if we set  $F_p(x) = \nabla_p H(x, p)$ , then  $F_p(x) \in F(x)$  and  $\langle F_p(x), p \rangle = H(x, p)$ . Moreover,  $F_p(\cdot)$  is Lipschitz continuous with Lipschitz constant  $K_1$ , i.e.,*

$$\|F_p(y) - F_p(x)\| \leq K_1 \cdot \|y - x\|, \quad \forall x, y \in \mathbb{R}^n.$$

(iii) *For every  $\xi \in \partial_x H(x, p)$ , it holds*

$$H(y, p) - H(x, p) - \langle \xi, y - x \rangle \geq -c_0 \cdot \|p\| \cdot \|y - x\|^2, \quad \forall x, y \in \mathbb{R}^n.$$

*Proof.* See Proposition 1 in [8]. □

**Proposition 3.7.** *Assume that  $F$  and  $H$  satisfy assumptions (F) and (H). Let  $T > 0$ ,  $p(\cdot) : [0, T] \rightarrow \mathbb{R}^n$  be an absolutely continuous arc such that  $p(t) \neq 0$  for all  $t \in [0, T]$ . Given  $x_0 \in \mathbb{R}^n$ , then the problem*

$$(13) \quad \begin{cases} \dot{x}(t) = F_{p(t)}(x(t)), & \text{for a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases}$$

*admits a unique solution  $y(\cdot, x_0)$ . Moreover, for all  $t > 0$ , the following holds:*

(i)  $x_0 \mapsto y(t, x_0)$  is Lipschitz continuous on  $\mathbb{R}^n$  and

$$\|y(t, z_0) - y(t, x_0)\| \leq e^{Kt} \cdot \|z_0 - x_0\|, \quad \forall z_0 \in \mathbb{R}^n,$$

(ii)  $\|y(t, x_0)\| \leq (\|x_0\| + 1) \cdot e^{K_2 t} - 1$ ,

(iii)  $\|y(t, x_0) - x_0\| \leq K_2 \cdot (\|x_0\| + 1) e^{K_2 t}$ ,

*where  $K_2$  is the constant in Remark 3.4.*

*Proof.* The proof is based on Gronwall's inequality, see Proposition 3.4, Lemma 3.5, and Lemma 3.7 in [5]. □

The following statement allows us to make local approximation of trajectories of (7) with smooth ones:

**Proposition 3.8.** *Assume  $F$  and  $H$  satisfy hypotheses (F) and (H). Let  $\mathcal{K} \subset \mathbb{R}^n$  be compact and let  $\delta > 0$  be given. Set*

$$M = \sup \{\|v\| : v \in F(x), x \in \mathcal{K} + \delta B(0, 1)\}, \quad \delta_1 = \frac{\delta}{M + 1}.$$

*Then for each  $x_0 \in \mathcal{K}$ , and  $v \in F(x_0)$  there exists a  $C^1$  trajectory  $y^{x_0}(\cdot)$  of (7) such that*

$$\|\dot{y}_0^x(t) - v\| \leq K M t, \quad \forall t \in [0, \delta_1].$$

*Proof.* See Proposition 2.3 in [9]. □

We recall the following result:

**Proposition 3.9.** *Assume that  $F$  and  $H$  satisfy (F) and (H). Suppose  $x(\cdot)$  is an optimal trajectory starting from  $x_0$  and  $\xi \neq 0$  is a unit proximal normal vector to  $\mathcal{S}$  at  $x(T(x_0))$  so that  $\mu \doteq H(x(T(x_0)), \xi)^{-1} \leq 0$ . Then there exists an absolutely continuous arc  $p : [0, T] \rightarrow \mathbb{R}^n$  satisfying*

$$(14) \quad \begin{cases} \dot{x}(t) \in F_{p(t)}(x(t)) \\ -\dot{p}(t) \in \partial_x H(x(t), p(t)) \end{cases} \quad \text{a.e. } t \in [0, T(x_0)]$$

and the transversality condition

$$-p(T) = \mu \cdot \xi.$$

*Proof.* See Theorem 2.1 in [9]. □

## 4. MAIN RESULTS

### 4.1. The dual arc inclusion.

**Theorem 4.1.** *Assume that  $F$  and  $H$  satisfy (F) and (H). Let  $x \in \mathbb{R}^n \setminus \mathcal{S}$ , let  $x(\cdot)$  be an optimal trajectory which starts from  $x$  and reaches  $\mathcal{S}$  in time  $T(x)$ , and let  $\xi$  be a unit proximal inner normal to  $\mathcal{S}$  at  $\bar{x} = x(T(x))$ . Consider the system:*

$$(15) \quad \begin{cases} \dot{x}(t) = F_{p(t)}(x(t)), \\ -\dot{p}(t) \in \partial_x H(x(t), p(t)) \end{cases} \quad \text{for a.e. } t \in [0, T(x)].$$

*Then:*

- (i) *if  $H(\bar{x}, \xi) \neq 0$  and we couple system (15) with the condition  $p(T(x)) = \frac{\xi}{H(\bar{x}, \xi)}$ , we have  $-p(t) \in \partial^P T(x(t))$  for all  $t \in [0, T(x)]$ ,*
- (ii) *if  $H(\bar{x}, \xi) = 0$  and we couple system (15) with the condition  $p(T(x)) = \xi$ , we have  $-p(t) \in \partial^\infty T(x(t))$  for all  $t \in [0, T(x)]$ .*

*Proof.* Applying the dynamic programming principle, one can see that if the conclusion of the theorem holds for  $t = 0$  then it will hold for all  $t \in [0, T(x)]$ . Therefore, we only need to prove the result for  $t = 0$ . We divide the proof into two main steps.

*Step 1 (near the target  $\mathcal{S}$ ):* There exists a constant  $C_1 \geq 0$  such that for all  $\bar{y} \in \overline{\mathcal{S}^c}$  and  $\beta \leq T(\bar{y})$  it holds:

$$(16) \quad \langle p(T(x)), \bar{y} - \bar{x} \rangle + \beta H(\bar{x}, p(T(x))) \leq C_1 \cdot (\|\bar{y} - \bar{x}\|^2 + \beta^2).$$

*Proof of Step 1:* Set  $\alpha = H(\bar{x}, p(T(x)))$ . Since  $p(T(x)) = \lambda \cdot \xi \in N_{\overline{\mathcal{S}^c}}^P(\bar{x})$ , we have that there exists  $C > 0$  such that:

$$(17) \quad \langle p(T(x)), \bar{y} - \bar{x} \rangle \leq C \cdot \|\bar{y} - \bar{x}\|^2, \quad \forall \bar{y} \in \overline{\mathcal{S}^c}.$$

Therefore, if  $\beta \leq 0$ , then (16) follows from (17) by choosing  $C_1 = C$ . Otherwise, if  $0 < \beta \leq T(\bar{y})$ , we consider the differential equation

$$(18) \quad \begin{cases} \dot{y}(t) &= F_{p(T(x))}(y(t)), \\ y(0) &= \bar{y}, \end{cases}$$

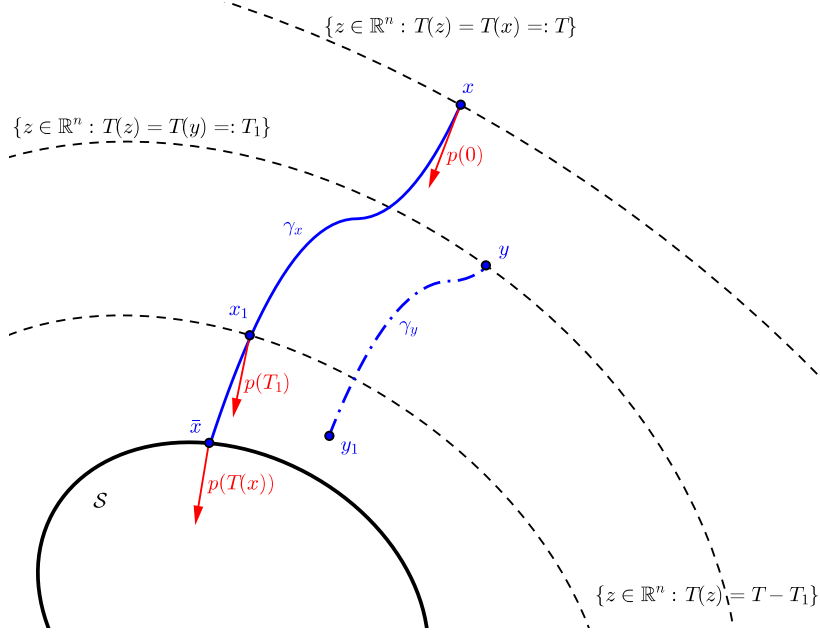


FIGURE 1. First case of the proof of Theorem 4.1, Step 2.

and set  $\bar{y}_1 := y(\beta)$ . Since  $\bar{y}_1 \in \overline{\mathcal{S}^c}$ , recalling (17) we have that

$$\begin{aligned} \langle p(T(x)), \bar{y}_1 - \bar{x} \rangle &\leq C \cdot \|\bar{y}_1 - \bar{x}\|^2 \leq 2C \cdot (\|\bar{y}_1 - \bar{y}\|^2 + \|\bar{y} - \bar{x}\|^2) \\ &\leq 2C \cdot (\|y(\beta) - y(0)\|^2 + \|\bar{y} - \bar{x}\|^2) \leq C_2 \cdot (\beta^2 + \|\bar{y} - \bar{x}\|^2), \end{aligned}$$

where  $C_2$  is a suitable positive constant. On the other hand, we compute

$$\begin{aligned} \langle p(T(x)), \bar{y} - \bar{y}_1 \rangle &= \left\langle p(T(x)), - \int_0^\beta F_{p(T(x))}(y(s)) ds \right\rangle \\ &= - \int_0^\beta \langle p(T(x)), F_{p(T(x))}(y(s)) \rangle ds \\ &= - \int_0^\beta \langle p(T(x)), F_{p(T(x))}(\bar{x}) \rangle ds + \\ &\quad - \int_0^\beta \langle p(T(x)), F_{p(T(x))}(y(s)) - F_{p(T(x))}(\bar{x}) \rangle ds \\ &\leq -\alpha \cdot \beta + C_3 \cdot (\|\bar{y} - \bar{x}\|^2 + \beta^2), \end{aligned}$$

where  $C_3$  is another suitable positive constant (we used also item (ii) in Proposition 3.6). Combining the two above inequalities, we obtain that

$$(19) \quad \langle p(T(x)), \bar{y} - \bar{x} \rangle \leq -\alpha \cdot \beta + (C_2 + C_3) \cdot (\|\bar{y} - \bar{x}\|^2 + \beta^2).$$

Therefore, (16) holds with  $C_1 = \max\{C, C_2 + C_3\}$ , and this concludes the proof of Step 1.  $\diamond$

*Step 2:* We show that  $(p(0), \alpha) \in N_{\text{hypo}(T)}^P(x, T(x))$ , i.e. there exists a positive constant  $C_5$  such that for all  $y \in \overline{\mathcal{S}^c}$  and  $\beta \leq T(y)$ :

$$(20) \quad \langle p(0), y - x \rangle + \alpha(\beta - T(x)) \leq C_5 \cdot (\|y - x\|^2 + |\beta - T(x)|^2)$$

*Proof of Step 2:* There are two possible cases:

*First case:* Suppose  $T_1 := T(y) \leq T(x) =: T$ . In this case, we only need to show that (20) holds for  $\beta = T_1$ , i.e.,

$$(21) \quad \langle p(0), y - x \rangle + \alpha(T_1 - T) \leq C_5 \cdot (\|y - x\|^2 + |T_1 - T|^2).$$

We consider the differential equation:

$$(22) \quad \begin{cases} \dot{y}(t) = F_{p(t)}(y(t)), \\ y(0) = y, \end{cases}$$

and set  $y_1 = y(T_1)$ ,  $x_1 = x(T_1)$ . Observing that  $T(x_1) = T - T_1$  and  $y_1 \in \overline{\mathcal{S}^c}$ . We first compute:

$$\begin{aligned} \langle p(T_1), y_1 - x_1 \rangle &= \langle p(0), y - x \rangle + \int_0^{T_1} \frac{d}{ds} \langle p(s), y(s) - x(s) \rangle ds \\ &= \langle p(0), y - x \rangle + \int_0^{T_1} \langle \dot{p}(s), y(s) - x(s) \rangle + \langle p(s), F_{p(s)}(y(s)) - F_{p(s)}(x(s)) \rangle ds \\ &= \langle p(0), y - x \rangle + \int_0^{T_1} \langle \dot{p}(s), y(s) - x(s) \rangle + H(y(s), p(s)) - H(x(s), p(s)) ds. \end{aligned}$$

Since  $-\dot{p}(s) \in \partial_x H(x(s), p(s))$ , by Proposition 3.6 we have that

$$\langle -\dot{p}(s), y(s) - x(s) \rangle + H(x(s), p(s)) - H(y(s), p(s)) \leq c_0 \cdot \|p(s)\| \cdot \|y(s) - x(s)\|^2.$$

Hence, for a suitable constant  $C_6 > 0$ , it holds

$$\langle -\dot{p}(s), y(s) - x(s) \rangle + H(x(s), p(s)) - H(y(s), p(s)) \leq C_6 \cdot \|y - x\|^2$$

Therefore,

$$(23) \quad \langle p(0), y - x \rangle \leq \langle p(T_1), y_1 - x_1 \rangle + C_6 \cdot \|y - x\|^2.$$

Now, we compute

$$\begin{aligned} \langle p(T_1), y_1 - x_1 \rangle &= \langle p(T), y_1 - x_1 \rangle + \langle p(T_1) - p(T), y_1 - x_1 \rangle \\ &\leq \langle p(T), y_1 - x_1 \rangle + C_7 \cdot (|T - T_1|^2 + \|y - x\|^2), \end{aligned}$$

and

$$\begin{aligned} \langle p(T), y_1 - x_1 \rangle &= \langle p(T), x(T) - x(T_1) \rangle + \langle p(T), y_1 - \bar{x} \rangle \\ &\leq \int_{T_1}^T \langle p(T), F_{p(s)}(x(s)) \rangle ds + \langle p(T), y_1 - \bar{x} \rangle \\ &= \int_{T_1}^T \langle p(T), F_{p(s)}(x(s)) - F_{p(T)}(x(T)) \rangle ds + \alpha \cdot (T - T_1) + \langle p(T), y_1 - \bar{x} \rangle. \end{aligned}$$

Recalling (16), we have that

$$\langle p(T), y_1 - \bar{x} \rangle \leq C \cdot \|y_1 - \bar{x}\|^2 \leq C_8 \cdot (|T - T_1|^2 + \|y_1 - \bar{x}\|^2),$$

for a suitable positive constant  $C_8$ . On the other hand, for any  $s \in [T_1, T]$ ,

$$\begin{aligned} \langle p(T), F_{p(s)}(x(s)) - F_{p(T)}(x(T)) \rangle &= \\ &= \langle p(T), F_{p(T)}(x(s)) - F_{p(T)}(x(T)) \rangle + \langle p(T), F_{p(s)}(x(s)) - F_{p(T)}(x(s)) \rangle \\ &\leq C_9 \cdot |s - T|, \end{aligned}$$

for some constant  $C_9 > 0$ . Hence,

$$\int_{T_1}^T \langle p(T), F_{p(s)}(x(s)) - F_{p(T)}(x(T)) \rangle ds \leq \frac{C_9}{2} \cdot |T - T_1|^2.$$

Therefore, we obtain that

$$\langle p(T), y_1 - x_1 \rangle \leq \alpha \cdot (T - T_1) + C_{10} \cdot (|T - T_1|^2 + \|y - x\|^2),$$

for a suitable constant  $C_{10}$ . Combining this estimate with (23), we finally get (21).

*Second case:*  $T_1 := T(y) > T(x) =: T$ . We consider the differential equation

$$(24) \quad \begin{cases} \dot{y}(t) = F_{p(t)}(y(t)), \\ y(0) = y, \end{cases}$$

and set  $\bar{y} = y(T)$ . By the same arguments of the first step, we have that

$$(25) \quad \langle p(0), y - x \rangle \leq \langle p(T), \bar{y} - \bar{x} \rangle + C \cdot \|y - x\|^2.$$

On the other hand, observing that  $\bar{y} \in \mathcal{S}^c$ , from (16) we obtain that

$$\langle p(T), \bar{y} - \bar{x} \rangle + \alpha \cdot \beta_1 \leq C_1 \cdot (\|\bar{y} - \bar{x}\|^2 + \beta_1^2)$$

for all  $\beta_1 \leq T(\bar{y})$ . Thus, for every  $\beta \leq T(y)$ , by the dynamic programming principle we have that  $\beta - T \leq T(\bar{y})$ . Hence, taking  $\beta_1 = \beta - T$  in the above inequality, we get that

$$\langle p(T), \bar{y} - \bar{x} \rangle + \alpha \cdot (\beta - T) \leq C_1 \cdot (\|\bar{y} - \bar{x}\|^2 + |\beta - T|^2).$$

By Gronwall's inequality, we finally obtain that there exists  $C_{11} > 0$  such that

$$\langle p(T), \bar{y} - \bar{x} \rangle + \alpha \cdot (\beta - T) \leq C_{11} \cdot (\|y - x\|^2 + |\beta - T|^2).$$

The proof is complete.  $\square$

#### 4.2. The constancy of $H$ .

**Proposition 4.2.** *Assume (F) and (H). Let  $x \in \mathbb{R}^n \setminus \mathcal{S}$ , let  $x(\cdot)$  be an optimal trajectory at  $x$  reaching  $\mathcal{S}$  in time  $T(x)$ , and let  $\xi$  be a proximal inner unit normal to  $\mathcal{S}$  at  $\bar{x} = x(T(x))$ . Consider the system:*

$$(26) \quad \begin{cases} \dot{x}(t) = F_{p(t)}(x(t)), \\ -\dot{p}(t) \in \partial_x H(x(t), p(t)) \end{cases} \quad \text{for a.e. } t \in [0, T(x)].$$

Then:

- (i) if  $H(\bar{x}, \xi) \neq 0$  and we couple system (26) with the condition  $p(T(x)) = \frac{\xi}{H(\bar{x}, \xi)}$ , we have  $H(x(t), p(t)) = 1$  for all  $t \in [0, T(x)]$ ,
- (ii) if  $H(\bar{x}, \xi) = 0$  and we couple system (26) with the condition  $p(T(x)) = \xi$ , we have  $H(x(t), p(t)) = 0$  for all  $t \in [0, T(x)]$ .

*Proof.*

- (i) Assume that  $H(\bar{x}, p(T(x))) = 1$ . Fixing  $t \in (0, T(x))$  we will show that  $H(x(t), p(t)) = 1$ . First, observe that, by Theorem 4.1,  $-p(t) \in \partial^P T(x(t))$  for all  $t \in [0, T(x))$ . Now, for every  $v \in F(x(t))$ , Proposition 3.8 ensures that there exists a trajectory  $y(\cdot) \in C^1$  starting from  $x(t)$  at time  $t$  such that

$$\|\dot{y}(t+s) - v\| \leq K \cdot s, \quad \forall s \in (0, \delta).$$

Since  $-p(t) \in \partial^P T(x(t))$ , we have that

$$\begin{aligned} \langle p(t), y(t+s) - x(t) \rangle + \beta - T(x(t)) &\leq \\ &\leq C \cdot (\|y(t+s) - x(t)\|^2 + |\beta - T(x(t))|^2), \end{aligned}$$

for all  $\beta \leq T(y(t+s))$ . Observe that  $T(y(t+s)) \geq T(x(t)) - s$  and choose  $\beta = T(x(t)) - s$  in the above inequality to obtain

$$\langle p(t), y(t+s) - x(t) \rangle - s \leq C_1 \cdot s^2.$$

Hence,

$$\langle p(t), v \rangle = \lim_{s \rightarrow 0^+} \frac{\langle p(t), y(t+s) - x(t) \rangle}{s} \leq 1.$$

This implies that  $H(x(t), p(t)) \leq 1$ .

On the other hand, we have

$$\begin{aligned} \langle p(t), x(t-s) - x(t) \rangle + T(x(t-s)) - T(x(t)) &\leq \\ &\leq C \cdot (\|x(t-s) - x(t)\|^2 + |T(x(t-s)) - T(x(t))|^2). \end{aligned}$$

Since  $T(x(t-s)) - T(x(t)) = s$ , we get that

$$\langle p(t), x(t-s) - x(t) \rangle + s \leq C_1 \cdot s^2.$$

thus

$$1 \leq \limsup_{s \rightarrow 0^+} \left\langle p(t), \frac{x(t) - x(t-s)}{s} \right\rangle \leq H(x(t), p(t)).$$

Therefore, we finally obtain that

$$H(x(t), p(t)) = 1, \quad \forall t \in (0, T(x)).$$

By the continuity of  $H$ , the proof of (i) is complete.

- (ii) Assume that  $H(\bar{x}, p(T(x))) = 0$ . Fixing  $t \in (0, T(x))$ , we will show that  $H(x(t), p(t)) = 0$ . For every  $v \in F(x(t))$ , there exists a trajectory  $y(\cdot) \in C^1$  starting from  $x(t)$  at time  $t$  such that

$$\|\dot{y}(t+s) - v\| \leq K \cdot s, \quad \forall s \in (0, \delta).$$

Since  $-p(t) \in \partial^\infty T(x(t))$ , we have that

$$\langle p(t), y(t+s) - x(t) \rangle \leq C \cdot (\|y(t+s) - x(t)\|^2 + |\beta - T(x(t))|^2),$$

for every  $\beta \leq T(y(t+s))$ . Since  $T(y(t+s)) \geq T(x(t)) - s$ , we can choose  $\beta = T(x(t)) - s$  to obtain, by the above inequality,

$$\langle p(t), y(t+s) - x(t) \rangle \leq C_1 \cdot s^2.$$

Hence

$$\langle p(t), v \rangle = \lim_{s \rightarrow 0^+} \left\langle p(t), \frac{y(t+s) - x(t)}{s} \right\rangle \leq 0.$$

Therefore,  $H(x(t), p(t)) \leq 0$ .

On the other hand,

$$\begin{aligned} \langle p(t), x(t-s) - x(t) \rangle &\leq \\ &\leq C \cdot (\|x(t-s) - x(t)\|^2 + |T(x(t-s)) - T(x(t))|^2). \end{aligned}$$

Since  $T(x(t-s)) - T(x(t)) = s$ , from the above inequality we deduce that

$$\langle p(t), x(t-s) - x(t) \rangle \leq C_1 \cdot s^2.$$

Thus,

$$0 \leq \limsup_{s \rightarrow 0^+} \left\langle p(t), \frac{x(t) - x(t-s)}{s} \right\rangle \leq H(x(t), p(t)).$$

Therefore,

$$H(x(t), p(t)) = 0, \quad \forall t \in (0, T(x)).$$

By the continuity of  $H$ , the proof of (ii) is complete.  $\square$

### 4.3. Optimality conditions.

**Theorem 4.3.** *Assume (F) and (H). Let  $x \in \mathbb{R}^n \setminus \mathcal{S}$  and let  $x(\cdot)$  be a trajectory starting from  $x$  such that  $x(T) \in \mathcal{S}$ . Suppose there exists a continuous function  $p(\cdot) : [0, T] \rightarrow \mathbb{R}^n$  such that*

$$(27) \quad \begin{cases} \dot{x}(t) = F_{p(t)}(x(t)) \\ -p(t) \in \partial^P T(x(t)) \\ H(x(t), p(t)) = 1 \end{cases} \quad \text{for a.e. } t \in [0, T].$$

Then  $x(\cdot)$  is an optimal trajectory.

*Proof.* Set  $\varphi(\cdot) = T(x(\cdot))$ . Owing to the dynamic programming principle, for every  $t \in [0, T]$  and  $0 < s < T - t$  we have

$$\varphi(t+s) \geq \varphi(t) - s.$$

Hence,  $\liminf_{s \rightarrow 0^+} \frac{\varphi(t+s) - \varphi(t)}{s} \geq -1$ .

On the other hand, since  $-p(t) \in \partial^P T(x(t))$ , we have that

$$\begin{aligned} \langle p(t), x(t+s) - x(t) \rangle + \varphi(t+s) - \varphi(t) &\leq \\ &\leq C \cdot (\|x(t+s) - x(t)\|^2 + |\varphi(t+s) - \varphi(t)|^2). \end{aligned}$$

This is equivalent to

$$\begin{aligned} C \cdot \frac{\|x(t+s) - x(t)\|^2}{s} &\geq \\ &\geq \left\langle p(t), \frac{x(t+s) - x(t)}{s} \right\rangle + \frac{\varphi(t+s) - \varphi(t)}{s} \cdot (1 - C \cdot (\varphi(t+s) - \varphi(t))) \\ &= \frac{1}{s} \int_0^s \langle p(t), F_{p(t+\tau)}(x(t+\tau)) \rangle d\tau + \frac{\varphi(t+s) - \varphi(t)}{s} \cdot (1 - C \cdot (\varphi(t+s) - \varphi(t))) \\ &= \frac{1}{s} \int_0^s \langle p(t+\tau), F_{p(t+\tau)}(x(t+\tau)) \rangle d\tau + \frac{1}{s} \int_0^s \langle p(t) - p(t+\tau), F_{p(t+\tau)}(x(t+\tau)) \rangle d\tau + \\ &\quad + \frac{\varphi(t+s) - \varphi(t)}{s} \cdot (1 - C \cdot (\varphi(t+s) - \varphi(t))) \\ &= \frac{1}{s} \int_0^s H(x(t+\tau), p(t)) d\tau + \frac{1}{s} \int_0^s \langle p(t) - p(t+\tau), F_{p(t+\tau)}(x(t+\tau)) \rangle d\tau + \\ &\quad + \frac{\varphi(t+s) - \varphi(t)}{s} \cdot (1 - C \cdot (\varphi(t+s) - \varphi(t))), \\ &= 1 + \frac{1}{s} \int_0^s \langle p(t) - p(t+\tau), F_{p(t+\tau)}(x(t+\tau)) \rangle d\tau + \\ &\quad + \frac{\varphi(t+s) - \varphi(t)}{s} \cdot (1 - C \cdot (\varphi(t+s) - \varphi(t))), \end{aligned}$$



where we used the Mean Value Theorem and the fact that

$$\langle p(t + \tau), F_{p(t+\tau)}(x(t + \tau)) \rangle = H(p(t + \tau), x(t + \tau)) = 1,$$

for a.e.  $\tau \in [0, s]$  and for every  $0 < s < T - t$ . The continuity of  $p(\cdot)$  and the estimate

$$\lim_{s \rightarrow 0^+} \frac{\|x(t + s) - x(t)\|^2}{s} = 0,$$

now yield for every  $t \in ]0, T[$ :

$$\limsup_{s \rightarrow 0^+} \frac{\varphi(t + s) - \varphi(t)}{s} \leq -1.$$

Thus,  $\lim_{s \rightarrow 0^+} \frac{\varphi(t + s) - \varphi(t)}{s} = -1$  for every  $0 < t < T$ .

Now, for  $s < 0$ , we have that  $\varphi(t + s) \leq \varphi(t) - s$ . Thus,  $\frac{\varphi(t + s) - \varphi(t)}{s} \geq -1$  for all  $0 < t < T$  and  $-t < s < 0$ . Hence,

$$\liminf_{s \rightarrow 0^-} \frac{\varphi(t + s) - \varphi(t)}{s} \geq -1.$$

On the other hand, we have

$$\begin{aligned} C \cdot \|x(t) - x(t + s)\|^2 &\geq \\ &\geq \langle p(t + s), x(t) - x(t + s) \rangle + (\varphi(t) - \varphi(t + s)) \cdot (1 - C \cdot (\varphi(t) - \varphi(t + s))). \end{aligned}$$

For  $s < 0$ , by performing similar computations as done in the case  $0 < s < T - t$ , from the above inequality we get that

$$\begin{aligned} C \cdot \frac{\|x(t) - x(t + s)\|^2}{s} &\geq \\ &\geq \left\langle p(t + s), \frac{x(t) - x(t + s)}{s} \right\rangle + \frac{\varphi(t) - \varphi(t + s)}{s} \cdot (1 - C \cdot (\varphi(t) - \varphi(t + s))) \\ &\geq \left\langle p(t + s) - p(t), \frac{x(t) - x(t + s)}{s} \right\rangle + \left\langle p(t), \frac{x(t) - x(t + s)}{s} \right\rangle + \\ &\quad + \frac{\varphi(t) - \varphi(t + s)}{s} \cdot (1 - C \cdot (\varphi(t) - \varphi(t + s))) \\ &\geq \left\langle p(t + s) - p(t), \frac{x(t) - x(t + s)}{s} \right\rangle - \frac{1}{s} \int_0^s \langle p(t), F_{p(t+\tau)}(x(t + \tau)) \rangle d\tau + \\ &\quad + \frac{\varphi(t) - \varphi(t + s)}{s} \cdot (1 - C \cdot (\varphi(t) - \varphi(t + s))) \\ &\geq \left\langle p(t + s) - p(t), \frac{x(t) - x(t + s)}{s} \right\rangle - \frac{1}{s} \int_0^s \langle p(t) - p(t + \tau), F_{p(t+\tau)}(x(t + \tau)) \rangle d\tau + \\ &\quad + \frac{\varphi(t) - \varphi(t + s)}{s} \cdot (1 - C \cdot (\varphi(t) - \varphi(t + s))) - \frac{1}{s} \int_0^s \langle p(t + \tau), F_{p(t+\tau)}(x(t + \tau)) \rangle d\tau. \end{aligned}$$

Thus, for every  $0 < t < T$  we have

$$\limsup_{s \rightarrow 0^-} \frac{\varphi(t + s) - \varphi(t)}{s} \leq -1.$$

This implies that  $\lim_{s \rightarrow 0^-} \frac{\varphi(t + s) - \varphi(t)}{s} = -1$ . Therefore, we finally have that  $\dot{\varphi}(t) = -1$  for all  $t \in ]0, T(x)[$ , so

$$T(x(t)) = \varphi(t) = \varphi(0) - t = T(x) - t,$$

that is,  $x(\cdot)$  is an optimal trajectory.  $\square$

#### 4.4. Regularity results.

In order to prove the regularity results of this section we need the following technical lemma, the proof of which is postponed to the Appendix.

**Lemma 4.4.** *Let  $D \subseteq \mathbb{R}^n$  be open, let  $f \in C^0(D)$  be such that  $\text{hypo } f$  satisfies an external sphere condition with a locally uniform radius, and let  $V \subseteq D$  be a bounded open set. If  $\nabla f$  is essentially bounded on  $V$ , then  $f$  is Lipschitz continuous on  $V$ .*

**Lemma 4.5.** *Let  $\mathcal{S} \subseteq \mathbb{R}^n$  be a closed nonempty set with a  $C^{1,1}$ -smooth boundary. Assume (F) and (H). Suppose  $H \in C^{1,1}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$  and  $T(\cdot)$  is continuous in  $\mathcal{R}$ . Then  $x_0 \in \mathcal{R} \setminus \mathcal{S}$  is a point at which  $T(\cdot)$  fails to be Lipschitz continuous if and only if there exists:*

- (i) an optimal trajectory  $x_0(\cdot)$  with  $x_0(0) = x_0$ ,  $\bar{x}_0 := x_0(T(x_0)) \in \mathcal{S}$ , and
- (ii) a unit vector  $\xi_0 \in N_{\mathbb{R}^n \setminus \mathcal{S}}^P(\bar{x}_0)$  such that  $H(\bar{x}_0, \xi_0) = 0$ .

*Proof.* First of all, observe that, if (i) and (ii) hold true at a point  $x_0 \in \mathcal{R} \setminus \mathcal{S}$ , then  $\partial^\infty T(x_0) \neq \emptyset$  thanks to Theorem 4.1. Therefore,  $T(\cdot)$  cannot be Lipschitz at  $x_0$ .

In order to prove the converse, let  $x_0 \in \mathcal{R} \setminus \mathcal{S}$  be a point at which  $T(\cdot)$  fails to be Lipschitz. Note that, owing to Theorem 4.1 in [5],  $\text{hypo } T$  satisfies an external sphere condition with a locally constant radius. Consequently, by Lemma 4.1 and Corollary 4.1 in [18],  $x \mapsto N_{\text{hypo } T}^P(x, T(x))$  has closed graph, and  $\partial^P T(x)$  reduces to the singleton  $\{\nabla T(x)\}$  at every  $x \in \text{dom } \nabla T$ . Moreover,  $T(\cdot)$  is differentiable a.e. in  $\mathcal{R} \setminus \mathcal{S}$ . Then, by Lemma 4.4,  $\nabla T$  must be unbounded on any neighborhood of  $x_0$ . So, there exists a sequence  $\{x_j\}_{j \geq 1} \subset \text{dom } \nabla T$  and a unit vector  $\zeta_0 \in \mathbb{R}^n$  such that

$$\lim_{j \rightarrow \infty} x_j = x_0, \quad \lim_{j \rightarrow \infty} |\nabla T(x_j)| = \infty \quad \text{and} \quad \lim_{j \rightarrow \infty} \frac{(-\nabla T(x_j), 1)}{|(-\nabla T(x_j), 1)|} = (\zeta_0, 0).$$

Notice that, since the map  $x \mapsto N_{\text{hypo } T}^P(x, T(x))$  has closed graph,  $\zeta_0 \in \partial^\infty T(x_0)$ .

Next, thanks to Proposition 3.9, we can construct sequences of absolutely continuous arcs  $\{x_j(\cdot)\}_{j \geq 1}$ ,  $\{p_j(\cdot)\}_{j \geq 1}$ , and unit vectors  $\{\xi_j\}_{j \geq 1}$  with the following properties:

- (a)  $x_j(0) = x_j$  and  $\bar{x}_j := x_j(T(x_j)) \in \mathcal{S}$ , i.e.,  $x_j(\cdot)$  is an optimal trajectory starting from  $x_j$  which reaches the target at  $\bar{x}_j$ ;
- (b)  $\xi_j \in N_{\mathbb{R}^n \setminus \mathcal{S}}^P(\bar{x}_j)$ , and  $H(\bar{x}_j, \xi_j) \neq 0$ ;
- (c)  $p_j := p_j(T(x_j)) = \frac{\xi_j}{H(\bar{x}_j, \xi_j)}$ ;
- (d)  $(x_j(\cdot), p_j(\cdot))$  satisfies

$$(28) \quad \begin{cases} \dot{x}_j(t) = \nabla_p H(x_j(t), p_j(t)) \\ -\dot{p}_j(t) = \nabla_x H(x_j(t), p_j(t)) \end{cases} \quad \text{for a.e. } t \in [0, T(x_j)].$$

Also, since  $x_j \in \text{dom } \nabla T$ , Theorem 4.1 ensures that  $-p_j(0) = \nabla T(x_j)$ .

Since  $x_j \rightarrow x_0$  and  $T(\cdot)$  is continuous, a well known compactness property of the trajectories of the differential inclusion

$$(29) \quad \dot{x}(t) \in F(x(t))$$

ensure that, up to extracting a subsequence, one can assume that  $\{x_j(\cdot)\}_{j \geq 1}$  converges uniformly to an absolutely continuous arc,  $x_0(\cdot)$ , which satisfies (29) together with  $x_0(0) = x_0$ . Again, since  $T(\cdot)$  is continuous and  $\mathcal{S}$  is closed, we have that

$\bar{x}_0 := x_0(T(x_0)) \in \mathcal{S}$ , i.e.  $x_0(\cdot)$  is optimal. We can also assume that  $\{\xi_j\}_{j \in \mathbb{N}}$  converges to the (proximal) inner unit normal  $\xi_0$  to  $\mathcal{S}$  at  $\bar{x}_0$  by the smoothness of  $\mathcal{S}$ .

We now claim that  $|p_j| \rightarrow \infty$  as  $j \rightarrow \infty$ . Indeed, since  $H$  is homogeneous of degree 1 in  $p$  and of class  $C^{1,1}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ , for some constant  $M > 0$

$$|\nabla_x H(x, p)| \leq M|p|$$

for all  $p \in \mathbb{R}^n \setminus \{0\}$  and all  $x$  in the bounded set

$$A := \{x_j(t) : t \in [0, T(x_j)]\} \cup \{x_0(t) : t \in [0, T(x_0)]\}.$$

Therefore, (28) yields, for all  $j \geq 1$  and a.e.  $t \in [0, T(x_j)]$ ,

$$-\frac{1}{2} \frac{d}{dt} |p_j(t)|^2 = \langle -\dot{p}_j(t), p_j(t) \rangle = \langle \nabla_x H(x_j(t), p_j(t)), p_j(t) \rangle \leq M|p_j(t)|^2.$$

The above differential inequality implies that

$$|p_j| \geq e^{-MT(x_j)} |p_j(0)|.$$

Since  $|p_j(0)| \rightarrow \infty$  and  $T(x_j) \rightarrow T(x_0)$  as  $j \rightarrow \infty$ , we conclude that also  $|p_j| \rightarrow \infty$ .

Finally, observe that  $p_j = |p_j| \cdot \xi_j$  and  $H(\bar{x}_j, p_j) = 1$  for every  $j \in \mathbb{N}$ . So,

$$1 = \lim_{j \rightarrow \infty} H(\bar{x}_j, p_j) = \lim_{j \rightarrow \infty} |p_j| \cdot H(\bar{x}_j, \xi_j).$$

Since  $|p_j| \rightarrow \infty$ , we must have that

$$\lim_{j \rightarrow \infty} H(\bar{x}_j, \xi_j) = H(\bar{x}_0, \xi_0) = 0.$$

This shows points (i) and (ii) above. Invoke Theorem 4.1 to complete the proof.  $\square$

**Theorem 4.6.** *Suppose  $H \in C^{1,1}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ , and let  $g \in C_b^2(\mathbb{R}^n)$ . Define the target set  $\mathcal{S}$  by*

$$\mathcal{S} = \{x \in \mathbb{R}^n : g(x) \leq 0\},$$

*and let  $\Sigma$  be a countably  $\mathcal{H}^{n-2}$ -rectifiable subset of  $\partial\mathcal{S}$ . In addition to (F) and (H), assume that:*

- (a)  $0 \in F(x)$  for every  $x \in \partial\mathcal{S}$ ,
- (b)  $\nabla g \neq 0$  on  $\partial\mathcal{S}$ ,
- (c)  $T(\cdot)$  is continuous in  $\mathcal{R}$ , and
- (d) for all  $x \in \partial\mathcal{S} \setminus \Sigma$  and  $\lambda \in \mathbb{R}$

$$(30) \quad \nabla_x H(x, -\nabla g(x)) - \nabla_p H(x, -\nabla g(x)) \cdot \nabla^2 g(x) \neq \lambda \nabla g(x).$$

*Then:*

- (i)  $T(\cdot)$  is a function of special bounded variation on  $\mathcal{R} \setminus \mathcal{S}$ , and
- (ii) there exists a closed countably  $\mathcal{H}^{n-1}$ -rectifiable set  $\mathcal{S} \subset \mathcal{R}$  such that  $T(\cdot)$  is locally semiconcave on  $\mathcal{R} \setminus (\mathcal{S} \cup \mathcal{S})$ .

*Proof.* We begin by showing the following property needed for the sequel of the proof:

- ( $\star$ ) if, for some point  $\bar{x} \in \partial\mathcal{S}$ , there exists a nonzero vector  $\xi \in N_{\mathbb{R}^n \setminus \mathcal{S}}^P(x)$  such that  $H(\bar{x}, \xi) = 0$ , then  $\bar{x} \in \Sigma$ .

Indeed, since  $0 \in F(x)$ , we have that  $H(x, p) \geq 0$  for all  $x \in \partial\mathcal{S}$  and  $p \in \mathbb{R}^n$ . Therefore,  $H$  attains a constrained minimum at  $(\bar{x}, \xi)$ . Moreover, since  $H(\bar{x}, \cdot)$  is positively homogeneous,  $(\bar{x}, r\xi)$  also minimizes  $H$  constrained to  $\partial\mathcal{S}$  for all  $r \geq 0$ . Now,  $-\nabla g(\bar{x})$  can be represented as  $r\xi$  for some  $r > 0$  because  $\mathcal{S}$  has a smooth

boundary. So,  $\bar{x}$  is a minimizer of the smooth function  $x \mapsto H(x, -\nabla g(x))$  on  $\partial\mathcal{S}$ . Then, by Lagrange multiplier rule and (30), we have  $\bar{x} \in \Sigma$ .

Next, let us prove the *SBV* property (i). Since hypo  $T$  satisfies an external sphere condition, owing to Proposition 6.1 in [17] we have that  $T \in BV_{loc}(\mathcal{R} \setminus \mathcal{S})$ . So, we just need to show that, on  $\mathcal{R} \setminus \mathcal{S}$ , the singular part of  $DT(\cdot)$  is concentrated on a set of  $\sigma$ -finite  $\mathcal{H}^{n-1}$ -measure. Recalling that the characteristics of the pde

$$H(x, -\nabla T) - 1 = 0$$

satisfy

$$(31) \quad \begin{cases} \dot{X}(t) = -\nabla_p H(X(t), P(t)) \\ \dot{P}(t) = \nabla_x H(X(t), P(t)), \end{cases}$$

consider the flow  $(X, P) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ , that is, for any  $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$ ,  $(X, P)(\cdot, x, p)$  is the solution pf (31) with initial conditions

$$(32) \quad X(0) = x, \quad P(0) = p.$$

Notice that  $(t, x) \mapsto (X, P)(t, x, -\nabla g(x))$  is Lipschitz continuous under our assumptions. Now, set

$$\Phi(t, x) = X(t, x, -\nabla g(x)) \quad t \geq 0, \quad x \in \mathbb{R}^n.$$

Then, by Lemma 4.5 we have that, for any  $t > 0$ ,

$$\Sigma_t := \{x \in \mathcal{R} \setminus \mathcal{S} : \partial^\infty T(x) \neq \emptyset, T(x) \leq t\} \subseteq \Phi((0, t] \times \Sigma)$$

is countably  $\mathcal{H}^{n-1}$ -rectifiable since  $\Sigma$  is countably  $\mathcal{H}^{n-2}$ -rectifiable by hypothesis. Thus,

$$\mathcal{S} := \{x \in \mathcal{R} \setminus \mathcal{S} : T(\cdot) \text{ fails to be Lipschitz at } x\} = \bigcup_{k=1}^{\infty} \Sigma_k$$

is countably  $\mathcal{H}^{n-1}$ -rectifiable as well. Moreover, owing to Lemma 4.2 and Corollary 4.2 in [18],  $\mathcal{S}$  is closed in  $\mathcal{R}$ . Therefore, in view of Proposition 2.7, we conclude that  $T(\cdot) \in SBV_{loc}(\mathcal{R} \setminus \mathcal{S})$ .

Finally, by Corollary 4.3 in [5],  $T(\cdot)$  is locally semiconcave on  $\mathcal{R} \setminus (\mathcal{S} \cup \mathcal{S})$  since it is locally Lipschitz on such a set.  $\square$

*Example 4.7* (Prescribing singularities). Let  $g \in C_b^2(\mathbb{R}^n)$ , let

$$\mathcal{S} := \{x \in \mathbb{R}^n : g(x) \leq 0\},$$

and assume that  $\nabla g(x) \neq 0$  at every  $x \in \mathcal{S}$ . Choose a countable set  $\Sigma \subseteq \mathcal{S}$  such that, for some function  $\psi : \Omega \rightarrow [0, +\infty[$  of class  $C^{1,1}$ , we have  $\Sigma = \{y \in \mathbb{R}^n : \psi(y) = 0\}$ . Define

$$H(x, p) := \psi(x) \text{dist}(p, \nabla g(x)^\perp),$$

where  $\nabla g(x)^\perp$  denotes the hyperplane in  $\mathbb{R}^n$  that is orthogonal to  $\nabla g(x)$ . For all  $x, p, q \in \mathbb{R}^n$ , we have that  $H(x, \lambda p) = \lambda H(x, p) \geq 0$  for every  $\lambda \geq 0$ , and  $H(x, p + q) \leq H(x, p) + H(x, q)$ . So,  $H(x, \cdot)$  is the support function of a closed convex subset  $F(x)$  of  $\mathbb{R}^n$ . With these choices, we have that (30) is satisfied.

## APPENDIX A. PROOF OF LEMMA 4.4

We now prove the technical Lemma 4.4.

*Proof.* Since hypo  $f$  satisfies the external sphere condition, for some  $\theta > 0$ , we have that for every  $y, z \in V$  there exists a unit vector  $v \in N_{\text{hypo } f}^P(z, f(z))$  such that

$$(33) \quad \langle v, (y - z, f(y) - f(z)) \rangle \leq \frac{1}{2\theta} (|y - z|^2 + |f(y) - f(z)|^2).$$

For  $f(y) \neq f(z)$ , dividing the above inequality by  $|f(y) - f(z)|$  we obtain

$$(34) \quad \left\langle v, \left( \frac{y - z}{|f(y) - f(z)|}, \frac{f(y) - f(z)}{|f(y) - f(z)|} \right) \right\rangle \leq \frac{1}{2\theta} \left( \frac{|y - z|}{\frac{|f(y) - f(z)|}{|y - z|}} + |f(y) - f(z)| \right).$$

By contradiction, suppose there exists  $L > 0$  such that  $|\nabla f(q)| \leq L$  for a.e.  $q \in V$  and there exists  $\bar{x} \in V$  such that

$$(35) \quad \limsup_{\substack{y, x \rightarrow \bar{x} \\ y \neq x}} \frac{|f(y) - f(x)|}{|y - x|} = \infty.$$

Let  $\{y_j\}_{j \in \mathbb{N}} \subseteq V$  and  $\{x_j\}_{j \in \mathbb{N}} \subseteq V$  be sequences converging to  $\bar{x}$  such that  $x_j \neq y_j$  for all  $j \in \mathbb{N}$ , and

$$\lim_{j \rightarrow \infty} \frac{|f(y_j) - f(x_j)|}{|y_j - x_j|} = \infty.$$

We claim that there exist  $\{z_j\}_{j \in \mathbb{N}}$  and  $\{w_j\}_{j \in \mathbb{N}}$  such that  $z_j \rightarrow \bar{x}$ ,  $w_j \rightarrow \bar{x}$ ,  $f$  is differentiable at  $z_j$  and at  $w_j$ ,  $|\nabla f(z_j)| \leq L$  and  $|\nabla f(w_j)| \leq L$  for every  $j \in \mathbb{N}$ , and

$$\lim_{j \rightarrow \infty} \frac{|f(z_j) - f(w_j)|}{|z_j - w_j|} = \infty.$$

Indeed, recalling that  $f \in C^0(V)$ , for every  $j \in \mathbb{N}$  there exists  $\delta_j > 0$  such that if  $|y_j - q| \leq \delta_j$ , we have  $|f(y_j) - f(q)| \leq |f(y_j) - f(x_j)|^2$ . Since  $f$  is a.e. differentiable on  $V$  by Theorem 3.1 in [18], and  $|\nabla f(q)| \leq L$  for a.e.  $q \in V$  by assumption, for every  $j \in \mathbb{N}$  we can find  $z_j \in B(y_j, \delta_j) \cap B(y_j, |y_j - x_j|^2)$  such that  $f$  is differentiable at  $z_j$ ,  $|\nabla f(z_j)| \leq L$ , and  $z_j \neq x_j$ . Notice that

$$\begin{aligned} |x_j - y_j| &\leq |x_j - z_j| + |z_j - y_j| \leq |x_j - z_j| + |x_j - y_j|^2 \\ |x_j - z_j| &\leq |x_j - y_j| + |y_j - z_j| \leq |x_j - y_j| + |x_j - y_j|^2. \end{aligned}$$

Dividing both inequalities by  $|x_j - y_j|$  and letting  $j \rightarrow \infty$  yields

$$\lim_{j \rightarrow \infty} \frac{|x_j - z_j|}{|x_j - y_j|} = 1.$$

Similarly, since

$$\begin{aligned} |f(x_j) - f(y_j)| &\leq |f(x_j) - f(z_j)| + |f(z_j) - f(y_j)| \\ &\leq |f(x_j) - f(z_j)| + |f(x_j) - f(y_j)|^2 \end{aligned}$$

$$\begin{aligned} |f(x_j) - f(z_j)| &\leq |f(x_j) - f(y_j)| + |f(y_j) - f(z_j)| \\ &\leq |f(x_j) - f(y_j)| + |f(x_j) - f(y_j)|^2, \end{aligned}$$

we obtain

$$\lim_{j \rightarrow \infty} \frac{|f(x_j) - f(z_j)|}{|f(x_j) - f(y_j)|} = 1.$$

So,

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{|f(z_j) - f(x_j)|}{|z_j - x_j|} &= \lim_{j \rightarrow \infty} \frac{|f(z_j) - f(x_j)|}{|f(y_j) - f(x_j)|} \frac{|y_j - x_j|}{|z_j - x_j|} \frac{|f(y_j) - f(x_j)|}{|y_j - x_j|} \\ &= \lim_{j \rightarrow \infty} \frac{|f(y_j) - f(x_j)|}{|y_j - x_j|} = \infty. \end{aligned}$$

To construct  $w_j$ , it suffices to apply the above reasoning to  $x_j, z_j$  instead of  $y_j, x_j$ .

Next, recall that, owing to Lemma 4.1 in [18],

$$\begin{aligned} N_{\text{hypo } f}^P(z_j, f(z_j)) &= \left\{ \lambda \frac{(-\nabla f(z_j), 1)}{|(-\nabla f(z_j), 1)|} : \lambda \geq 0 \right\}, \\ N_{\text{hypo } f}^P(w_j, f(w_j)) &= \left\{ \lambda \frac{(-\nabla f(w_j), 1)}{|(-\nabla f(w_j), 1)|} : \lambda \geq 0 \right\}. \end{aligned}$$

We can assume that, as  $j \rightarrow +\infty$ ,

$$\frac{(-\nabla f(z_j), 1)}{|(-\nabla f(z_j), 1)|} \rightarrow (\zeta, \xi) \quad \text{and} \quad \frac{(-\nabla f(w_j), 1)}{|(-\nabla f(w_j), 1)|} \rightarrow (\eta, \sigma) \quad \text{with} \quad \xi, \sigma \geq \frac{1}{\sqrt{1+C^2}} > 0$$

since  $\nabla f(z_j)$  and  $\nabla f(w_j)$  are bounded. Moreover,  $(\zeta, \xi), (\eta, \sigma) \in N_{\text{hypo } f}^P(\bar{x}, f(\bar{x}))$  (since  $\theta$  does not depend on  $z_j$  or  $w_j$ ), and, for every  $y \in V$ , we have that

$$\langle (\zeta, \xi), (y - \bar{x}, f(y) - f(\bar{x})) \rangle \leq \frac{1}{2\theta} (|y - \bar{x}|^2 + |f(y) - f(\bar{x})|^2).$$

We now distinguish two cases.

- a. Assume that  $\{f(z_j) - f(x_j)\}_{j \in \mathbb{N}}$  has a monotone increasing subsequence. In this case, since it tends to 0, after relabeling, we have  $f(x_j) - f(z_j) \geq 0$  for every  $j \in \mathbb{N}$ . Choosing, in (34),

$$y = x_j, \quad z = z_j, \quad v = \frac{(-\nabla f(z_j), 1)}{|(-\nabla f(z_j), 1)|},$$

we have

$$\begin{aligned} \left\langle \frac{(-\nabla f(z_j), 1)}{|(-\nabla f(z_j), 1)|}, \left( \frac{x_j - z_j}{|f(x_j) - f(z_j)|}, \frac{f(x_j) - f(z_j)}{|f(x_j) - f(z_j)|} \right) \right\rangle &\leq \\ &\leq \frac{1}{2\theta} \left( \frac{|x_j - z_j|}{\frac{|f(x_j) - f(z_j)|}{|x_j - z_j|}} + |f(x_j) - f(z_j)| \right). \end{aligned}$$

Passing to the limit in the above inequality yields  $\xi \leq 0$ , which is a contradiction with the fact that  $\xi > 0$ .

- b. Assume that  $\{f(z_j) - f(x_j)\}_{j \in \mathbb{N}}$  has a monotone decreasing subsequence. In this case, we have that  $\{f(x_j) - f(z_j)\}_{j \in \mathbb{N}}$  has a monotone increasing subsequence, and we apply the argument of the previous case by switching the roles of  $x_j$  and  $z_j$ , and replacing  $\xi$  with  $\sigma$ .

Since we have reached a contradiction assuming (35), the proof is complete.  $\square$

## REFERENCES

- [1] Paolo Albano, Piermarco Cannarsa, Khai T. Nguyen, and Carlo Sinestrari, *Singular gradient flow of the distance function and homotopy equivalence*, Math. Ann. **356** (2013), no. 1, 23–43.
- [2] Jean-Pierre Aubin and Arrigo Cellina, *Differential inclusions*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 264, Springer-Verlag, Berlin, 1984. Set-valued maps and viability theory. MR755330 (85j:49010)
- [3] Jean-Pierre Aubin and Halina Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.
- [4] Luigi Ambrosio, Nicola Fusco, and Diego Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000. MR1857292 (2003a:49002)
- [5] Piermarco Cannarsa and Khai T. Nguyen, *Exterior sphere condition and time optimal control for differential inclusions*, SIAM J. Control Optim. **49** (2011), no. 6, 2258–2576.
- [6] Piermarco Cannarsa and Carlo Sinestrari, *Convexity properties of the minimum time function*, Calc. Var. Partial Differential Equations **3** (1995), no. 3, 273–298, DOI 10.1007/s005260050016. MR1385289 (97f:49032)
- [7] ———, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, Progress in Nonlinear Differential Equations and their Applications, 58, Birkhäuser Boston Inc., Boston, MA, 2004. MR2041617 (2005e:49001)
- [8] Piermarco Cannarsa and Peter R. Wolenski, *Semiconcavity of the value function for a class of differential inclusions*, Discrete Contin. Dyn. Syst. **29** (2011), no. 2, 453–466, DOI 10.3934/dcds.2011.29.453. MR2728465 (2012a:49046)
- [9] Piermarco Cannarsa, Francesco Marino, and Peter Wolenski, *The dual arc inclusion with differential inclusions*, Nonlinear Anal. **79** (2013), 176–189, DOI 10.1016/j.na.2012.11.021. MR3005035
- [10] Giovanni Colombo and Antonio Marigonda, *Differentiability properties for a class of non-convex functions*, Calc. Var. Partial Differential Equations **25** (2006), no. 1, 1–31, DOI 10.1007/s00526-005-0352-7. MR2183853 (2006h:49027)
- [11] ———, *Singularities for a class of non-convex sets and functions, and viscosity solutions of some Hamilton-Jacobi equations*, J. Convex Anal. **15** (2008), no. 1, 105–129. MR2389006 (2009i:49023)
- [12] Giovanni Colombo, Antonio Marigonda, and Peter R. Wolenski, *Some new regularity properties for the minimal time function*, SIAM J. Control Optim. **44** (2006), no. 6, 2285–2299 (electronic), DOI 10.1137/050630076. MR2248184 (2008d:49021)
- [13] ———, *The Clarke generalized gradient for functions whose epigraph has positive reach*, Math. Op. Res. **38** (2013), no. 3, 451–468, DOI 10.1287/moor.1120.0580.
- [14] Giovanni Colombo and Khai T. Nguyen, *On the structure of the minimum time function*, SIAM J. Control Optim. **48** (2010), no. 7, 4776–4814, DOI 10.1137/090774549. MR2720234
- [15] Giovanni Colombo and Lionel Thibault, *Prox-regular sets and applications*, in *Handbook of Nonconvex Analysis*, D.Y. Gao, D. Motreanu eds., International Press, Somerville, MA, 2010.
- [16] Giovanni Colombo, Khai T. Nguyen, and Luong V. Nguyen, *SBV regularity for minimum time function*, Calc. Var. Partial Differential Equations, posted on 2013, DOI 10.1007/s00526-013-0682-9, (to appear in print).
- [17] Antonio Marigonda, Khai T. Nguyen, and Davide Vittone, *Some regularity properties for a class of upper semicontinuous functions*, Indiana Univ. Math. Journal. **62** (2013), no. 1.
- [18] Khai T. Nguyen, *Hypographs satisfying an external sphere condition and the regularity of the minimum time function*, J. Math. Anal. Appl. **372** (2010), 611–628, DOI 10.1016/j.jmaa.2010.07.010.
- [19] Khai T. Nguyen and Davide Vittone, *Rectifiability of siguralities of non-lipschitz functions*, J. of Convex Analysis **19** (2012), no. 1, 159–170.

PIERMARCO CANNARSA: DIPARTIMENTO DI MATEMATICA,  
UNIVERSITÀ DI ROMA 'TOR VERGATA'  
VIA DELLA RICERCA SCIENTIFICA 1, I-00133 ROMA, ITALY.  
*E-mail address:* `cannarsa@mat.uniroma2.it`

ANTONIO MARIGONDA: DEPARTMENT OF COMPUTER SCIENCE,  
UNIVERSITY OF VERONA  
STRADA LE GRAZIE 15, I-37134 VERONA, ITALY.  
*E-mail address:* `antonio.marigonda@univr.it`

KHAI T. NGUYEN: DEPARTMENT OF MATHEMATICS,  
PENN STATE UNIVERSITY,  
UNIVERSITY PARK, PA. 16802, U.S.A.  
*E-mail address:* `ktn2@psu.edu`



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