A relaxation result for state constrained inclusions in infinite dimension

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A RELAXATION RESULT FOR STATE CONSTRAINED INCLUSIONS IN INFINITE DIMENSION

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Abstract. In this paper we consider a state constrained differential inclusion $\dot{x} \in \mathbb{A}x + F(t,x)$, with $\mathbb{A}$ generator of a strongly continuous semigroup in an infinite dimensional separable Banach space. Under an "inward pointing condition" we prove a relaxation result stating that the set of trajectories lying in the interior of the constraint is dense in the set of constrained trajectories of the convexified inclusion $\dot{x} \in \mathbb{A}x + \overline{\text{co}}F(t,x)$. Some applications to control problems involving PDEs are given.

1. Introduction

We study a class of infinite dimensional differential inclusions subject to state constraints. Interest in these kind of equations arises in several contexts. Differential inclusions find a natural application in a research area of great development, the control theory, and the infinite dimensional setting allows to apply our results to control problems involving PDEs. Hence, models describing many physical phenomena such as diffusion, vibration of strings, fluid dynamics, may be included in our analysis.

In this paper we are concerned with the differential inclusion

$$\dot{x}(t) \in \mathbb{A}x(t) + F(t,x(t)) \quad \text{a.e. } t \in [t_0, 1]\,, \tag{1.1}$$

and the convexified differential inclusion

$$\dot{x}(t) \in \mathbb{A}x(t) + \overline{\text{co}}F(t,x(t)) \quad \text{a.e. } t \in [t_0, 1]\,, \tag{1.2}$$

with $\overline{\text{co}}F(t,x(t))$ the closed convex hull of $F(t,x(t))$. The operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous semigroup $S(t) : X \to X$, X is an infinite dimensional separable Banach space, $F : I \times X \rightsquigarrow X$ is a set-valued map with closed non-empty images, $I = [0,1]$ and $t_0 \in I$. The trajectories of the differential inclusion (1.1) are understood in the mild sense (see [25]) and are subject to the state constraint. Namely given a set $K \subset X$, we restrict our attention to the trajectories satisfying

$$x(t) \in K, \qquad \text{for } t \in [t_0, 1]\,. \tag{1.3}$$

In this paper we shall always assume that K is the closure of an open subset of X. When satisfying the constraint, a trajectory x is called *feasible*.

Differential inclusions, and control systems, in presence of state constraints, are largely employed in applied sciences. One of the tools playing a key role in this context consists in approximating feasible trajectories by trajectories lying in the interior of the constraints. It is used for instance to establish regularity properties of value functions, to justify the

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use of the Maximum Principle in normal form, to prove existence and regularity results of optimal solutions. The classical technique employed to construct the approximating trajectories relies on the possibility of directing the velocity into the interior of the constraint K whenever approaching the boundary of K. To this aim, in the finite dimensional setting, an "inward pointing condition" was proposed by Soner, see [28], to get continuity of the value function associated to an optimal control problem with dynamics $\dot{x} \in F(x)$ independent of t. Since then, this subject has received considerable attention, a partial list of references includes [5, 6, 10, 16, 18, 19].

Defining the oriented distance from $x \in X$ to K by

$$d_K(x) = \begin{cases} \inf_{k\in K} \|x-k\|_X & \text{if } x \notin K \\ -\inf_{k\in (X\setminus K)} \|x-k\|_X & \text{otherwise,} \end{cases}$$

the inward pointing condition, in the case of time independent F and state constraints with a locally $C^{1,1}$ boundary, takes the following form:

$$\min_{v\in F(\bar{x})} \langle \nabla d_K(\bar{x}), v\rangle \leq -\rho, \qquad \forall\, \bar{x} \in \partial K\,, \tag{1.4}$$

cf. [5, 18]. As in many applied models state constraints having nonsmooth boundary are present, a number of papers made extensions of (1.4) to the nonsmooth setting. However, contrary to the smooth case, here some regularity of the dynamics $F(t,x)$ is usually required both in t and in x. On the other hand, it may happen in some applications that the dynamics depends on t in a merely measurable way. In order to extend the theory to this situation, in the recent works [16, 17] a new inward pointing condition (equivalent to the classical one if K has smooth boundary) is proposed: for any "bad" velocity v pointing outside the constraint, there exists a "good" one $\bar{v}$ such that the difference $\bar{v}-v$ points inside in a uniform way, namely:

$$\begin{cases} \forall\, \bar{x} \in \partial K,\ \exists\, \rho > 0,\ \forall\, t \in I,\ \forall\, v \in F(t,\bar{x}) \text{ with } \sup\limits_{\xi\in\partial d_K(\bar{x})} \langle \xi,\, v\rangle \geq 0, \\ \exists\, \bar{v} \in F(t,\bar{x}) \ \text{ satisfying } \sup\limits_{\xi\in\partial d_K(\bar{x})} \big\langle \xi, \bar{v}-v\big\rangle < -\rho\,. \end{cases} \tag{1.5}$$

Here $\partial d_K(\bar{x})$ denotes the Clarke generalized gradient of d_K at the point $\bar{x} \in X$. Under this assumption, in [16, 17] some approximation results were proved in order to get uniqueness of solutions for a constrained Hamilton-Jacobi-Bellman equation.

The purpose of the present paper is to perform the analysis in the infinite dimensional setting, the natural framework for many phenomena described by PDEs. Also in this case we need results which permit to approximate feasible trajectories by trajectories staying in the interior of the state constraints. Assuming an inward pointing condition, Theorem 3.2 below guarantees the existence of the required approximation. Notice that, although the literature dealing with infinite dimensional control theory (and infinite dimensional differential inclusions) is quite rich, see e.g. the books [2, 3, 4, 14, 21, 22], the recent paper [11] and the bibliography therein, to our knowledge, no similar results are known in this setting. As an application, we obtain our main result, a relaxation theorem in infinite dimension (see Theorem 3.1).

We deal with great generality, allowing the state space X to be a separable Banach space. Hence, our analysis applies to some interesting and delicate frameworks as the space of essentially bounded functions and the space of continuous functions. For this reason, in this context, the relaxation theorem is obtained under a version of condition (1.5), requiring some uniformity on a neighborhood of ∂K and with respect to the semigroup. Nevertheless, as illustrated in Section 3, if some compactness conditions are satisfied, a much more simple condition, analogue to the finite dimensional (1.5) is sufficient.

We consider the following inward pointing condition:

$$\begin{gathered}\forall\, \bar{x} \in \partial K,\ \exists\, \eta,\ \rho,\ M > 0 \text{ such that } \forall\, t \in I,\ \forall\, x \in K \cap B(\bar{x}, \eta),\\ \forall\, v \in \overline{\mathrm{co}}F(t,x) \text{ satisfying } \sup_{\tau \le \eta,\ \xi \in \partial d_K(z_0)} \langle \xi,\, S(\tau)\, v\rangle \ge 0, \quad \text{for some } z_0 \in B(x,\eta),\\ \exists\, \bar{v} \in \overline{\mathrm{co}}F(t,x) \cap B(v, M) \text{ satisfying } \sup_{\substack{z \in B(S(\tau)x, \eta)\\ \tau \le \eta, \xi \in \partial d_K(z)}} \big\langle \xi, S(\tau)\,(\bar{v} - v)\big\rangle < -\rho.\end{gathered} \tag{1.6}$$

Notice that condition (1.6) deals with the set-valued map $\overline{\mathrm{co}}F$, since, in order to prove the relaxation theorem, we must be able to approximate relaxed trajectories by relaxed trajectories lying in the interior of K. However, under additional compactness conditions, the first convex hull can be removed from (1.6).

Quite interesting for the applications is the case when X is a Hilbert space, see Section 5. In this framework we will provide an alternative version of condition (1.6), which drastically simplifies the analysis when the set of constraints K is convex. The inward pointing condition needed here involves projections on convex sets rather than generalized gradients of the oriented distance function which belong to the dual space X^*.

1.1. **Outline of the paper.** Section 2 contains a list of notations, definitions, and assumptions in use. The main theorems are stated in Section 3. Some results which allow to simplify the inward pointing condition are also proposed. The Hilbert setting is analyzed in Section 4, while Section 5 is devoted to some applications involving PDEs and integrodifferential equations.

2. Preliminaries

In this Section we list the notation and the main assumptions in use throughout the paper.

2.1. **Notation.**

- $B(x,r)$ denotes the closed ball of center $x \in X$ and radius $r > 0$; B is the closed unit ball in X centered at 0;
- given a Banach space Y, $\mathbb{L}(X,Y)$ denotes the Banach space of bounded linear operators from X into Y, $\mathcal{C}(I,X)$ the space of continuous functions from I to X, $L^1(I,X)$ the space of Bochner integrable functions from I to X, and $L^\infty(I,X)$ the space of measurable essentially bounded functions from I to X;
- $\langle \cdot, \cdot \rangle$ stands for the duality pairing on $X^* \times X$;
- μ is the Lebesgue measure on the real line.

We will use the following notion of solution.

Definition 2.1. Let $t_0 \in I$ and $x_0 \in X$. A function $x \in \mathcal{C}([t_0,1],X)$ is a (mild) solution of (1.1) with initial datum $x(t_0) = x_0$ if there exists a function $f \in L^1([t_0,1],X)$ such that

$$f(t) \in F(t, x(t)), \qquad \text{for a.e. } t \in (t_0, 1) \tag{2.1}$$

and

$$x(t) = S(t-t_0)\,x_0 + \int_{t_0}^{t} S(t-s)\,f(s)\,\mathrm{d}s, \qquad \text{for any } t \in [t_0, 1], \tag{2.2}$$

i.e. f is an integrable selection of the set valued map $t \rightsquigarrow F(t, x(t))$ and x is a mild solution (see [25]) of the initial value problem

$$\begin{cases} \dot{x}(t) = \mathbb{A}x(t) + f(t), & \text{for a.e. } t \in [t_0, 1] \\ x(t_0) = x_0. \end{cases} \tag{2.3}$$

In order to simplify the notation, for $t_0 \in I$ and $x_0 \in X$, we denote by $S^K(t_0, x_0)$ the set of mild solutions x of (1.1) satisfying $x(t_0) = x_0$ and (1.3), and by f_x the corresponding measurable selection in (2.3).

Notice that, since $S(t)$ is a strongly continuous semigroup, there exists $M_S > 0$ such that

$$\|S(t)\|_{\mathbb{L}(X,X)} \leq M_S, \qquad \text{for any } t \in I. \tag{2.4}$$

The differential inclusion (1.1) is a convenient tool to investigate for example the semilinear control system

$$\begin{cases} \dot{x}(t) = \mathbb{A}x(t) + f(t, x(t), u(t)) & \text{a.e. } t \in [t_0, 1] \\ u(t) \in U \end{cases} \tag{2.5}$$

where U is an appropriate separable metric space of controls. Setting $F(t, x) = f(t, x, U)$, we can reduce (2.5) to (1.1) by applying a measurable selection theorem.

2.2. **Assumptions.** In our main theorems, we will assume the following conditions:

- positive invariance of K by the semigroup:

$$S(t)\,K \subset K \qquad \forall\, t \in I; \tag{2.6}$$

- $\forall\, t \in I$ and any $x \in X$, $F(t, x)$ is closed, and, for any $x \in X$,

$$\text{the set-valued map } F(\cdot, x) \text{ is Lebesgue measurable;} \tag{2.7}$$

- $F(t, \cdot)$ is locally Lipschitz in the following sense: for any $R > 0$, there exists $k_R \in L^1(I; \mathbb{R}^+)$ such that, for a.e. $t \in I$ and any $x, y \in RB$,

$$F(t, x) \subset F(t, y) + k_R(t)\|x - y\|_X B; \tag{2.8}$$

- there exists $\phi \in L^1(I; \mathbb{R}^+)$ such that, for a.e. $t \in I$ and any $x \in X$,

$$F(t, x) \subset \phi(t)(1 + \|x\|_X)B. \tag{2.9}$$

3. The main results

In this section we state the main results of the paper. The first is a relaxation theorem.

Theorem 3.1. *Assume* (1.6) *and* (2.6)–(2.9). *Then, for any $\varepsilon > 0$ and any feasible trajectory $\hat{x}$ of* (1.2), (1.3), *there exists a trajectory x of* (1.1) *satisfying*

$$x(t_0) = \hat{x}(t_0), \qquad x(t) \in \operatorname{Int} K, \quad \text{for any } t \in (t_0, 1] \tag{3.1}$$

and

$$\|\hat{x} - x\|_{L^\infty(t_0,1;X)} \le \varepsilon. \tag{3.2}$$

The key point in the proof of Theorem 3.1, is a result on approximation of feasible trajectories, by trajectories lying in the interior of the constraint K.

Theorem 3.2. *Assume* (2.6)–(2.9) *and that*

$$\begin{gathered} \forall\, \bar{x} \in \partial K,\ \exists\, \eta,\, \rho,\, M > 0 \text{ such that } \forall\, t \in I,\ \forall\, x \in K \cap B(\bar{x}, \eta), \\ \forall\, v \in F(t,x) \text{ satisfying } \sup_{\tau \le \eta,\ \xi \in \partial d_K(z_0)} \langle \xi,\, S(\tau)\, v\rangle \ge 0, \qquad \text{for some } z_0 \in B(x,\eta), \\ \exists\, \bar{v} \in F(t,x) \cap B(v, M) \text{ satisfying } \sup_{\substack{z \,\in\, B(S(\tau)x,\eta) \\ \tau \le \eta, \xi \in \partial d_K(z)}} \big\langle \xi, S(\tau)\,(\bar{v} - v)\big\rangle < -\rho. \end{gathered} \tag{3.3}$$

Then, for any $\varepsilon > 0$ and any feasible trajectory $\hat{x}$ of (1.1), (1.3), *there exists a trajectory x of* (1.1) *satisfying* (3.1) *and* (3.2).

In the following propositions pointwise versions of the inward pointing condition (3.3) are proposed, see the applications in Section 5.

Proposition 3.3. *Assume that* (2.8)–(2.9) *hold true with time independent $k_R, \phi \in \mathbb{R}^+$,*

$$F(\cdot, x) \text{ is continuous for any } x \in X, \tag{3.4}$$

and

$$F(t, \bar{x}) \text{ is compact, for any } t \in I \text{ and any } \bar{x} \in \partial K. \tag{3.5}$$

Then, assumption (1.5) *implies* (3.3). *Consequently, if* (1.5) *holds true with F replaced by $\overline{co}\, F$, then (1.6) is satisfied.*

In the next proposition the convexity of values of F is needed on the boundary of K.

Proposition 3.4. *Assume that* (3.4) *and* (2.8)–(2.9)*, with time independent $k_R, \phi \in \mathbb{R}^+$, hold true, that X is reflexive, for any $\bar{x} \in \partial K$ and $t \in I$, $F(t, \bar{x})$ is convex, and*

(3.6) *the set-valued map $\partial d_K(\cdot)$ is upper semicontinuous in $\bar{x}$, and $\partial d_K(\bar{x})$ is compact.*

Then, assumption (1.5) *implies* (3.3) *and* (1.6).

Notice that when d_K is C^1 on a neighborhood of ∂K, then condition (3.6) is satisfied. In the proof of Theorem 3.1, we need to approximate relaxed trajectories by relaxed trajectories lying in the interior of K. This is the reason why the inward pointing condition (1.6) required in this case involves the set-valued map $\overline{co} F$. By the way, the first convex hull in (1.6) can be removed in some special cases, as indicated in the next proposition.

Proposition 3.5. *Suppose that for every $t \in I$ and $\bar{x} \in \partial K$, $coF(t, \bar{x})$ is closed and the set-valued map*

$$[0, 1] \ni t \rightsquigarrow co\Big\{ v \in F(t, \bar{x}) : \sup_{\xi \in \partial d_K(\bar{x})} \langle \xi, v\rangle \le 0 \Big\} \tag{3.7}$$

is upper semicontinuous with closed values. Assume that (3.4) *and* (2.8)–(2.9)*, with time independent $k_R, \phi \in \mathbb{R}^+$, hold true. Further, suppose that either* (3.5) *is satisfied, or that*

X is a reflexive space and (3.6) is satisfied. Then, the following assumption: for any $\bar{x} \in \partial K$, there exists $\rho > 0$ such that

$$\forall\, t \in I \text{ and any } v \in F(t,\bar{x}) \text{ satisfying } \sup_{\xi \in \partial d_K(\bar{x})} \langle \xi,\, v\rangle \geq 0, \tag{3.8}$$
$$\exists\, \bar{v} \in co\, F(t,\bar{x}) \text{ satisfying } \sup_{\xi \in \partial d_K(\bar{x})} \big\langle \xi, \bar{v} - v \big\rangle < -\rho\,;$$

implies (1.6).

In the following remark, the special case of affine forcing terms is analyzed, providing further simplification.

Remark 3.6. If d_K is C^1 on a neighborhood of ∂K and, for a subset $U \subset Y$,

$$F(t,x) = f_0(t,x) + g(t,x)U, \tag{3.9}$$

where Y is a Banach space,

$$f_0 : I \times X \to X \qquad \text{and} \qquad g : I \times X \to \mathbb{L}(Y,X),$$

then the classical inward pointing condition implies (1.5) with F and also with $\overline{co}\, F$ whenever either U is compact or Y is reflexive. Namely, assume (3.4), and (2.8), (2.9) for time independent $k_R, \phi \in \mathbb{R}^+$, with F replaced by f_0 and g. If either U is compact or Y is reflexive, then $\overline{co}\, F(t,x) = f_0(t,x) + g(t,x)\overline{co}\, U$. Then the inward pointing condition:

$$\forall\, \bar{x} \in \partial K,\ \forall\, t \in I,\ \exists\, \bar{u} \in U \text{ such that } \langle \nabla d_K(\bar{x}), f_0(t,\bar{x}) + g(t,\bar{x})\bar{u}\rangle < 0 \tag{3.10}$$

implies (1.5) both with F and with $\overline{co}\, F$. Indeed, by compactness of $[0,1]$ and continuity of $f_0(\cdot,\bar{x})$ and $g(\cdot,\bar{x})$, assumption (3.10) yields:

$$\forall\, \bar{x} \in \partial K,\ \exists\, \rho > 0,\ \forall\, t \in I,\ \exists\, \bar{u} \in U \text{ satisfying } \langle \nabla d_K(\bar{x}), f_0(t,\bar{x}) + g(t,\bar{x})\bar{u}\rangle < -\rho. \tag{3.11}$$

Let $t \in I$ and $u \in \overline{co}\, U$ be so that $\langle \nabla d_K(\bar{x}), f_0(t,\bar{x}) + g(t,\bar{x})u\rangle \geq 0$. Thus

$$\langle \nabla d_K(\bar{x}), f_0(t,\bar{x})\rangle \geq -\langle \nabla d_K(\bar{x}), g(t,\bar{x})u\rangle.$$

Then, taking $\bar{u}$ as in (3.11), we obtain

$$\begin{aligned}\langle \nabla d_K(\bar{x}), & f_0(t,\bar{x}) + g(t,\bar{x})\bar{u} - \big(f_0(t,\bar{x}) + g(t,\bar{x})u\big)\rangle \\ &= \langle \nabla d_K(\bar{x}), g(t,\bar{x})\bar{u} - g(t,\bar{x})u\rangle \leq \langle \nabla d_K(\bar{x}), g(t,\bar{x})\bar{u} + f_0(t,\bar{x})\rangle < -\rho,\end{aligned}$$

yielding (1.5) with F and also with $\overline{co}\, F$.

Under the same assumptions and F given by (3.9), let us consider two examples, where condition (3.10) can be further simplified.

Case 1: $0 \in U$. If $\langle \nabla d_K(\bar{x}), f_0(t,\bar{x})\rangle < 0$, for any $\bar{x} \in \partial K$ and $t \in I$, then (3.10) holds for $\bar{u} = 0$.

Case 2: U is the unit sphere in Y. Here, if

$$\langle \nabla d_K(\bar{x}), f_0(t,\bar{x})\rangle < \|g(t,\bar{x})^* \nabla d_K(\bar{x})\|_Y \neq 0, \tag{3.12}$$

for any $\bar{x} \in \partial K$ and $t \in I$, then (3.10) holds for

$$\bar{u} = -\frac{g(t,\bar{x})^* \nabla d_K(\bar{x})}{\|g(t,\bar{x})^* \nabla d_K(\bar{x})\|_Y}, \tag{3.13}$$

where $g(t,\bar{x})^*$ is the adjoint of $g(t,\bar{x})$.

Indeed, in this case, for any $\bar{x} \in \partial K$,

$$\langle \nabla d_K(\bar{x}), f_0(t,\bar{x}) + g(t,\bar{x})\bar{u}\rangle = \langle \nabla d_K(\bar{x}), f_0(t,\bar{x})\rangle - \|g(t,\bar{x})^* \nabla d_K(\bar{x})\|_Y < 0,$$

yielding (3.10).

4. The case of Hilbert spaces

Here we analyze the case when the state space X is Hilbert. In this setting, we show that if the state constraint is convex then the inward pointing condition can be drastically simplified by involving projections on convex sets instead of generalized gradients of the oriented distance function which do belong to the dual space X^*. This turns out to be very useful in the applications, as we will show in Section 5.

Let $\langle \cdot, \cdot \rangle_X$ be the scalar product in X and let K be a proper closed subset of X such that $K = \overline{\operatorname{Int} K}$. Denote by Z the set of points $z \in X \setminus \partial K$ admitting a unique projection $P_{\partial K}(z)$ on ∂K. This set is dense in X (see [26]). For every $z \in Z$, set

$$n_z = \frac{z - P_{\partial K}(z)}{\|z - P_{\partial K}(z)\|_X} \operatorname{sgn}(d_K(z))\,.$$

A new inward pointing condition involving n_z is proposed in this Hilbert framework in order to obtain results analogous to those from Section 3.

Theorem 4.1. *Assume* (2.6)–(2.9). *Then,*

(i) *the assertions of Theorem 3.1 are valid under the following inward pointing condition:*

$$\begin{gathered}\forall\, \bar{x} \in \partial K,\ \exists\, \eta,\ \rho,\ M > 0 \text{ such that } \forall\, t \in I,\ \forall\, x \in K \cap B(\bar{x},\eta),\\ \forall\, v \in \overline{co}\, F(t,x) \text{ satisfying } \sup_{\tau \le \eta,\, z \in Z \cap B(x,\eta)} \langle n_z,\, S(\tau)\, v\rangle_X \ge 0,\\ \exists\, \bar{v} \in \overline{co}\, F(t,x) \cap B(v,M) \text{ satisfying } \sup_{\tau \le \eta,\, z \in Z \cap B(S(\tau)x,\eta)} \big\langle \xi, S(\tau)\,(\bar{v} - v)\big\rangle_X < -\rho.\end{gathered} \tag{4.1}$$

(ii) *the assertions of Theorem 3.2 are valid under the following inward pointing condition:*

$$\begin{gathered}\forall\, \bar{x} \in \partial K,\ \exists\, \eta,\ \rho,\ M > 0 \text{ such that } \forall\, t \in I,\ \forall\, x \in K \cap B(\bar{x},\eta),\\ \forall\, v \in F(t,x) \text{ satisfying } \sup_{\tau \le \eta,\, z \in Z \cap B(x,\eta)} \langle n_z,\, S(\tau)\, v\rangle_X \ge 0, \qquad \text{for some } z_0 \in B(x,\eta),\\ \exists\, \bar{v} \in F(t,x) \cap B(v,M) \text{ satisfying } \sup_{\tau \le \eta,\, z \in Z \cap B(S(\tau)x,\eta)} \big\langle n_z, S(\tau)\,(\bar{v} - v)\big\rangle_X < -\rho.\end{gathered} \tag{4.2}$$

Again, these conditions can be simplified when the data satisfy some compactness assumptions.

Proposition 4.2. *Assume that* (2.8)–(2.9) *with time independent $k_R, \phi \in \mathbb{R}^+$ and* (3.4) *hold true. Further suppose that either* (3.5) *is valid, or $F(t,\bar{x})$ is convex for any $t \in I$ and $\bar{x} \in \partial K$, and*

$$\forall\, \bar{x} \in \partial K,\ \exists\, r > 0 \text{ such that the set } \big\{n_z : z \in Z \cap B(\bar{x},r)\big\} \text{ is pre-compact.} \tag{4.3}$$

Then, the following assumption: for any $\bar{x} \in \partial K$, there exists $\rho > 0$ such that

$$\text{for any } t \in I \text{ and } v \in F(t,\bar{x}) \text{ satisfying } \inf_{\varepsilon>0} \sup_{z\in Z\cap B(\bar{x},\varepsilon)} \langle n_z,\, v\rangle_X \geq 0, \tag{4.4}$$
$$\text{there exists } \bar{v} \in F(t,\bar{x}) \text{ satisfying } \inf_{\varepsilon>0} \sup_{z\in Z\cap B(\bar{x},\varepsilon)} \big\langle n_z, \bar{v} - v\big\rangle_X < -\rho\,.$$

implies (4.2).

Remark 4.3. The proof of Proposition 4.2 provided in Section 6 implies that it is still valid if (4.3) is replaced by the following less restrictive assumption:

for $\bar{x} \in \partial K$ define $\mathcal{N}(\bar{x}) :=\mathrm{Limsup}_{z\to\bar{x},\, z\in Z}\, \{n_z\}$ (the Kuratowski upper limit) and assume that for all $\bar{x} \in \partial K$ the set $\mathcal{N}(\bar{x})$ is compact and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$n_z \in \mathcal{N}(\bar{x}) + \varepsilon B \quad \forall\, z \in Z \cap B(\bar{x}, \delta).$$

In particular, if ∂K is of class C^1, then the above holds true.

4.1. **Convex state constraints.** If K convex, then the inward pointing conditions (4.1), (4.2), and (4.4) can be weakened by replacing Z with $K^c := X \setminus K$. Indeed, any $z \in K^c$ admits a unique projection on ∂K and, as proved in the following proposition, for any $z \in \mathrm{Int}\, K \cap Z$ we can find an element $w \in K^c$ such that $n_z = n_w$.

Proposition 4.4. *Let K be a closed convex set such that $K = \overline{\mathrm{Int}\, K}$. Then, for any $z \in \mathrm{Int}\, K \cap Z$, there exists $w \in K^c$ such that $z - P_{\partial K} z = P_{\partial K} w - w$. In particular, $n_z = n_w$.*

Proof. Let $z \in \mathrm{Int}\, K \cap Z$ and $P_{\partial K}(z)$ be its unique projection on ∂K. By the Hahn-Banach theorem, there exists $p \in X$ such that $\|p\|_X = 1$ and

$$\langle p, P_{\partial K}(z)\rangle_X \leq \langle p, k\rangle_X, \qquad \text{for any } k \in K.$$

Let

$$\mathcal{M}^+ = \big\{x \in X : \langle p, x - P_{\partial K}(z)\rangle_X \geq 0\big\} \supseteq K$$

and

$$\mathcal{M} = \partial\mathcal{M}^+ = \big\{x \in X : \langle p, x - P_{\partial K}(z)\rangle_X = 0\big\}.$$

Then $\mathcal{M}$ is a closed hyperplane in X and there exists a unique projection $P_{\mathcal{M}}(z)$ of z on $\mathcal{M}$. Actually, since $P_{\partial K}(z) \in \mathcal{M}$,

$$\|z - P_{\mathcal{M}}(z)\|_X \leq \|z - P_{\partial K}(z)\|_X,$$

and, since $K \subset \mathcal{M}^+$ and z lies in the interior of K,

$$\|z - P_{\mathcal{M}}(z)\|_X \geq \|z - P_{\partial K}(z)\|_X,$$

we deduce that $P_{\mathcal{M}}(z) = P_{\partial K}(z)$. Take $w = z + 2(P_{\partial K}(z) - z)$. As $z \in \mathrm{Int}\, K \subset \mathrm{Int}\, \mathcal{M}^+$, we have

$$\langle p, w - P_{\partial K}(z)\rangle_X = \langle p, P_{\partial K}(z) - z\rangle_X < 0,$$

yielding $w \in X \setminus \mathcal{M}^+ \subset K^c$. Further, for any $x \in \mathcal{M}$,

$$0 = \langle z - P_{\mathcal{M}}(z), x - P_{\mathcal{M}}(z)\rangle_X = \langle z - P_{\partial K}(z), x - P_{\partial K}(z)\rangle_X = \langle P_{\partial K}(z) - w, x - P_{\partial K}(z)\rangle_X.$$

This implies that $P_{\partial K}(z) = P_{\mathcal{M}}(w)$. Finally, since $\mathcal{M}^+$ is a closed convex set, w admits a unique projection $P_{\mathcal{M}^+}(w) = P_{\mathcal{M}}(w) = P_{\partial K}(z)$. So, for any $k \in K \subset \mathcal{M}^+$,

$$\langle w - P_{\partial K}(z), k - P_{\partial K}(z)\rangle_X \leq 0,$$

implying $P_{\partial K}(z) = P_{\partial K}(w)$. This ends the proof. $\square$

5. Examples

The examples analyzed in this section describe some classical models involving partial differential equations and integrodifferential equations, to which we may apply our abstract results. In all the examples the state constraints satisfy the positive invariance property (2.6).

5.1. **A one-dimensional heat equation.** The first example is a one-dimensional parabolic equation describing the heat flux in a cylindrical bar, whose lateral surface is perfectly insulated and whose length is much larger than its cross-section. The Neumann boundary conditions are assumed, corresponding to the requirement that the heat flux at the two ends of the bar is zero. For $x = x(t,s) : [0,1] \times [0,1] \to \mathbb{R}$ we consider the following inclusion (we omit the variable s in the sequel)

$$\begin{cases} \dot{x}(t) \in \mathbb{A}x(t) + F(t, x(t)), & t \in [0,1] \\ x(0) = x_0. \end{cases} \tag{5.1}$$

The state space is $X = H^1(0,1)$ and the linear operator acting as $\mathbb{A}x = x'' - x$ with domain $D(\mathbb{A}) = \{x \in H^2(0,1;\mathbb{R}) : x'(0) = x'(1) = 0\}$ is the infinitesimal generator of a strongly continuous semigroup $S(t) : X \to X$, see e.g. [29, chapter II]. (The notation prime stands for the distributional derivative.) Classical results in PDEs ensure that, if the initial datum x_0 takes nonnegative values, then the solution x to $\dot{x}(t) = \mathbb{A}x(t)$, $x(0) = x_0$ takes nonnegative values. The reader is referred to [1] or [27], containing a number of examples of sets invariant under the action of the semigroup associated with $\mathbb{A}$. In particular, if the state constraint is the cone of nonnegative functions:

$$x(t) \in K = \{x \in X : x \geq 0\},$$

then the invariance property (2.6) is satisfied. Moreover, K is convex and $\operatorname{Int} K \neq \emptyset$. The state space is endowed with the scalar product

$$\langle x, y\rangle_X = x(0)y(0) + \langle x', y'\rangle_{L^2(0,1)}, \qquad \text{for any } x, y \in X,$$

whose associated norm

$$\|x\|_X^2 = |x(0)|^2 + \|x'\|^2_{L^2(0,1)} \qquad \text{for any } x \in X,$$

turns out to be equivalent to the usual one $\|\cdot\|_{H^1(0,1)}$. We show next that the set

$$\{n_z : z \in K^c\} \ \text{ is pre-compact.} \tag{5.2}$$

Since K is a closed convex cone, then any $z \in K^c$ can be uniquely represented as

$$z = P_{\partial K}(z) + b(z),$$

with $b(z) \in K^-$, here K^- is the negative polar cone to K. By [32],

$$K^- = \{p \in X : p' \ \text{ is nondecreasing and } p(0) \leq p'(s) \leq 0, \ \text{for a.e. } s \in [0,1]\},$$

see also [24] where an explicit formula for b is provided. To prove (5.2), notice that

$$\Big\{n_z = \frac{b(z)}{\|b(z)\|_X} : z \in K^C\Big\} \subset Q := \Big\{\frac{p}{\|p\|_X} : p \in K^-, p \neq 0\Big\} \subset \partial B.$$

Any $y \in Q$ satisfies y' nondecreasing and

$$-1 \leq y(0) \leq y'(s) \leq 0, \qquad \text{for a.e. } s \in [0,1].$$

So, taking a sequence $\{y_n\}$ in Q,

$$\|y_n\|_{W^{1,\infty}(0,1)} := \|y_n\|_{L^\infty(0,1)} + \|y_n'\|_{L^\infty(0,1)} \leq 3,$$

implying that $y_n(0) \to y(0)$ (up to a subsequence). Since y_n' is nondecreasing, Helly's selection theorem, see [20], allows to deduce that, (again up to a subsequence),

$$y_n'(s) \to g(s), \qquad \text{for a.e. } s, \text{ with } g \in L^\infty(0,1),$$

and, applying Lebesgue dominated theorem, we deduce that $y_n \to g$ in $L^2(0,1)$. Further,

$$y_n(s) = y_n(0) + \int_0^s y_n'(\tau)\mathrm{d}\tau \to y(s) := y(0) + \int_0^s g(\tau)\mathrm{d}\tau, \qquad \text{for any } s \in [0,1].$$

Hence $g = y'$, $y \in W^{1,\infty}(0,1) \subset H^1(0,1)$, yielding the required pre-compactness.

Let F satisfy assumptions (2.8)–(2.9) with time independent $k_R, \phi \in \mathbb{R}^+$, (3.4), and let $F(t,\bar{x})$ be convex, for any $\bar{x} \in \partial K$ and $t \in I$. Taking into account Proposition 4.2 and the results in subsection 4.1, the inward pointing condition (4.2) is implied by the following assumption: for any $\bar{x} \in \partial K$ there exists $\rho > 0$ such that

$$\text{for any } t \in I \text{ and } v \in F(t,\bar{x}) \text{ satisfying } \inf_{\varepsilon>0} \sup_{z \in K^c \cap B(\bar{x},\varepsilon)} \langle n_z, v\rangle_X \geq 0,$$

$$\text{there exists } \bar{v} \in F(t,\bar{x}) \text{ satisfying } \inf_{\varepsilon>0} \sup_{z \in K^c \cap B(\bar{x},\varepsilon)} \big\langle n_z, \bar{v} - v\big\rangle_X < -\rho\,.$$

5.2. **Fourier's problem of the ring.** In the second example we consider the temperature distribution in a homogeneous isotropic circular ring with diameter small in comparison with its length and perfectly insulated lateral surfaces. This problem can be modeled by a one-dimensional equation with periodic boundary conditions

$$\begin{cases} \dot{x}(t) \in \mathbb{A}x(t) + F(t,x(t)), & t \in [0,1] \\ x(0) = x_0, \end{cases} \tag{5.3}$$

where $x = x(t,s) : [0,1] \times [0,1] \to \mathbb{R}$ (s is omitted as in the previous example), the state space is $X = H^1_{per}(0,1) := \big\{x \in H^1(0,1;\mathbb{R}) : x(0) = x(1)\big\}$, the linear operator acting as $\mathbb{A}x = x''$ with domain $D(\mathbb{A}) = H^2(0,1;\mathbb{R}) \cap H^1_{per}(0,1)$ is the infinitesimal generator of a strongly continuous semigroup $S(t) : X \to X$, see e.g. [9]. As before we supplement inclusion (5.3) with the state constraint

$$x(t) \in K = \big\{x \in X : x \geq 0\big\}.$$

Then, K satisfies condition (2.6), see for instance [23] dealing with invariant sets for semigroups. Again, K is a closed and convex set with non empty interior. Hence, by the results contained in subsection 4.1, the inward pointing conditions (4.1), (4.2), and (4.4) hold true with Z replaced by K^c.

5.3. **A model for Boltzmann viscoelasticity.** The last example deals with the phenomena of isothermal viscoelasticity. An integrodifferential inclusion is involved, since, as outlined in the seminal works of Boltzmann and Volterra [7, 8, 30, 31], a correct description of the mechanical behavior of elastic bodies requires the notion of memory. The key assumption in this theory is that both the instantaneous stress and the past stresses influence the evolution of the displacement function $y = y(\boldsymbol{x}, t) : \Omega \times \mathbb{R} \to \mathbb{R}$. Here $\Omega \subset \mathbb{R}^3$, a bounded domain with smooth boundary $\partial\Omega$, represents the region occupied by the elastic body. Omitting the variable $\boldsymbol{x}$ in the sequel, we study the following inclusion

$$\ddot{y}(t) + A\Big[y(t) - \int_0^\infty \mu(s) y(t-s)\,\mathrm{d}s\Big] \in \mathcal{F}(t, y(t)), \qquad t > 0, \tag{5.4}$$

where, $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, according to the assumption that the body is kept fixed at the boundary of Ω, the *memory kernel* μ, taking into account the viscoelastic behavior, is supposed to be a (nonnegative) nonincreasing and summable function on $\mathbb{R}^+$, with total mass

$$\kappa = \int_0^\infty \mu(s)\mathrm{d}s \in (0,1),$$

piecewise absolutely continuous, and thus differentiable almost everywhere with $\mu' \leq 0$. Equation (5.4) is supplemented with the following initial condition

$$y(0) = y_0, \qquad \dot{y}(0) = z_0, \qquad y(-s)_{|s>0} = \phi_0(s),$$

for some prescribed data y_0, z_0, ϕ_0, the latter taking into account the past history of y. Applying Dafermos' *history approach*, see [13], we can write (5.4) as a differential inclusion of type (1.1). To this aim, we first introduce an auxiliary variable which contains all the information about the unknown function up to the actual time

$$\eta^t(s) = y(t) - y(t-s), \qquad t \geq 0,\ s > 0$$

and we recast problem (5.4) as the system of two variables $y = y(t)$ and $\eta = \eta^t(s)$

$$\begin{cases} \ddot{y}(t) + A\Big[(1-\kappa)y(t) + \displaystyle\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\Big] \in \mathcal{F}(t, y(t)), \\ \dot{\eta}^t = T\eta^t + \dot{y}(t). \end{cases} \tag{5.5}$$

with initial conditions

$$y(0) = y_0, \qquad \dot{y}(0) = z_0, \qquad \eta^0 = \eta_0 = y_0 - \phi_0.$$

Here the operator T is the infinitesimal generator of the right-translation semigroup on the *memory space* $\mathcal{M} = L^2_\mu(\mathbb{R}^+; H_0^1(\Omega))$, namely,

$$T\eta = -\eta' \qquad \text{with domain} \qquad \mathrm{dom}(T) = \big\{\eta \in \mathcal{M} : \eta' \in \mathcal{M},\ \eta(0) = 0\big\}.$$

The notation prime standing for the distributional derivative, and $\eta(0) = \lim_{s\to 0} \eta(s)$ in $H_0^1(\Omega)$. In [11], details and related bibliography can be found, jointly with some applications of this model to optimal control problems. Now, defining the linear operator $\mathbb{A}$ on the state space $X = L^2(\Omega) \times H_0^1(\Omega) \times \mathcal{M}$, acting as

$$\mathbb{A}(y, z, \eta) = \Big(z, -A\big[(1-\kappa)y + \int_0^\infty \mu(s)\eta(s)\,\mathrm{d}s\big], T\eta + z\Big)$$

with domain

$$\mathrm{dom}(\mathbb{A}) = \left\{ (y,z,\eta) \in X \;\middle|\; \begin{array}{l} z \in H_0^1(\Omega)\,,\ \eta \in \mathrm{dom}(T), \\ (1-\kappa)y + \int_0^\infty \mu(s)\eta(s)\mathrm{d}s \in H^2(\Omega) \cap H_0^1(\Omega) \end{array} \right\}$$

and setting

$$x(t) = (y(t), z(t), \eta(t)), \qquad x_0 = (y_0, z_0, \eta_0), \qquad F(t, x(t)) = \big(0, \mathcal{F}(t, y(t)), 0\big),$$

we view (5.5) as the following problem in X:

$$\left\{ \begin{array}{ll} \dot{x}(t) \in \mathbb{A}x(t) + F(t, x(t)), & \text{for } t \in I \\ x(0) = x_0. \end{array} \right. \tag{5.6}$$

The operator $\mathbb{A}$ generates a strongly continuous semigroup of contractions $S(t) : X \to X$ whose first component is a solution of (5.4) for $\mathcal{F} = 0$. Here, taking as a state constraint in (5.6)

$$K = B,$$

we deduce that (2.6) is satisfied. Further, let U, f_0 and g be as in Remark 3.6. Since $K = B$ in a Hilbert space, an appropriate adaptation on the basis of Section 4 of the inward pointing condition (3.10) reads: for any $\bar{x} \in X$ with $\|\bar{x}\| = 1$ and any $t \in I$, there exists $\bar{u} \in U$ such that

$$\langle \bar{x}, f_0(t, \bar{x}) + g(t, \bar{x})\bar{u} \rangle < 0.$$

In particular in the case when U is the unit sphere in $\mathbb{R}^N$, the condition

$$\langle \bar{x}, f_0(t, \bar{x}) \rangle < \|g(t, \bar{x})^* \bar{x}\|_{\mathbb{R}^N} \neq 0,$$

implies the above inward pointing condition, for $\bar{u}$ defined by $\bar{u} = -\frac{g(t,\bar{x})^*\bar{x}}{\|g(t,\bar{x})^*\bar{x}\|_{\mathbb{R}^N}}$. As discussed in Remark 3.6, Proposition 3.5 holds in this case implying the validity of Theorems 3.1 and 3.2.

Appendix

This section contains some technical results needed in the course of the investigation. The first is an infinite dimensional version of the Filippov Theorem, see [15, Theorem 1.2], modified here in a suitable form for our scopes.

Lemma A.1. *Let $\delta_0 \geq 0$ and $t_0 \in I$. Assume* (2.7)–(2.8)*, let y be a solution to* (2.3) *in* $[t_0, 1]$*, for some $f \in L^1(t_0, 1; X)$. Set $R = \frac{1}{2}\max_{t \in [t_0,1]} \|y(t)\|_X$,*

$$\gamma(t) = \mathrm{dist}\big(f(t), F(t, y(t))\big) \qquad \textit{and} \qquad m(t) = M_S \mathrm{e}^{M_S \int_{t_0}^t k_R(s)\mathrm{d}s}.$$

If $m(1)\big(\delta_0 + \int_{t_0}^1 \gamma(s)\mathrm{d}s\big) < \frac{R}{2}$, then, for any $y_0 \in y(t_0) + \delta_0 B$ and any $\beta > 1$, there exists a solution x to (1.1) *in* $[t_0, 1]$*, satisfying $x(t_0) = y_0$,*

$$\|x(t) - y(t)\|_X \leq m(t)\Big(\delta_0 + \beta \int_{t_0}^t \gamma(s)\mathrm{d}s\Big), \qquad \textit{for any } t \in [t_0, 1],$$

and

$$\|f_x(t) - f(t)\|_X \leq k_R(t)m(t)\Big(\delta_0 + \beta \int_{t_0}^t \gamma(s)\mathrm{d}s\Big) + \beta\gamma(t), \qquad \textit{for a.e. } t \in [t_0, 1],$$

where $f_x \in L^1(t_0, 1; X)$ *is so that* (2.1) *and* (2.2) *hold true for* $f = f_x$.

Proof. The proof proceeds exactly as in [15, Theorem 1.2]. The only difference is in the first line of page 109, while applying Lemma 1.3 of [15]. The assumption $\beta > 1$ is needed to ensure that, for a.e. t, the set

$$F(t, y(t)) \cap \{f(t) + \beta\gamma(t)B\} \neq \emptyset.$$

Indeed, if $\gamma(t) = 0$, the definition of γ ensures that $f(t) \in F(t, y(t))$, while, if $\gamma(t) > 0$, since $\beta > 1$, from the very definition of distance and the measurable selection theorem we get that there exists a measurable selection $w(t) \in F(t, y(t))$ such that $\|f(t) - w(t)\|_X \leq \beta\gamma(t)$. Thus

$$w(t) \in F(t, y(t)) \cap \{f(t) + \beta\gamma(t)B\}.$$

Notice that in finite dimension we can take $\beta = 1$, recovering the original Filippov Theorem. □

The second result is a version of the mean value theorem for the oriented distance d_K in Hilbert spaces. Here we make use of the notations introduced in Section 4.

Lemma A.2. *Let* $(X, \langle\, , \,\rangle_X)$ *be a Hilbert space. For every* $x, y \in X$ *we have*

$$d_K(y) - d_K(x) \leq \inf_{\varepsilon>0} \sup_{z\in Z\cap([x,y]+\varepsilon B)} \langle n_z, y - x\rangle_X \,.$$

Proof. Let x, $y \in X$ and fix $\varepsilon > 0$. It is enough to consider the case $x \neq y$. For every $s \in [0, 1]$ set $\gamma(s) = x + s(y - x)$. Then $d_K(\gamma(\cdot))$ is absolutely continuous. It is sufficient to prove that for almost every $s \in [0, 1]$

$$\frac{\mathrm{d}}{\mathrm{d}s} d_K(\gamma(s)) \leq \sup_{z\in Z\cap B(\gamma(s),\varepsilon)} \langle n_z, y - x\rangle_X \,.$$

Indeed, then it follows that

$$d_K(y) - d_K(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} d_K(\gamma(s))\,\mathrm{d}s \leq \sup_{z\in Z\cap([x,y]+\varepsilon B)} \langle n_z, y - x\rangle_X \,.$$

The maps $d_K(\gamma(\cdot))$ and $d_K^2(\gamma(\cdot))$ are almost everywhere differentiable. Denote by $D \subset (0, 1)$ the set on which both functions are differentiable and fix $s \in D$. We distinguish three cases.

Case 1: $\gamma(s) \notin K$. Let $L > 0$ be a Lipschitz constant for d_K^2 on $\gamma([0, 1]) + B$. Recall that $|d_K(x)| = \mathrm{dist}(x, \partial K)$, for all $x \in X$. Then for each $0 < h < \min\{\varepsilon, 1 - s\}$ and every $z \in Z \cap (B(\gamma(s), h^2) \smallsetminus K)$ we have

$$d_K^2(\gamma(s)) \geq \|z - P_{\partial K}(z)\|_X^2 - L\,h^2 \tag{A.1}$$

and

$$\begin{aligned} d_K^2(\gamma(s + h)) &\leq \|\gamma(s + h) - P_{\partial K}(z)\|_X^2 = \|\gamma(s) - z\|_X^2 \\ &+ 2\langle \gamma(s) - z\,,\, z - P_{\partial K}(z) + h(y - x)\rangle_X + \|z - P_{\partial K}(z) + h(y - x)\|_X^2 \\ &= \|z - P_{\partial K}(z) + h(y - x)\|_X^2 + o(h)\,. \end{aligned}$$

Consequently, for all $h > 0$ small enough,

$$\begin{aligned}
\frac{d_K^2(\gamma(s+h))-d_K^2(\gamma(s))}{h} &\leq \inf_{z\in Z\cap(B(\gamma(s),h^2)\setminus K)} \frac{\|z-P_{\partial K}(z)+h(y-x)\|_X^2-\|z-P_{\partial K}(z)\|_X^2}{h} + o(1) \\
&= \inf_{z\in Z\cap(B(\gamma(s),h^2)\setminus K)} 2\,\langle z - P_{\partial K}(z)\,,\, y-x\rangle_X + o(1) \\
&= \inf_{z\in Z\cap(B(\gamma(s),h^2)\setminus K)} 2\,d_K(\gamma(s))\,\langle n_z\,,\, y-x\rangle_X + o(1) \\
&\leq \sup_{z\in Z\cap B(\gamma(s),\varepsilon)} 2\,d_K(\gamma(s))\,\langle n_z\,,\, y-x\rangle_X + o(1)\,.
\end{aligned}$$

Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s} d_K^2(\gamma(s)) \leq \sup_{z\in Z\cap B(\gamma(s),\varepsilon)} 2\,d_K(\gamma(s))\,\langle n_z\,,\, y-x\rangle_X$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s} d_K(\gamma(s)) = \frac{1}{2\,d_K(\gamma(s))}\cdot\frac{\mathrm{d}}{\mathrm{d}s} d_K^2(\gamma(s)) \leq \sup_{z\in Z\cap B(\gamma(s),\varepsilon)} \langle n_z\,,\, y-x\rangle_X\,.$$

Case 2: $\gamma(s) \in \operatorname{Int} K$. Similarly to Case 1, for every $0 < h < \min\{\varepsilon, s\}$ and every $z \in Z \cap (B(\gamma(s), h^2) \setminus K^c)$ we have (A.1) and

(A.2)
$$\begin{aligned}
&d_K^2(\gamma(s-h)) \leq \|\gamma(s-h) - P_{\partial K}(z)\|_X^2 \\
&= \|\gamma(s)-z\|_X^2 + 2\langle\gamma(s)-z\,,\, z-P_{\partial K}(z)-h(y-x)\rangle_X + \|z-P_{\partial K}(z)-h(y-x)\|_X^2 \\
&= \|z-P_{\partial K}(z)-h(y-x)\|_X^2 + o(h)\,.
\end{aligned}$$

Inequalities (A.1) and (A.2) yield

$$\begin{aligned}
\frac{d_K^2(\gamma(s-h))-d_K^2(\gamma(s))}{h} &\leq \inf_{z\in Z\cap(B(\gamma(s),h^2)\setminus K^c)} 2\,\langle P_{\partial K}(z)-z\,,\, y-x\rangle_X + o(1) \\
&= \inf_{z\in Z\cap(B(\gamma(s),h^2)\setminus K^c)} 2\,(-d_K(\gamma(s)))\,\langle n_z\,,\, y-x\rangle_X + o(1) \\
&\leq 2\,(-d_K(\gamma(s))) \sup_{z\in Z\cap B(\gamma(s),\varepsilon)} \langle n_z\,,\, y-x\rangle_X + o(1)\,.
\end{aligned}$$

Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s} d_K^2(\gamma(s)) \geq 2\,d_K(\gamma(s)) \sup_{z\in Z\cap B(\gamma(s),\varepsilon)} \langle n_z\,,\, y-x\rangle_X$$

and again

$$\frac{\mathrm{d}}{\mathrm{d}s} d_K(\gamma(s)) = \frac{1}{2\,d_K(\gamma(s))}\cdot\frac{\mathrm{d}}{\mathrm{d}s} d_K^2(\gamma(s)) \leq \sup_{z\in Z\cap B(\gamma(s),\varepsilon)} \langle n_z\,,\, y-x\rangle_X\,.$$

Case 3: $\gamma(s) \in \partial K$. Let us first suppose that $|\frac{\mathrm{d}}{\mathrm{d}s} d_K(\gamma(s))| = 2C > 0$. Then for all $h > 0$ small enough we have $|d_K(\gamma(s+h))| \geq Ch$, so that the point s is isolated in the set $\{s \in D \,:\, \gamma(s) \in \partial K\}$. Consequently,

$$\mu\left(\left\{s \in D \,:\, \gamma(s) \in \partial K\,,\, \frac{\mathrm{d}}{\mathrm{d}s} d_K(\gamma(s)) \neq 0\right\}\right) = 0\,.$$

It remains to verify that in the case where $\frac{\mathrm{d}}{\mathrm{d}s}d_K(\gamma(s))=0$, we have

$$\sup_{z\in Z\cap B(\gamma(s),\varepsilon)}\langle n_z\,,\,y-x\rangle_X\geq 0\,.$$

Assume by contradiction that there exists $\alpha>0$ such that

$$\sup_{z\in Z\cap B(\gamma(s),\varepsilon)}\langle n_z\,,\,y-x\rangle_X\leq -\alpha\,.$$

Let $0<h<\varepsilon/4\|y-x\|_X$, $\{\xi_i\}_{i\in\mathbb{N}}\subset B(\gamma(s),\frac{\varepsilon}{2})\cap\mathrm{Int}\,K$ converge to $\gamma(s)$ as $i\to+\infty$ and set, for every $0\leq r\leq h$, $\gamma_i(r)=\xi_i+r(y-x)$. Fix $i\in\mathbb{N}$ and define

$$h_i=\sup\{0<\tilde{h}<h\ :\ \gamma_i([0,\tilde{h}])\subset\mathrm{Int}\,K\}$$

Proceeding as in Case 2, we can prove that

$$\frac{\mathrm{d}}{\mathrm{d}r}d_K(\gamma_i(r))\leq\sup_{z\in Z\cap B(\gamma_i(r),\frac{\varepsilon}{4})}\langle n_z\,,\,y-x\rangle_X\leq-\alpha\qquad\text{for a.e. } r\in[0,h_i]\,.$$

Then, $\gamma_i(h_i)\in\mathrm{Int}\,K$ and, consequently, $h_i=h$. Summarizing, we have obtained that for every $i\in\mathbb{N}$

$$\frac{\mathrm{d}}{\mathrm{d}r}d_K(\gamma_i(r))\leq\sup_{z\in Z\cap B(\gamma_i(r),\frac{\varepsilon}{4})}\langle n_z\,,\,y-x\rangle_X\leq-\alpha\qquad\text{for a.e. } r\in[0,h]\,.$$

Therefore, for every $i\in\mathbb{N}$ and $h\in(0,\frac{\varepsilon}{4\|y-x\|_X})$ we have $d_K(\gamma_i(h))-d_K(\gamma_i(0))\leq-\alpha\,h$. Taking the limit as $i\to+\infty$, we obtain

$$d_K(\gamma(s+h))-d_K(\gamma(s))\leq-\alpha\,h\,.$$

This is impossible, since $\frac{\mathrm{d}}{\mathrm{d}s}d_K(\gamma(s))=0$.

□

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