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1 Local Stabilization of Nonlinear Systems Through the 2 Reduction Model Approach

3 Frederic Mazenc and Michael Malisoff

4 **Abstract**—We study a general class of nonlinear systems with input
5 delays of arbitrary size. We adapt the reduction model approach to prove
6 local asymptotic stability of the closed loop input delayed systems, using
7 feedbacks that may be nonlinear. Our Lyapunov-Krasovskii functionals
8 make it possible to determine estimates of the basins of attraction for the
9 closed loop systems.

10 **Index Terms**—Delay, nonlinear, reduction model, stabilization.

11 I. INTRODUCTION

12 The reduction model approach is a well-known stabilization tech-
13 nique for systems with input delays. It originated in [1] and has been
14 studied in many works, e.g., [2]–[6]. It is effective for stabilizing
15 linear time-invariant systems with arbitrarily long pointwise or dis-
16 tributed input delays. However, the approach does not directly apply
17 to nonlinear systems; it is extended by introducing an extra dynamic
18 (which gives the ‘state predictor’) whose initial condition is given by
19 an implicit equation (as is done in [7]–[9], and [6, Chapt. 6, p. 128]),
20 and only a few recent works adapt it to time varying systems [10]. This
21 is a limitation, because many systems are nonlinear and lead to the
22 stabilization of time varying nonlinear systems when a trajectory has
23 to be tracked. Moreover, the work [11] is limited to globally Lipschitz
24 nonlinear systems, and it has a restriction on the size of the delays. See
25 also [12] and [13] for stabilization of nonlinear systems with arbitrarily
26 long input delays when the systems have special structures, and [14]
27 for compensation of arbitrarily long input delays under input sampling
28 based on prediction.

29 These remarks motivate our work. We show that the reduction
30 model approach can be used to locally asymptotically stabilize a
31 large family of nonlinear time varying systems of the form $\dot{x}(t) =$
32 $A(t)x(t) + B(t)u(t - \tau) + F(t, x(t))$, with arbitrarily long constant
33 known input delays τ , where F is of order 2 in x at the origin.
34 Our key assumption is the stabilizability of a linear approximation of
35 the closed loop system at 0. Under this assumption, the result seems
36 intuitively obvious. However, to the best of the authors’ knowledge,
37 it has never been rigorously established. In particular, the stability
38 of the closed loop system we obtain cannot be proven by applying
39 the Hartman-Grobman theorem, which only applies to ordinary dif-
40 ferential equations; see [15, Chapt. 1]. One of the crucial benefits
41 offered by our result is that it yields asymptotically stable closed
42 loop systems for which one can determine a suitable subset of the
43 basin of attraction of the closed loop systems. This information is
44 valuable, because it gives a guarantee that some solutions converge to
45 the origin. We estimate the basin of attraction by building a Lyapunov-
46 Krasovskii functional. It is different from the one in [16], but can be
47 combined with it to establish ISS results. See also [17] for estimates

of the basins of attraction for time invariant nonlinear systems with 48
predictor feedbacks, under an ISS assumption on the closed loop 49
systems with undelayed controllers. The predictor feedbacks in [17] 50
can be implemented using numerical methods but are totally different 51
from ours, so our work can be viewed as complementary to [17]. 52
Our work is mainly a methodological development, rather than a 53
specific real-world application or experiments. However, input delays 54
naturally arise from measurement and transport phenomena, and our 55
assumptions are very general, so we anticipate that our work can 56
have considerable benefits when applied to mechanical systems where 57
latencies commonly occur. 58

The rest of this note is organized as follows. We give our definitions 59
in Section II. In Section III, we show how the class of systems we 60
study naturally arises in tracking problems. We state our main result in 61
Section IV, and we prove it in Section V. In Section VI, we discuss a 62
large class of examples where the estimates of the basins of attraction 63
become arbitrarily large when the input delays converge to zero. In 64
Section VII, we illustrate our result in a worked out example. We 65
conclude in Section VIII with a summary of our findings. 66

67 II. DEFINITIONS AND NOTATION

We let $n \in \mathbb{N}$ be arbitrary and I_n denote the identity matrix in 68
 $\mathbb{R}^{n \times n}$, and $|\cdot|$ be the usual Euclidean norm of matrices and vectors. 69
For square matrices M_1 and M_2 of the same size, we write $M_1 \geq M_2$ 70
to mean that $M_1 - M_2$ is nonnegative definite. For each integer $r \geq 1$, 71
let C^r denote the set of all functions whose partial derivatives up 72
through order r exist and are continuous, and C^0 denotes the set of all 73
continuous functions, when the domains and ranges are clear from the 74
context. When we want to emphasize the domains and ranges, we use 75
 $C^r(\mathcal{U}, \mathcal{V})$ to denote the set of all C^r functions having domain \mathcal{U} and 76
range \mathcal{V} , where \mathcal{U} and \mathcal{V} are suitable subsets of Euclidean spaces. For 77
any constant $\tau \geq 0$ and any continuous function $\varphi : [-\tau, \infty) \rightarrow \mathbb{R}^n$ 78
and all $t \geq 0$, we define the function φ_t by $\varphi_t(\theta) = \varphi(t + \theta)$ for all 79
 $\theta \in [-\tau, 0]$, i.e., the translation operator. Let \mathcal{K}_∞ be the set of all C^0 80
functions $\gamma : [0, \infty) \rightarrow [0, \infty)$ such that $\gamma(0) = 0$ and γ is strictly 81
increasing and unbounded. Given subsets S_1 and S_2 of Euclidean 82
spaces, we say that a function $J : S_1 \times S_2 \rightarrow \mathbb{R}^p$ is locally Lipschitz 83
with respect to its second argument provided for each compact set $E \subseteq$ 84
 S_2 , there is a constant L_E such that $|J(p, x) - J(p, y)| \leq L_E|x - y|$ 85
for all $p \in S_1$ and all $x \in E$ and $y \in E$. We say that J is strictly in- 86
creasing in its second argument provided the function $Y(x) = J(p, x)$ 87
is strictly increasing for each $p \in S_1$; we define strictly increasing and 88
nondecreasing in either argument in a similar way. We say that J has 89
order 2 in y at the origin provided there is a continuous function α such 90
that $|J(p, y)| \leq |y|^2\alpha(|y|)$ for all $(p, y) \in S_1 \times S_2$. We sometimes 91
omit arguments of functions when the arguments are clear from the 92
context. 93

94 III. MOTIVATION: TRACKING PROBLEM

In this section, we explain how the problem of tracking a trajectory 95
may lead to the problem we solve in the next section. Consider a time 96
varying nonlinear system 97

$$\dot{\xi}(t) = g(t, \xi(t)) + B(t)\mu(t - \tau) \quad (1)$$

where the state ξ is valued in \mathbb{R}^n , the control μ is valued in \mathbb{R}^p , 98
 $\tau \geq 0$ is a known constant delay, $g = (g_1, g_2, \dots, g_n)^\top$ is a nonlinear 99
function of class C^2 , and B is a continuous function. The dimensions 100
 n and p are arbitrary. We assume that (1) is forward complete for 101
all measurable locally essentially bounded choices for μ , so $\xi(t)$ is 102

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103 defined for all nonnegative times for all such μ 's. We also assume that
104 there is a nondecreasing function γ such that

$$\max \left\{ \left| \frac{\partial^2}{\partial \xi^2} g_i(t, \xi) \right| : |\xi| \leq q, t \geq 0, i \in \{1, 2, \dots, n\} \right\} \leq \gamma(q) \quad (2)$$

105 for all $q \geq 0$, which exists when g is C^2 and periodic in t .

106 The objective is to follow an admissible trajectory ξ_r of class C^1 ,
107 meaning the dynamics for $x = \xi - \xi_r$ should be asymptotically stable.
108 By admissible, we mean that there is a known continuous function
109 $\mu_r(t)$ such that $\dot{\xi}_r(t) = g(t, \xi_r(t)) + B(t)\mu_r(t)$ for all $t \geq 0$. In
110 particular, this means that $\xi_r(t)$ is defined for all $t \geq 0$. We assume
111 that ξ_r is a known bounded function.

112 Let $x(t) = \xi(t) - \xi_r(t)$ and $\mu(t - \tau) = u(t - \tau) + \mu_r(t)$. Then
113 the error equation is

$$\dot{x}(t) = G(t, x(t)) + B(t)u(t - \tau) \quad (3)$$

114 where $G(t, x) = g(t, x + \xi_r(t)) - g(t, \xi_r(t))$. Notice that $G(t, x) =$
115 $\int_0^1 (\partial g / \partial x)(t, lx + \xi_r(t)) x dl$, so $G(t, x) = (\partial g / \partial x)(t, \xi_r(t))x +$
116 $F(t, x)$, where

$$F(t, x) = \int_0^1 \left(\frac{\partial g}{\partial x}(t, lx + \xi_r(t)) - \frac{\partial g}{\partial x}(t, \xi_r(t)) \right) x dl \quad (4)$$

117 holds for all t and x .

118 Applying the Mean Value Theorem and using (2) and the bound-
119 edness of ξ_r , we can find a function $\alpha \in C^0$ such that $|F(t, x)| \leq$
120 $|x|^2 \alpha(|x|)$. Since ξ_r can depend on t , the system (3) is time varying,
121 even if g is time-invariant and B is constant. This motivates the study
122 of systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t - \tau) + F(t, x(t)) \quad (5)$$

123 where F is of order 2 in x at the origin, which will be our focus for the
124 rest of this note.

125 IV. STATEMENT OF MAIN RESULT

126 We state our main result for (5), where x is valued in \mathbb{R}^n , the
127 control u is valued in \mathbb{R}^p and is to be specified, $\tau \geq 0$ is a given
128 constant delay, and F is a nonlinear function. The dimensions n and
129 p are arbitrary. The functions A , B and F are continuous, and F is
130 locally Lipschitz with respect to x . The set of all initial conditions we
131 consider is $E_0 = \{(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)\}$, so the initial
132 times for our trajectories are always 0. Let $\lambda : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ be
133 the fundamental solution associated with A . Then $\lambda(t_0, t_0) = I_n$ and
134 $(\partial \lambda / \partial t)(t, t_0) = A(t)\lambda(t, t_0)$ hold for all real numbers t and t_0 . We
135 introduce the following assumptions:

136 *Assumption 1:*

137 (i) There is a continuous, positive valued, nondecreasing function
138 h such that

$$|\lambda(t, l)B(l)| \leq h(\tau) \text{ for all } t \in \mathbb{R} \text{ and } l \in [t, t + \tau]. \quad (6)$$

139 (ii) There is a constant $a^+ \geq 0$ such that $\sup_{t \in \mathbb{R}} |A(t)| \leq a^+$. \square

140 Assumption 1 always holds when B is bounded and A is constant,
141 so for instance, it holds for the one-dimensional system

$$\dot{x}(t) = x(t) + u(t - \tau) + lx^2(t) \sin(x(t)) \quad (7)$$

142 where $u \in \mathbb{R}$ is the input, τ is a positive constant delay, and l is
143 a positive constant. In the case of (7), we can take $A = 1$, $B = 1$,

$\lambda(t, t_0) = e^{t-t_0}$, and $F(t, x) = lx^2 \sin(x)$, so Assumption 1 holds
144 with $h(\tau) = 1$. To ease the readability of our technical assumptions,
145 we will explain how the example (7) satisfies our assumptions, after
146 we introduce each of our three assumptions. Our next assumption is:

Assumption 2: There are a continuous function $K : [0, \infty)^2 \rightarrow$
148 $\mathbb{R}^{p \times n}$, a nondecreasing continuous function $k : [0, \infty) \rightarrow (0, \infty)$, an
149 everywhere positive definite and symmetric function $Q : [0, \infty)^2 \rightarrow$
150 $\mathbb{R}^{n \times n}$ of class C^1 with respect to its first argument, and continuous
151 functions $q_i : [0, \infty) \rightarrow (0, \infty)$ for $i = 1, 2, 3$ such that $|K(t, \tau)| \leq$
152 $k(\tau)$ for all $(t, \tau) \in [0, \infty)^2$, and such that with the choices $H(t, \tau) =$
153 $A(t) + \lambda(t, t + \tau)B(t + \tau)K(t, \tau)$ and $R(t, \tau, s) = s^\top Q(t, \tau)s$, the
154 following two conditions are satisfied for all $\tau \geq 0$: (i) Along
155 all trajectories of $\dot{s}(t) = H(t, \tau)s(t)$, we have $\dot{R}(t, \tau, s(t)) \leq$
156 $-q_1(\tau)R(t, \tau, s(t))$ and (ii) the bounds
157

$$q_2(\tau)I_n \leq Q(t, \tau) \text{ and } |Q(t, \tau)| \leq q_3(\tau) \quad (8)$$

are satisfied for all $t \geq 0$. \square 158

Assumption 2 holds for (7) as well. In fact, by choosing $K(t, \tau) =$
159 $-2e^\tau$, we obtain $H(t, \tau) = 1 - e^{-\tau}2e^\tau = -1$, so Assumption 2 is
160 satisfied with $Q(t, \tau) = 1/2$, $q_1(\tau) = 2$, $q_2(\tau) = q_3(\tau) = 1/2$, and
161 $k(\tau) = 2e^\tau$. Finally, we assume:

Assumption 3: There are two continuous functions f_1 and f_2 that
163 are locally Lipschitz with respect to their last argument, and continu-
164 ous functions α_1 and α_2 , such that
165

$$F(t, x) = \lambda(t, t + \tau)B(t + \tau)f_1(t, \tau, x) + f_2(t, x) \text{ and} \quad (9)$$

$$|f_1(t, \tau, x)| \leq |x|^2 \alpha_1(\tau, |x|^2) \text{ and}$$

$$|f_2(t, x)| \leq |x|^2 \alpha_2(|x|^2) \quad (10)$$

for all $t \in \mathbb{R}$, $\tau \geq 0$, and $x \in \mathbb{R}^n$. Also, $\beta_3(\tau, m) = m\alpha_1(\tau, m^2)$
166 is strictly increasing and unbounded in m , and $\beta_4(m) = m\alpha_2(m^2)$
167 is nondecreasing in m . Finally, there are continuous functions $\theta_1 :$
168 $[0, \infty)^2 \rightarrow (0, \infty)$ and $\theta_2 : [0, \infty) \rightarrow (0, \infty)$ such that
169

$$|\alpha_1(\tau, b + c) - \alpha_1(\tau, c)| \leq b\theta_1(\tau, b + c) \quad (11)$$

$$|\alpha_2(b + c) - \alpha_2(c)| \leq b\theta_2(b + c) \quad (12)$$

are satisfied for all $b \geq 0$ and $c \geq 0$. \square 170

To see why (7) satisfies Assumption 3, note that for (7), the fact
171 that $\lambda(t, t + \tau)B$ is invertible implies that one can choose $f_2 = 0$ and
172 $f_1(t, \tau, x) = le^\tau x^2 \sin(x)$. Then we can satisfy Assumption 3 for (7)
173 by taking $\alpha_1(\tau, m) = le^\tau$ and $\alpha_2(m) = 0$ for all m and τ .
174

Returning to the general system (5), it follows from Assumptions
175 2–3 that for any constant $\tau > 0$ and:
176

$$\alpha_3(\tau, m) = \frac{q_3(\tau)}{\sqrt{q_2(\tau)}} \alpha_2(m) + 2a\alpha_1(\tau, m), \quad (13)$$

where a is any constant such that
177

$$0 < a \leq \frac{q_1(\tau)\sqrt{q_2(\tau)}}{8k(\tau)} \quad (14)$$

there are unique positive values $v_1(\tau)$ and $v_2(\tau)$ (which also depend
178 on a) such that
179

$$v_1(\tau)\alpha_3\left(\tau, \frac{4}{q_2(\tau)}v_1^2(\tau)\right) = \frac{q_1(\tau)q_2(\tau)}{16} \text{ and} \quad (15)$$

$$v_2(\tau)\alpha_3\left(\tau, \frac{4h^2(\tau)}{a^2}v_2^2(\tau)\right) = \frac{a^2}{4\tau h^2(\tau)}. \quad (16)$$

180 The existence of unique values $v_1(\tau)$ and $v_2(\tau)$ follows because
 181 $\beta_3(\tau, m)$ is strictly increasing and unbounded in m and $\beta_4(m)$ is
 182 nondecreasing, so $m\alpha_3(\tau, m^2)$ is strictly increasing and unbounded
 183 in m . The choice of α_3 in (13) will become clear when we prove:

184 *Theorem 1:* Let $\tau > 0$ be any constant and Assumptions 1–3
 185 hold. Let a be any constant satisfying (14), and set $v(\tau) =$
 186 $\min\{v_1(\tau), v_2(\tau)\}$ where v_1 and v_2 are as above. Then, for each
 187 initial function $(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)$ satisfying

$$\sqrt{q_3(\tau)} \left| \phi_x(0) + \int_{-\tau}^0 \lambda(0, m + \tau) B(m + \tau) \phi_u(m) dm \right| + \frac{a}{\tau} \int_{-\tau}^0 (m + 2\tau) |\phi_u(m)| dm < v(\tau) \quad (17)$$

188 the unique solution of (5), in closed loop with

$$u(t) = -f_1(t, \tau, x(t)) + K(t, \tau) \left[x(t) + \int_{t-\tau}^t \lambda(t, m + \tau) B(m + \tau) u(m) dm \right] \quad (18)$$

189 converges to 0 as $t \rightarrow \infty$. Moreover, (18) locally asymptotically
 190 stabilizes (5) to 0. \square

191 *Remark 1:* We comment that our control (18) agrees with the
 192 standard predictor controller in the linear time invariant case where
 193 $f_1 = f_2 = 0$ and A and B are constant. The extra term $-f_1(t, \tau, x(t))$
 194 is used to compensate part of the nonlinearity of the system (5).
 195 Assumption 2 is a generalization of the standard assumption that
 196 (A, B) is a stabilizable pair, which is the special case of Assumption
 197 2 where $\tau = 0$, A and B are constant, and where K and Q can also be
 198 taken to be constant. However, we allow the delay $\tau > 0$ to be as large
 199 as we want. On the other hand, since the q_i 's are continuous positive
 200 valued functions of the delay, they have positive upper and lower
 201 bounds over all $\tau \in [0, \tau_M]$ for any constant τ_M . Also, the function
 202 k from Assumption 2 is nondecreasing in τ . Hence, if we are only
 203 concerned with a bounded set $[0, \tau_M]$ of possible values for τ , then we
 204 can assume in Assumption 2 that the q_i 's and k are all positive con-
 205 stants, by replacing them by the constants $\min\{q_1(\tau) : 0 \leq \tau \leq \tau_M\}$,
 206 $\min\{q_2(\tau) : 0 \leq \tau \leq \tau_M\}$, $\max\{q_3(\tau) : 0 \leq \tau \leq \tau_M\}$, and $k(\tau_M)$
 207 without relabeling. These observations will be key to our proof in
 208 Section VI that for important special cases, our estimate of the domain
 209 of attraction becomes arbitrarily large when $\tau \rightarrow 0^+$. \square

210 *Remark 2:* Assumptions 1–2 always hold when A and B are con-
 211 stant provided (A, B) is stabilizable. Indeed, in that case $\lambda(t, t_0) =$
 212 $e^{(t-t_0)A}$, so the stabilizability of (A, B) is equivalent to the stabiliz-
 213 ability of $(A, \lambda(t, t + \tau)B)$. Also, when the α_i 's are C^1 , the existence
 214 of functions θ_i satisfying the requirements from Assumption 3 follows
 215 from the Mean Value Theorem, since Assumption 2 only requires
 216 (11), (12) for nonnegative b 's and c 's. Since F is of order 2 in x
 217 at 0, we can always satisfy Assumption 3 with $f_1 = 0$ and $f_2 = F$.
 218 However, these choices may lead to a conservative estimate of the size
 219 of the basin of attraction; see the example in Section VII. Our use
 220 of a feedback control with distributed terms is motivated by the facts
 221 that τ is arbitrary and $\xi(t) = A(t)\xi(t)$ may be exponentially unstable.
 222 In general, the explicit expression for λ is unknown, but it can be
 223 computed in many important cases. This is the case in particular if
 224 A is constant or $n = 1$. We illustrate Theorem 1 in Section VII. \square

225 *Remark 3:* In conjunction with our local asymptotic stability result,
 226 we have boundedness of the control from Theorem 1, along all of the
 227 closed loop trajectories. \square

V. PROOF OF THEOREM 1

228

Throughout the proof, we consider any solution of (5) in closed loop 229
 with (18) for any initial condition satisfying the requirements (17) of 230
 Theorem 1, and any constant delay $\tau \geq 0$. 231

First Part: New Representation of the System: Let t_e be any positive 232
 real number or ∞ such that the solution is defined over $[-\tau, t_e)$. Such 233
 a $t_e > 0$ exists, because the dynamics (5) grows linearly in x in any 234
 bounded open neighborhood of $x(0)$. Later we show that t_e can always 235
 be taken to be ∞ for all of the trajectories we are considering. We 236
 introduce the operators 237

$$z(t) = x(t) + \Gamma(t, u_t), \text{ where} \\ \Gamma(t, u_t) = \int_{t-\tau}^t \lambda(t, m + \tau) B(m + \tau) u(m) dm. \quad (19)$$

In all of what follows, we assume that $t \in [0, t_e)$ is arbitrary, unless 238
 otherwise noted, and we omit some of the arguments of the time 239
 derivatives when they are clear, so $\dot{\Gamma}(t)$ means $(d/dt)\Gamma(t, u_t)$. Then 240
 the properties of the fundamental matrix give $\dot{\Gamma}(t) = A(t)\Gamma(t, u_t) +$ 241
 $\lambda(t, t + \tau)B(t + \tau)u(t) - B(t)u(t - \tau)$. Using the formula (5) and 242
 our decomposition (9) for $F(t, x)$, we obtain 243

$$\dot{z}(t) = A(t)z(t) + \lambda(t, t + \tau)B(t + \tau)[u(t) + f_1(t, \tau, x(t))] + f_2(t, x(t)). \quad (20)$$

Also, our feedback (18) satisfies $u(t) = -f_1(t, \tau, x(t)) +$ 244
 $K(t, \tau)z(t)$. Consequently, in terms of our function H from 245
 Assumption 2, (20) becomes 246

$$\dot{z}(t) = H(t, \tau)z(t) + f_2(t, x(t)). \quad (21)$$

Assumption 2 ensures global asymptotic stability of the linearizations 247
 $\dot{z}(t) = H(t, \tau)z(t)$ of (21) at 0. Moreover, the equality 248

$$x(t) = z(t) - \int_{t-\tau}^t \lambda(t, m + \tau) B(m + \tau) u(m) dm \quad (22)$$

is satisfied. 249

Second Part: Decay Conditions: We study the stability of the closed 250
 loop system using its representation as (21) coupled with (22). We 251
 introduce the operator 252

$$\Xi(u_t) = \frac{1}{\tau} \int_{t-\tau}^t (m - t + 2\tau) |u(m)| dm. \quad (23)$$

Observe for later use that 253

$$\int_{t-\tau}^t |u(m)| dm \leq \Xi(u_t) \leq 2 \int_{t-\tau}^t |u(m)| dm. \quad (24)$$

Then, for all $t \geq 0$, we have 254

$$\dot{\Xi}(t) \leq 2 |u(t)| - \frac{1}{\tau} \int_{t-\tau}^t |u(m)| dm. \quad (25)$$

Also, we can use the upper bound on f_1 from (10), the bound for 255
 K given in Assumption 2 and the formula $u(t) = -f_1(t, \tau, x(t)) +$ 256
 $K(t, \tau)z(t)$ to get $|u(t)| \leq k(\tau)|z(t)| + |x(t)|^2 \alpha_1(\tau, |x(t)|^2)$. More- 257
 over, (8) implies that for all $t \geq 0$ and all $z \in \mathbb{R}^n$, we have $q_2(\tau)|z|^2 \leq$ 258

259 $z^\top Q(t, \tau)z$. Taking square roots of both sides of the preceding in-
260 equality and the dividing by $\sqrt{q_2(\tau)} > 0$ gives

$$|z| \leq \frac{1}{\sqrt{q_2(\tau)}} \sqrt{R(t, \tau, z)}. \quad (26)$$

261 Combining the last two estimates with (25) gives

$$\begin{aligned} \dot{\Xi}(t) \leq & -\frac{1}{\tau} \int_{t-\tau}^t |u(m)| dm + \frac{2k(\tau)}{\sqrt{q_2(\tau)}} \sqrt{R(t, \tau, z(t))} \\ & + 2|x(t)|^2 \alpha_1(\tau, |x(t)|^2). \end{aligned} \quad (27)$$

262 We deduce from Assumptions 2–3 that the time derivative of R
263 along all trajectories of (21) satisfies

$$\begin{aligned} \dot{R}(t) \leq & -q_1(\tau)R(t, \tau, z(t)) + 2z(t)^\top Q(t, \tau)f_2(t, x(t)) \\ \leq & -q_1(\tau)R(t, \tau, z(t)) + 2|z(t)|q_3(\tau)|f_2(t, x(t))|. \end{aligned} \quad (28)$$

264 From (26), we deduce that

$$\begin{aligned} \dot{R}(t) \leq & -q_1(\tau)R(t, \tau, z(t)) \\ & + 2q_3(\tau) \frac{\sqrt{R(t, \tau, z(t))}}{\sqrt{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2). \end{aligned} \quad (29)$$

265 Consider the family of functions $S_\varepsilon(t, \tau, z) = \sqrt{R(t, \tau, z)} + \varepsilon -$
266 $\sqrt{\varepsilon}$ parameterized by the constant $\varepsilon \in [0, 1]$ and let $S = S_0$. Since R
267 is of class C^1 with respect to t and z , it follows that for all $\varepsilon \in (0, 1)$,
268 the function S_ε is of class C^1 with respect to t and z , while S is only
269 continuous. Also, (29) and Lemma 1 in the Appendix (applied with the
270 choice $r = R(t, \tau, z)$) give

$$\begin{aligned} \dot{S}_\varepsilon(t) \leq & -q_1(\tau) \frac{R(t, \tau, z(t))}{2\sqrt{R(t, \tau, z(t))} + \varepsilon} \\ & + q_3(\tau) \frac{\sqrt{R(t, \tau, z(t))} |x(t)|^2 \alpha_2(|x(t)|^2)}{\sqrt{R(t, \tau, z(t))} + \varepsilon\sqrt{q_2(\tau)}} \\ \leq & -\frac{q_1(\tau)}{2} S(t, \tau, z(t)) \\ & + \frac{q_3(\tau)}{\sqrt{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2) \\ & + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))] \end{aligned} \quad (30)$$

271 along all trajectories of (21).

272 *Third Part: Lyapunov-Krasovskii Functionals:* Let us consider the
273 family of functions

$$V_\varepsilon(t, z, u_t) = a\Xi(u_t) + S_\varepsilon(t, \tau, z) \quad (31)$$

274 where the constant a satisfies (14) and we omit the argument τ in V_ε
275 to simplify the notation. Then, (27) and (30) give

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & \left(\frac{2ak(\tau)}{\sqrt{q_2(\tau)}} - \frac{q_1(\tau)}{2} \right) S(t, \tau, z(t)) \\ & + \frac{q_3(\tau)}{\sqrt{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2) \end{aligned}$$

$$\begin{aligned} & -\frac{a}{\tau} \int_{t-\tau}^t |u(m)| dm + 2a|x(t)|^2 \alpha_1(\tau, |x(t)|^2) \\ & + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))]. \end{aligned} \quad (32)$$

Since a satisfies (14), we get

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & -\frac{q_1(\tau)}{4} S(t, \tau, z(t)) + |x(t)|^2 \alpha_3(\tau, |x(t)|^2) \\ & -\frac{a}{\tau} \int_{t-\tau}^t |u(m)| dm + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))] \end{aligned} \quad (33)$$

where α_3 was defined in (13).

Next, we find a suitable upper bound on the term
 $|x(t)|^2 \alpha_3(\tau, |x(t)|^2)$ from (33). Our formula (22) for $x(t)$,
Assumption 1, and our bound (26) on $|z|$ give

$$\begin{aligned} |x(t)| \leq & |z(t)| + h(\tau) \int_{t-\tau}^t |u(m)| dm \\ \leq & \frac{1}{\sqrt{q_2(\tau)}} S(t, \tau, z(t)) + h(\tau) \int_{t-\tau}^t |u(m)| dm. \end{aligned} \quad (34)$$

Recall that our monotonicity properties of β_3 and β_4 from Assumption
3 imply that $m\alpha_3(\tau, m^2)$ is strictly increasing as a function of m
for each τ . Therefore, by separately considering the cases where
 $S(t, \tau, z(t))/\sqrt{q_2(\tau)} \leq h(\tau) \int_{t-\tau}^t |u(m)| dm$ and where the reverse
inequality holds, we get

$$\begin{aligned} |x(t)|^2 \alpha_3(\tau, |x(t)|^2) & \leq \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \alpha_3\left(\tau, \frac{4}{q_2(\tau)} S^2(t, \tau, z(t))\right) \\ & + 4h^2(\tau) \left[\int_{t-\tau}^t |u(m)| dm \right]^2 \alpha_3\left(\tau, 4h^2(\tau) \left[\int_{t-\tau}^t |u(m)| dm \right]^2\right). \end{aligned} \quad (35)$$

We can combine this inequality with (33) to get

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & \left[-\frac{q_1(\tau)}{4} + \frac{4}{q_2(\tau)} S(t, \tau, z(t)) \alpha_3 \right. \\ & \times \left. \left(\tau, \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \right) \right] S(t, \tau, z(t)) \\ & + \left[-\frac{a}{\tau} + 4h^2(\tau) \int_{t-\tau}^t |u(m)| dm \alpha_3 \right. \\ & \times \left. \left(\tau, 4h^2(\tau) \left[\int_{t-\tau}^t |u(m)| dm \right]^2 \right) \right] \\ & \times \int_{t-\tau}^t |u(m)| dm + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))]. \end{aligned}$$

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287 Also, $V_\varepsilon(t, z(t), u_t) \geq \sqrt{R(t, \tau, z(t)) + \varepsilon} - \sqrt{\varepsilon} \geq S(t, \tau, z(t)) -$
 288 $\sqrt{\varepsilon}$ and $V_\varepsilon(t, z(t), u_t) \geq a \int_{t-\tau}^t |u(m)| dm$ hold for all $\varepsilon \in [0, 1]$, by
 289 (24). Since $m\alpha_3(\tau, m^2)$ is increasing in m for each τ , it follows that:

$$\begin{aligned} S(t, \tau, z(t)) \alpha_3 \left(\tau, \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \right) \\ \leq [V_\varepsilon(t, z(t), u_t) + \sqrt{\varepsilon}] \alpha_3 \\ \times \left(\tau, \frac{4}{q_2(\tau)} [2V_\varepsilon(t, z(t), u_t) \sqrt{\varepsilon} + \varepsilon] + \frac{4}{q_2(\tau)} V_\varepsilon^2(t, z(t), u_t) \right). \end{aligned}$$

290 We now apply (11), (12), with $b = (4/q_2(\tau))(2V_\varepsilon(t, z(t), u_t)\sqrt{\varepsilon} +$
 291 $\varepsilon)$ and $c = 4V_\varepsilon^2(t, z(t), u_t)/q_2(\tau)$, and use the fact that $\varepsilon \leq \sqrt{\varepsilon} \leq$
 292 $\varepsilon^{1/4}$ for all $\varepsilon \in [0, 1]$, to find a continuous positive valued and non-
 293 decreasing function φ_c (also depending on τ , but independent of ε)
 294 such that

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & \left[-\frac{q_1(\tau)}{4} + \frac{4}{q_2(\tau)} V_\varepsilon(t, z(t), u_t) \right. \\ & \times \alpha_3 \left(\tau, \frac{4}{q_2(\tau)} V_\varepsilon^2(t, z(t), u_t) \right) \left. \right] S(t, \tau, z(t)) \\ & + \frac{1}{a} \left[-\frac{a^2}{\tau} + 4h^2(\tau) V_\varepsilon(t, z(t), u_t) \right. \\ & \times \alpha_3 \left(\tau, 4h^2(\tau) \frac{V_\varepsilon^2(t, z(t), u_t)}{a^2} \right) \left. \right] \int_{t-\tau}^t |u(m)| dm \\ & + \varepsilon^{\frac{1}{4}} \varphi_c(V_\varepsilon(t, z(t), u_t)). \end{aligned} \quad (36)$$

295 Next, recall that our assumption (17) implies that $\sqrt{q_3(\tau)}|z(0)| +$
 296 $(a/\tau) \int_{-\tau}^0 (m+2\tau)|u(m)| dm < v(\tau)$, where $v(\tau) = \min\{v_1(\tau),$
 297 $v_2(\tau)\}$ as before. Then (8) from Assumption 2 gives $V_0(0, z(0), u_0) <$
 298 $v(\tau)$. Since $\sqrt{c_1 + c_2} \leq \sqrt{c_1} + \sqrt{c_2}$ holds for all nonnegative con-
 299 stants c_1 and c_2 , we know that $V_\varepsilon \leq V_0$ holds pointwise for all
 300 $\varepsilon \in (0, 1]$. It follows that $V_\varepsilon(0, z(0), u_0) \leq V_0(0, z(0), u_0) < \bar{v}$ hold
 301 for all $\varepsilon \in (0, 1]$, where $\bar{v} = [V_0(0, z(0), u_0) + v(\tau)]/2 > 0$. Then
 302 $\bar{v} < v(\tau)$.

303 Set $\bar{v}_a = (v(\tau) + \bar{v})/2$. Since $m\alpha_3(\tau, m^2)$ is strictly increasing in
 304 m , and since $\bar{v}_a < v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$, it follows from our
 305 conditions (15), (16) on $v_1(\tau)$ and $v_2(\tau)$ that the constants:

$$\begin{aligned} \bar{p}_1 &= \frac{q_1(\tau)}{4} - \frac{4}{q_2(\tau)} \bar{v}_a \alpha_3 \left(\tau, \frac{4}{q_2(\tau)} \bar{v}_a^2 \right) \text{ and} \\ \bar{p}_2 &= \frac{a^2}{\tau} - 4h^2(\tau) \bar{v}_a \alpha_3 \left(\tau, 4h^2(\tau) \frac{\bar{v}_a^2}{a^2} \right) \end{aligned} \quad (37)$$

306 are positive for all $\tau > 0$. Fix any value of $\varepsilon \in (0, 1]$ satisfying

$$\varepsilon \in \left(0, \left(\frac{\min\{\bar{p}_1, \bar{p}_2\} \bar{v}}{4\varphi_c(\bar{v}_a) \max\{a^2, 1\}} \right)^4 \right) \quad (38)$$

307 where the left endpoint is omitted because we need $\varepsilon > 0$.
 308 Next, we prove by contradiction that $V_\varepsilon(t, z(t), u_t) \leq \bar{v}$ for all
 309 $t \geq 0$. Assume that this property does not hold. Then, since
 310 $\bar{v}_a > \bar{v}$ and $V_\varepsilon(0, z(0), u_0) < \bar{v}$, we can find a $t_2 > 0$ such that
 311 $V_\varepsilon(t, z(t), u_t) \leq \bar{v}_a$ for all $t \in [0, t_2]$ and $V_\varepsilon(t_2, z(t_2), u_{t_2}) > \bar{v}$. Set
 312 $t_1 = \inf\{t \leq t_2 : V_\varepsilon(p, z(p), u_p) \geq \bar{v} \text{ for all } p \in [t, t_2]\}$. Then, since
 313 $t \mapsto V_\varepsilon(t, z(t), u_t)$ is continuous, we get $V_\varepsilon(t, z(t), u_t) \in [\bar{v}, \bar{v}_a]$ for
 314 all $t \in [t_1, t_2]$, $V_\varepsilon(t_1, z(t_1), u_{t_1}) = \bar{v}$, and $\dot{V}_\varepsilon(t_1) \geq 0$.

By (36) and the fact that $l\alpha_3(\tau, l^2)$ is strictly increasing in l 315

$$\dot{V}_\varepsilon(t) \leq -\bar{p}_1 S(t, \tau, z(t)) - \frac{1}{a} \bar{p}_2 \int_{t-\tau}^t |u(m)| dm + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \quad (39)$$

for all $t \in [t_1, t_2]$. It follows from our lower bound on Ξ from (24) 316
 that: 317

$$\begin{aligned} \dot{V}_\varepsilon(t) &\leq -\frac{1}{2 \max\{a^2, 1\}} \min\{\bar{p}_1, \bar{p}_2\} V_0(t, z(t), u_t) + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \\ &\leq -\frac{1}{2 \max\{a^2, 1\}} \min\{\bar{p}_1, \bar{p}_2\} V_\varepsilon(t, z(t), u_t) + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \end{aligned} \quad (40)$$

for all $t \in [t_1, t_2]$. Since $V_\varepsilon(t, z(t), u_t) \in [\bar{v}, \bar{v}_a]$ for all $t \in [t_1, t_2]$, we 318
 deduce that 319

$$\begin{aligned} \dot{V}_\varepsilon(t) &\leq -\frac{1}{2 \max\{a^2, 1\}} \min\{\bar{p}_1, \bar{p}_2\} \bar{v} + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \\ &\leq -\frac{\min\{\bar{p}_1, \bar{p}_2\}}{4 \max\{a^2, 1\}} \bar{v} < 0 \end{aligned} \quad (41)$$

for all $t \in [t_1, t_2]$ when ε satisfies (38). It follows that $\dot{V}_\varepsilon(t_1) < 0$. 320
 This yields a contradiction with the choice of t_1 . Hence, when (38) 321
 holds, we get $V_\varepsilon(t, z(t), u_t) \leq \bar{v}$ for all $t \geq 0$, which implies that we 322
 can choose $t_e = \infty$. Also, arguing as we did before, we get 323

$$\dot{V}_\varepsilon(t) \leq -\frac{\min\{\bar{p}_1, \bar{p}_2\}}{2 \max\{a^2, 1\}} V_\varepsilon(t, z(t), u_t) + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}). \quad (42)$$

for all $t \geq 0$. This gives a value $t_c > 0$ such that for all $t \geq t_c$, we have 324

$$V_\varepsilon(t, z(t), u_t) \leq \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{\min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \quad (43)$$

(since V_ε is nonnegative valued), and therefore also 325

$$\begin{aligned} \Xi(u_t) &\leq \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{a \min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \text{ and} \\ S_\varepsilon(t, \tau, z) &\leq \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{\min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \end{aligned} \quad (44)$$

Since $S_\varepsilon(t, \tau, z) = \sqrt{R(t, \tau, z) + \varepsilon} - \sqrt{\varepsilon} \geq \sqrt{q_2(\tau)}|z| - \sqrt{\varepsilon}$ holds 326
 for all t, τ , and z , (24) gives 327

$$\begin{aligned} \max \left\{ \int_{t-\tau}^t |u(m)| dm, \sqrt{q_2(\tau)}|z| \right\} \\ \leq \sqrt{\varepsilon} + \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{\min\{a, 1\} \min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \end{aligned} \quad (45)$$

for all $t \geq t_c$. Set 328

$$\Delta = \max \left\{ \frac{1}{\sqrt{q_2(\tau)}}, 1 \right\} \left(1 + \frac{4\varphi_c(\bar{v})}{\min\{a, 1\} \min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \right). \quad (46)$$

Then, since $\varepsilon \in [0, 1]$, it follows that for all $t \geq t_c$, the inequalities 329

$$|z(t)| \leq \Delta \varepsilon^{\frac{1}{4}} \text{ and } \int_{t-\tau}^t |u(m)| dm \leq \Delta \varepsilon^{\frac{1}{4}} \quad (47)$$

are satisfied. Since ε is arbitrarily small, we deduce that $|z(t)|$ and 330
 $\int_{t-\tau}^t |u(m)| dm$ converge to zero when $t \rightarrow \infty$. This and the first 331
 inequality in (34) imply that $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Also, by letting 332
 ε depend on the maximum of V_0 on a suitable neighborhood of the 333
 origin, we can prove the local stability part. This proves the theorem. 334

335 VI. ARBITRARILY LARGE DOMAINS OF ATTRACTION

336 Theorem 1 applies for all $\tau > 0$. On the other hand, consider the
 337 special case where $f_2 = 0$ in the decomposition (9) of F . Then,
 338 setting $\tau = 0$ in (9) and in our control (18) produces the uniformly
 339 globally asymptotically stable closed loop system $\dot{x}(t) = [A(t) +$
 340 $B(t)K(t, 0)]x(t)$ from Assumption 2. This suggests that the domain
 341 of attraction should become arbitrarily large as $\tau \rightarrow 0^+$ when $f_2 = 0$.
 342 Our next theorem implies that this is indeed the case. We will assume
 343 that the functions q_i and k from Assumption 2 are constant, so we
 344 omit their arguments τ . This is not restrictive, since now we only need
 345 to consider τ 's on a bounded interval; see Remark 1.

346 *Corollary 1:* Let Assumptions 1–3 hold with $f_2 = 0$ and the q_i 's
 347 and k all constant. Then for each constant $v_* > 0$, we can find values
 348 $a \in (0, q_1 \sqrt{q_2} / (8k))$ and $\tau_M > 0$ (both depending on v_*) such that:
 349 For each initial condition $(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)$ satisfying

$$\begin{aligned} & \left| \sqrt{q_3(\tau)} \left| \phi_x(0) + \int_{-\tau}^0 \lambda(0, m + \tau) B(m + \tau) \phi_u(m) dm \right| \right. \\ & \left. + \frac{a}{\tau} \int_{-\tau}^0 (m + 2\tau) |\phi_u(m)| dm < v_* \right. \end{aligned} \quad (48)$$

350 and each constant delay $\tau \in (0, \tau_M)$, the trajectory of (5) in closed
 351 loop with (18) converges to 0 as $t \rightarrow \infty$. \square

352 *Proof:* We set $\alpha_2 = 0$, so we have $\alpha_3 = 2a\alpha_1$. Then (15), (16)
 353 become

$$\begin{aligned} v_1(\tau)\alpha_1 \left(\tau, \frac{4}{q_2} v_1^2(\tau) \right) &= \frac{q_1 q_2}{32a} \text{ and} \\ v_2(\tau)\alpha_1 \left(\tau, \frac{4h^2(\tau)}{a^2} v_2^2(\tau) \right) &= \frac{a}{8\tau h^2(\tau)}. \end{aligned} \quad (49)$$

354 For each constant $\tau_M > 0$, Assumption 3 provides a function $\bar{\gamma}$ of
 355 class \mathcal{K}_∞ such that $m\alpha_1(\tau, m^2) \leq \bar{\gamma}(m)$ for all $\tau \in [0, \tau_M]$ and
 356 $m \geq 0$. Then, replacing $\alpha_1(\tau, m^2)$ in (49) by $\bar{\gamma}(m)/m$ gives

$$\begin{aligned} \bar{\gamma} \left(\sqrt{\frac{4}{q_2}} v_1(\tau) \right) &= \frac{q_1 \sqrt{q_2}}{16a} \text{ and} \\ \bar{\gamma} \left(\frac{2h(\tau)}{a} v_2(\tau) \right) &= \frac{1}{4\tau h(\tau)} \end{aligned} \quad (50)$$

357 for all $\tau \in (0, \tau_M)$. Our proof of Theorem 1 shows that the conclusions
 358 of that theorem remain true when $v_1(\tau)$ and $v_2(\tau)$ are defined to be the
 359 solutions of (50). Therefore

$$\begin{aligned} v_1(\tau) &= \frac{\sqrt{q_2}}{2} \bar{\gamma}^{-1} \left(\frac{q_1 \sqrt{q_2}}{16a} \right) \text{ and} \\ v_2(\tau) &= \frac{a}{2h(\tau)} \bar{\gamma}^{-1} \left(\frac{1}{4\tau h(\tau)} \right). \end{aligned} \quad (51)$$

360 Also, when τ is sufficiently small, the choice

$$a = \frac{1}{\sqrt{\bar{\gamma}^{-1} \left(\frac{1}{4\tau h(\tau)} \right)}} \quad (52)$$

will satisfy our requirements (14) on a , because (52) converges to 0
 as $\tau \rightarrow 0^+$ and because we are now assuming that the q_i 's and k are
 positive constants. Then (51) become

$$\begin{aligned} v_1(\tau) &= \frac{\sqrt{q_2}}{2} \bar{\gamma}^{-1} \left(\frac{q_1 \sqrt{q_2}}{16} \sqrt{\bar{\gamma}^{-1} \left(\frac{1}{4\tau h(\tau)} \right)} \right) \text{ and} \\ v_2(\tau) &= \frac{1}{2h(\tau)} \sqrt{\bar{\gamma}^{-1} \left(\frac{1}{4\tau h(\tau)} \right)}. \end{aligned} \quad (53)$$

Therefore, both $v_1(\tau)$ and $v_2(\tau)$ converge to ∞ when $\tau \rightarrow 0^+$. It
 follows that $v(\tau) \rightarrow \infty$ as $\tau \rightarrow 0^+$, so we can satisfy (48) for small
 enough $\tau > 0$ by choosing τ such that $v(\tau) > v_*$. The corollary now
 follows from Theorem 1. \blacksquare

368 VII. ILLUSTRATIVE EXAMPLE

We illustrate Theorem 1 using the 1 dimensional system from (7),
 which is

$$\dot{x}(t) = x(t) + u(t - \tau) + lx^2(t) \sin(x(t)) \quad (54)$$

where $u \in \mathbb{R}$ is the input, τ is a positive constant delay, and l is
 a positive constant. This system is not globally Lipschitz in the
 state x . With the notation of the previous sections, we have $A = 1$,
 $B = 1$, $\lambda(t, t_0) = e^{t-t_0}$, and $F(t, x) = lx^2 \sin(x)$. As we saw in
 Section IV, (54) satisfies our assumptions with $h(\tau) = 1$, $K(t, \tau) =$
 $-2e^\tau$, $Q(t, \tau) = 1/2$, $q_1(\tau) = 2$, $q_2(\tau) = q_3(\tau) = 1/2$, $k(\tau) = 2e^\tau$,
 $f_2 = 0$, $f_1(t, \tau, x) = le^\tau x^2 \sin(x)$, $\alpha_1(\tau, m) = le^\tau$ and $\alpha_2(m) = 0$.
 According to (14), the inequalities $0 < a \leq 1/(8\sqrt{2}e^\tau)$ have to be
 satisfied and, by the expression of α_3 in (13), $\alpha_3(\tau, m) = 2ale^\tau$.

Choosing

$$a = \frac{1}{8\sqrt{2}e^\tau} \quad (55)$$

we can straightforwardly derive an estimate of the basin of attraction
 from Theorem 1 by using $v = \min\{v_1, v_2\}$, where

$$v_1(\tau) = \frac{1}{2\sqrt{2}l} \quad (56)$$

and

$$v_2(\tau) = \frac{1}{64\sqrt{2}\tau e^{2\tau} l} \quad (57)$$

which converge to ∞ as $l \rightarrow 0$ for each $\tau > 0$. On the other hand,
 when $\tau \in (0, 1]$, we can take

$$a = \frac{\sqrt{\tau}}{8\sqrt{2}e^\tau} \quad (58)$$

to obtain the values

$$v_1(\tau) = \frac{1}{2l\sqrt{2\tau}} \quad (59)$$

and

$$v_2(\tau) = \frac{1}{64le^{2\tau}\sqrt{2\tau}} \quad (60)$$

so $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$ converges to ∞ as l converges to zero
 for fixed $\tau > 0$, or as τ converges to zero for fixed l , so the basin
 of attraction becomes arbitrarily large. This gives convergence of the
 closed loop solution to 0.

392 If, on the other hand, we had chosen, $f_1 = 0$ and $f_2(t, x) =$
 393 $lx^2 \sin(x)$, then one could choose $\alpha_1 = c_0$ for any constant $c_0 > 0$
 394 and $\alpha_2(m) = l$. This gives $\alpha_3(\tau, m) = 2ac_0 + (1/\sqrt{2})l$. Then the
 395 corresponding solutions of (15), (16) with the choice

$$a = \frac{1}{8\sqrt{2}e^\tau} \quad (61)$$

396 satisfy

$$v_1(\tau) \leq \frac{\sqrt{2}}{16l} \quad (62)$$

397 and

$$v_2(\tau) \leq \frac{1}{256\sqrt{2}e^{2\tau}\tau l} \quad (63)$$

398 which would mean that $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$ does not converge
 399 to ∞ as τ goes to zero. Thus, the choice $f_1 = 0$ and $f_2(t, x) =$
 400 $lx^2 \sin(x)$ is conservative.

401 VIII. CONCLUSION

402 Stabilization of nonlinear systems with input delays is a central
 403 problem that has been studied by many authors using model reduction,
 404 prediction, and other methods. Here we adapted the reduction model
 405 approach to the problem of locally asymptotically stabilizing the origin
 406 of time varying nonlinear systems with pointwise input delays. Our
 407 method of proof makes it possible to determine an estimate of the basin
 408 of attraction. The result can be adapted to the case where the delay in
 409 the input is distributed. Our results can be combined with those of [5]
 410 and [10].

411 APPENDIX

412 TECHNICAL LEMMA

413 We used the following to get (30) in the second part of the proof of
 414 Theorem 1:

415 *Lemma 1:* Let $\varepsilon \in (0, 1]$ be a positive real number. Then

$$-\frac{r}{\sqrt{r+\varepsilon}} \leq -\sqrt{r} + \varepsilon^{\frac{1}{4}}[1 + \sqrt{r}] \quad (64)$$

416 holds for all $r \geq 0$.

417 *Proof:* Let $r \geq 0$ be given. We first prove that

$$\frac{r}{\sqrt{r+\varepsilon}} \geq \frac{1}{\sqrt{1+\sqrt{\varepsilon}}}\sqrt{r} - \varepsilon^{\frac{1}{4}}. \quad (65)$$

418 If $\sqrt{r}/(\sqrt{1+\sqrt{\varepsilon}}) - \varepsilon^{1/4} \leq 0$, then (65) is satisfied. On the
 419 other hand, if $\sqrt{r}/(\sqrt{1+\sqrt{\varepsilon}}) - \varepsilon^{1/4} \geq 0$, then $r \geq (1+\sqrt{\varepsilon})\sqrt{\varepsilon}$.
 420 It follows that $(\sqrt{\varepsilon}+1)r \geq (1+\sqrt{\varepsilon})\varepsilon + r \geq \varepsilon + r$. Consequently,
 421 $r/(r+\varepsilon) \geq 1/(\sqrt{\varepsilon}+1)$. Taking the square root, and then multiply-
 422 ing through by \sqrt{r} , we obtain

$$r\sqrt{\frac{1}{r+\varepsilon}} \geq \frac{\sqrt{r}}{\sqrt{\varepsilon}+1}. \quad (66)$$

Therefore, (65) holds in both cases. Next, observe that (65) implies 423
 that 424

$$\begin{aligned} -\frac{r}{\sqrt{r+\varepsilon}} &\leq -\sqrt{r} + \left[1 - \frac{1}{\sqrt{1+\sqrt{\varepsilon}}}\right]\sqrt{r} + \varepsilon^{\frac{1}{4}} \\ &\leq -\sqrt{r} + \left[\sqrt{1+\sqrt{\varepsilon}} - 1\right]\sqrt{r} + \varepsilon^{\frac{1}{4}}. \end{aligned} \quad (67)$$

Hence, the relation $\sqrt{b+c} \leq \sqrt{b} + \sqrt{c}$ for nonnegative values b 425
 and c gives $-r/\sqrt{r+\varepsilon} \leq -\sqrt{r} + \varepsilon^{1/4}\sqrt{r} + \varepsilon^{1/4}$. This gives the 426
 conclusion. \square 427

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