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ROBUSTNESS OF PERFORMANCE AND STABILITY FOR MULTISTEP AND UPDATED MULTISTEP MPC SCHEMES

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Abstract. We consider a model predictive control approach to approximate the solution of infinite horizon optimal control problems for perturbed nonlinear discrete time systems. By reducing the number of re-optimizations, the computational load can be lowered considerably at the expense of reduced robustness of the closed-loop solution against perturbations. In this paper, we propose and analyze an update strategy based on re-optimizations on shrinking horizons which is computationally less expensive than that based on full horizon re-optimization, and at the same time allowing for rigorously quantifiable robust performance estimates.

1. **Introduction.** The paper deals with solving infinite horizon optimal control problems (OCPs) for perturbed nonlinear systems by model predictive control (MPC). MPC provides an algorithmic synthesis of an approximately optimal feedback law by iteratively solving finite horizon OCPs. Due to its feedback nature, MPC has good inherent robustness properties in the perturbed setting considered in this paper, although the optimization in each iteration is performed for a nominal model, i.e., without taking into account perturbations.

The computational load of MPC can be lowered considerably by performing re-optimizations less often, resulting in a so-called multistep feedback law. In the nominal (i.e., unperturbed) case, only a mild difference between the quality of the solutions can be observed when using multistep feedback laws instead of a standard MPC scheme. For a system subject to perturbations, however, the multistep feedback does not allow the controller to react, for an extended period of time, against the deviation of the real state to the predicted state. Hence, multistep feedback laws are in general considerably less robust against perturbations. To address the challenge of maintaining robustness while keeping the computational cost low, in this paper we propose and analyze an updating approach based on re-optimizations on shrinking horizons which are computationally less expensive than re-optimizations on the full horizon.

Our analysis builds upon the work presented in [5] in which, for a finite horizon optimal control problem setting for systems under perturbations, the application of the nominal control strategy and the shrinking horizon strategy are analyzed. The

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shrinking horizon strategy consists of performing re-optimization, for the nominal model, at each sampling instant using the current perturbed state. The evident performance improvement brought about by the re-optimization is quantified in [5] using moduli of continuity of value functions. From the finite horizon optimal control setting, in this paper we switch attention to infinite horizon optimal control. Since we treat undiscounted problems, one of the key challenges when passing from finite to infinite horizon is that typically asymptotic stability of the approximately optimal solution must be established before we can even talk about approximately optimal performance. Since for perturbed systems asymptotic stability is often too strong a property to expect, in this paper we develop our results using the notion of practical asymptotic stability.

Our approach has similarities to [14] in the sense that updates are applied in order to cope with the nominal and real model disparity. Moreover, as in [14] we consider MPC without stabilizing terminal constraints or costs, i.e., the simplest possible MPC variant. However, while in [14] the main result states that reasonable updates do not negatively affect stability and performance, our main result in this paper shows that the particular shrinking horizon updates do indeed allow for improved stability and performance estimates compared to non-updated MPC. Although the particular shrinking horizon updates considered in this paper are quite specific, we expect that the results of our analysis can be extended to the so-called *sensitivity-based multistep MPC* [13] (based on [3, 12, 15]) wherein re-optimization is replaced by a sensitivity-based update viewing the latter as an approximation to the former. This expectation is supported by the fact that shrinking horizon updates and sensitivity updates yield almost identical closed loop behaviour in the numerical example in this paper. Eventually, our analysis may thus pave the way to a — to our knowledge first — rigorous closed-loop robustness analysis of fast MPC variants using real-time iteration [17] and hierarchical updates [2].

The paper is organized as follows. In Sections 2, 3 and 4, we provide the setup, describe the MPC algorithms used in this paper and summarize established stability and performance results for nominal multistep MPC. In Section 5, perturbations are introduced to the system, a weaker concept of stability is defined and a notation needed for the analysis of trajectories with undergoing perturbations and re-optimizations is introduced. Section 6 gives analogous statements to some properties in Section 4 in the perturbed and possibly re-optimized setting. In Section 7, we examine suboptimality performance indices of the perturbed schemes under consideration. These results serve as ingredients for the main stability and performance result formulated and proved in Section 8. Our results are illustrated by a numerical example in Section 9. Finally, Section 10 concludes the paper.

2. **Setting and Preliminaries.**

We consider the nonlinear discrete time control system

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

where x is the state and u is the control value. Let the normed vector spaces X and U be state and control spaces, respectively. For a given state constraint set $\mathbb{X}$ and control constraint sets $\mathbb{U}(x)$, $x \in \mathbb{X}$, we require $x \in \mathbb{X} \subseteq X$ and $u \in \mathbb{U}(x) \subseteq U$. The notation $x_u(\cdot, x_0)$ (or briefly $x_u(\cdot)$) denotes the state trajectory when the initial state x_0 is driven by control sequence $u(\cdot)$. We refer to (1) as the nominal model. In Section 5 we will incorporate perturbations into this model.

A time-dependent feedback law $\mu : \mathbb{X} \times \mathbb{N} \to \mathbb{U}$ yields the feedback controlled system

$$x(k+1) = f(x(k), \mu(x(\tilde{k}), k)). \tag{2}$$

Here, the next state at time instant $k+1$ depends on the current state at k and the feedback value $\mu(x(\tilde{k}), k)$, which enters the system as a control value. The feedback value, in turn, depends on the system state $x(\tilde{k})$ at a time $\tilde{k} = \tilde{k}(k) \leq k$ which may be strictly smaller than k. We refer to (2) as the closed-loop system.

The classical MPC method is motivated by the following problem. We aim to find a feedback law μ that 'solves' the infinite horizon OCP

$$\min_{u(\cdot)\in\mathbb{U}^\infty(x_0)} J_\infty\left(x_0, u(\cdot)\right) \tag{3}$$

where the objective function is given by

$$J_\infty\left(x_0, u(\cdot)\right) := \sum_{k=0}^{\infty} \ell\left(x_u(k, x_0), u(k)\right)$$

which is an infinite sum of stage costs $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_0^+$ along the trajectory with x_0 as the initial value steered by the control sequence $u(\cdot) \in \mathbb{U}^\infty(x_0)$. This type of objective is often related to feedback stabilization problems, see Section 4 for details. The objective is minimized over all infinite admissible control sequences, i.e., all control sequences $u(\cdot)$ satisfying

$$\mathbb{U}^\infty(x_0) := \left\{ u(\cdot) \in U^\infty \;\middle|\; \begin{array}{c} x_u(k+1, x_0) \in \mathbb{X} \text{ and} \\ u(k) \in \mathbb{U}(x_u(k, x_0)) \text{ for all } k = 0, 1, \ldots \end{array} \right\}$$

Its optimal value function is given by

$$V_\infty(x_0) := \inf_{u(\cdot)\in\mathbb{U}^\infty(x_0)} J_\infty\left(x_0, u\right)$$

and the infinite horizon closed-loop performance of a given time-dependent feedback μ is given by

$$J_\infty^{\mathrm{cl}}(x_0, \mu) := \sum_{k=0}^{\infty} \ell\left(x_\mu(k, x_0), \mu(x_\mu(\tilde{k}, x_0), k)\right) \tag{4}$$

which is the infinite sum of costs along the trajectory driven by the feedback law. Given an initial state, we would like to solve the infinite horizon optimal control problem and obtain an optimal control in feedback form, i.e., to find a feedback μ with $J_\infty^{\mathrm{cl}}(x_0, \mu) = V_\infty(x_0)$. In the general nonlinear setting, however, this problem is often computationally intractable, so we circumvent it by considering the finite horizon minimization problem

$$\min_{u(\cdot)\in\mathbb{U}^N(x_0)} J_N\left(x_0, u(\cdot)\right) \tag*{$\mathcal{P}_N(x_0)$}$$

for an objective function

$$J_N\left(x_0, u(\cdot)\right) := \sum_{k=0}^{N-1} \ell\left(x_u(k, x_0), u(k)\right)$$

representing a cost associated with an initial state x_0, a control sequence $u(\cdot)$ and optimization horizon N. The minimization is performed over all control sequences $u(\cdot) \in \mathbb{U}^N(x_0)$ where

$$\mathbb{U}^N(x_0) := \left\{ u(\cdot) \in U^N \;\middle|\; \begin{array}{c} x_u(k+1, x_0) \in \mathbb{X} \text{ and} \\ u(k) \in \mathbb{U}(x_u(k, x_0)) \text{ for all } k = 0, \ldots, N-1 \end{array} \right\}$$

One can observe that $\mathcal{P}_N(x_0)$ is parametric with respect to the initial value x_0, hence, the reason for the notation. We define the optimal value function associated with the initial state value x_0 by

$$V_N(x_0) := \inf_{u(\cdot)\in\mathbb{U}^N(x_0)} J_N\left(x_0, u(\cdot)\right)$$

In this paper, we assume there exists a (not necesssarily unique) control sequence $u^*(\cdot) \in \mathbb{U}^N(x_0)$ satisfying $V_N(x_0) = J_N\left(x_0, u^*(\cdot)\right)$, which is called the optimal control sequence. Alternatively, statements could be formulated using ε-optimal control sequences, at the expense of a considerably more technical presentation.

The dynamic programming principle, an important concept that we will be using in our analysis, relates the optimal value functions of OCPs of different optimization horizon length for different points along a trajectory, see [1] or [7, Section 3.4].

Theorem 2.1. *(Dynamic programming principle) Let x_0 be an initial state value. Let $u^*(0), u^*(1), \ldots, u^*(N-1)$ be an optimal control sequence for $\mathcal{P}_N(x_0)$ and $x_{u^*}(0) = x_0, x_{u^*}(1), \ldots, x_{u^*}(N)$ denote the corresponding optimal state trajectory. Then for any i, $i = 0, 1, \ldots, N-1$, the control sequence $u^*(i), u^*(i+1), \ldots, u^*(N-1)$ is an optimal control sequence for $\mathcal{P}_{N-i}(x_{u^*}(i))$.*

3. MPC algorithms.

In this section, we explain how the finite horizon OCP $\mathcal{P}_N(x_0)$ can be used in order to construct an approximately optimal feedback law for the infinite horizon problem (3).

The 'usual' or 'standard' MPC algorithm proceeds iteratively as follows.

Algorithm 3.1. (Standard MPC)

(1) measure the state $x(k) \in \mathbb{X}$ of the system at time instant k
(2) set $x_0 := x(k)$ and solve the finite horizon problem $\mathcal{P}_N(x_0)$. Let u^* denote the optimal control sequence and define the MPC feedback $\mu_N(x(k)) := u^*(0)$
(3) apply the control value $\mu_N(x(k))$ to the system, set $k := k+1$ and go to (1)

This iteration, also known as a receding horizon strategy, gives rise to a (non-time-dependent) feedback μ_N which — under appropriate conditions, see Section 4 — approximately solves the infinite horizon problem. It generates a nominal closed-loop trajectory $x_{\mu_N}(k)$ according to the rule

$$x_{\mu_N}(k+1) = f(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \tag{5}$$

In this work, we consider two other variants of MPC controllers. First, we consider multistep feedback MPC [8] in which the optimization in Step (2) is performed less often, by applying the first $m \in \{2, \ldots, N-1\}$ elements of the optimal control sequence obtained after optimization.

Algorithm 3.2. (m-step MPC)

(1) measure the state $x(k) \in \mathbb{X}$ of the system at time instant k
(2) set $x_0 := x(k)$ and solve the finite horizon problem $\mathcal{P}_N(x_0)$. Let u^* denote the optimal control sequence and define the time-dependent MPC feedback

$$\mu_{N,m}(x(k), k+j) := u^*(j), \quad j = 0, \ldots, m-1 \tag{6}$$

(3) apply the control values $\mu_{N,m}(x(k), k+j)$, $j = 0, \ldots, m-1$, to the system, set $k := k+m$ and go to (1)

Here, the value m is called the control horizon. The resulting nominal closed-loop system is given by

$$x_{\mu_{N,m}}(k+1) = f(x_{\mu_{N,m}}(k), \mu_{N,m}(x_{\mu_{N,m}}(\lfloor k \rfloor_m), k)) \tag{7}$$

where $\lfloor k \rfloor_m$ denotes the largest integer multiple of m less than or equal to k. The motivation behind considering m-step MPC is that the number of optimizations is reduced by the factor $1/m$, thus the computational effort decreases accordingly.

Second, we also consider an updated multistep feedback MPC which, similar to the usual MPC, entails performing optimization every time step, but unlike the standard MPC, wherein we perform optimization over full horizon N, we re-optimize over shrinking horizons.

Algorithm 3.3. (updated m-step MPC)

(1) measure the state $x(k) \in \mathbb{X}$ of the system at time instant k
(2) set $j := k - \lfloor k \rfloor_m$, $x_j := x(k)$ and solve the finite horizon problem $\mathcal{P}_{N-j}(x_j)$. Let u^* denote the optimal control sequence and define the MPC feedback

$$\hat{\mu}_{N,m}(x(k), k) := u^*(0) \tag{8}$$

(3) apply the control value $\hat{\mu}_{N,m}(x(k), k)$ to the system, set $k := k + 1$ and go to (1)

The nominal updated multistep MPC closed loop is then described by

$$x_{\hat{\mu}_{N,m}}(k+1) = f(x_{\hat{\mu}_{N,m}}(k), \hat{\mu}_{N,m}(x_{\hat{\mu}_{N,m}}(k), k)) \tag{9}$$

We note that due to the dynamic programming principle in Theorem 2.1, in the nominal setting the closed loop generated by the multistep feedback (7) and by the updated multistep feedback MPC closed-loop system (9) coincide. For this reason, (8) is only useful in the presence of perturbations. These will be formalized in Section 5.

In presence of perturbations, however, we expect the updated multistep feedback to provide more robustness, in the sense that stability is maintained for larger perturbations and performance degradation is less pronounced as for the non-updated case. This will be rigorously analyzed in the remainder of this paper. Compared to standard MPC, the optimal control problems on shrinking horizon needed for the updates are faster to solve than the optimal control problems on full horizon. Moreover, for small perturbations the updates may also be replaced by approximative updates in which wherein re-optimizations are approximated by a sensitivity approach [13], as illustrated by our numerical example in Section 9. This leads to another significant reduction of the computation time.

4. **Nominal stability and performance.**

Before we analyze the properties of the feedback laws under perturbation, we briefly summarize the main steps of the analysis of nominal MPC without terminal conditions from [4, 8] (see also [7, Chapter 6]) which we will later adapt to the perturbed situation.

Suppose x_* is an equilibrium of (1). MPC is typically used as an algorithm to find $\mu_N : \mathbb{X} \times \mathbb{N} \to \mathbb{U}$ that approximately solves the infinite horizon OCP such that x_* is asymptotically stable for the feedback controlled system (5) in the following sense.

Definition 4.1. An equilibrium $x_* \in \mathbb{X}$ is asymptotically stable for the closed-loop system (2) if there exists $\beta \in \mathcal{KL}$ such that[1]

$$\|x_\mu(k, x_0)\|_{x_*} \leq \beta(\|x_0\|_{x_*}, k)$$

holds for all $x_0 \in \mathbb{X}$ and all $k \in \mathbb{N}_0$ where $\|x\|_{x_*} := \|x - x_*\|$. In this case, we say that the feedback law μ asymptotically stabilizes x_*.

Asymptotic stability is enforced by choosing the stage cost ℓ to penalize the distance to the desired equilibrium. Formally, we assume that there exist $\mathcal{K}_\infty$-functions α_1, α_2 such that the inequality

$$\alpha_1(\|x\|_{x_*}) \leq \ell^*(x) \leq \alpha_2(\|x\|_{x_*}) \tag{10}$$

holds for all $x \in X$, where $\ell^*(x) := \inf_{u \in U} \ell(x, u)$.

Conditions needed so that an MPC feedback law asymptotically stabilizes a nominal system have been well understood in the literature. On one hand, the use of stabilizing terminal constraints or Lyapunov function terminal costs added to the objective function is employed in order to ensure asymptotic stability of the MPC closed loop, see, e.g., [16] or [7, Chapter 5] and references therein. In this paper, we do not use such terminal conditions but rather consider MPC without terminal constraints and costs. Due to its simplicity in design and implementation, this variant is often preferred in practice which is why we are interested in analyzing its properties. The key for the analysis of such MPC schemes is the following proposition.

Proposition 4.2. *(i) Consider a time-dependent feedback law $\mu : \mathbb{X} \times \mathbb{N} \to U$, the corresponding solution $x_\mu(k)$ with $x_\mu(0) = x_0$ of (2), and a function $V : X \to \mathbb{R}_0^+$ satisfying the relaxed dynamic programming inequality*

$$V(x_0) \geq V(x_\mu(m)) + \alpha \sum_{k=0}^{m-1} \ell(x_\mu(k, x_0), \mu(x_\mu(k, x_0), k)) \tag{11}$$

for some $\alpha \in (0, 1]$, some $m \geq 1$ and all $x_0 \in \mathbb{X}$. Then for all $x \in \mathbb{X}$ the estimate

$$V_\infty(x) \leq J_\infty^{cl}(x, \mu) \leq V(x)/\alpha \tag{12}$$

holds.

(ii) If, moreover, (10) holds and there exists $\alpha_4 \in \mathcal{K}_\infty$ with $V(x) \leq \alpha_4(\|x\|_{x_*})$, then the equilibrium x_* is asymptotically stable for the closed-loop system.

Proof. See Proposition 2.4 and Theorem 5.2 of [4]. □

In (12), the value α is a performance bound which indicates how good the feedback μ approximates the solution of the infinite horizon problem: for $\alpha = 1$, the feedback is infinite horizon optimal and the smaller $\alpha > 0$ is the larger the suboptimality gap becomes. Moreover, the existence of an $\alpha > 0$ ensures asymptotic stability. In the sequel, we will present a constructive approach to compute α. To this end, we assume that there exists $B_k \in \mathcal{K}_\infty$ such that the optimal value functions of $\mathcal{P}_N(x_0)$ satisfy

$$V_k(x) \leq B_k(\ell^*(x)) \quad \text{for all} \quad x \in \mathbb{X} \text{ and all } \quad k = 2, \ldots, N \tag{13}$$

[1] A continuous function $\rho : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a $\mathcal{K}$-function if $\rho(0) = 0$ and is strictly increasing. ρ is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function that is unbounded. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a $\mathcal{KL}$-function if for each r, $\lim_{t\to\infty} \beta(r, t) = 0$ and for each $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}_\infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a $\mathcal{KL}_0$-function if for each r, $\lim_{t\to\infty} \beta(r, t) = 0$ and for each $t \geq 0$ we either have $\beta(\cdot, t) \in \mathcal{K}_\infty$ or $\beta(\cdot, t) \equiv 0$.

The existence of the functions B_k can be concluded from asymptotic controllability properties of the system, for details see [8, 13] and [7, Chapter 6].

The following proposition considers arbitrary values λ_n, $n = 0, \ldots, N-1$, and ν and gives necessary conditions which hold if these values coincide with optimal stage costs $\ell(x_{u^*}(n), u^*(n))$ and optimal values $V_N(x_{u^*}(m))$, respectively.

Proposition 4.3. *Assume (13) and consider $N \geq 1, m \in \{1, \ldots, N-1\}$, a sequence $\lambda_n > 0$, $n = 0, \ldots, N-1$, a value $\nu > 0$. Consider $x_0 \in X$ and assume that there exists an optimal control function $u^*(\cdot) \in \mathbb{U}$ for the finite horizon problem $\mathcal{P}_N(x_0)$ with horizon length N, such that*

$$\lambda_n = \ell(x_{u^*}(n), u^*(n)), \quad n = 0, \ldots, N-1$$

holds. Then

$$\sum_{n=k}^{N-1} \lambda_n \leq B_{N-k}(\lambda_k), \quad k = 0, \ldots, N-2 \tag{14}$$

holds. If, furthermore,

$$\nu = V_N(x_{u^*}(m))$$

holds, then

$$\nu \leq \sum_{n=0}^{j-1} \lambda_{n+m} + B_{N-j}(\lambda_{j+m}), \quad j = 0, \ldots, N-m-1 \tag{15}$$

holds.

Proof. See Proposition 4.1 and Remark 6.15 of [4]. □

By using the proposition, we arrive at the following theorem giving sufficient conditions for suboptimality and stability of the multistep MPC feedback law $\mu_{N,m}$ and an approach to compute the suboptimality index α.

Theorem 4.4. *Let (13) hold and assume that the optimization problem*

$$\begin{array}{l} \alpha := \inf\limits_{\lambda_0, \ldots, \lambda_{N-1}, \nu} \dfrac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\sum_{n=0}^{m-1} \lambda_n} \\ \text{subject to the constraints (14) and (15)} \\ \text{and } \lambda_0, \ldots, \lambda_{N-1}, \nu > 0 \end{array} \tag*{$\mathcal{P}_\alpha$}$$

has an optimal value $\alpha \in (0, 1]$. Then, the optimal value function V_N of $\mathcal{P}_N(x)$ and the multistep MPC feedback law $\mu_{N,m}$ satisfy the assumptions of Proposition 4.2(i) and, in particular, the inequality

$$\alpha V_\infty(x) \leq \alpha J_\infty^{cl}(x, \mu_{N,m}) \leq V_N(x)$$

holds for all $x \in \mathbb{X}$. If, moreover, (10) holds then the closed loop is asymptotically stable.

Proof. For the first assertion, see the proof of Corollary 4.5 of [4]. The second assertion follows from Proposition 4.2(ii) setting $\alpha_4 := B_N$. □

Remark 4.5. Theorem 4.4 particularly shows inequality (11) for $V = V_N$ and $\mu = \mu_{N,m}$, i.e.,

$$V_N(x_{\mu_{N,m}}(m, x_0)) \leq V_N(x_0) - \alpha \sum_{k=0}^{m-1} \ell(x_{\mu_{N,m}}(k, x_0), \mu_{N,m}(x_{\mu_{N,m}}(k, x_0), k)) \tag{16}$$

for all $x_0 \in \mathbb{X}$. This inequality can be seen as a Lyapunov inequality and shows that V_N is an m-step Lyapunov function. Condition (13) may be relaxed if we only intend to establish (16) for states $x_0 \in Y$ for a subset $Y \subseteq \mathbb{X}$, cf. Remark 6.15(ii) of [7].

The optimization problem $\mathcal{P}_\alpha$ becomes a linear program if the $B_k(r)$ are linear in r. In this case, an explicit formula for α can be derived.

Theorem 4.6. *Let B_K, $K = 2, \ldots, N$, be linear functions and define $\gamma_K := B_K(r)/r$. Then the optimal value α of problem $\mathcal{P}_\alpha$ for given optimization horizon N, control horizon m satisfies satisfies $\alpha = 1$ if and only if $\gamma_{m+1} \leq 1$ and*

$$\alpha \geq \frac{(\gamma_{m+1}-1)\prod_{i=m+2}^{N}(\gamma_i - 1)\prod_{i=N-m+1}^{N}(\gamma_i - 1)}{\left(\prod_{i=m+1}^{N}\gamma_i - (\gamma_{m+1}-1)\prod_{i=m+2}^{N}(\gamma_i-1)\right)\left(\prod_{i=N-m+1}^{N}\gamma_i - \prod_{i=N-m+1}^{N}(\gamma_i-1)\right)} \tag{17}$$

otherwise. If, moreover, the B_K are of the form $B_K(r) := \sum_{k=0}^{K-1}\beta(r,k)$ for some $\beta \in \mathcal{KL}_0$ satisfying $\beta(r, n+m) \leq \beta(\beta(r,n),m)$ for all $r \geq 0, n, m \in \mathbb{N}_0$, then equality holds in (17).

Proof. See Theorem 5.4 and Remark 5.5 of [8]. □

An analysis of Formula (17) reveals that $\alpha \to 1$ as $N \to \infty$ if there exists $\bar\gamma \in \mathbb{R}$ with $\gamma_k \leq \bar\gamma$ [8, Corollary 6.1]. Hence, under this condition, stability and performance arbitrarily close to the infinite horizon optimal performance can always be achieved by choosing N sufficiently large. Moreover, the value delivered by Formula (17) for $m = 1$ is always less or equal than the value for $m \geq 2$ [8, Proposition 7.3]. This means that if Theorem 4.6 guarantees asymptotic stability (i.e., $\alpha > 0$) of standard MPC $m = 1$ (Algorithm 3.1), then it also guarantees stability of multistep MPC for arbitrary $m = 2, \ldots, N-1$ (Algorithms 3.2 and 3.3).

To summarize, the reasoning in this section is as follows: Inequality (13) allows us to formulate the optimization problem $\mathcal{P}_\alpha$. If this problem has a solution $\alpha > 0$ then the assumptions of Proposition 4.2 are satisfied from which asymptotic stability and performance estimates can be obtained. In case the B_K in (13) are linear, an explicit formula for the solution of $\mathcal{P}_\alpha$ is provided by (17). This is the setting and approach we are going to extend to perturbed systems in the remainder of this paper.

5. **Perturbations and Robust Stability.** Mathematical models are approximations of real systems, hence a mismatch is inevitable between the predicted states and those that are measured from the real plant. The results discussed in the previous section are based on a nominal setting in which the mathematical model coincides with the real system. Taking into account the presence of perturbations, we consider the perturbed closed-loop model

$$\tilde{x}(k+1) = f\left(\tilde{x}(k), \mu(\tilde{x}(k), k)\right) + d(k) \tag{18}$$

where $d(k) \in X$ represents external perturbation and modeling errors.

Remark 5.1. For brevity of exposition, we use in our analysis the perturbed closed-loop model (18) instead of the more general model

$$\tilde{x}(k+1) = f\left(\tilde{x}(k), \mu(\tilde{x}(k)+e(k),k)\right) + d(k) \tag{19}$$

where $e(k) \in X$ represents state measurement errors. Stability and performance statements for this model can be derived from respective statements for (18) using the techniques from [7, Proof of Theorem 8.36] or [10, Proof of Proposition 1].

In the following discussion, we use the notation $\tilde{x}_\mu(\cdot, x_0)$ to denote a solution of (18) in order to distinguish it from the nominal trajectory $x_\mu(\cdot, x_0)$. Furthermore, we consider the set

$$S_{\overline{d}}(x_0) := \left\{ \tilde{x}_\mu(\cdot, x_0) \mid \|d(k)\| \leq \overline{d} \text{ for all } k \in \mathbb{N}_0 \right\}$$

of all possible solutions starting in x_0 with perturbations bounded by $\overline{d}$.

Remark 5.2. In the remainder of this paper, we assume that for the initial values x_0, perturbation levels $\overline{d}$ and feedback laws μ under consideration, any trajectory $\tilde{x}_\mu(\cdot, x_0) \in S_{\overline{d}}(x_0)$ exists and satisfies $\tilde{x}_\mu(k, x_0) \in \mathbb{X}$ for all $k \in \mathbb{N}$. Techniques which allow to rigorously ensure this property are discussed, e.g., in Sections 8.8–8.9 of [7] and the references therein.

Asymptotic stability is in general too strong a property to hold under perturbations. However, it is often still possible to prove suitable relaxed stability properties. Here, we make use of the so-called semiglobal practical stability.

Definition 5.3. We say that x_* is *semi-globally practically asymptotically stable with respect to perturbation* d if there exists $\beta \in \mathcal{KL}$ such that the following property holds: for each $\delta > 0$ and $\Delta > \delta$ there exists $\overline{d} > 0$ such that

$$\|\tilde{x}_\mu(k, x_0)\|_{x_*} \leq \max\{\beta(\|x_0\|_{x_*}, k), \delta\} \tag{20}$$

holds for all $x_0 \in \mathbb{X}$ with $\|x_0\|_{x_*} \leq \Delta$, all $\tilde{x}_\mu(\cdot, x_0) \in S_{\overline{d}}(x_0)$ and all $k \in \mathbb{N}_0$.

In words, this definition demands that for initial values not too far away from x_* the system behaves like an asymptotically stable system provided the state is not too close to x_*. Here, "not too far away" and "not too close" are quantified via Δ and δ, respectively, and determine the admissible bound $\overline{d}$ on the perturbation. In what follows, we will establish this property via the following definition and lemma.

Definition 5.4. Consider sets $\widehat{P} \subset Y \subseteq \mathbb{X}$. A point $x_* \in \widehat{P}$ is called *$\widehat{P}$-practically uniform asymptotically stable on Y* if there exists $\beta \in \mathcal{KL}$ such that

$$\|\tilde{x}_\mu(k, x_0)\|_{x_*} \leq \beta(\|x_0\|_{x_*}, k)$$

holds for all $x_0 \in Y$, all $\tilde{x}_\mu(\cdot, x_0) \in S_{\overline{d}}(x_0)$ and all k with $\tilde{x}(k, x_0) \notin \widehat{P}$.

Lemma 5.5. *The m-step MPC closed-loop system (7) is semi-globally practically asymptotically stable with respect to d if for every $\delta > 0$ and every $\Delta > \delta$ there exists $\overline{d} > 0$ and sets $\widehat{P} \subset Y \subseteq \mathbb{X}$ with*

$$\overline{\mathcal{B}}_\Delta(x_*) \cap \mathbb{X} \subseteq Y \quad \text{and} \quad \widehat{P} \subseteq \overline{\mathcal{B}}_\delta(x_*)$$

such that for each solution $\tilde{x}_\mu(\cdot, x_0) \in S_{\overline{d}}(x_0)$ the system is $\widehat{P}$-practically uniform asymptotically stable on Y in the sense of Definition 5.4.

Proof. The proof follows from the fact that according to Definition 5.4 for each $k \in \mathbb{N}_0$ either $\|\tilde{x}_\mu(k, x_0)\|_{x_*} \leq \beta(\|x_0\|_{x_*}, k)$ or $\tilde{x}_\mu(k, x_0) \in \widehat{P}$. Since the latter implies $\|\tilde{x}_\mu(k, x_0)\|_{x_*} \leq \delta$, we observe the assertion. □

Now that we have defined the appropriate stability notion we can also define the appropriate performance measure. To this end, note that the set $\widehat{P}$ in Definition 5.4 can be interpreted as the region of the state space in which the perturbations become predominant. Hence, when considering the performance of such a solution, it only makes sense to consider the trajectory until it first hits the set $\widehat{P}$. Thus, we need to truncate the infinite horizon closed loop cost $J_\infty^{\mathrm{cl}}(x_0, \mu)$ from (4) as follows.

Definition 5.6. Consider a set $\widehat{P} \subset \mathbb{X}$. Then the performance associated to a perturbed solution $\tilde{x}_\mu(\cdot, x_0)$ of a closed-loop system outside $\widehat{P}$ is defined as

$$J_{\widehat{P}}^{\mathrm{cl}}(\tilde{x}_\mu(\cdot, x_0), \mu) := \sum_{k=0}^{k^*-1} \ell\left(\tilde{x}_\mu(k, x_0), \mu(\tilde{x}_\mu(k, x_0), k)\right), \tag{21}$$

where $k^* \in \mathbb{N}_0$ is minimal with $\tilde{x}_\mu(k^*, x_0) \in \widehat{P}$.

As a technical ingredient, we additionally need the following set properties.

Definition 5.7. Let $m \in \mathbb{N}$. A set $Y \subseteq \mathbb{X}$ is said to be *m-step forward invariant for (18) with respect to $\overline{d}$* if for all $x_0 \in Y$ and all $\tilde{x}_\mu(\cdot, x_0) \in S_{\overline{d}}(x_0)$, it holds that $\tilde{x}_\mu(pm, x_0) \in Y$ for all $p \in \mathbb{N}$.

For an m-step forward invariant set Y with respect to $\overline{d}$ we call $\widehat{Y} \supseteq Y$ an *intermediate set* if $\tilde{x}_\mu(k, x_0) \in \widehat{Y}$ for all $k \in \mathbb{N}$ and all $x_0 \in Y$.

Based on these definitions, we have the following theorem extending Proposition 4.2 to the perturbed setting.

Theorem 5.8. *(i) Consider a stage cost $\ell : X \times U \to \mathbb{R}_0^+$, an integer $m \in \mathbb{N}$ and a function $V : X \to \mathbb{R}_0^+$. Let $\mu : \mathbb{X} \times \mathbb{N} \to U$ be an admissible m-step feedback law of the form (6) or (8) and let $Y \subseteq \mathbb{X}$ and $P \subset Y$ be m-step forward invariant for (18) with respect to some $\overline{d} > 0$. Let $\widehat{P} \supseteq P$ be an intermediate set for P. Assume there exists $\alpha \in (0, 1]$ such that the relaxed dynamic programming inequality*

$$V(x_0) \geq V(\tilde{x}_\mu(m, x_0)) + \alpha \sum_{k=0}^{m-1} \ell(\tilde{x}_\mu(k, x_0), \mu(\tilde{x}_\mu(k, x_0), k)) \tag{22}$$

holds for all $x_0 \in Y \setminus P$ and all $\tilde{x}_\mu(\cdot, x_0) \in S_{\overline{d}}(x_0)$. Then the suboptimality estimate

$$J_{\widehat{P}}^{cl}(\tilde{x}_\mu(k, x_0), \mu) \leq V(x_0)/\alpha \tag{23}$$

holds for all $x_0 \in Y \setminus \widehat{P}$ and all $\tilde{x}_\mu(k, x_0) \in S_{\overline{d}}(x_0)$.

(ii) If, moreover, (10) holds and there exists $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ with $\alpha_3(\|x\|_{x_*}) \leq V(x) \leq \alpha_4(\|x\|_{x_*})$, then the closed-loop system (18) is $\widehat{P}$-practically asymptotically stable on Y in the sense of Definition 5.4.

Proof. (i) For proving (23), by a straightforward induction from (22) we obtain

$$\alpha \sum_{k=0}^{pm-1} \ell(\tilde{x}_\mu(k, x_0), (\tilde{x}_\mu(k, x_0), k)) \leq V(x_0) - V(\tilde{x}_\mu(pm, x_0)) \leq V(x_0)$$

for all $p \in \mathbb{N}$ for which $\tilde{x}_\mu(k, x_0) \notin P$ for $k = 0, m, 2m, \ldots, (p-1)m$. In particular, since $P \subseteq \widehat{P}$, this inequality holds for the smallest p satisfying $pm \geq k^*$ for k^* from Definition 5.6, implying

$$J_{\widehat{P}}^{\mathrm{cl}}(\tilde{x}_\mu(k, x_0), \mu) \leq \sum_{k=0}^{pm-1} \ell(\tilde{x}_\mu(k, x_0), (\tilde{x}_\mu(k, x_0), k)) \leq V(x_0)/\alpha.$$

(ii) For proving practical asymptotic stability, analogous to the first part of the proof of [4, Theorem 5.2] we find a function $\rho \in \mathcal{KL}$ such that $V_N(x_\mu(pm, x_0)) \leq \rho(V_N(x_0), p)$ holds for all $x_0 \in Y$ and all $p \in \mathbb{N}$ with $pm \leq k^*$ for k^* from Definition 5.6. Now for $k \in \{1, \dots, k^*\}$ which is not an integer multiple of m, (22) with $\tilde{x}_\mu(\lfloor k \rfloor_m, x_0)$ in place of x_0 and nonnegativity of ℓ imply

$$\ell(\tilde{x}_\mu(k, x_0)) \leq V_N(\tilde{x}_\mu(\lfloor k \rfloor_m, x_0))/\alpha.$$

Since $V_N(x) \leq \alpha_4 \circ \alpha_1^{-1}(\ell(x, u))$ holds for all u, this yields

$$V_N(\tilde{x}_\mu(k, x_0)) \leq \alpha_4 \circ \alpha_1^{-1}(V_N(\tilde{x}_\mu(\lfloor k \rfloor_m, x_0))/\alpha) \leq \alpha_4 \circ \alpha_1^{-1}(\rho(V(x_0), \lfloor k \rfloor_m)/\alpha).$$

From this we obtain

$$\|\tilde{x}_\mu(k, x_0)\|_{x_*} \leq \alpha_3^{-1} \circ \alpha_4 \circ \alpha_1^{-1}(\rho(\alpha_4(\|x_0\|_{x_*}), \lfloor k \rfloor_m)/\alpha).$$

This implies $\|\tilde{x}_\mu(k, x_0)\|_{x_*} \leq \beta(\|x_0\|_{x_*}, k)$ for all $k = 0, \dots, k^*$ with

$$\beta(r, k) := \alpha_3^{-1} \circ \alpha_4 \circ \alpha_1^{-1}(\rho(\alpha_4(r), \lfloor k \rfloor_m)/\alpha) + e^{-k}$$

which is easily extended to a $\mathcal{KL}$-function by linear interpolation in its second argument. Since $\tilde{x}_\mu(k^*, x_0) \in P$ implies that for all $k \geq k^*$ we have $\tilde{x}_\mu(k, x_0) \in \widehat{P}$, this shows the claimed $\widehat{P}$-practical asymptotic stability. □

As already discussed at the end of Section 3, we expect the shrinking horizon update mechanism of the updated MPC algorithm to enhance robustness of the closed loop. Formally, this can be expressed via the parameter α, whose sign determines asymptotic stability and whose absolute value (if positive) determines the degree of suboptimality of the closed loop. Since larger values of α indicate both stability for larger ranges of $\overline{d}$ and better performance for identical values of $\overline{d}$, we would expect that the updated MPC variant allows for more optimistic estimates for α.

For finite horizon problems, comparisons between the nominal open-loop control applied to the perturbed system and the shrinking horizon RHC are examined in [5]. There, potential improvements due to re-optimization are investigated and are revealed to depend on the moduli of continuity of the optimization objective on the one hand and of the optimal value function on the other hand. Particularly, in the case where the system is open-loop unstable but controllable, the latter modulus of continuity is much smaller, thus explaining the significant benefit of re-optimization.

Our analysis in this paper builds upon the framework of [5] and in the remainder of this section we summarize and extend the results from this reference. We focus our attention to the evolution described by the perturbed multistep MPC closed-loop system

$$\tilde{x}_{\mu_{N,m}}(k+1) = f(\tilde{x}_{\mu_{N,m}}(k), \mu_{N,m}(\tilde{x}_{\mu_{N,m}}(\lfloor k \rfloor_m), k)) \;+\; d(k) \tag{24}$$

and the perturbed updated multistep MPC closed-loop system

$$\tilde{x}_{\hat{\mu}_{N,m}}(k+1) = f(\tilde{x}_{\hat{\mu}_{N,m}}(k), \hat{\mu}_{N,m}(\tilde{x}_{\hat{\mu}_{N,m}}(k), k)) \;+\; d(k) \tag{25}$$

where perturbation occurs and re-optimization is performed. The feedback controls $\mu_{N,m}$ and $\hat{\mu}_{N,m}$ are defined in (6) and (8), respectively.

In the following, we introduce an intuitive and rigorous notation for the trajectories generated by (7), (24) and (25) reflecting perturbations and performed re-optimizations during the first m steps of its evolution. As before, let N be the optimization horizon and m be the control horizon.

Notation 5.9. Let $x_{j,p,r}$ denote the state trajectory elements at time $j \in \{0, \dots, N\}$ that have gone through $p \le j$ perturbations at time instants $k = 1, \dots, p$ and along which $r \le p$ re-optimizations with shrinking horizons $N - k$ have been performed at time instants $k = 1, \dots, r$.

Remark 5.10. For $j = 0, 1, \dots, m$ and $x_{0,0,0} = x_0$, the trajectories of the nominal m-step, the perturbed m-step and the perturbed updated m-step MPC closed-loop system as defined in (7), (24) and (25), respectively, can be expressed in the new notation as

$$x_{\mu_{N,m}}(j, x_0) = x_{j,0,0}, \quad \tilde{x}_{\mu_{N,m}}(j, x_0) = x_{j,j,0} \quad \text{and} \quad \tilde{x}_{\hat{\mu}_{N,m}}(j, x_0) = x_{j,j,j}.$$

Notation 5.11. Let $u^*_{j,p,r}$ denote the optimal control sequence obtained by performing a re-optimization with initial value $x_{j,p,r-1}$ and optimization horizon $N-j$, i.e., $u^*_{j,p,r}$ is obtained by solving $\mathcal{P}_{N-j}(x_{j,p,r-1})$.

Since the initial value does not change when performing a re-optimization, the identity $x_{j,p,r-1} = x_{j,p,r}$ holds. We also remark that for our analysis it is sufficient to consider states of the form $x_{j,p,r}$ with $r = 0, p, p-1$.

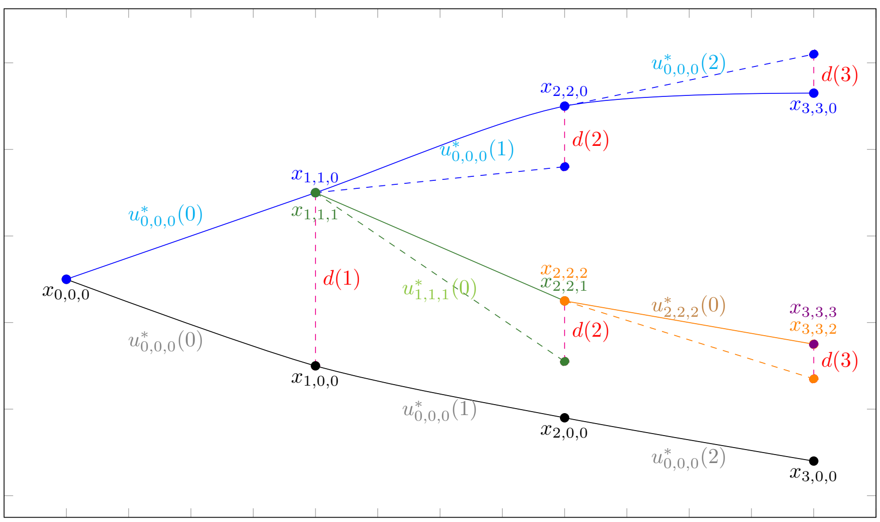

Figure 1. Trajectories through time where perturbations occur and re-optimizations are performed

Figure 1 illustrates the trajectories through time where perturbations occur and re-optimizations are performed for the control horizon $m = 3$. At time $t = 0$, by solving $\mathcal{P}_3(x_{0,0,0})$, we obtain an open-loop optimal control sequence $u^*_{0,0,0}(j) = u^*(j)$, $j = 0, 1, 2$ for which we can generate a nominal multistep trajectory $x_{j,0,0}$, $j = 0, \dots, 3$ via (7) shown in black in the sketch. For an additive perturbation $d(\cdot)$, the blue trajectory in Figure 1 indicates the perturbed multistep trajectory $x_{j,j,0}$, $j = 0, \dots, 3$ generated by (24). Here each transition (shown in solid blue) is composed of the nominal transition $f(x_{j,j,0}, u^*_{0,0,0}(j))$ (blue dashed) followed by the addition of the perturbation $d(1), d(2), d(3)$ (red dashed). Finally, the trajectory $x_{j,j,j}$ obtained by re-optimization in each step and generated by (25) with perturbation d is shown piecewise in blue, green and orange, with the different colors indicating the different control sequences $u^*_{j,j,j}$, $j = 0, \ldots, 2$ whose first pieces are used in the transition. Again, the nominal transition and the effect of the perturbation $d(j)$ are indicated as dashed lines and the resulting perturbed transitions from $x_{j,j,j}$ to $x_{j+1,j+1,j} = x_{j+1,j+1,j+1}$ as solid lines.

Similar to how $x_{j,p,r}$ was defined, we define the following stage cost.

Notation 5.12. For time instants $j \in \{0, \ldots, N-1\}$ and for $j \geq p,\ p \geq r,\ r = 0, p, p-1$ we define

$$\lambda_{j,p,r} = \ell\left(x_{j,p,r}, u^*_{r,r,r}(j-r)\right) \tag{26}$$

Observe that in order to determine the control needed to evaluate the stage cost for the state $x_{j,p,r}$, we go back to the last instant of the optimization, namely to time r and use the optimal control sequence obtained there for horizon $N-r$ and initial value $x_{r,r,r}$.

In order to simplify the numbering in the subsequent computations, we extend (26) to give meaning to the notation when $j < p,\ p \geq r,\ r = 0, p, p-1$ through

$$\lambda_{j,p,r} := \begin{cases} \lambda_{j,j,j} & \text{if } r \neq 0 \\ \lambda_{j,j,0} & \text{if } r = 0. \end{cases} \tag{27}$$

Remark 5.13. Although the previous discussion yields $x_{j,j,j-1} = x_{j,j,j}$, we see that $\lambda_{j,j,j-1} \neq \lambda_{j,j,j}$ since $\lambda_{j,j,j-1} = \ell\left(x_{j,j,j-1}, u^*_{j-1,j-1,j-1}(1)\right)$ while $\lambda_{j,j,j} = \ell\left(x_{j,j,j}, u^*_{j,j,j}(0)\right)$.

6. Properties resulting from perturbations and re-optimizations.

Our goal in this section is to provide a counterpart of Proposition 4.3 for the perturbed closed-loop. To this end, using the notation introduced, we derive a number of inequalities along the different trajectories.

6.1. Estimates involving $V_N(x_{m,m,0})$ and $V_N(x_{m,m,m})$.

We derive in this subsection some implications of inequality (13) on trajectories involving occurrence of perturbation and re-optimization. The following lemmas provide an upper bound for $V_N(x_{m,m,0})$ and for $V_N(x_{m,m,m})$

Lemma 6.1. *Assume (13) and consider $x_{0,0,0} = x \in \mathbb{X}$ and an optimal control $u^*(\cdot) \in \mathbb{U}^N$ for the finite horizon optimal control problem $\mathcal{P}_N(x)$ with optimization horizon N. Then for each $m = 1, \ldots, N-1$ and each $j = 0, \ldots, N-m-1$,*

$$V_N(x_{m,m,0}) \leq \sum_{n=0}^{j-1} \lambda_{n+m,m,0} + B_{N-j}(\lambda_{j+m,m,0}) \tag{28}$$

$$V_N(x_{m,m,m}) \leq \sum_{n=0}^{j-1} \lambda_{n+m,m,m} + B_{N-j}(\lambda_{j+m,m,m}) \tag{29}$$

Proof. To show (28), we take the trajectory element $x_{m,m,0}$ whose evolution is steered by the optimal control $u^*(\cdot)$ along the perturbed system (24) within m-steps. We consider $x_{j+m,m,0}$ for some $j \in \{m, \ldots, N-1\}$.

We define

$$\tilde{u}(n) = \begin{cases} u^*(n+m) & n \in \{0, \ldots, j-1\} \\ u_{\tilde{x}}(n-j) & n \in \{j, \ldots, N-1\} \end{cases} \tag{30}$$

where $u_{\check{x}}(\cdot)$ results from solving the optimization problem $\mathcal{P}_{N-j}(\check{x})$ with initial value $\check{x} = x_{j+m,m,0} = x_{u^*}(j+m,x) = x_{u^*(\cdot+m)}(j, x_{m,m,0})$. This yields

$$\begin{aligned} V_N(x_{m,m,0}) &\leq J_N(x_{m,m,0}, \tilde{u}(\cdot)) \\ &= J_j(x_{m,m,0}, u^*(\cdot + m)) + J_{N-j}(x_{j+m,m,0},\ u_{\check{x}}(\cdot)) \\ &= \sum_{n=0}^{j-1} \ell(x_{n+m,m,0}, u^*(n+m)) + \sum_{n=0}^{N-j-1} \ell(x_{u_{\check{x}}}(n,\check{x}), u_{\check{x}}(n)) \\ &\leq \sum_{n=0}^{j-1} \lambda_{n+m,m,0} + V_{N-j}(\check{x}) \ \leq \sum_{n=0}^{j-1} \lambda_{n+m,m,0} + B_{N-j}(\ell^*(\check{x})) \\ &= \sum_{n=0}^{j-1} \lambda_{n+m,m,0} + B_{N-j}(\lambda_{j+m,m,0}). \end{aligned}$$

To show (29), we proceed analogously with $\check{x} = x_{j+m,m,m} = x_{u_{m,m,m}}(j, x_{m,m,m})$. □

6.2. **Estimates involving uniform continuity.** The following are generalizations of Theorems 6 and 8 in [5] allowing an arbitrary time instant $k \in \{0, 1, \ldots, N-1\}$ to be the reference point in place of $k = 0$. These results eventually provide a basis for comparing, in the finite horizon OCP setting, the nominal system, the perturbed system controlled by the nominal optimal control and the perturbed system under the shrinking horizon updated feedback controller.

Theorem 6.2. *Given $k \in \{0, \ldots, N-1\}$. For any $p \in \{1, \ldots, N-k-1\}$,*

$$\left| \sum_{j=k}^{N-1} \lambda_{j,k,0} - \sum_{j=k}^{N-1} \lambda_{j,k+p,0} \right| \leq \sum_{j=1}^{p} |J_{N-k-j}(x_{k+j,k+j-1,0}, u^*(\cdot + k + j)) - J_{N-k-j}(x_{k+j,k+j,0}, u^*(\cdot + k + j))| \tag{31}$$

and

$$\left| \sum_{j=k}^{N-1} \lambda_{j,k,k} - \sum_{j=k}^{N-1} \lambda_{j,k+p,k+p} \right| \leq \sum_{j=1}^{p} |V_{N-k-j}(x_{k+j,k+j-1,k+j-1}) - V_{N-k-j}(x_{k+j,k+j,k+j})| \tag{32}$$

Proof. The proof follows using the same technique as the proofs of Theorems 6 and 8 in [5] with the appropriate changes in the indices. □

For the next corollary we need the following definition.

Definition 6.3. (i) The optimal value function V_N is said to be uniformly continuous on a set $A \subseteq \mathbb{X}$ if there exists a $\mathcal{K}$-function ω_{V_N} such that for all $x_1, x_2 \in A$

$$|V_N(x_1) - V_N(x_2)| \leq \omega_{V_N}(\|x_1 - x_2\|).$$

(ii) The cost functional J_N is said to be uniformly continuous on $A \subseteq \mathbb{X}$ uniformly in $u \in \mathbb{U}^N$ if there exists a function $\omega_{J_N} \in \mathcal{K}$ such that for all $x_1, x_2 \in A$ and all $u \in \mathbb{U}^N$

$$|J_N(x_1, u) - J_N(x_2, u)| \leq \omega_{J_N}(\|x_1 - x_2\|).$$

The functions ω_{V_N} and ω_{J_N} are called moduli of continuity. Analogous uniform continuity definitions can be defined for f, ℓ and B_K with the corresponding moduli of continuity.

Following directly is a corollary that sizes up the differences among values associated with the tails of the nominal trajectory, the tails of the perturbed trajectory with nominal control and the tails of the perturbed trajectory with re-optimized control.

Corollary 6.4. *Let $k \in \{0, \ldots, N-1\}$. Suppose V_i, $i = 1, \ldots, N$, is uniformly continuous on a set A containing $x_{j,k,0}$ and $x_{j,j,0}$ for $j = k, \ldots, N-1$ with modulus of continuity ω_{V_i}. Suppose J_i, $i = 1, \ldots, N$, is uniformly continuous on a set A containing $x_{j,k,k}$ and $x_{j,j,j}$ for $j = k, \ldots, N-1$ uniformly in u on $\mathbb{X}$ with modulus of continuity ω_{J_i}. Then*

$$\left| \sum_{j=k}^{N-1} \lambda_{j,k,0} - \sum_{j=k}^{N-1} \lambda_{j,j,0} \right| \leq \sum_{j=1}^{N-k-1} \omega_{J_{N-k-j}} \left(\|d(k+j)\| \right) \tag{33}$$

and

$$\left| \sum_{j=k}^{N-1} \lambda_{j,k,k} - \sum_{j=k}^{N-1} \lambda_{j,j,j} \right| \leq \sum_{j=1}^{N-k-1} \omega_{V_{N-k-j}} \left(\|d(k+j)\| \right) \tag{34}$$

Proof. Straightforward from (32) and (31) with $p = N - k - 1$. □

Note that on the right hand side of the estimates the perturbations that occur before time step k do not appear since in both schemes they have cancelled each other.

For the special case of $k = 0$, Corollary 6.4 is one of the central results of [5]. It shows that on the finite horizon N, the performance difference between the nominal and perturbed system controlled by the nominal optimal control is determined by ω_{J_N} while the difference between the nominal and the shrinking horizon updated feedback controller is determined by ω_{V_N}. Since for open loop unstable and controllable systems ω_{V_N} is considerably smaller than ω_{J_N} [5, Section V], this explains the significant benefit of updating in this case.

In the next lemma, we combine the preceding results to derive an upper bound for the values corresponding to the tails of the perturbed trajectory with nominal control and for the tails of the perturbed trajectory with re-optimized control.

Lemma 6.5. *Let the assumptions of Corollary 6.4 hold. Suppose further B_K, $K = 1, \ldots, N$, is uniformly continuous on $\mathbb{R}_0^+$ with modulus of continuity ω_{B_K}. Then for $k = 0, \ldots, N-2$ the inequalities*

$$\begin{aligned} \sum_{j=k}^{N-1} \lambda_{j,j,0} \quad &\leq \quad B_{N-k}(\lambda_{k,k,0}) + \omega_{B_{N-k}}(\lambda_{k,k,0} - \lambda_{k,0,0}) \\ &\qquad + \omega_{J_{N-k}}(x_{k,k,0} - x_{k,0,0}) + \sum_{j=1}^{N-k-1} \omega_{J_{N-k-j}} \left(\|d(k+j)\| \right) \end{aligned} \tag{35}$$

hold and

$$\sum_{j=k}^{N-1} \lambda_{j,j,j} \quad \leq \quad B_{N-k}(\lambda_{k,k,k}) + \sum_{j=1}^{N-k-1} \omega_{V_{N-k-j}} \left(\|d(k+j)\| \right). \tag{36}$$

Proof. Inequality (35) follows since

$$\begin{aligned}
\sum_{j=k}^{N-1} \lambda_{j,j,0} &\leq \sum_{j=k}^{N-1} \lambda_{j,k,0} + \sum_{j=1}^{N-k-1} \omega_{J_{N-k-j}}\left(\|d(k+j)\|\right) \\
&= J_{N-k}(x_{k,k,0}, u^*_{0,0,0}(k+\cdot)) + \sum_{j=1}^{N-k-1} \omega_{J_{N-k-j}}\left(\|d(k+j)\|\right) \\
&= J_{N-k}(x_{k,0,0}, u^*_{0,0,0}(k+\cdot)) + \omega_{J_{N-k}}(x_{k,k,0} - x_{k,0,0}) \\
&\quad + \sum_{j=1}^{N-k-1} \omega_{J_{N-k-j}}\left(\|d(k+j)\|\right) \\
&\leq B_{N-k}(\ell^*(x_{k,0,0})) + \omega_{J_{N-k}}(x_{k,k,0} - x_{k,0,0}) \\
&\quad + \sum_{j=1}^{N-k-1} \omega_{J_{N-k-j}}\left(\|d(k+j)\|\right) \\
&= B_{N-k}(\lambda_{k,0,0}) + \omega_{J_{N-k}}(x_{k,k,0} - x_{k,0,0}) \\
&\quad + \sum_{j=1}^{N-k-1} \omega_{J_{N-k-j}}\left(\|d(k+j)\|\right) \\
&\leq B_{N-k}(\lambda_{k,k,0}) + \omega_{B_{N-k}}(\lambda_{k,k,0} - \lambda_{k,0,0}) \\
&\quad + \omega_{J_{N-k}}(x_{k,k,0} - x_{k,0,0}) + \sum_{j=1}^{N-k-1} \omega_{J_{N-k-j}}\left(\|d(k+j)\|\right).
\end{aligned}$$

To show (36) we compute

$$\begin{aligned}
\sum_{j=k}^{N-1} \lambda_{j,j,j} &\leq \sum_{j=k}^{N-1} \lambda_{j,k,k} + \sum_{j=1}^{N-k-1} \omega_{V_{N-k-j}}\left(\|d(k+j)\|\right) \\
&= V_{N-k}(x_{k,k,k}) + \sum_{j=1}^{N-k-1} \omega_{V_{N-k-j}}\left(\|d(k+j)\|\right) \\
&= J_{N-k}(x_{k,k,k}, u^*_{k,k,k}(\cdot)) + \sum_{j=1}^{N-k-1} \omega_{V_{N-k-j}}\left(\|d(k+j)\|\right) \\
&= J_{N-k}(x_{k,k,k}, u_{x_{k,k,k}}(\cdot)) + \sum_{j=1}^{N-k-1} \omega_{V_{N-k-j}}\left(\|d(k+j)\|\right) \\
&\leq B_{N-k}(\ell^*(x_{k,k,k})) + \sum_{j=1}^{N-k-1} \omega_{V_{N-k-j}}\left(\|d(k+j)\|\right) \\
&= B_{N-k}(\lambda_{k,k,k}) + \sum_{j=1}^{N-k-1} \omega_{V_{N-k-j}}\left(\|d(k+j)\|\right).
\end{aligned}$$

□

6.3. **Counterpart of Proposition 4.3.** By combining the results from this section we can now state the following counterpart of Proposition 4.3. It yields necessary conditions which hold if these values λ_n coincide with either $\lambda_{n,n,0}$ or $\lambda_{n,n,n},\ n = 0,\dots,N-1$, and ν with either $V_N(x_{m,m,0})$ or $V_N(x_{m,m,m})$.

Corollary 6.6. *Consider $N \geq 1, m \in \{1,\dots,N-1\}$ and let the assumptions of Lemmas 6.1 and 6.5 hold. Let $x = x_{0,0,0} \in \mathbb{X}$ and consider a perturbation sequence $d(\cdot)$ where $d(k) = 0$ for $k \geq m$ generating the trajectories $x_{\mu_{N,N-1}}(n,x) = x_{n,n,0}$ and $x_{\hat{\mu}_{N,N-1}}(n,x) = x_{n,n,n}$, cf. Remark 5.10. Consider a sequence $\lambda_n > 0,\ n = 0,\dots,N-1$ and a value $\nu > 0$ such that either*

(i) $\lambda_n = \lambda_{n,n,0},\quad n = 0,\dots,N-1$ *and* $\nu = V_N(x_{m,m,0})$ *or*

(ii) $\lambda_n = \lambda_{n,n,n},\quad n = 0,\dots,N-1$ *and* $\nu = V_N(x_{m,m,m})$ *holds.*

Then the inequalities

$$\sum_{n=k}^{N-1} \lambda_n \leq B_{N-k}(\lambda_k) + \xi_k, \quad k = 0,\dots,N-2 \tag{37}$$

$$\nu \leq \sum_{n=0}^{j-1} \lambda_{n+m} + B_{N-j}(\lambda_{j+m}), \quad j = 0,\dots,N-m-1 \tag{38}$$

hold for

$$\begin{aligned}
(i)\quad \xi_k &= \xi_k^{pmult} = \textstyle\sum_{j=1}^{N-k-1} \omega_{J_{N-k-j}}(\|d(k+j)\|)\\
&\qquad + \omega_{B_{N-k}}(\lambda_{k,k,0} - \lambda_{k,0,0}) + \omega_{J_{N-k}}(x_{k,k,0} - x_{k,0,0})\\
(ii)\quad \xi_k &= \xi_k^{upd} = \textstyle\sum_{j=1}^{N-k-1} \omega_{V_{N-k-j}}(\|d(k+j)\|)\,.
\end{aligned}$$

Proof. For case (i), inequality (38) follows immediately from (28) while (37) follows directly from (35). For case (ii), (38) follows from (29), and (37) from (36). ☐

Remark 6.7. We will later use Corollary 6.6 in order to establish inequality (22). Since this inequality only depends on the perturbation values $d(0),\dots,d(m-1)$, we could make the simplifying assumption $d(k) = 0$ for $k \geq m$ in Corollary 6.6.

7. The perturbed versions of $\mathcal{P}_\alpha$.

Inequalities (14) and (15) comprise the constraints in the minimization problem $\mathcal{P}_\alpha$ for finding the suboptimality index of the nominal m-step MPC scheme with respect to the infinite horizon problem. For the perturbed and the perturbed updated m-step MPC, the preceding corollary yields analogous 'perturbed' inequalities (37) and (38). In this section, we investigate how much the values α resulting from the corresponding perturbed versions of $\mathcal{P}_\alpha$ may differ from the nominal case. To this end, we first state the three problems under consideration. Here, for the subsequent analysis it turns out beneficial to include perturbation terms in both inequalities (37) and (38).

First, the optimization problem $\mathcal{P}_\alpha$ corresponding to the nominal multistep MPC can be written in terms of the latterly introduced notation as

$$\alpha^{\text{nmult}} := \inf_{\lambda_{n,0,0}, n=0,\dots,N-1, \nu^{\text{nmult}}} \frac{\sum_{n=0}^{N-1} \lambda_{n,0,0} - \nu^{\text{nmult}}}{\sum_{n=0}^{m-1} \lambda_{n,0,0}}$$

subject to

$$\begin{aligned} &\sum_{n=k}^{N-1} \lambda_{n,0,0} \le B_{N-k}(\lambda_{k,0,0}), \quad k = 0, \dots, N-2 \\ &\nu^{\text{nmult}} \le \sum_{n=0}^{j-1} \lambda_{n+m,0,0} + B_{N-j}(\lambda_{j+m,0,0}), \quad j = 0, \dots, N-m-1 \\ &\sum_{n=0}^{m-1} \lambda_{n,0,0} \ge 0, \ \lambda_{m,0,0}, \ \dots, \ \lambda_{N-1,0,0}, \nu^{\text{nmult}} \ge 0. \end{aligned} \qquad \mathcal{P}_\alpha^{\text{nmult}}$$

For the perturbed multistep MPC without update, we define α^{pmult} via

$$\alpha^{\text{pmult}} := \inf_{\lambda_{n,n,0}, n=0,\dots,N-1, \nu^{\text{pmult}}} \frac{\sum_{n=0}^{N-1} \lambda_{n,n,0} - \nu^{\text{pmult}}}{\sum_{n=0}^{m-1} \lambda_{n,n,0}}$$

subject to

$$\begin{aligned} &\sum_{n=k}^{N-1} \lambda_{n,n,0} \le B_{N-k}(\lambda_{k,k,0}) + \xi^{\text{pmult}}, \quad k = 0, \dots, N-2 \\ &\nu^{\text{pmult}} \le \sum_{n=0}^{j-1} \lambda_{n+m,m,0} + B_{N-j}(\lambda_{j+m,m,0}) + \xi^{\text{pmult}}, \quad j = 0, \dots, N-m-1 \\ &\sum_{n=0}^{m-1} \lambda_{n,n,0} \ge \zeta, \ \lambda_{m,m,0}, \ \dots, \lambda_{N-1,N-1,0}, \ \nu^{\text{pmult}} \ge 0 \end{aligned} \qquad \mathcal{P}_\alpha^{\text{pmult}}$$

where

$$\xi^{\text{pmult}} := \max_{k \in \{0,\dots,N-2\}} \xi_k^{\text{pmult}} \quad \text{with } \xi_k^{\text{pmult}} \text{ from Corollary 6.6(i)} \tag{39}$$

Finally, for the perturbed updated multistep MPC, we define α^{upd} by

$$\alpha^{\text{upd}} := \inf_{\lambda_{n,n,n}, n=0,\dots,N-1, \nu^{\text{upd}}} \frac{\sum_{n=0}^{N-1} \lambda_{n,n,n} - \nu^{\text{upd}}}{\sum_{n=0}^{m-1} \lambda_{n,n,n}}$$

subject to

$$\begin{aligned} &\sum_{n=k}^{N-1} \lambda_{n,n,n} \le B_{N-k}(\lambda_{k,k,k}) + \xi^{\text{upd}}, \quad k = 0, \dots, N-2 \\ &\nu^{\text{upd}} \le \sum_{n=0}^{j-1} \lambda_{n+m,m,m} + B_{N-j}(\lambda_{j+m,m,m}) + \xi^{\text{upd}}, \quad j = 0, \dots, N-m-1 \\ &\sum_{n=0}^{m-1} \lambda_{n,n,n} \ge \zeta, \ \lambda_{m,m,m}, \ \dots, \lambda_{N-1,N-1,N-1}, \ \nu^{\text{upd}} \ge 0 \end{aligned} \qquad \mathcal{P}_\alpha^{\text{upd}}$$

with

$$\xi^{\text{upd}} = \max_{k \in \{0,\dots,N-2\}} \xi_k^{\text{upd}} \quad \text{with } \xi_k^{\text{upd}} \text{ from Corollary 6.6(ii)} \tag{40}$$

The next lemma is the key technical step to show how α^{nmult}, α^{pmult} and α^{upd} are related. It provides an estimate for the difference between the solutions to two abstract optimization problems of the type introduced above.

Lemma 7.1. *Consider increasing functions $B_k^i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ for $k \in \mathbb{N}$ and $i = 1, 2$ for which $B_k^2(r)$ is linear. Assume that these functions satisfy $B_k^i(r) \geq r$ for all $k \in \mathbb{N}, r \geq 0$ and that there exists a real constant $\xi > 0$ with*

$$B_k^1(r) \leq B_k^2(r) + \xi \tag{41}$$

For $i = 1, 2$ and a constant $\zeta \geq 0$ consider the optimization problems

$$\alpha^i := \inf_{\lambda_0,\ldots,\lambda_{N-1},\nu} \frac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\sum_{n=0}^{m-1} \lambda_n}$$

$$\text{subject to} \quad \sum_{n=k}^{N-1} \lambda_n \leq B_{N-k}^i(\lambda_k), \quad k = 0, \ldots, N-2 \tag{42}$$

$$\nu \leq \sum_{n=0}^{j-1} \lambda_{n+m} + B_{N-j}^i(\lambda_{j+m}), \quad j = 0, \ldots, N-m-1 \tag{43}$$

$$\sum_{n=0}^{m-1} \lambda_n \geq \zeta, \lambda_0, \ldots, \lambda_{N-1}, \nu > 0 \tag{44}$$

Then the following holds.

(i) If $\zeta > 0$, then the inequality $\alpha^2 \leq \alpha^1 + \dfrac{B_{m+1}^2(\xi) + \xi}{\zeta}$ holds.

(ii) If $\zeta = 0$ and $\alpha^2 \geq 0$, then for all values $\lambda_0, \ldots, \lambda_{N-1}, \nu$ satisfying (42)–(44) for $i = 1$ the inequality $\nu \leq \sum_{n=0}^{N-1} \lambda_n + B_{m+1}^2(\xi) + \xi$ holds.

Proof. (i) Fix $\varepsilon > 0$. Consider ε-optimal values $\lambda_0^1, \ldots, \lambda_{N-1}^1, \nu^1$ satisfying the constraints (42)–(44) for $i = 1$ and

$$\frac{\sum_{n=0}^{N-1} \lambda_n^1 - \nu^1}{\sum_{n=0}^{m-1} \lambda_n^1} \leq \alpha^1 + \varepsilon$$

Case 1: Suppose $\lambda_{N-1}^1 - \xi > 0$. In the following we construct $\lambda_0^2, \ldots, \lambda_{N-1}^2, \nu^2$ satisfying the constraints (42)–(44) for $i = 2$ and

$$\frac{\sum_{n=0}^{N-1} \lambda_n^2 - \nu^2}{\sum_{n=0}^{m-1} \lambda_n^2} \leq \alpha^1 + \varepsilon + \frac{B_{m+1}^2(\xi)}{\zeta}$$

Set $\lambda_n^2 := \lambda_n^1$, $n = 0, \ldots, N-2$, $\lambda_{N-1}^2 := \lambda_{N-1}^1 - \xi$. Set $\nu^2 := \max\{0, \nu^1 - B_{m+1}^2(\xi) - \xi\}$. Notice that by this construction, $\lambda_0^2, \ldots, \lambda_{N-1}^2, \nu^2$ satisfies constraint (44). For $k = 0, \ldots, N-2$ this implies

$$\sum_{n=k}^{N-1} \lambda_n^2 = \sum_{n=k}^{N-1} \lambda_n^1 - \xi \leq B_{N-k}^1(\lambda_k^1) - \xi \leq B_{N-k}^2(\lambda_k^1) + \xi - \xi = B_{N-k}^2(\lambda_k^2)$$

where the last equality holds since k ranges only from 0 to $N-2$. This implies (42) for $B_k = B_k^2$.

Next observe that for $j = 0, \ldots, N-m-2$

$$\begin{aligned}\nu^1 &\leq \sum_{n=0}^{j-1} \lambda^1_{n+m} + B^1_{N-j}(\lambda^1_{j+m}) \leq \sum_{n=0}^{j-1} \lambda^1_{n+m} + B^2_{N-j}(\lambda^1_{j+m}) + \xi \\ &= \sum_{n=0}^{j-1} \lambda^2_{n+m} + B^2_{N-j}(\lambda^2_{j+m}) + \xi\end{aligned}$$

holds. Further observe that for $j = N-m-1$ we have

$$\begin{aligned}\nu^1 &\leq \sum_{n=0}^{N-m-2} \lambda^1_{n+m} + B^1_{m+1}(\lambda^1_{N-1}) \leq \sum_{n=0}^{N-m-2} \lambda^1_{n+m} + B^2_{m+1}(\lambda^1_{N-1}) + \xi \\ &= \sum_{n=0}^{N-m-2} \lambda^2_{n+m} + B^2_{m+1}(\lambda^2_{N-1} + \xi) + \xi \\ &= \sum_{n=0}^{N-m-2} \lambda^2_{n+m} + B^2_{m+1}(\lambda^2_{N-1}) + B^2_{m+1}(\xi) + \xi\end{aligned}$$

with the last equality due to linearity of B^2_{N-k}. In case $\nu^2 = 0$ we get

$$\nu^2 \leq \sum_{n=0}^{j-1} \lambda^2_{n+m} + B^2_{N-j}(\lambda^2_{j+m}), \ = 0, \ldots, N-m-2$$

and in case $\nu^2 = \nu^1 - B^2_{m+1}(\xi) - \xi$ the inequalities

$$\begin{aligned}\nu^2 &\leq \sum_{n=0}^{N-m-2} \lambda^2_{n+m} + B^2_{m+1}(\lambda^2_{N-1}) \\ \nu^2 &\leq \nu^1 - \xi \leq \sum_{n=0}^{j-1} \lambda^2_{n+m} + B^2_{N-j}(\lambda^2_{j+m}), \quad j = 0, \ldots, N-m-2\end{aligned}$$

hold. Thus, for $j = 0, \ldots, N-m-1$, we have $\nu^2 \leq \sum_{n=0}^{j-1} \lambda^2_{n+m} + B^2_{N-j}(\lambda^2_{j+m})$. This implies (43) for $B_k = B^2_k$.

Since $\sum_{n=0}^{m-1} \lambda^1_n = \sum_{n=0}^{m-1} \lambda^2_n \geq \zeta > 0$ and $\xi > 0$, the values $\lambda^2_m, \ldots, \lambda^2_{N-1}, \nu^2$ satisfy all constraints (42)–(44) for $i = 2$ and we obtain

$$\begin{aligned}\alpha^2 &\leq \frac{\sum_{n=0}^{N-1} \lambda^2_n - \nu^2}{\sum_{n=0}^{m-1} \lambda^2_n} = \frac{\sum_{n=0}^{N-1} \lambda^1_n - \xi - \nu^2}{\sum_{n=0}^{m-1} \lambda^2_n} \leq \frac{\sum_{n=0}^{N-1} \lambda^1_n - \xi - \nu^1 + B^2_{m+1}(\xi) + \xi}{\sum_{n=0}^{m-1} \lambda^1_n} \\ &\leq \alpha^1 + \varepsilon + \frac{B^2_{m+1}(\xi)}{\zeta}.\end{aligned}$$

Case 2: Now suppose $\lambda^1_{N-1} - \xi \le 0$. Let $\mu := \sum_{n=0}^{N-m-2} \lambda^1_{n+m} + B^1_{m+1}(\lambda^1_{N-1})$. Then

$$\begin{aligned}
\alpha^1 + \varepsilon &\ge \frac{\sum_{n=0}^{N-1} \lambda^1_n - \nu^1}{\sum_{n=0}^{m-1} \lambda^1_n} \ge \frac{\sum_{n=0}^{N-1} \lambda^1_n - \mu}{\sum_{n=0}^{m-1} \lambda^1_n} \\
&= \frac{\sum_{n=0}^{m-1} \lambda^1_n + \sum_{n=m}^{N-2} \lambda^1_n + \lambda^1_{N-1} - \mu}{\sum_{n=0}^{m-1} \lambda^1_n} \\
&= 1 + \frac{\mu - B^1_{m+1}(\lambda^1_{N-1}) + \lambda^1_{N-1} - \mu}{\sum_{n=0}^{m-1} \lambda^1_n} \\
&= 1 + \frac{B^1_{m+1}(\lambda^1_{N-1}) - \lambda^1_{N-1}}{-\sum_{n=0}^{m-1} \lambda^1_n} \ge 1 + \frac{B^1_{m+1}(\lambda^1_{N-1}) - \lambda^1_{N-1}}{-\zeta} \\
&\ge 1 - \frac{B^1_{m+1}(\lambda^1_{N-1})}{\zeta} \ge 1 - \frac{B^1_{m+1}(\xi)}{\zeta} \ge \alpha^2 - \frac{B^1_{m+1}(\xi)}{\zeta} \\
&\ge \alpha^2 - \frac{B^2_{m+1}(\xi) + \xi}{\zeta}.
\end{aligned}$$

Hence, in both cases we obtain $\alpha^2 \le \alpha^1 + \varepsilon + \frac{B^2_{m+1}(\xi)+\xi}{\zeta}$ which shows the assertion since $\varepsilon > 0$ was arbitrary.

(ii) We proceed by contradiction. Assume there are values $\lambda^1_0, \dots, \lambda^1_{N-1}, \nu^1$ satisfying (42)–(44) for $i = 1$ and $\nu^1 > \sum_{n=0}^{N-1} \lambda^1_n + B^2_{m+1}(\xi) + \xi$. Then the same construction as in (i) yields $\lambda^2_0, \dots, \lambda^2_{N-1}, \nu^2$ satisfying (42)–(44) for $i = 2$ and

$$\alpha^2 \le \frac{\sum\limits_{n=0}^{N-1} \lambda^2_n - \nu^2}{\sum\limits_{n=0}^{m-1} \lambda^2_n} \le \frac{\sum\limits_{n=0}^{N-1} \lambda^1_n - \nu^1 + B^2_{m+1}(\xi) + \xi}{\sum\limits_{n=0}^{m-1} \lambda^1_n} < 0$$

which contradicts the assumption $\alpha^2 \ge 0$. □

The following theorem finally applies Lemma 7.1 to the problems $\mathcal{P}^{\mathrm{nmult}}_\alpha$, $\mathcal{P}^{\mathrm{pmult}}_\alpha$ and $\mathcal{P}^{\mathrm{upd}}_\alpha$.

Theorem 7.2. *Consider problems $\mathcal{P}^{nmult}_\alpha$, $\mathcal{P}^{pmult}_\alpha$ and $\mathcal{P}^{upd}_\alpha$, let the assumptions of Theorem 4.4 hold and assume that the B_k, $k \in \mathbb{N}$ from $\mathcal{P}^{nmult}_\alpha$ are linear functions. Then*

$$\alpha^{pmult} \ge \alpha^{nmult} - \frac{B_{m+1}(\xi^{pmult}) + \xi^{pmult}}{\zeta}$$

$$\alpha^{upd} \ge \alpha^{nmult} - \frac{B_{m+1}(\xi^{upd}) + \xi^{upd}}{\zeta}.$$

Here, α^{nmult} can be replaced by the right hand side of Equation (17).

Proof. We apply Lemma 7.1 setting $\alpha^2 := \alpha^{\mathrm{nmult}}$, $B^2_k(r) := B_k(r)$, $\alpha^1 := \alpha^{\mathrm{pmult}}$ and $B^1_k(r) := B_k(r) + \xi^{\mathrm{pmult}}$. This yields $\alpha^{\mathrm{nmult}} \le \alpha^{\mathrm{pmult}} + \frac{B^2_{m+1}(\xi^{\mathrm{pmult}})+\xi^{\mathrm{pmult}}}{\zeta}$. Similarly, taking $\alpha^2 := \alpha^{\mathrm{nmult}}$, $B^2_k(r) := B_k(r)$, $\alpha^1 := \alpha^{\mathrm{upd}}$ and $B^1_k(r) := B_k(r) + \xi^{\mathrm{upd}}$, we have that $\alpha^{\mathrm{nmult}} \le \alpha^{\mathrm{upd}} + \frac{B^2_{m+1}(\xi^{\mathrm{upd}})+\xi^{\mathrm{upd}}}{\zeta}$. The fact that α^{nmult} can be replaced by the right hand side of (17) follows immediately from Theorem 4.6. □

The preceding theorem gives lower bounds for the values α^{pmult} and α^{upd} of the perturbed problems in terms of the performance index α^{nmult} of the nominal problem. Recall that a larger value of the suboptimality index α indicates better performance of the scheme. Thus, the theorem limits the performance loss to the values $\frac{B_{m+1}(\xi^{\text{pmult}})+\xi^{\text{pmult}}}{\zeta}$ and $\frac{B_{m+1}(\xi^{\text{upd}})+\xi^{\text{upd}}}{\zeta}$, respectively.

Remark 7.3. The constraint bound ζ is needed in order to prevent the quotient $\frac{B^2_{m+1}(\xi)+\xi}{\zeta}$ from blowing up. In the next section we relate this bound to the parameter $\delta > 0$ in the semiglobal practical asymptotic stability property.

8. **Asymptotic stability and performance.**

In this section we combine all of the previous results in order to prove the 'perturbed' counterpart to Theorem 4.4. To this end, we start with a preparatory lemma.

Lemma 8.1. *Let the assumptions of Corollary 6.6 hold.*

(a) Consider a perturbation sequence $d(\cdot)$ with $d(k) = 0$ for all $k \geq m$ and a trajectory $\tilde{x}_{\mu_{N,m}}(\cdot, x_0)$ of (24) which corresponds to a perturbation sequence $\tilde{d}(\cdot)$ with $\tilde{d}(k) = d(k)$ for $k = 0, \dots, m-1$,

(i) Let α^{pmult} be the solution of $\mathcal{P}_\alpha^{pmult}$ for $d(\cdot)$ and some $\zeta > 0$ and assume $\sum_{k=0}^{m-1} \ell(\tilde{x}_{\mu_{N,m}}(k, x_0), \mu_{N,m}(\tilde{x}_{\mu_{N,m}}(k, x_0), k)) \geq \zeta$. Then the inequality

$$V_N(x_{\mu_{N,m}}(m, x_0)) \leq V_N(x_0) - \tilde{\alpha}^{pmult} \sum_{k=0}^{m-1} \ell(\tilde{x}_{\mu_{N,m}}(k, x_0), \mu_{N,m}(\tilde{x}_{\mu_{N,m}}(k, x_0), k)) \tag{45}$$

holds for

$$\tilde{\alpha}^{pmult} = \alpha^{pmult} - \frac{\sigma}{\zeta} \qquad \text{where } \sigma = \sum_{j=1}^{m-1} \omega_{J_{N-j}}(\|d(j)\|) \tag{46}$$

(ii) Assume that all values $\lambda_0, \dots, \lambda_{N-1}, \nu^{pmult}$ satisfying the constraints from $\mathcal{P}_\alpha^{pmult}$ satisfy $\nu \leq \sum_{n=0}^{N-1} \lambda_n + B_{m+1}(\xi^{pmult}) + \xi^{pmult}$. Then the inequality

$$V_N(x_{\mu_{N,m}}(m, x_0)) \leq V_N(x_0) + B_{m+1}(\xi^{pmult}) + \xi^{pmult} + \sigma$$

holds for σ from (i).

(b) The analogous statements holds for the trajectories $\tilde{x}_{\hat{\mu}_{N,m}}(\cdot, x_0)$ of (25) with $\mathcal{P}_\alpha^{pmult}$, $\tilde{\alpha}^{pmult}$ etc. replaced by $\mathcal{P}_\alpha^{upd}$, $\tilde{\alpha}^{upd}$ etc. and $\sigma = \sum_{j=1}^{N-1} \omega_{V_{N-j}}(\|d(j)\|)$.

Proof. (a)(i) Consider the trajectory $x_{j,j,0}$ corresponding to the perturbation $d(\cdot)$ starting in $x_{0,0,0} = x_0$, and the corresponding values $\lambda_{j,j,0}$. Note that for $j = 0, \dots, m$ the identities $\tilde{x}_{\mu_{N,m}}(j, x_0) = x_{j,j,0}$ and for $j = 0, \dots, m-1$ the identities $\ell(\tilde{x}_{\mu_{N,m}}(j, x_0), \mu_{N,m}(\tilde{x}_{\mu_{N,m}}(j, x_0), j)) = \lambda_{j,j,0}$ hold.

By Corollary 6.6(i), the values $\lambda_n = \lambda_{n,n,0}$ and $\nu = V_N(x_{m,m,0})$ satisfy the constraints of $\mathcal{P}_\alpha^{\text{pmult}}$. This implies

$$\nu^{\text{pmult}} \leq \sum_{n=0}^{N-1} \lambda_{n,n,0} - \alpha^{\text{pmult}} \sum_{n=0}^{m-1} \lambda_{n,n,0}$$

from which using (33) we obtain

$$\begin{aligned} V_N(x_{\mu_{N,m}}(m,x_0)) &\le \sum_{n=0}^{N-1}\lambda_{n,n,0} - \alpha^{\mathrm{pmult}}\sum_{n=0}^{m-1}\lambda_{n,n,0} \\ &\le \underbrace{\sum_{n=0}^{N-1}\lambda_{n,0,0}}_{=V_N(x)} + \underbrace{\sum_{n=1}^{N-1}\omega_{J_{N-n}}(\|d(n)\|)}_{=\sigma\le\sigma\frac{\zeta}{\sum_{n=0}^{m-1}\lambda_{n,n,0}}} - \alpha^{\mathrm{pmult}}\sum_{n=0}^{m-1}\lambda_{n,n,0} \\ &\le V_N(x) - \tilde\alpha^{\mathrm{pmult}}\sum_{n=0}^{m-1}\lambda_{n,n,0}, \end{aligned}$$

i.e., the assertion, since $d(m) = \ldots, d(N-1) = 0$.

(a)(ii) Similar to (i) we obtain

$$V_N(x_{\mu_{N,m}}(m,x_0)) \le \sum_{n=0}^{N-1}\lambda_{n,n,0} + B_{m+1}(\xi^{\mathrm{pmult}}) + \xi^{\mathrm{pmult}}.$$

From this the assertion follows using the same estimates as in (i).

(b) Follows by analogous arguments using $x_{j,j,j}$, $\lambda_{j,j,j}$, Corollary 6.6(ii) and (34). $\square$

The following theorem — together with the subsequent remark — constitutes the main result of this paper. For its formulation we need one more property of f.

Definition 8.2. We say that f is *uniformly bounded on each ball* $\overline{\mathcal{B}}_\Delta(x_*)$ if for any $\Delta > 0$ the value $\sup_{\|x\|_{x_*}\le\Delta, u\in\mathbb{U}(x)}\|f(x,u)\|$ is finite.

Theorem 8.3. *(i) Let $N \ge 1$ and consider the MPC Algorithm 3.2 with stage cost $\ell : X \times U \to \mathbb{R}_0^+$ satisfying (10), yielding the multistep feedback law $\mu_{N,m}$. Assume that f is uniformly bounded on each ball $\overline{\mathcal{B}}_\Delta(x_*)$ and that J_K, $K = 1, \ldots, N$, f and ℓ are uniformly continuous uniformly in u on each ball $A = \overline{\mathcal{B}}_\eta(x_*)$ around x_* with their respective moduli of continuity $\omega^\eta_{J_K}$, ω^η_f and ω^η_ℓ. Assume that (13) holds with B_K being linear and that the optimization problem $\mathcal{P}^{nmult}_\alpha$ has an optimal value $\alpha^{nmult} \in (0, 1]$, implying that the nominal closed-loop system is asymptotically stable.*

Then the perturbed m-step closed-loop system (24) with feedback law $\mu_{N,m}$ is semi-globally practically asymptotically stable on $\mathbb{X}$ with respect to d. The bound $\overline{d}$ depending on Δ and δ in Definition 5.3 can be chosen to satisfy the condition $\tilde\alpha^{pmult} > \kappa\alpha^{nmult}$ for arbitrary $\kappa \in (0, 1)$, with $\tilde\alpha^{pmult}$ from Lemma 8.1(a)(i). Here, the moduli of continuity ω_{J_N} involved in the estimates for $\tilde\alpha^{pmult}$ and α^{pmult} are chosen as $\omega_{J_N} = \omega^\eta_{J_N}$ with η depending on Δ. The value ζ in these estimates depends on δ.

Moreover, for $\tilde\alpha^{pmult} > 0$ the performance estimate

$$J^{cl}_{k^*}(\tilde x_{\mu_{N,m}}(\cdot,x),\mu_{N,m}) \le V_N(x)/\tilde\alpha^{pmult}.$$

holds for all $\tilde x_{\mu_{N,m}}(\cdot,x) \in S_{\overline{d}}(x)$.

(ii) The same statements hold for the MPC Algorithm 3.3 and the corresponding closed-loop system (25) when we replace the moduli of continuity $\omega^\eta_{J_K}$ by $\omega^\eta_{V_K}$ and $\tilde\alpha^{pmult}$, α^{pmult} by $\tilde\alpha^{upd}$, α^{upd}, respectively.

Proof. (i) Fix $\Delta > \delta > 0$ and an arbitrary $\kappa \in (0,1)$. We prove the assertion using Lemma 5.5 and Theorem 5.8. To this end, we will construct m-step forward invariant sets Y and P with respect to $\overline{d}$ with intermediate set $\widehat{P}$ for P satisfying

$$\overline{\mathcal{B}}_\Delta(x_*) \subseteq Y \text{ and } \widehat{P} \subseteq \overline{\mathcal{B}}_\delta(x_*).$$

and such that (22) holds with $V = V_N$, $\mu = \mu_{N,m}$ and $\alpha := \kappa\alpha^{\text{nmult}}$ for all $x_0 \in Y \backslash P$.

First, observe that by taking $\alpha_3 := \alpha_1$ and $\alpha_4 := B_N \circ \alpha_2$ with α_2 from (10) we obtain

$$\alpha_3(\|x_*\|) \leq \ell^*(x) \leq V_N(x) \leq B_N(\ell^*(x)) \leq B_N(\alpha_2(\|x\|_{x_*})) = \alpha_4(\|x\|_{x_*}) \qquad (47)$$

Construction of Y: Fixing some arbitrary $\tilde{d} > 0$, due to the uniform continuity of f on balls around x_*, there exists $\eta_1 > 0$ such that $f(x,u) + d \in B_{\eta_1}(x_*)$ holds for all $x \in \overline{\mathcal{B}}_\Delta(x_*)$ and all $\|d\| \leq \tilde{d}$. Continuing inductively for $i = 2, \dots, N$ with η_{i-1} in place of Δ, we find η_N such that any solution $\tilde{x}_\mu(\cdot, x_0) \in S_{\tilde{d}}(x_0)$ for any $x_0 \in \overline{\mathcal{B}}_\Delta(x_*)$ satisfies $\tilde{x}_\mu(k, x_0) \in \overline{\mathcal{B}}_{\eta_N}(x_*)$ for all $k = 0, \dots, N$.

We set $L := \alpha_4(\eta_N)$ which implies that for any $x \in \overline{\mathcal{B}}_{\eta_N}(x_*) \cap \mathbb{X}$ we have $V_N(x) \leq \alpha_4(\|x\|_{x_*}) \leq \alpha_4(\eta_N) = L$ and thus

$$Y := V_N^{-1}([0,L]) \supseteq \overline{\mathcal{B}}_{\eta_N}(x_*) \cap \mathbb{X} \supseteq \overline{\mathcal{B}}_\Delta(x_*) \cap \mathbb{X}.$$

Setting $\eta = \alpha_1^{-1}(L)$ implies $Y \subset \overline{\mathcal{B}}_\eta(x_*)$. We let $\omega_{J_K} = \omega^\eta_{J_K}$, $K = 0, \dots, N$, $\omega_f = \omega^\eta_f$ and $\omega_\ell = \omega^\eta_\ell$ denote the moduli of continuity of J_K, f and ℓ, respectively, on $A = \overline{\mathcal{B}}_\eta(x_*)$.

Construction of P and $\widehat{P}$: We set $p := \alpha \cdot \alpha_1 \circ \alpha_4^{-1} \circ \alpha_3(\delta)$ with $\alpha = \kappa\alpha^{\text{nmult}}$ and define

$$P := V_N^{-1}([0,p]) \subseteq \overline{\mathcal{B}}_{\alpha_3^{-1}(p)}(x_*).$$

In addition, we define $\widehat{P} := \overline{\mathcal{B}}_\delta(x_*)$. For later use, we also define $q := p/2$, $Q := V_N^{-1}([0,q]) \subset P$ and $\zeta = \alpha_1(\alpha_4^{-1}(q))$. Observe that if $x \notin Q$, then $\alpha_4(\|x\|_{x_*}) \geq V_N(x) \geq q$ which yields $\ell^*(x) \geq \alpha_1(\|x\|_{x_*}) \geq \alpha_1(\alpha_4^{-1}(q))$, i.e., the choice of ζ ensures $\ell^*(x) \geq \zeta$.

Choice of $\overline{d}$: We choose $\overline{d} \in (0, \min\{\tilde{d}, q\}]$ maximal such that the two conditions

$$B_{m+1}(\xi^{\text{pmult}}) + \xi^{\text{pmult}} + \sigma \leq q \qquad \text{and} \qquad \tilde{\alpha}^{\text{pmult}} \geq \kappa\alpha^{\text{nmult}}$$

hold for ξ^{pmult} from Corollary 6.6(i), and σ and $\tilde{\alpha}^{\text{pmult}}$ from Lemma 8.1(a)(i) with ζ from above. Such $\overline{d} > 0$ exists due to Lemma 8.1 and Theorem 7.2: Due to the uniform continuity assumption on the J_K, f and ℓ and the linearity of B_K, all terms in the definition of ξ^{pmult} in Corollary 6.6(i) vanish as $\overline{d} \to 0$. We note that $\overline{d}$ depends on δ via q and ζ (which depends on δ via the construction of P) and on Δ via the moduli of continuity ω_{J_K}, ω_f and ω_ℓ (which depend on Δ via the construction of Y). By Lemma 8.1, this choice of $\overline{d}$ ensures (45) and thus (22) with $V = V_N$, $\mu = \mu_{N,m}$ and $\alpha = \tilde{\alpha}^{\text{pmult}} = \kappa\alpha^{\text{nmult}} > 0$ for all $x_0 \in Y$ with $\ell^*(x_0) \geq \zeta$. By the choice of ζ, this includes all $x_0 \in Y \setminus Q$.

m-step forward invariance of Y: It is sufficient to show the implication $x_0 \in Y \Rightarrow \tilde{x}_{\mu_{N,m}}(m, x_0) \in Y$ for all $\tilde{x}_{\mu_{N,m}}(\cdot, x_0) \in S_{\overline{d}}(x_0)$ since $\tilde{x}_{\mu_{N,m}}(rm, x_0) \in Y$ for $r \geq 2$ then follows by induction. For $x_0 \in Y \setminus Q$, we know that (45) applies, yielding $V_N(\tilde{x}_{\mu_{N,m}}(m, x_0)) \leq V_N(x_0)$ which implies $\tilde{x}_{\mu_{N,m}}(m, x_0) \in Y$. For $x_0 \in Q$, we know that $\|x_0\|_{x_*} \leq \delta < \Delta$. By construction of Y, all perturbed trajectories starting in $\overline{\mathcal{B}}_\Delta(x_*)$ remain in Y for at least N steps, which implies $\tilde{x}_{\mu_{N,m}}(m, x_0) \in Y$ since $m < N$.

***m*-step forward invariance of P:** Again, it is sufficient to show the implication $x_0 \in P \;\Rightarrow\; \tilde{x}_{\mu_{N,m}}(m, x_0) \in P$ for all $\tilde{x}_{\mu_{N,m}}(\cdot, x_0) \in S_{\overline{d}}(x_0)$. We thus consider arbitrary $x_0 \in P$ and $\tilde{x}_{\mu_{N,m}}(\cdot, x_0) \in S_{\overline{d}}(x_0)$ and distinguish two cases:

Case 1: $x_0 \notin Q$. Then (45) applies, yielding $V_N(\tilde{x}_{\mu_{N,m}}(m, x_0)) \leq V_N(x_0)$ which implies $\tilde{x}_{\mu_{N,m}}(m, x_0) \in P$.

Case 2: $x_0 \in Q$. Since $\alpha^{nmult} > 0$, Lemma 7.1(ii) applies and ensures that the assumptions of Lemma 8.1(a)(ii) are satisfied. Then the choice of Q, q and $\overline{d}$ yields

$$V_N(\tilde{x}_{\mu_{N,m}}(m, x_0)) \leq V_N(x_0) + B_{m+1}(\xi^{\text{pmult}}) + \xi^{\text{pmult}} + \sigma \leq q + q = p$$

which again implies $\tilde{x}_{\mu_{N,m}}(m, x_0) \in P$.

$\widehat{P}$ is an intermediate set: It remains to show that $\tilde{x}_{\mu_{N,m}}(k, x_0) \in \widehat{P} = \overline{\mathcal{B}}_\delta(x_*)$ for all $k \geq 0$ and $x_0 \in P$. To this end, we use the inequality

$$V_N(\tilde{x}_{\mu_{N,m}}(k, x_0)) \leq \alpha_4 \circ \alpha_1^{-1}(V_N(\tilde{x}_{\mu_{N,m}}(\lfloor k \rfloor_m, x_0))/\alpha)$$

derived in the proof of Theorem 5.8(ii). Since P is m-step forward invariant, we know $\tilde{x}_\mu(\lfloor k \rfloor_m, x_0) \in P$ and thus

$$V_N(\tilde{x}_{\mu_{N,m}}(k, x_0)) \leq \alpha_4 \circ \alpha_1^{-1}(p/\alpha)$$

which by (47) and choice of p implies

$$\|\tilde{x}_{\mu_{N,m}}(k, x_0)\|_{x_*} \leq \alpha_3^{-1} \circ \alpha_4 \circ \alpha_1^{-1}(p/\alpha) = \delta$$

and thus shows $\tilde{x}_{\mu_{N,m}}(k, x_0) \in \widehat{P}$.

(ii) The proof is completely identical to (i), observing that throughout the proof of (i), we have only used properties of Algorithm 3.2 and system (24) which have also been proven for Algorithm 3.3 and system (25). $\square$

Remark 8.4. The decisive difference between the cases (i) and (ii) in Theorem 8.3 is that the error terms — which determine both the bound for $\overline{d}$ and the suboptimality index α — depend on ω_{J_K} for Algorithm 3.2 and on ω_{V_K} for Algorithm 3.3. Since typically the latter is smaller than the former, cf. [5, Theorem 3], with the difference being significant, e.g., in case of open loop unstable and controllabe systems, cf. [5, Section V], this explains and quantifies the better robustness properties of the updated MPC scheme.

9. **Numerical Example: An inverted pendulum.** In order to illustrate our results, we consider a nonlinear inverted pendulum model consisting of a cart mounted on a track where it can move and attached to it is a rigid pendulum that is able to rotate freely. We use the different MPC controllers discussed in this paper to swing up the pendulum to the unstable upright or inverted position. We consider the model

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{g}{\ell}\sin(x_1(t)) - \frac{k_L}{l}\arctan(1000x_2(t))x_2^2(t) - \frac{u(t)}{l}\cos(x_1(t)) \\
&\quad -k_R\left(\frac{4ax_2(t)}{1+4(ax_2(t))^2} + \frac{2\arctan(bx_2(t))}{\pi}\right) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= u(t)
\end{aligned}$$

where $x_i, i = 1, \ldots, 4$ represents pendulum angular displacement, angular velocity, cart position and cart velocity, respectively, with gravitational constant $g = 9.81$, pendulum length $l = 1.25$ and friction parameters $k_L = 0.007$ and $k_R = 0.197$. In order to convert the continuous time system to a discrete time model (1) we sample it with zero order hold and sampling period $T = 0.2$. To stabilize the upright position $x_* = ((2k+1)\pi, 0, 0, 0),\ k \in \mathbb{N}$, we consider the stage cost

$$\begin{aligned}\ell(x(i), u(i)) \quad = \quad & \int_{t_i}^{t_{i+1}} 10^{-4} u(t)^2 + \big(3.51 \sin(x_1(t) - \pi)^2 + 4.82 \sin(x_1(t) - \pi) x_2(t) \\ & + 2.31 x_2(t)^2 + 0.1 \left((1 - \cos(x_1(t) - \pi)) \cdot (1 + \cos(x_2(t))^2)\right)^2 \\ & + \ 0.01 x_3(t)^2 + 0.1 x_4(t)^2\big)^2 \, dt\end{aligned}$$

where $t_i = iT$, leading to a cost functional of $J_N(x_0, u) = \sum_{i=0}^{N-1} \ell(x(i), u(i))$. We aim to compare simulations resulting from the multistep, updated multistep and sensitivity-based multistep feedback controllers both on nominal and perturbed setting. We set the length of the optimization horizon to $N = 15$, set the initial value $x_0 = (-\pi - 0.1, 0, -0.1, 0)$ and for the perturbed system (18) we use a fixed randomly generated perturbation sequence of the form $d(k) = [0, 0, d_3(k), 0]^\top, k \in \mathbb{N}$, (i.e., perturbations occur on the cart position x_3 and are identical for each simulation) with values in the interval $[-\overline{d}_3, 0]$ for $\overline{d}_3 = 0.05$.

Figure 2 illustrates that the trajectories for $m = 1$ where the 1-step MPC scheme (shown in blue) renders the nominal system asymptotically stable at $(-\pi, 0, 0, 0)$ while, as expected, the 1-step perturbed solution (cyan) is only practically asymptotically stable, i.e., only converges to a neighborhood of x_*. We remark that for $m = 1$, the trajectories generated by (24) and (25) coincide, hence only the former is shown in the figure. For $m = 7$, trajectories resulting from the nominal 7-step (blue), perturbed 7-step (red), and perturbed updated 7-step (green) are plotted in Figure 3. The larger m is chosen, the longer the multistep controller does not counteract the effect of the perturbation preventing the trajectory to arrive closer to the equilibrium which is exactly what we see in plots (shown in red). Improvement is manifested by applying the updates to the multistep scheme allowing the trajectory to shoot towards the equilibrium against the perturbations (shown in green). In addition, the black lines show the solution for the updated 7-step scheme where the shrinking horizon optimizations were replaced by sensitivity updates, cf. [13]. We note that this solution is barely distinguishable from the updated 7-step scheme (green), thus showing that the analysis in this paper also provides valuable information for the sensitivity updated scheme. Finally, the figure also illustrates how all the schemes mentioned compare to the 1-step scheme — the most robust scheme (shown in cyan).

Table 1 shows the comparison of time requirements in CPU time among the multistep and the updated multistep schemes for increasing multisteps m. To allow comparison, time instants 0 to 100 are considered for which for each scheme, `floor`$(100/m)$ optimizations with full horizon N are performed and the times needed are recorded. As expected, since neither a control has to be computed nor an optimization has to be performed for the multistep scheme, the larger m is chosen, the larger the savings in time becomes. For each m, due to the sequence of optimization with shrinking horizon that has to be performed, the corresponding updated scheme requires more time which one can easily notice in the table. Although optimization for each time step is still required for the updated multistep scheme, savings in time is nevertheless achieved in contrast to the 1-step MPC —

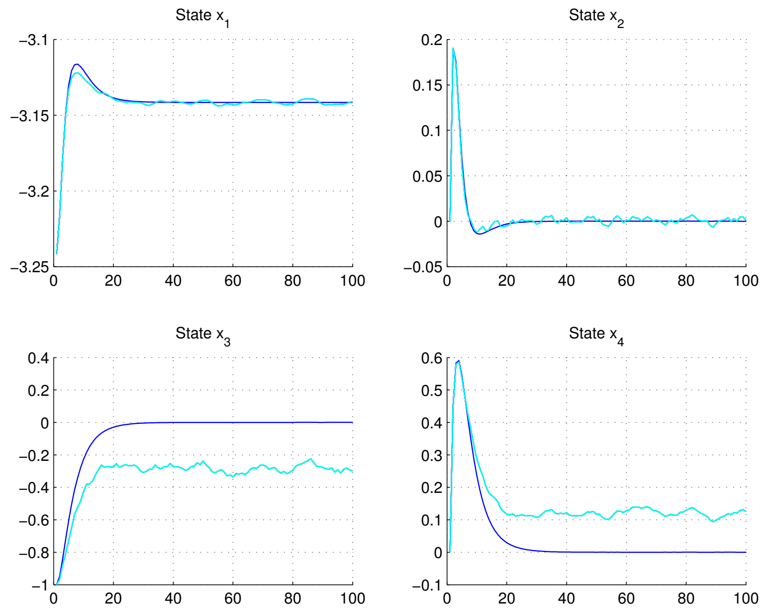

FIGURE 2. State trajectories driven by the 1-step MPC scheme for nominal (blue) and perturbed system (cyan)

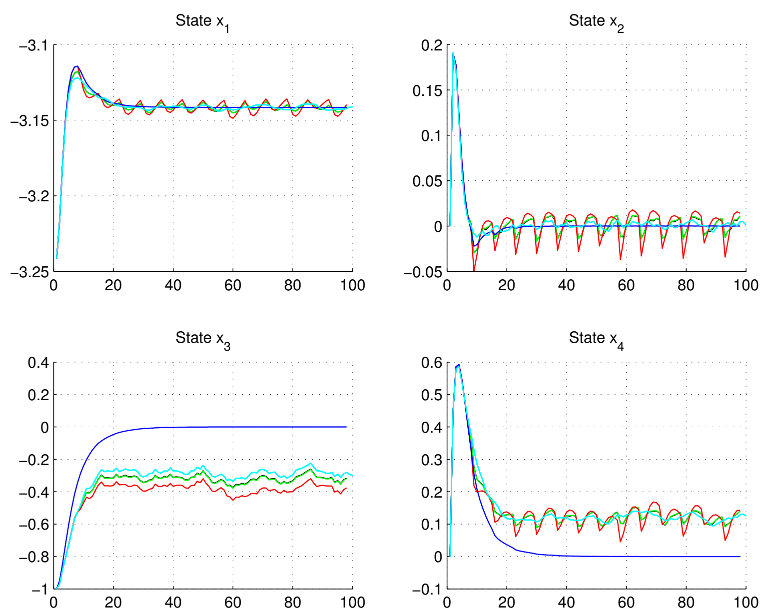

FIGURE 3. State trajectories driven by the 7-step MPC scheme for nominal system (blue), the 1-step (cyan), 7-step (red), updated 7-step (green) and sensitivity-based 7-step (black) MPC schemes for the perturbed system

the most expensive scheme — which performs optimization with full horizon N at each time instant.

TABLE 1. Comparison of time requirements in CPU time

m	multistep	updated
1	11.0447	11.0967
2	5.6484	10.4687
3	3.6762	10.3646
4	2.5522	10.1046
5	2.1921	9.3766
6	1.8241	8.6125
7	1.5801	7.7765
8	1.2321	7.7845
9	1.0881	7.2405
10	1.0641	6.5404
11	0.9521	6.1124
12	0.8601	5.7884
13	0.8681	5.2243

Finally, Table 2 presents performance indices α of the schemes which are computed from the generated trajectories using the approach presented in [6]. We vary m and list the values of α for the first three iterations of each scheme. In our simulation, the values of α for the nominal multistep scheme indicates that the feedback is 'close' to being infinite horizon optimal having values $\alpha > 0.9$. Furthermore, along increasing m, the α values increase, peak and then deteriorate exemplifying the parabolic profile of the α's of the multistep MPC scheme reported in [8]. For the perturbed system with $\overline{d}_3 = 0.05$, for the multistep scheme, α values are observably lower and even worsen on the second and third iteration where negative values are also seen. These negative values indicate that the region $\widehat{P}$ of practical asymptotic stability has been reached, cf. [6, Section 4]. Most importantly, Table 2 shows a noticeable improvement to the values of α for the updated multistep brought about by the re-optimization that counteracts the effect of the perturbation as seen in the last three columns of the table. Weighing in all benefits after examining the time requirements and suboptimality estimates, by updating the multistep feedback for the perturbed system, we clearly gain time savings compared to the classical MPC scheme, and improve robustness in comparison with the multistep feedback scheme.

TABLE 2. Suboptimality index α of the schemes for various m and iterations

	nominal multistep			perturbed multistep			updated multistep		
m	0	2m	3m	0	2m	3m	0	2m	3m
1	0.9908	0.9917	0.9935	0.8667	0.8699	0.6032	0.8667	0.8699	0.6032
2	0.9911	0.9937	0.9950	0.8678	0.6322	0.8479	0.8681	0.6383	0.8538
3	0.9915	0.9944	0.9948	0.7936	0.7713	0.5857	0.7955	0.7810	0.6203
4	0.9917	0.9942	0.9937	0.7672	0.6870	0.5282	0.7729	0.7139	0.5647
5	0.9916	0.9933	0.9916	0.7632	0.6898	0.4171	0.7734	0.7307	0.4882
6	0.9913	0.9916	0.9880	0.7724	0.3915	0.3810	0.7868	0.4974	0.4037
7	0.9908	0.9887	0.9829	0.7404	0.4850	-0.0954	0.7629	0.5695	-0.0251
8	0.9902	0.9843	0.9755	0.7103	0.4233	-0.0370	0.7414	0.4981	0.0228
9	0.9895	0.9778	0.9662	0.7066	0.1941	-0.0328	0.7423	0.2845	-0.0129
10	0.9888	0.9698	0.9561	0.6988	0.0840	-0.2314	0.7379	0.1718	-0.2125
11	0.9883	0.9622	0.9461	0.6477	0.1414	-0.0467	0.6953	0.1394	0.0009
12	0.9880	0.9576	0.9400	0.6183	0.1227	-0.1213	0.6688	0.0776	-0.0356
13	0.9879	0.9584	0.9372	0.6133	-0.0139	-0.1130	0.6609	-0.0474	-0.0468

10. **Conclusion and Outlook.** Estimates and statements analogous to those of the nominal multistep MPC closed-loop system can be obtained for the perturbed closed-loop setting. The approach comprises showing that a relaxed Lyapunov inequality holds for the multistep feedback along perturbed trajectories. Assuming a suboptimality performance index for the nominal setting, performance estimates for the perturbed closed-loop with nominal controller and with updated controller are derived. From the respective performance indices, practical stability properties of both schemes are proved. The enhanced robustness of Algorithm 3.3 induced by the shrinking horizon updates becomes visible in our estimates via the moduli of continuity ω_{V_N}, which are often considerably smaller than their counterparts ω_{J_N} appearing in the respective estimates for the non-updated Algorithm 3.2. For future work, this result will be applied to the sensitivity-based feedback MPC where the sensitivity-based updates can be viewed as a less costly approximation of the updates of the updated multistep feedback MPC.

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