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# 1 Local Stabilization of Nonlinear Systems Through the 2 Reduction Model Approach

3 Frederic Mazenc and Michael Malisoff

4 **Abstract**—We study a general class of nonlinear systems with input  
5 delays of arbitrary size. We adapt the reduction model approach to prove  
6 local asymptotic stability of the closed loop input delayed systems, using  
7 feedbacks that may be nonlinear. Our Lyapunov-Krasovskii functionals  
8 make it possible to determine estimates of the basins of attraction for the  
9 closed loop systems.

10 **Index Terms**—Delay, nonlinear, reduction model, stabilization.

## 11 I. INTRODUCTION

12 The reduction model approach is a well-known stabilization tech-  
13 nique for systems with input delays. It originated in [1] and has been  
14 studied in many works, e.g., [2]–[6]. It is effective for stabilizing  
15 linear time-invariant systems with arbitrarily long pointwise or dis-  
16 tributed input delays. However, the approach does not directly apply  
17 to nonlinear systems; it is extended by introducing an extra dynamic  
18 (which gives the ‘state predictor’) whose initial condition is given by  
19 an implicit equation (as is done in [7]–[9], and [6, Chapt. 6, p. 128]),  
20 and only a few recent works adapt it to time varying systems [10]. This  
21 is a limitation, because many systems are nonlinear and lead to the  
22 stabilization of time varying nonlinear systems when a trajectory has  
23 to be tracked. Moreover, the work [11] is limited to globally Lipschitz  
24 nonlinear systems, and it has a restriction on the size of the delays. See  
25 also [12] and [13] for stabilization of nonlinear systems with arbitrarily  
26 long input delays when the systems have special structures, and [14]  
27 for compensation of arbitrarily long input delays under input sampling  
28 based on prediction.

29 These remarks motivate our work. We show that the reduction  
30 model approach can be used to locally asymptotically stabilize a  
31 large family of nonlinear time varying systems of the form  $\dot{x}(t) =$   
32  $A(t)x(t) + B(t)u(t - \tau) + F(t, x(t))$ , with arbitrarily long constant  
33 known input delays  $\tau$ , where  $F$  is of order 2 in  $x$  at the origin.  
34 Our key assumption is the stabilizability of a linear approximation of  
35 the closed loop system at 0. Under this assumption, the result seems  
36 intuitively obvious. However, to the best of the authors’ knowledge,  
37 it has never been rigorously established. In particular, the stability  
38 of the closed loop system we obtain cannot be proven by applying  
39 the Hartman-Grobman theorem, which only applies to ordinary dif-  
40 ferential equations; see [15, Chapt. 1]. One of the crucial benefits  
41 offered by our result is that it yields asymptotically stable closed  
42 loop systems for which one can determine a suitable subset of the  
43 basin of attraction of the closed loop systems. This information is  
44 valuable, because it gives a guarantee that some solutions converge to  
45 the origin. We estimate the basin of attraction by building a Lyapunov-  
46 Krasovskii functional. It is different from the one in [16], but can be  
47 combined with it to establish ISS results. See also [17] for estimates

of the basins of attraction for time invariant nonlinear systems with 48  
predictor feedbacks, under an ISS assumption on the closed loop 49  
systems with undelayed controllers. The predictor feedbacks in [17] 50  
can be implemented using numerical methods but are totally different 51  
from ours, so our work can be viewed as complementary to [17]. 52  
Our work is mainly a methodological development, rather than a 53  
specific real-world application or experiments. However, input delays 54  
naturally arise from measurement and transport phenomena, and our 55  
assumptions are very general, so we anticipate that our work can 56  
have considerable benefits when applied to mechanical systems where 57  
latencies commonly occur. 58

The rest of this note is organized as follows. We give our definitions 59  
in Section II. In Section III, we show how the class of systems we 60  
study naturally arises in tracking problems. We state our main result in 61  
Section IV, and we prove it in Section V. In Section VI, we discuss a 62  
large class of examples where the estimates of the basins of attraction 63  
become arbitrarily large when the input delays converge to zero. In 64  
Section VII, we illustrate our result in a worked out example. We 65  
conclude in Section VIII with a summary of our findings. 66

## 67 II. DEFINITIONS AND NOTATION

We let  $n \in \mathbb{N}$  be arbitrary and  $I_n$  denote the identity matrix in 68  
 $\mathbb{R}^{n \times n}$ , and  $|\cdot|$  be the usual Euclidean norm of matrices and vectors. 69  
For square matrices  $M_1$  and  $M_2$  of the same size, we write  $M_1 \geq M_2$  70  
to mean that  $M_1 - M_2$  is nonnegative definite. For each integer  $r \geq 1$ , 71  
let  $C^r$  denote the set of all functions whose partial derivatives up 72  
through order  $r$  exist and are continuous, and  $C^0$  denotes the set of all 73  
continuous functions, when the domains and ranges are clear from the 74  
context. When we want to emphasize the domains and ranges, we use 75  
 $C^r(\mathcal{U}, \mathcal{V})$  to denote the set of all  $C^r$  functions having domain  $\mathcal{U}$  and 76  
range  $\mathcal{V}$ , where  $\mathcal{U}$  and  $\mathcal{V}$  are suitable subsets of Euclidean spaces. For 77  
any constant  $\tau \geq 0$  and any continuous function  $\varphi : [-\tau, \infty) \rightarrow \mathbb{R}^n$  78  
and all  $t \geq 0$ , we define the function  $\varphi_t$  by  $\varphi_t(\theta) = \varphi(t + \theta)$  for all 79  
 $\theta \in [-\tau, 0]$ , i.e., the translation operator. Let  $\mathcal{K}_\infty$  be the set of all  $C^0$  80  
functions  $\gamma : [0, \infty) \rightarrow [0, \infty)$  such that  $\gamma(0) = 0$  and  $\gamma$  is strictly 81  
increasing and unbounded. Given subsets  $S_1$  and  $S_2$  of Euclidean 82  
spaces, we say that a function  $J : S_1 \times S_2 \rightarrow \mathbb{R}^p$  is locally Lipschitz 83  
with respect to its second argument provided for each compact set  $E \subseteq$  84  
 $S_2$ , there is a constant  $L_E$  such that  $|J(p, x) - J(p, y)| \leq L_E|x - y|$  85  
for all  $p \in S_1$  and all  $x \in E$  and  $y \in E$ . We say that  $J$  is strictly in- 86  
creasing in its second argument provided the function  $Y(x) = J(p, x)$  87  
is strictly increasing for each  $p \in S_1$ ; we define strictly increasing and 88  
nondecreasing in either argument in a similar way. We say that  $J$  has 89  
order 2 in  $y$  at the origin provided there is a continuous function  $\alpha$  such 90  
that  $|J(p, y)| \leq |y|^2\alpha(|y|)$  for all  $(p, y) \in S_1 \times S_2$ . We sometimes 91  
omit arguments of functions when the arguments are clear from the 92  
context. 93

## 94 III. MOTIVATION: TRACKING PROBLEM

In this section, we explain how the problem of tracking a trajectory 95  
may lead to the problem we solve in the next section. Consider a time 96  
varying nonlinear system 97

$$\dot{\xi}(t) = g(t, \xi(t)) + B(t)\mu(t - \tau) \quad (1)$$

where the state  $\xi$  is valued in  $\mathbb{R}^n$ , the control  $\mu$  is valued in  $\mathbb{R}^p$ , 98  
 $\tau \geq 0$  is a known constant delay,  $g = (g_1, g_2, \dots, g_n)^\top$  is a nonlinear 99  
function of class  $C^2$ , and  $B$  is a continuous function. The dimensions 100  
 $n$  and  $p$  are arbitrary. We assume that (1) is forward complete for 101  
all measurable locally essentially bounded choices for  $\mu$ , so  $\xi(t)$  is 102

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103 defined for all nonnegative times for all such  $\mu$ 's. We also assume that  
104 there is a nondecreasing function  $\gamma$  such that

$$\max \left\{ \left| \frac{\partial^2}{\partial \xi^2} g_i(t, \xi) \right| : |\xi| \leq q, t \geq 0, i \in \{1, 2, \dots, n\} \right\} \leq \gamma(q) \quad (2)$$

105 for all  $q \geq 0$ , which exists when  $g$  is  $C^2$  and periodic in  $t$ .

106 The objective is to follow an admissible trajectory  $\xi_r$  of class  $C^1$ ,  
107 meaning the dynamics for  $x = \xi - \xi_r$  should be asymptotically stable.  
108 By admissible, we mean that there is a known continuous function  
109  $\mu_r(t)$  such that  $\dot{\xi}_r(t) = g(t, \xi_r(t)) + B(t)\mu_r(t)$  for all  $t \geq 0$ . In  
110 particular, this means that  $\xi_r(t)$  is defined for all  $t \geq 0$ . We assume  
111 that  $\xi_r$  is a known bounded function.

112 Let  $x(t) = \xi(t) - \xi_r(t)$  and  $\mu(t - \tau) = u(t - \tau) + \mu_r(t)$ . Then  
113 the error equation is

$$\dot{x}(t) = G(t, x(t)) + B(t)u(t - \tau) \quad (3)$$

114 where  $G(t, x) = g(t, x + \xi_r(t)) - g(t, \xi_r(t))$ . Notice that  $G(t, x) =$   
115  $\int_0^1 (\partial g / \partial x)(t, lx + \xi_r(t)) x dl$ , so  $G(t, x) = (\partial g / \partial x)(t, \xi_r(t))x +$   
116  $F(t, x)$ , where

$$F(t, x) = \int_0^1 \left( \frac{\partial g}{\partial x}(t, lx + \xi_r(t)) - \frac{\partial g}{\partial x}(t, \xi_r(t)) \right) x dl \quad (4)$$

117 holds for all  $t$  and  $x$ .

118 Applying the Mean Value Theorem and using (2) and the bound-  
119 edness of  $\xi_r$ , we can find a function  $\alpha \in C^0$  such that  $|F(t, x)| \leq$   
120  $|x|^2 \alpha(|x|)$ . Since  $\xi_r$  can depend on  $t$ , the system (3) is time varying,  
121 even if  $g$  is time-invariant and  $B$  is constant. This motivates the study  
122 of systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t - \tau) + F(t, x(t)) \quad (5)$$

123 where  $F$  is of order 2 in  $x$  at the origin, which will be our focus for the  
124 rest of this note.

#### 125 IV. STATEMENT OF MAIN RESULT

126 We state our main result for (5), where  $x$  is valued in  $\mathbb{R}^n$ , the  
127 control  $u$  is valued in  $\mathbb{R}^p$  and is to be specified,  $\tau \geq 0$  is a given  
128 constant delay, and  $F$  is a nonlinear function. The dimensions  $n$  and  
129  $p$  are arbitrary. The functions  $A$ ,  $B$  and  $F$  are continuous, and  $F$  is  
130 locally Lipschitz with respect to  $x$ . The set of all initial conditions we  
131 consider is  $E_0 = \{(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)\}$ , so the initial  
132 times for our trajectories are always 0. Let  $\lambda : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  be  
133 the fundamental solution associated with  $A$ . Then  $\lambda(t_0, t_0) = I_n$  and  
134  $(\partial \lambda / \partial t)(t, t_0) = A(t)\lambda(t, t_0)$  hold for all real numbers  $t$  and  $t_0$ . We  
135 introduce the following assumptions:

136 *Assumption 1:*

137 (i) There is a continuous, positive valued, nondecreasing function  
138  $h$  such that

$$|\lambda(t, l)B(l)| \leq h(\tau) \text{ for all } t \in \mathbb{R} \text{ and } l \in [t, t + \tau]. \quad (6)$$

139 (ii) There is a constant  $a^+ \geq 0$  such that  $\sup_{t \in \mathbb{R}} |A(t)| \leq a^+$ .  $\square$

140 Assumption 1 always holds when  $B$  is bounded and  $A$  is constant,  
141 so for instance, it holds for the one-dimensional system

$$\dot{x}(t) = x(t) + u(t - \tau) + lx^2(t) \sin(x(t)) \quad (7)$$

142 where  $u \in \mathbb{R}$  is the input,  $\tau$  is a positive constant delay, and  $l$  is  
143 a positive constant. In the case of (7), we can take  $A = 1$ ,  $B = 1$ ,

$\lambda(t, t_0) = e^{t-t_0}$ , and  $F(t, x) = lx^2 \sin(x)$ , so Assumption 1 holds  
144 with  $h(\tau) = 1$ . To ease the readability of our technical assumptions,  
145 we will explain how the example (7) satisfies our assumptions, after  
146 we introduce each of our three assumptions. Our next assumption is:

*Assumption 2:* There are a continuous function  $K : [0, \infty)^2 \rightarrow$   
148  $\mathbb{R}^{p \times n}$ , a nondecreasing continuous function  $k : [0, \infty) \rightarrow (0, \infty)$ , an  
149 everywhere positive definite and symmetric function  $Q : [0, \infty)^2 \rightarrow$   
150  $\mathbb{R}^{n \times n}$  of class  $C^1$  with respect to its first argument, and continuous  
151 functions  $q_i : [0, \infty) \rightarrow (0, \infty)$  for  $i = 1, 2, 3$  such that  $|K(t, \tau)| \leq$   
152  $k(\tau)$  for all  $(t, \tau) \in [0, \infty)^2$ , and such that with the choices  $H(t, \tau) =$   
153  $A(t) + \lambda(t, t + \tau)B(t + \tau)K(t, \tau)$  and  $R(t, \tau, s) = s^\top Q(t, \tau)s$ , the  
154 following two conditions are satisfied for all  $\tau \geq 0$ : (i) Along  
155 all trajectories of  $\dot{s}(t) = H(t, \tau)s(t)$ , we have  $\dot{R}(t, \tau, s(t)) \leq$   
156  $-q_1(\tau)R(t, \tau, s(t))$  and (ii) the bounds  
157

$$q_2(\tau)I_n \leq Q(t, \tau) \text{ and } |Q(t, \tau)| \leq q_3(\tau) \quad (8)$$

are satisfied for all  $t \geq 0$ .  $\square$  158

Assumption 2 holds for (7) as well. In fact, by choosing  $K(t, \tau) =$   
159  $-2e^\tau$ , we obtain  $H(t, \tau) = 1 - e^{-\tau}2e^\tau = -1$ , so Assumption 2 is  
160 satisfied with  $Q(t, \tau) = 1/2$ ,  $q_1(\tau) = 2$ ,  $q_2(\tau) = q_3(\tau) = 1/2$ , and  
161  $k(\tau) = 2e^\tau$ . Finally, we assume: 162

*Assumption 3:* There are two continuous functions  $f_1$  and  $f_2$  that  
163 are locally Lipschitz with respect to their last argument, and continu-  
164 ous functions  $\alpha_1$  and  $\alpha_2$ , such that 165

$$F(t, x) = \lambda(t, t + \tau)B(t + \tau)f_1(t, \tau, x) + f_2(t, x) \text{ and} \quad (9)$$

$$|f_1(t, \tau, x)| \leq |x|^2 \alpha_1(\tau, |x|^2) \text{ and}$$

$$|f_2(t, x)| \leq |x|^2 \alpha_2(|x|^2) \quad (10)$$

for all  $t \in \mathbb{R}$ ,  $\tau \geq 0$ , and  $x \in \mathbb{R}^n$ . Also,  $\beta_3(\tau, m) = m\alpha_1(\tau, m^2)$  166  
is strictly increasing and unbounded in  $m$ , and  $\beta_4(m) = m\alpha_2(m^2)$  167  
is nondecreasing in  $m$ . Finally, there are continuous functions  $\theta_1 : [0, \infty)^2 \rightarrow (0, \infty)$  and  $\theta_2 : [0, \infty) \rightarrow (0, \infty)$  such that 169

$$|\alpha_1(\tau, b + c) - \alpha_1(\tau, c)| \leq b\theta_1(\tau, b + c) \quad (11)$$

$$|\alpha_2(b + c) - \alpha_2(c)| \leq b\theta_2(b + c) \quad (12)$$

are satisfied for all  $b \geq 0$  and  $c \geq 0$ .  $\square$  170

To see why (7) satisfies Assumption 3, note that for (7), the fact  
171 that  $\lambda(t, t + \tau)B$  is invertible implies that one can choose  $f_2 = 0$  and  
172  $f_1(t, \tau, x) = le^\tau x^2 \sin(x)$ . Then we can satisfy Assumption 3 for (7)  
173 by taking  $\alpha_1(\tau, m) = le^\tau$  and  $\alpha_2(m) = 0$  for all  $m$  and  $\tau$ . 174

Returning to the general system (5), it follows from Assumptions  
175 2–3 that for any constant  $\tau > 0$  and: 176

$$\alpha_3(\tau, m) = \frac{q_3(\tau)}{\sqrt{q_2(\tau)}} \alpha_2(m) + 2a\alpha_1(\tau, m), \quad (13)$$

where  $a$  is any constant such that 177

$$0 < a \leq \frac{q_1(\tau)\sqrt{q_2(\tau)}}{8k(\tau)} \quad (14)$$

there are unique positive values  $v_1(\tau)$  and  $v_2(\tau)$  (which also depend  
178 on  $a$ ) such that 179

$$v_1(\tau)\alpha_3\left(\tau, \frac{4}{q_2(\tau)}v_1^2(\tau)\right) = \frac{q_1(\tau)q_2(\tau)}{16} \text{ and} \quad (15)$$

$$v_2(\tau)\alpha_3\left(\tau, \frac{4h^2(\tau)}{a^2}v_2^2(\tau)\right) = \frac{a^2}{4\tau h^2(\tau)}. \quad (16)$$

180 The existence of unique values  $v_1(\tau)$  and  $v_2(\tau)$  follows because  
 181  $\beta_3(\tau, m)$  is strictly increasing and unbounded in  $m$  and  $\beta_4(m)$  is  
 182 nondecreasing, so  $m\alpha_3(\tau, m^2)$  is strictly increasing and unbounded  
 183 in  $m$ . The choice of  $\alpha_3$  in (13) will become clear when we prove:

184 *Theorem 1:* Let  $\tau > 0$  be any constant and Assumptions 1–3  
 185 hold. Let  $a$  be any constant satisfying (14), and set  $v(\tau) =$   
 186  $\min\{v_1(\tau), v_2(\tau)\}$  where  $v_1$  and  $v_2$  are as above. Then, for each  
 187 initial function  $(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)$  satisfying

$$\sqrt{q_3(\tau)} \left| \phi_x(0) + \int_{-\tau}^0 \lambda(0, m + \tau) B(m + \tau) \phi_u(m) dm \right| + \frac{a}{\tau} \int_{-\tau}^0 (m + 2\tau) |\phi_u(m)| dm < v(\tau) \quad (17)$$

188 the unique solution of (5), in closed loop with

$$u(t) = -f_1(t, \tau, x(t)) + K(t, \tau) \left[ x(t) + \int_{t-\tau}^t \lambda(t, m + \tau) B(m + \tau) u(m) dm \right] \quad (18)$$

189 converges to 0 as  $t \rightarrow \infty$ . Moreover, (18) locally asymptotically  
 190 stabilizes (5) to 0.  $\square$

191 *Remark 1:* We comment that our control (18) agrees with the  
 192 standard predictor controller in the linear time invariant case where  
 193  $f_1 = f_2 = 0$  and  $A$  and  $B$  are constant. The extra term  $-f_1(t, \tau, x(t))$   
 194 is used to compensate part of the nonlinearity of the system (5).  
 195 Assumption 2 is a generalization of the standard assumption that  
 196  $(A, B)$  is a stabilizable pair, which is the special case of Assumption  
 197 2 where  $\tau = 0$ ,  $A$  and  $B$  are constant, and where  $K$  and  $Q$  can also be  
 198 taken to be constant. However, we allow the delay  $\tau > 0$  to be as large  
 199 as we want. On the other hand, since the  $q_i$ 's are continuous positive  
 200 valued functions of the delay, they have positive upper and lower  
 201 bounds over all  $\tau \in [0, \tau_M]$  for any constant  $\tau_M$ . Also, the function  
 202  $k$  from Assumption 2 is nondecreasing in  $\tau$ . Hence, if we are only  
 203 concerned with a bounded set  $[0, \tau_M]$  of possible values for  $\tau$ , then we  
 204 can assume in Assumption 2 that the  $q_i$ 's and  $k$  are all positive con-  
 205 stants, by replacing them by the constants  $\min\{q_1(\tau) : 0 \leq \tau \leq \tau_M\}$ ,  
 206  $\min\{q_2(\tau) : 0 \leq \tau \leq \tau_M\}$ ,  $\max\{q_3(\tau) : 0 \leq \tau \leq \tau_M\}$ , and  $k(\tau_M)$   
 207 without relabeling. These observations will be key to our proof in  
 208 Section VI that for important special cases, our estimate of the domain  
 209 of attraction becomes arbitrarily large when  $\tau \rightarrow 0^+$ .  $\square$

210 *Remark 2:* Assumptions 1–2 always hold when  $A$  and  $B$  are con-  
 211 stant provided  $(A, B)$  is stabilizable. Indeed, in that case  $\lambda(t, t_0) =$   
 212  $e^{(t-t_0)A}$ , so the stabilizability of  $(A, B)$  is equivalent to the stabiliz-  
 213 ability of  $(A, \lambda(t, t + \tau)B)$ . Also, when the  $\alpha_i$ 's are  $C^1$ , the existence  
 214 of functions  $\theta_i$  satisfying the requirements from Assumption 3 follows  
 215 from the Mean Value Theorem, since Assumption 2 only requires  
 216 (11), (12) for nonnegative  $b$ 's and  $c$ 's. Since  $F$  is of order 2 in  $x$   
 217 at 0, we can always satisfy Assumption 3 with  $f_1 = 0$  and  $f_2 = F$ .  
 218 However, these choices may lead to a conservative estimate of the size  
 219 of the basin of attraction; see the example in Section VII. Our use  
 220 of a feedback control with distributed terms is motivated by the facts  
 221 that  $\tau$  is arbitrary and  $\xi(t) = A(t)\xi(t)$  may be exponentially unstable.  
 222 In general, the explicit expression for  $\lambda$  is unknown, but it can be  
 223 computed in many important cases. This is the case in particular if  
 224  $A$  is constant or  $n = 1$ . We illustrate Theorem 1 in Section VII.  $\square$

225 *Remark 3:* In conjunction with our local asymptotic stability result,  
 226 we have boundedness of the control from Theorem 1, along all of the  
 227 closed loop trajectories.  $\square$

## V. PROOF OF THEOREM 1

228

Throughout the proof, we consider any solution of (5) in closed loop 229  
 with (18) for any initial condition satisfying the requirements (17) of 230  
 Theorem 1, and any constant delay  $\tau \geq 0$ . 231

*First Part: New Representation of the System:* Let  $t_e$  be any positive 232  
 real number or  $\infty$  such that the solution is defined over  $[-\tau, t_e)$ . Such 233  
 a  $t_e > 0$  exists, because the dynamics (5) grows linearly in  $x$  in any 234  
 bounded open neighborhood of  $x(0)$ . Later we show that  $t_e$  can always 235  
 be taken to be  $\infty$  for all of the trajectories we are considering. We 236  
 introduce the operators 237

$$z(t) = x(t) + \Gamma(t, u_t), \text{ where} \\ \Gamma(t, u_t) = \int_{t-\tau}^t \lambda(t, m + \tau) B(m + \tau) u(m) dm. \quad (19)$$

In all of what follows, we assume that  $t \in [0, t_e)$  is arbitrary, unless 238  
 otherwise noted, and we omit some of the arguments of the time 239  
 derivatives when they are clear, so  $\dot{\Gamma}(t)$  means  $(d/dt)\Gamma(t, u_t)$ . Then 240  
 the properties of the fundamental matrix give  $\dot{\Gamma}(t) = A(t)\Gamma(t, u_t) +$  241  
 $\lambda(t, t + \tau)B(t + \tau)u(t) - B(t)u(t - \tau)$ . Using the formula (5) and 242  
 our decomposition (9) for  $F(t, x)$ , we obtain 243

$$\dot{z}(t) = A(t)z(t) + \lambda(t, t + \tau)B(t + \tau)[u(t) + f_1(t, \tau, x(t))] + f_2(t, x(t)). \quad (20)$$

Also, our feedback (18) satisfies  $u(t) = -f_1(t, \tau, x(t)) +$  244  
 $K(t, \tau)z(t)$ . Consequently, in terms of our function  $H$  from 245  
 Assumption 2, (20) becomes 246

$$\dot{z}(t) = H(t, \tau)z(t) + f_2(t, x(t)). \quad (21)$$

Assumption 2 ensures global asymptotic stability of the linearizations 247  
 $\dot{z}(t) = H(t, \tau)z(t)$  of (21) at 0. Moreover, the equality 248

$$x(t) = z(t) - \int_{t-\tau}^t \lambda(t, m + \tau) B(m + \tau) u(m) dm \quad (22)$$

is satisfied. 249

*Second Part: Decay Conditions:* We study the stability of the closed 250  
 loop system using its representation as (21) coupled with (22). We 251  
 introduce the operator 252

$$\Xi(u_t) = \frac{1}{\tau} \int_{t-\tau}^t (m - t + 2\tau) |u(m)| dm. \quad (23)$$

Observe for later use that 253

$$\int_{t-\tau}^t |u(m)| dm \leq \Xi(u_t) \leq 2 \int_{t-\tau}^t |u(m)| dm. \quad (24)$$

Then, for all  $t \geq 0$ , we have 254

$$\dot{\Xi}(t) \leq 2 |u(t)| - \frac{1}{\tau} \int_{t-\tau}^t |u(m)| dm. \quad (25)$$

Also, we can use the upper bound on  $f_1$  from (10), the bound for 255  
 $K$  given in Assumption 2 and the formula  $u(t) = -f_1(t, \tau, x(t)) +$  256  
 $K(t, \tau)z(t)$  to get  $|u(t)| \leq k(\tau)|z(t)| + |x(t)|^2 \alpha_1(\tau, |x(t)|^2)$ . More- 257  
 over, (8) implies that for all  $t \geq 0$  and all  $z \in \mathbb{R}^n$ , we have  $q_2(\tau)|z|^2 \leq$  258



259  $z^\top Q(t, \tau)z$ . Taking square roots of both sides of the preceding in-  
260 equality and the dividing by  $\sqrt{q_2(\tau)} > 0$  gives

$$|z| \leq \frac{1}{\sqrt{q_2(\tau)}} \sqrt{R(t, \tau, z)}. \quad (26)$$

261 Combining the last two estimates with (25) gives

$$\begin{aligned} \dot{\Xi}(t) \leq & -\frac{1}{\tau} \int_{t-\tau}^t |u(m)| dm + \frac{2k(\tau)}{\sqrt{q_2(\tau)}} \sqrt{R(t, \tau, z(t))} \\ & + 2|x(t)|^2 \alpha_1(\tau, |x(t)|^2). \end{aligned} \quad (27)$$

262 We deduce from Assumptions 2–3 that the time derivative of  $R$   
263 along all trajectories of (21) satisfies

$$\begin{aligned} \dot{R}(t) \leq & -q_1(\tau)R(t, \tau, z(t)) + 2z(t)^\top Q(t, \tau)f_2(t, x(t)) \\ \leq & -q_1(\tau)R(t, \tau, z(t)) + 2|z(t)|q_3(\tau)|f_2(t, x(t))|. \end{aligned} \quad (28)$$

264 From (26), we deduce that

$$\begin{aligned} \dot{R}(t) \leq & -q_1(\tau)R(t, \tau, z(t)) \\ & + 2q_3(\tau) \frac{\sqrt{R(t, \tau, z(t))}}{\sqrt{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2). \end{aligned} \quad (29)$$

265 Consider the family of functions  $S_\varepsilon(t, \tau, z) = \sqrt{R(t, \tau, z)} + \varepsilon -$   
266  $\sqrt{\varepsilon}$  parameterized by the constant  $\varepsilon \in [0, 1]$  and let  $S = S_0$ . Since  $R$   
267 is of class  $C^1$  with respect to  $t$  and  $z$ , it follows that for all  $\varepsilon \in (0, 1)$ ,  
268 the function  $S_\varepsilon$  is of class  $C^1$  with respect to  $t$  and  $z$ , while  $S$  is only  
269 continuous. Also, (29) and Lemma 1 in the Appendix (applied with the  
270 choice  $r = R(t, \tau, z)$ ) give

$$\begin{aligned} \dot{S}_\varepsilon(t) \leq & -q_1(\tau) \frac{R(t, \tau, z(t))}{2\sqrt{R(t, \tau, z(t))} + \varepsilon} \\ & + q_3(\tau) \frac{\sqrt{R(t, \tau, z(t))} |x(t)|^2 \alpha_2(|x(t)|^2)}{\sqrt{R(t, \tau, z(t))} + \varepsilon\sqrt{q_2(\tau)}} \\ \leq & -\frac{q_1(\tau)}{2} S(t, \tau, z(t)) \\ & + \frac{q_3(\tau)}{\sqrt{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2) \\ & + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))] \end{aligned} \quad (30)$$

271 along all trajectories of (21).

272 *Third Part: Lyapunov-Krasovskii Functionals:* Let us consider the  
273 family of functions

$$V_\varepsilon(t, z, u_t) = a\Xi(u_t) + S_\varepsilon(t, \tau, z) \quad (31)$$

274 where the constant  $a$  satisfies (14) and we omit the argument  $\tau$  in  $V_\varepsilon$   
275 to simplify the notation. Then, (27) and (30) give

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & \left( \frac{2ak(\tau)}{\sqrt{q_2(\tau)}} - \frac{q_1(\tau)}{2} \right) S(t, \tau, z(t)) \\ & + \frac{q_3(\tau)}{\sqrt{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2) \end{aligned}$$

$$\begin{aligned} & -\frac{a}{\tau} \int_{t-\tau}^t |u(m)| dm + 2a|x(t)|^2 \alpha_1(\tau, |x(t)|^2) \\ & + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))]. \end{aligned} \quad (32)$$

Since  $a$  satisfies (14), we get

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & -\frac{q_1(\tau)}{4} S(t, \tau, z(t)) + |x(t)|^2 \alpha_3(\tau, |x(t)|^2) \\ & -\frac{a}{\tau} \int_{t-\tau}^t |u(m)| dm + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))] \end{aligned} \quad (33)$$

where  $\alpha_3$  was defined in (13).

Next, we find a suitable upper bound on the term  
278  $|x(t)|^2 \alpha_3(\tau, |x(t)|^2)$  from (33). Our formula (22) for  $x(t)$ ,  
279 Assumption 1, and our bound (26) on  $|z|$  give

$$\begin{aligned} |x(t)| \leq & |z(t)| + h(\tau) \int_{t-\tau}^t |u(m)| dm \\ \leq & \frac{1}{\sqrt{q_2(\tau)}} S(t, \tau, z(t)) + h(\tau) \int_{t-\tau}^t |u(m)| dm. \end{aligned} \quad (34)$$

Recall that our monotonicity properties of  $\beta_3$  and  $\beta_4$  from Assumption  
281 3 imply that  $m\alpha_3(\tau, m^2)$  is strictly increasing as a function of  $m$   
282 for each  $\tau$ . Therefore, by separately considering the cases where  
283  $S(t, \tau, z(t))/\sqrt{q_2(\tau)} \leq h(\tau) \int_{t-\tau}^t |u(m)| dm$  and where the reverse  
284 inequality holds, we get

$$\begin{aligned} |x(t)|^2 \alpha_3(\tau, |x(t)|^2) & \leq \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \alpha_3\left(\tau, \frac{4}{q_2(\tau)} S^2(t, \tau, z(t))\right) \\ & + 4h^2(\tau) \left[ \int_{t-\tau}^t |u(m)| dm \right]^2 \alpha_3\left(\tau, 4h^2(\tau) \left[ \int_{t-\tau}^t |u(m)| dm \right]^2\right). \end{aligned} \quad (35)$$

We can combine this inequality with (33) to get

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & \left[ -\frac{q_1(\tau)}{4} + \frac{4}{q_2(\tau)} S(t, \tau, z(t)) \alpha_3 \right. \\ & \times \left. \left( \tau, \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \right) \right] S(t, \tau, z(t)) \\ & + \left[ -\frac{a}{\tau} + 4h^2(\tau) \int_{t-\tau}^t |u(m)| dm \alpha_3 \right. \\ & \times \left. \left( \tau, 4h^2(\tau) \left[ \int_{t-\tau}^t |u(m)| dm \right]^2 \right) \right] \\ & \times \int_{t-\tau}^t |u(m)| dm + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))]. \end{aligned}$$

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287 Also,  $V_\varepsilon(t, z(t), u_t) \geq \sqrt{R(t, \tau, z(t)) + \varepsilon} - \sqrt{\varepsilon} \geq S(t, \tau, z(t)) -$   
 288  $\sqrt{\varepsilon}$  and  $V_\varepsilon(t, z(t), u_t) \geq a \int_{t-\tau}^t |u(m)| dm$  hold for all  $\varepsilon \in [0, 1]$ , by  
 289 (24). Since  $m\alpha_3(\tau, m^2)$  is increasing in  $m$  for each  $\tau$ , it follows that:

$$\begin{aligned} S(t, \tau, z(t)) \alpha_3 \left( \tau, \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \right) \\ \leq [V_\varepsilon(t, z(t), u_t) + \sqrt{\varepsilon}] \alpha_3 \\ \times \left( \tau, \frac{4}{q_2(\tau)} [2V_\varepsilon(t, z(t), u_t) \sqrt{\varepsilon} + \varepsilon] + \frac{4}{q_2(\tau)} V_\varepsilon^2(t, z(t), u_t) \right). \end{aligned}$$

290 We now apply (11), (12), with  $b = (4/q_2(\tau))(2V_\varepsilon(t, z(t), u_t)\sqrt{\varepsilon} +$   
 291  $\varepsilon)$  and  $c = 4V_\varepsilon^2(t, z(t), u_t)/q_2(\tau)$ , and use the fact that  $\varepsilon \leq \sqrt{\varepsilon} \leq$   
 292  $\varepsilon^{1/4}$  for all  $\varepsilon \in [0, 1]$ , to find a continuous positive valued and non-  
 293 decreasing function  $\varphi_c$  (also depending on  $\tau$ , but independent of  $\varepsilon$ )  
 294 such that

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & \left[ -\frac{q_1(\tau)}{4} + \frac{4}{q_2(\tau)} V_\varepsilon(t, z(t), u_t) \right. \\ & \times \alpha_3 \left( \tau, \frac{4}{q_2(\tau)} V_\varepsilon^2(t, z(t), u_t) \right) \left. \right] S(t, \tau, z(t)) \\ & + \frac{1}{a} \left[ -\frac{a^2}{\tau} + 4h^2(\tau) V_\varepsilon(t, z(t), u_t) \right. \\ & \times \alpha_3 \left( \tau, 4h^2(\tau) \frac{V_\varepsilon^2(t, z(t), u_t)}{a^2} \right) \left. \right] \int_{t-\tau}^t |u(m)| dm \\ & + \varepsilon^{\frac{1}{4}} \varphi_c(V_\varepsilon(t, z(t), u_t)). \end{aligned} \quad (36)$$

295 Next, recall that our assumption (17) implies that  $\sqrt{q_3(\tau)}|z(0)| +$   
 296  $(a/\tau) \int_{-\tau}^0 (m+2\tau)|u(m)| dm < v(\tau)$ , where  $v(\tau) = \min\{v_1(\tau),$   
 297  $v_2(\tau)\}$  as before. Then (8) from Assumption 2 gives  $V_0(0, z(0), u_0) <$   
 298  $v(\tau)$ . Since  $\sqrt{c_1} + c_2 \leq \sqrt{c_1} + \sqrt{c_2}$  holds for all nonnegative con-  
 299 stants  $c_1$  and  $c_2$ , we know that  $V_\varepsilon \leq V_0$  holds pointwise for all  
 300  $\varepsilon \in (0, 1]$ . It follows that  $V_\varepsilon(0, z(0), u_0) \leq V_0(0, z(0), u_0) < \bar{v}$  hold  
 301 for all  $\varepsilon \in (0, 1]$ , where  $\bar{v} = [V_0(0, z(0), u_0) + v(\tau)]/2 > 0$ . Then  
 302  $\bar{v} < v(\tau)$ .

303 Set  $\bar{v}_a = (v(\tau) + \bar{v})/2$ . Since  $m\alpha_3(\tau, m^2)$  is strictly increasing in  
 304  $m$ , and since  $\bar{v}_a < v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$ , it follows from our  
 305 conditions (15), (16) on  $v_1(\tau)$  and  $v_2(\tau)$  that the constants:

$$\begin{aligned} \bar{p}_1 &= \frac{q_1(\tau)}{4} - \frac{4}{q_2(\tau)} \bar{v}_a \alpha_3 \left( \tau, \frac{4}{q_2(\tau)} \bar{v}_a^2 \right) \text{ and} \\ \bar{p}_2 &= \frac{a^2}{\tau} - 4h^2(\tau) \bar{v}_a \alpha_3 \left( \tau, 4h^2(\tau) \frac{\bar{v}_a^2}{a^2} \right) \end{aligned} \quad (37)$$

306 are positive for all  $\tau > 0$ . Fix any value of  $\varepsilon \in (0, 1]$  satisfying

$$\varepsilon \in \left( 0, \left( \frac{\min\{\bar{p}_1, \bar{p}_2\} \bar{v}}{4\varphi_c(\bar{v}_a) \max\{a^2, 1\}} \right)^4 \right) \quad (38)$$

307 where the left endpoint is omitted because we need  $\varepsilon > 0$ .  
 308 Next, we prove by contradiction that  $V_\varepsilon(t, z(t), u_t) \leq \bar{v}$  for all  
 309  $t \geq 0$ . Assume that this property does not hold. Then, since  
 310  $\bar{v}_a > \bar{v}$  and  $V_\varepsilon(0, z(0), u_0) < \bar{v}$ , we can find a  $t_2 > 0$  such that  
 311  $V_\varepsilon(t, z(t), u_t) \leq \bar{v}_a$  for all  $t \in [0, t_2]$  and  $V_\varepsilon(t_2, z(t_2), u_{t_2}) > \bar{v}$ . Set  
 312  $t_1 = \inf\{t \leq t_2 : V_\varepsilon(p, z(p), u_p) \geq \bar{v} \text{ for all } p \in [t, t_2]\}$ . Then, since  
 313  $t \mapsto V_\varepsilon(t, z(t), u_t)$  is continuous, we get  $V_\varepsilon(t, z(t), u_t) \in [\bar{v}, \bar{v}_a]$  for  
 314 all  $t \in [t_1, t_2]$ ,  $V_\varepsilon(t_1, z(t_1), u_{t_1}) = \bar{v}$ , and  $\dot{V}_\varepsilon(t_1) \geq 0$ .

By (36) and the fact that  $l\alpha_3(\tau, l^2)$  is strictly increasing in  $l$  315

$$\dot{V}_\varepsilon(t) \leq -\bar{p}_1 S(t, \tau, z(t)) - \frac{1}{a} \bar{p}_2 \int_{t-\tau}^t |u(m)| dm + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \quad (39)$$

for all  $t \in [t_1, t_2]$ . It follows from our lower bound on  $\Xi$  from (24) 316  
 that: 317

$$\begin{aligned} \dot{V}_\varepsilon(t) &\leq -\frac{1}{2 \max\{a^2, 1\}} \min\{\bar{p}_1, \bar{p}_2\} V_0(t, z(t), u_t) + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \\ &\leq -\frac{1}{2 \max\{a^2, 1\}} \min\{\bar{p}_1, \bar{p}_2\} V_\varepsilon(t, z(t), u_t) + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \end{aligned} \quad (40)$$

for all  $t \in [t_1, t_2]$ . Since  $V_\varepsilon(t, z(t), u_t) \in [\bar{v}, \bar{v}_a]$  for all  $t \in [t_1, t_2]$ , we 318  
 deduce that 319

$$\begin{aligned} \dot{V}_\varepsilon(t) &\leq -\frac{1}{2 \max\{a^2, 1\}} \min\{\bar{p}_1, \bar{p}_2\} \bar{v} + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \\ &\leq -\frac{\min\{\bar{p}_1, \bar{p}_2\}}{4 \max\{a^2, 1\}} \bar{v} < 0 \end{aligned} \quad (41)$$

for all  $t \in [t_1, t_2]$  when  $\varepsilon$  satisfies (38). It follows that  $\dot{V}_\varepsilon(t_1) < 0$ . 320  
 This yields a contradiction with the choice of  $t_1$ . Hence, when (38) 321  
 holds, we get  $V_\varepsilon(t, z(t), u_t) \leq \bar{v}$  for all  $t \geq 0$ , which implies that we 322  
 can choose  $t_e = \infty$ . Also, arguing as we did before, we get 323

$$\dot{V}_\varepsilon(t) \leq -\frac{\min\{\bar{p}_1, \bar{p}_2\}}{2 \max\{a^2, 1\}} V_\varepsilon(t, z(t), u_t) + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}). \quad (42)$$

for all  $t \geq 0$ . This gives a value  $t_c > 0$  such that for all  $t \geq t_c$ , we have 324

$$V_\varepsilon(t, z(t), u_t) \leq \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{\min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \quad (43)$$

(since  $V_\varepsilon$  is nonnegative valued), and therefore also 325

$$\begin{aligned} \Xi(u_t) &\leq \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{a \min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \text{ and} \\ S_\varepsilon(t, \tau, z) &\leq \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{\min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \end{aligned} \quad (44)$$

Since  $S_\varepsilon(t, \tau, z) = \sqrt{R(t, \tau, z) + \varepsilon} - \sqrt{\varepsilon} \geq \sqrt{q_2(\tau)}|z| - \sqrt{\varepsilon}$  holds 326  
 for all  $t, \tau$ , and  $z$ , (24) gives 327

$$\begin{aligned} \max \left\{ \int_{t-\tau}^t |u(m)| dm, \sqrt{q_2(\tau)}|z| \right\} \\ \leq \sqrt{\varepsilon} + \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{\min\{a, 1\} \min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \end{aligned} \quad (45)$$

for all  $t \geq t_c$ . Set 328

$$\Delta = \max \left\{ \frac{1}{\sqrt{q_2(\tau)}}, 1 \right\} \left( 1 + \frac{4\varphi_c(\bar{v})}{\min\{a, 1\} \min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \right). \quad (46)$$

Then, since  $\varepsilon \in [0, 1]$ , it follows that for all  $t \geq t_c$ , the inequalities 329

$$|z(t)| \leq \Delta \varepsilon^{\frac{1}{4}} \text{ and } \int_{t-\tau}^t |u(m)| dm \leq \Delta \varepsilon^{\frac{1}{4}} \quad (47)$$

are satisfied. Since  $\varepsilon$  is arbitrarily small, we deduce that  $|z(t)|$  and 330  
 $\int_{t-\tau}^t |u(m)| dm$  converge to zero when  $t \rightarrow \infty$ . This and the first 331  
 inequality in (34) imply that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Also, by letting 332  
 $\varepsilon$  depend on the maximum of  $V_0$  on a suitable neighborhood of the 333  
 origin, we can prove the local stability part. This proves the theorem. 334

## 335 VI. ARBITRARILY LARGE DOMAINS OF ATTRACTION

336 Theorem 1 applies for all  $\tau > 0$ . On the other hand, consider the  
 337 special case where  $f_2 = 0$  in the decomposition (9) of  $F$ . Then,  
 338 setting  $\tau = 0$  in (9) and in our control (18) produces the uniformly  
 339 globally asymptotically stable closed loop system  $\dot{x}(t) = [A(t) +$   
 340  $B(t)K(t, 0)]x(t)$  from Assumption 2. This suggests that the domain  
 341 of attraction should become arbitrarily large as  $\tau \rightarrow 0^+$  when  $f_2 = 0$ .  
 342 Our next theorem implies that this is indeed the case. We will assume  
 343 that the functions  $q_i$  and  $k$  from Assumption 2 are constant, so we  
 344 omit their arguments  $\tau$ . This is not restrictive, since now we only need  
 345 to consider  $\tau$ 's on a bounded interval; see Remark 1.

346 *Corollary 1:* Let Assumptions 1–3 hold with  $f_2 = 0$  and the  $q_i$ 's  
 347 and  $k$  all constant. Then for each constant  $v_* > 0$ , we can find values  
 348  $a \in (0, q_1 \sqrt{q_2} / (8k))$  and  $\tau_M > 0$  (both depending on  $v_*$ ) such that:  
 349 For each initial condition  $(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)$  satisfying

$$\begin{aligned} & \left| \sqrt{q_3(\tau)} \left| \phi_x(0) + \int_{-\tau}^0 \lambda(0, m + \tau) B(m + \tau) \phi_u(m) dm \right| \right. \\ & \left. + \frac{a}{\tau} \int_{-\tau}^0 (m + 2\tau) |\phi_u(m)| dm < v_* \right. \quad (48) \end{aligned}$$

350 and each constant delay  $\tau \in (0, \tau_M)$ , the trajectory of (5) in closed  
 351 loop with (18) converges to 0 as  $t \rightarrow \infty$ .  $\square$

352 *Proof:* We set  $\alpha_2 = 0$ , so we have  $\alpha_3 = 2a\alpha_1$ . Then (15), (16)  
 353 become

$$\begin{aligned} v_1(\tau)\alpha_1 \left( \tau, \frac{4}{q_2} v_1^2(\tau) \right) &= \frac{q_1 q_2}{32a} \text{ and} \\ v_2(\tau)\alpha_1 \left( \tau, \frac{4h^2(\tau)}{a^2} v_2^2(\tau) \right) &= \frac{a}{8\tau h^2(\tau)}. \quad (49) \end{aligned}$$

354 For each constant  $\tau_M > 0$ , Assumption 3 provides a function  $\bar{\gamma}$  of  
 355 class  $\mathcal{K}_\infty$  such that  $m\alpha_1(\tau, m^2) \leq \bar{\gamma}(m)$  for all  $\tau \in [0, \tau_M]$  and  
 356  $m \geq 0$ . Then, replacing  $\alpha_1(\tau, m^2)$  in (49) by  $\bar{\gamma}(m)/m$  gives

$$\begin{aligned} \bar{\gamma} \left( \sqrt{\frac{4}{q_2}} v_1(\tau) \right) &= \frac{q_1 \sqrt{q_2}}{16a} \text{ and} \\ \bar{\gamma} \left( \frac{2h(\tau)}{a} v_2(\tau) \right) &= \frac{1}{4\tau h(\tau)} \quad (50) \end{aligned}$$

357 for all  $\tau \in (0, \tau_M)$ . Our proof of Theorem 1 shows that the conclusions  
 358 of that theorem remain true when  $v_1(\tau)$  and  $v_2(\tau)$  are defined to be the  
 359 solutions of (50). Therefore

$$\begin{aligned} v_1(\tau) &= \frac{\sqrt{q_2}}{2} \bar{\gamma}^{-1} \left( \frac{q_1 \sqrt{q_2}}{16a} \right) \text{ and} \\ v_2(\tau) &= \frac{a}{2h(\tau)} \bar{\gamma}^{-1} \left( \frac{1}{4\tau h(\tau)} \right). \quad (51) \end{aligned}$$

360 Also, when  $\tau$  is sufficiently small, the choice

$$a = \frac{1}{\sqrt{\bar{\gamma}^{-1} \left( \frac{1}{4\tau h(\tau)} \right)}} \quad (52)$$

will satisfy our requirements (14) on  $a$ , because (52) converges to 0  
 as  $\tau \rightarrow 0^+$  and because we are now assuming that the  $q_i$ 's and  $k$  are  
 positive constants. Then (51) become

$$\begin{aligned} v_1(\tau) &= \frac{\sqrt{q_2}}{2} \bar{\gamma}^{-1} \left( \frac{q_1 \sqrt{q_2}}{16} \sqrt{\bar{\gamma}^{-1} \left( \frac{1}{4\tau h(\tau)} \right)} \right) \text{ and} \\ v_2(\tau) &= \frac{1}{2h(\tau)} \sqrt{\bar{\gamma}^{-1} \left( \frac{1}{4\tau h(\tau)} \right)}. \quad (53) \end{aligned}$$

Therefore, both  $v_1(\tau)$  and  $v_2(\tau)$  converge to  $\infty$  when  $\tau \rightarrow 0^+$ . It  
 follows that  $v(\tau) \rightarrow \infty$  as  $\tau \rightarrow 0^+$ , so we can satisfy (48) for small  
 enough  $\tau > 0$  by choosing  $\tau$  such that  $v(\tau) > v_*$ . The corollary now  
 follows from Theorem 1.  $\blacksquare$

## 368 VII. ILLUSTRATIVE EXAMPLE

We illustrate Theorem 1 using the 1 dimensional system from (7),  
 which is

$$\dot{x}(t) = x(t) + u(t - \tau) + lx^2(t) \sin(x(t)) \quad (54)$$

where  $u \in \mathbb{R}$  is the input,  $\tau$  is a positive constant delay, and  $l$  is  
 a positive constant. This system is not globally Lipschitz in the  
 state  $x$ . With the notation of the previous sections, we have  $A = 1$ ,  
 $B = 1$ ,  $\lambda(t, t_0) = e^{t-t_0}$ , and  $F(t, x) = lx^2 \sin(x)$ . As we saw in  
 Section IV, (54) satisfies our assumptions with  $h(\tau) = 1$ ,  $K(t, \tau) =$   
 $-2e^\tau$ ,  $Q(t, \tau) = 1/2$ ,  $q_1(\tau) = 2$ ,  $q_2(\tau) = q_3(\tau) = 1/2$ ,  $k(\tau) = 2e^\tau$ ,  
 $f_2 = 0$ ,  $f_1(t, \tau, x) = le^\tau x^2 \sin(x)$ ,  $\alpha_1(\tau, m) = le^\tau$  and  $\alpha_2(m) = 0$ .  
 According to (14), the inequalities  $0 < a \leq 1/(8\sqrt{2}e^\tau)$  have to be  
 satisfied and, by the expression of  $\alpha_3$  in (13),  $\alpha_3(\tau, m) = 2ale^\tau$ .

Choosing

$$a = \frac{1}{8\sqrt{2}e^\tau} \quad (55)$$

we can straightforwardly derive an estimate of the basin of attraction  
 from Theorem 1 by using  $v = \min\{v_1, v_2\}$ , where

$$v_1(\tau) = \frac{1}{2\sqrt{2}l} \quad (56)$$

and

$$v_2(\tau) = \frac{1}{64\sqrt{2}\tau e^{2\tau} l} \quad (57)$$

which converge to  $\infty$  as  $l \rightarrow 0$  for each  $\tau > 0$ . On the other hand,  
 when  $\tau \in (0, 1]$ , we can take

$$a = \frac{\sqrt{\tau}}{8\sqrt{2}e^\tau} \quad (58)$$

to obtain the values

$$v_1(\tau) = \frac{1}{2l\sqrt{2\tau}} \quad (59)$$

and

$$v_2(\tau) = \frac{1}{64le^{2\tau}\sqrt{2\tau}} \quad (60)$$

so  $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$  converges to  $\infty$  as  $l$  converges to zero  
 for fixed  $\tau > 0$ , or as  $\tau$  converges to zero for fixed  $l$ , so the basin  
 of attraction becomes arbitrarily large. This gives convergence of the  
 closed loop solution to 0.

392 If, on the other hand, we had chosen,  $f_1 = 0$  and  $f_2(t, x) =$   
 393  $lx^2 \sin(x)$ , then one could choose  $\alpha_1 = c_0$  for any constant  $c_0 > 0$   
 394 and  $\alpha_2(m) = l$ . This gives  $\alpha_3(\tau, m) = 2ac_0 + (1/\sqrt{2})l$ . Then the  
 395 corresponding solutions of (15), (16) with the choice

$$a = \frac{1}{8\sqrt{2}e^\tau} \quad (61)$$

396 satisfy

$$v_1(\tau) \leq \frac{\sqrt{2}}{16l} \quad (62)$$

397 and

$$v_2(\tau) \leq \frac{1}{256\sqrt{2}e^{2\tau}\tau l} \quad (63)$$

398 which would mean that  $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$  does not converge  
 399 to  $\infty$  as  $\tau$  goes to zero. Thus, the choice  $f_1 = 0$  and  $f_2(t, x) =$   
 400  $lx^2 \sin(x)$  is conservative.

#### 401 VIII. CONCLUSION

402 Stabilization of nonlinear systems with input delays is a central  
 403 problem that has been studied by many authors using model reduction,  
 404 prediction, and other methods. Here we adapted the reduction model  
 405 approach to the problem of locally asymptotically stabilizing the origin  
 406 of time varying nonlinear systems with pointwise input delays. Our  
 407 method of proof makes it possible to determine an estimate of the basin  
 408 of attraction. The result can be adapted to the case where the delay in  
 409 the input is distributed. Our results can be combined with those of [5]  
 410 and [10].

#### 411 APPENDIX

#### 412 TECHNICAL LEMMA

413 We used the following to get (30) in the second part of the proof of  
 414 Theorem 1:

415 *Lemma 1:* Let  $\varepsilon \in (0, 1]$  be a positive real number. Then

$$-\frac{r}{\sqrt{r+\varepsilon}} \leq -\sqrt{r} + \varepsilon^{\frac{1}{4}}[1 + \sqrt{r}] \quad (64)$$

416 holds for all  $r \geq 0$ .

417 *Proof:* Let  $r \geq 0$  be given. We first prove that

$$\frac{r}{\sqrt{r+\varepsilon}} \geq \frac{1}{\sqrt{1+\sqrt{\varepsilon}}}\sqrt{r} - \varepsilon^{\frac{1}{4}}. \quad (65)$$

418 If  $\sqrt{r}/(\sqrt{1+\sqrt{\varepsilon}}) - \varepsilon^{1/4} \leq 0$ , then (65) is satisfied. On the  
 419 other hand, if  $\sqrt{r}/(\sqrt{1+\sqrt{\varepsilon}}) - \varepsilon^{1/4} \geq 0$ , then  $r \geq (1+\sqrt{\varepsilon})\sqrt{\varepsilon}$ .  
 420 It follows that  $(\sqrt{\varepsilon}+1)r \geq (1+\sqrt{\varepsilon})\varepsilon + r \geq \varepsilon + r$ . Consequently,  
 421  $r/(r+\varepsilon) \geq 1/(\sqrt{\varepsilon}+1)$ . Taking the square root, and then multiply-  
 422 ing through by  $\sqrt{r}$ , we obtain

$$r\sqrt{\frac{1}{r+\varepsilon}} \geq \frac{\sqrt{r}}{\sqrt{\varepsilon}+1}. \quad (66)$$

Therefore, (65) holds in both cases. Next, observe that (65) implies 423  
 that 424

$$\begin{aligned} -\frac{r}{\sqrt{r+\varepsilon}} &\leq -\sqrt{r} + \left[1 - \frac{1}{\sqrt{1+\sqrt{\varepsilon}}}\right]\sqrt{r} + \varepsilon^{\frac{1}{4}} \\ &\leq -\sqrt{r} + \left[\sqrt{1+\sqrt{\varepsilon}} - 1\right]\sqrt{r} + \varepsilon^{\frac{1}{4}}. \end{aligned} \quad (67)$$

Hence, the relation  $\sqrt{b+c} \leq \sqrt{b} + \sqrt{c}$  for nonnegative values  $b$  425  
 and  $c$  gives  $-r/\sqrt{r+\varepsilon} \leq -\sqrt{r} + \varepsilon^{1/4}\sqrt{r} + \varepsilon^{1/4}$ . This gives the 426  
 conclusion.  $\square$  427

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