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-Continuous Kleene ω -Algebras

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Abstract

We define and study basic properties of *-continuous Kleene ω -algebras that involve a *-continuous Kleene algebra with a *-continuous action on a semimodule and an infinite product operation that is also *-continuous. We show that *-continuous Kleene ω -algebras give rise to iteration semiring-semimodule pairs. We show how our work can be applied to solve certain energy problems for hybrid systems.

1 Introduction

A *Kleene algebra* [14] is an idempotent semiring $S = (S, \vee, \cdot, \perp, 1)$ equipped with a star operation $*$: $S \rightarrow S$ such that for all $x, y \in S$, yx^* is the least solution of the fixed point equation $z = zx \vee y$ and x^*y is the least solution of the fixed point equation $z = xz \vee y$.

Examples of Kleene algebras include the language semiring $P(A^*) = (P(A^*), \vee, \cdot, \perp, 1)$ over an alphabet A , whose elements are the subsets of the set A^* of all finite words over A , and whose operations are set union and concatenation, with the languages \emptyset and $\{\epsilon\}$ serving as \perp and 1 . Here, ϵ denotes the empty word. The star operation is the usual Kleene star: $X^* = \bigcup_{n \geq 0} X^n = \{u_1 \dots u_n : u_1, \dots, u_n \in L, n \geq 0\}$, for all $X \subseteq A^*$.

Another example is the Kleene algebra $P(A \times A) = (P(A \times A), \vee, \cdot, 0, 1)$ of binary relations over any set A , whose operations are union, relational composition (written in diagrammatic order), and where the empty relation \emptyset and the identity relation id serve as the constants \perp and 1 . The star operation is the formation of the reflexive-transitive closure, so that $R^* = \bigcup_{n \geq 0} R^n$ for all $R \in P(A \times A)$.

The above examples are in fact *continuous Kleene algebras*, i.e., idempotent semirings S such that equipped with the natural order, they are all complete lattices (hence all suprema exist), and the product operation preserves arbitrary suprema in either argument:

$$y(\bigvee X) = \bigvee yX \quad \text{and} \quad (\bigvee X)y = \bigvee Xy$$

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for all $X \subseteq S$ and $y \in S$. The star operation is given by $x^* = \bigvee_{n \geq 0} x^n$, so that x^* is the supremum of the set $\{x^n : n \geq 0\}$ of all powers of x . It is well-known that the language semirings $P(A^*)$ may be identified as the *free* continuous Kleene algebras (in a suitable category of continuous Kleene algebras).

A larger class of models is given by the **-continuous Kleene algebras* [14]. By the definition of *-continuous Kleene algebra $S = (S, \vee, \cdot, \perp, 1)$, only suprema of sets of the form $\{x^n : n \geq 0\}$ need to exist, where x is any element of S , and x^* is given by this supremum. Moreover, product preserves such suprema in both of its arguments:

$$y\left(\bigvee_{n \geq 0} x^n\right) = \bigvee_{n \geq 0} yx^n \quad \text{and} \quad \left(\bigvee_{n \geq 0} x^n\right)y = \bigvee_{n \geq 0} x^n y.$$

For any alphabet A , the collection $R(A^*)$ of all regular languages over A is an example of a *-continuous Kleene algebra which is not a continuous Kleene algebra. The Kleene algebra $R(A^*)$ may be identified as the free *-continuous Kleene algebra on A . It is also the free Kleene algebra on A , cf. [13]. There are several other characterizations of $R(A^*)$, the most general of which identifies $R(A^*)$ as the free iteration semiring on A satisfying the identity $1^* = 1$, cf. [2, 15].

For non-idempotent extensions of the notions of continuous Kleene algebras, *-continuous Kleene algebras and Kleene algebras, we refer to [7, 8].

When A is an alphabet, let A^ω denote the set of all ω -words (sequences) over A . An ω -language over A is a subset of A^ω . It is natural to consider the set $P(A^\omega)$ of all languages of ω -words over A , equipped with the operation of set union as \vee and the empty language \emptyset as \perp , and the left action of $P(A^*)$ on $P(A^\omega)$ defined by $XV = \{xv : x \in X, v \in V\}$ for all $X \subseteq A^*$ and $V \subseteq A^\omega$. It is clear that $(P(A^\omega), \vee, \perp)$ is in fact a $P(A^*)$ -semimodule and thus $(P(A^*), P(A^\omega))$ is a semiring-semimodule pair. We may also equip $(P(A^*), P(A^\omega))$ with an infinite product operation mapping an ω -sequence (X_0, X_1, \dots) over $P(A^*)$ to the ω -language $\prod_{n \geq 0} X_n = \{x_0 x_1 \dots \in A^\omega : x_n \in X_n\}$. (Thus, an infinite number of the x_n must be different from ϵ . Note that $1^\omega = \perp$ holds.) The semiring-semimodule pair so obtained is a continuous Kleene ω -algebra.

More generally, we call a semiring-semimodule pair (S, V) a *continuous Kleene ω -algebra* if S is a continuous Kleene algebra (hence S and V are idempotent), V is a complete lattice with the natural order, and the action preserves all suprema in either argument. Moreover, there is an infinite product operation which is compatible with the action and associative in the sense that the following hold for all $x, x_0, x_1, \dots \in S$:

- $x(\prod_{n \geq 0} x_n) = \prod_{n \geq 0} y_n$, where $y_0 = x$ and $y_n = x_{n-1}$ for all $n > 0$,
- $\prod_{n \geq 0} x_n = \prod_{k \geq 0} y_k$, whenever there is a sequence of integers $0 = i_0 \leq i_1 \leq \dots$ increasing without a bound such that $y_k = x_{i_k} \cdots x_{i_{k+1}-1}$ for all $k \geq 0$.

Moreover, the infinite product operation preserves all suprema:

- $\prod_{n \geq 0} (\bigvee X_n) = \bigvee \{\prod_{n \geq 0} x_n : x_n \in X_n, n \geq 0\}$,

for all $X_0, X_1, \dots \subseteq S$.

The above notion of continuous Kleene ω -algebra may be seen as a special case of the not necessarily idempotent complete semiring-semimodule pairs of [10].

Our aim in this paper is to provide an extension of the notion of continuous Kleene ω -algebras to **-continuous Kleene ω -algebras* which are semiring-semimodule pairs (S, V) consisting of a *-continuous Kleene algebra $S = (S, \vee, \cdot, \perp, 1)$ acting on a necessarily idempotent semimodule $V = (V, \vee, \perp)$, such that the action preserves certain suprema in its first argument, and which are equipped with an infinite product operation satisfying the above compatibility and associativity conditions and some weaker forms of the last axiom. We will define both a finitary and a non-finitary version of *-continuous Kleene ω -algebras.

We will establish several properties of *-continuous Kleene ω -algebras, including the existence of the supremum of certain subsets related to regular ω -languages. Then we will use these results in our characterization of the free *-continuous Kleene ω -algebras, at least in the finitary version.

When (S, V) is a *-continuous Kleene algebra, we may define an omega operation $S \rightarrow V$ by $x^\omega = \prod_{n \geq 0} x_n$, where $x_n = x$ for each $n \geq 0$. One of our results will show that equipped with this omega operation, each *-continuous Kleene ω -algebra gives rise to an iteration semiring-semimodule pair. This extends the known result that each *-continuous Kleene algebra is an iteration semiring.

Part of this work is motivated by an application to so-called *energy problems* for hybrid systems. When modeling systems with constraints on energy consumption, it is of interest to know whether certain states are reachable under the given energy constraints, or whether the system admits an infinite run. In the context of formal modeling and verification, such problems were first taken up in [3], which spurred a number of other papers on similar problems.

We have shown in [6] that most of these problems can be covered by a new abstract notion of *energy automaton*, which is a finite automaton in which the transitions are labeled with functions which map input energy to output energy. The intuition is that the transitions of an energy automaton are *energy transformations*, so given an initial energy, one can ask whether a final state is reachable when starting with this initial energy, or whether there is a Büchi run given this initial energy.

We will show in the later parts of this paper that energy functions form a finitary *-continuous Kleene ω -algebra, hence that energy problems can be solved using the techniques developed here.

2 Semirings and semiring-semimodule pairs

Although we already mentioned semirings and semimodules in the introduction, in this section we briefly recall the main definitions from [1, 12] in order to make the paper self-contained.

Recall from [1, 12] that a *semiring* $S = (S, +, \cdot, 0, 1)$ consists of a commutative monoid $(S, +, 0)$ and a monoid $(S, \cdot, 1)$ such that the distributive laws

$$\begin{aligned} x(y + z) &= xy + xz \\ (y + z)x &= yx + zx \end{aligned}$$

and the zero laws

$$0 \cdot x = 0 = x \cdot 0$$

hold for all $x, y, z \in S$. It follows that the product operation distributes over all finite sums.

An *idempotent semiring* is a semiring S whose sum operation is idempotent, so that $x + x = x$ for all $x \in S$. Each idempotent semiring S is partially ordered by the relation $x \leq y$ iff $x + y = y$, and then sum and product preserve the partial order and 0 is the least element. Moreover, for all $x, y \in S$, $x + y$ is the least upper bound of the set $\{x, y\}$. Accordingly, in an idempotent semiring S , we will usually denote the sum operation by \vee and 0 by \perp .

In addition to semirings, we will also consider semiring-semimodule pairs. Suppose that $S = (S, +, \cdot, 0, 1)$ is a semiring and $V = (V, +, 0)$ a commutative monoid. When V is equipped with a left S -action $S \times V \rightarrow V$, $(s, v) \mapsto sv$, satisfying

$$\begin{aligned}(s + s')v &= sv + s'v \\ s(v + v') &= sv + sv' \\ (ss')v &= s(s'v) \\ 0s &= 0 \\ s0 &= 0 \\ 1v &= v\end{aligned}$$

for all $s, s' \in S$ and $v \in V$, then we call V a (unitary left) S -semimodule and (S, V) a *semiring-semimodule pair*. Note that when S is idempotent, then V is necessarily idempotent, so that we use the notation (V, \vee, \perp) .

3 Free continuous Kleene ω -algebras

We have defined continuous Kleene ω -algebras in the introduction as idempotent semiring-semimodule pairs (S, V) such that $S = (S, \vee, \cdot, \perp, 1)$ is a continuous Kleene algebra and $V = (V, \vee, \perp)$ is a continuous S -semimodule. Thus, equipped with the natural order, \leq , S and V are complete lattices and the product and the action preserve all suprema in either argument. Moreover, there is an infinite product operation, satisfying the compatibility and associativity conditions, which preserves all suprema.

In this section, we offer descriptions of the free continuous Kleene ω -algebras and the free continuous Kleene ω -algebras satisfying the identity $1^\omega = \perp$.

A homomorphism between continuous Kleene algebras preserves all operations. A homomorphism is continuous if it preserves all suprema. We recall the following basic result.

Theorem 3.1 *For each set A , the language semiring $(P(A^*), \vee, \cdot, \perp, 1)$ where \vee is set union, \cdot is concatenation and the constants \perp and 1 are \emptyset and $\{\epsilon\}$, respectively, is the free continuous Kleene algebra on A .*

Thus, if S is a continuous Kleene algebra and $h : A \rightarrow S$ is any function, then there is a unique continuous homomorphism $h^\sharp : A^* \rightarrow S$ extending h .

In view of Theorem 3.1, it is not surprising that the continuous Kleene ω -algebras can be described using languages of finite and ω -words.

Suppose that A is a set. Let A^ω denote the set of all ω -words over A , and let $A^\infty = A^* \cup A^\omega$. Let $P(A^*)$ denote the language semiring over A , and let $P(A^\infty)$ denote the semimodule of all subsets of A^∞ equipped with the action of $P(A^*)$ defined by $XY = \{xy : x \in X, y \in Y\}$ for all

$X \subseteq A^*$ and $Y \subseteq A^\infty$. We also define an infinite product by $\prod_{n \geq 0} X_n = \{u_0 u_1 \dots : u_n \in X_n\}$. Note that $P(A^\omega)$ is a subsemimodule of $P(A^\infty)$.

Proposition 3.2 *For each set A , $(P(A^*), P(A^\infty))$ is a continuous Kleene ω -algebra.*

Homomorphisms between continuous Kleene ω -algebras (S, V) and (S', V') consist of two functions $h_S : S \rightarrow S'$ and $h_V : V \rightarrow V'$ that together preserve all operations. A homomorphism (h_S, h_V) is continuous if h_S and h_V preserve all suprema.

Theorem 3.3 *$(P(A^*), P(A^\infty))$ is the free continuous Kleene ω -algebra, freely generated by A .*

Proof. Suppose that (S, V) is any continuous Kleene ω -algebra and let $h : A \rightarrow S$ be a mapping. We want to show that there is a unique extension of h to a continuous homomorphism (h_S^\sharp, h_V^\sharp) from $(P(A^\omega), P(A^\infty))$ to (S, V) .

For each $u = a_0 \dots a_{n-1}$ in A^* , define $h_S(u) = h(a_0) \dots h(a_{n-1})$ and $h_V(u) = h(a_0) \dots h(a_{n-1})1^\omega = \prod_{k \geq 0} b_k$, where $b_k = a_k$ for all $k < n$ and $b_k = 1$ for all $k \geq n$. When $u = a_0 a_1 \dots \in A^\omega$, define $h_V(u) = \prod_{k \geq 0} h(a_k)$. Note that we have $h_S(uv) = h_S(u)h_S(v)$ for all $u, v \in A^*$ and $h_S(\epsilon) = 1$. Also, $h_V(uv) = h_S(u)h_V(v)$ for all $u \in A^*$ and $v \in A^\infty$. Thus, $h_V(XY) = h_S(X)h_V(Y)$ for all $X \subseteq A^*$ and $Y \subseteq A^\infty$. Moreover, for all u_0, u_1, \dots in A^* , if $u_i \neq \epsilon$ for infinitely many i , then $h_V(u_0 u_1 \dots) = \prod_{k \geq 0} h_S(u_k)$. If on the other hand, $u_k = \epsilon$ for all $k \geq n$, then $h_V(u_0 u_1 \dots) = h_S(u_0) \dots h_S(u_{n-1})1^\omega$. In either case, if $X_0, X_1, \dots \subseteq A^*$, then $h_V(\prod_{n \geq 0} X_n) = \prod_{n \geq 0} h_S(X_n)$.

Suppose now that $X \subseteq A^*$ and $Y \subseteq A^\infty$. We define $h_S^\sharp(X) = \bigvee h_S(X)$ and $h_V^\sharp(Y) = \bigvee h_V(Y)$. It is well-known that h_S^\sharp is a continuous semiring morphism $P(A^*) \rightarrow S$. Also, h_V^\sharp preserves arbitrary suprema, since $h_V^\sharp(\bigcup_{i \in I} Y_i) = \bigvee h_V(\bigcup_{i \in I} Y_i) = \bigvee \bigcup_{i \in I} h_V(Y_i) = \bigvee_{i \in I} \bigvee h_V(Y_i) = \bigvee_{i \in I} h_V^\sharp(Y_i)$.

We prove that the action is preserved. Let $X \subseteq A^*$ and $Y \subseteq A^\infty$. Then $h_V^\sharp(XY) = \bigvee h_V(XY) = \bigvee h_S(X)h_V(Y) = \bigvee h_S(X) \bigvee h_V(Y) = h_S^\sharp(X)h_V^\sharp(Y)$.

Finally, we prove that the infinite product is preserved. Let $X_0, X_1, \dots \subseteq A^*$. Then $h_V^\sharp(\prod_{n \geq 0} X_n) = \bigvee h_V(\prod_{n \geq 0} X_n) = \bigvee \prod_{n \geq 0} h_S(X_n) = \prod_{n \geq 0} \bigvee h_S(X_n) = \prod_{n \geq 0} h_S^\sharp(X_n)$.

It is clear that h_S extends h , and that (h_S, h_V) is unique. □

Consider now $(P(A^*), P(A^\omega))$ with infinite product defined by $\prod_{n \geq 0} X_n = \{u_0 u_1 \dots \in A^\omega : u_n \in X_n, n \geq 0\}$. It is also a continuous Kleene ω -algebra. Moreover, it satisfies $1^\omega = \perp$.

Lemma 3.4 *$(P(A^*), P(A^\omega))$ is a quotient of $(P(A^*), P(A^\infty))$ under the continuous homomorphism (φ_S, φ_V) such that φ_S is the identity on $P(A^*)$ and φ_V maps $Y \subseteq A^\infty$ to $Y \cap A^\omega$.*

Proof. Suppose that $Y_i \subseteq A^\infty$ for all $i \in I$. It holds $\varphi_V(\bigcup_{i \in I} Y_i) = A^\omega \cap \bigcup_{i \in I} Y_i = \bigcup_{i \in I} (A^\omega \cap Y_i) = \bigcup_{i \in I} \varphi_V(Y_i)$.

Let $X \subseteq A^*$ and $Y \subseteq A^\infty$. Then $h_V(XY) = XY \cap A^\omega = X(Y \cap A^\omega) = \varphi_S(X)\varphi_V(Y)$.

Finally, let $X_0, X_1, \dots \subseteq A^*$. Then $h_V(\prod_{n \geq 0} X_n) = \{u_0 u_1 \dots \in A^\omega : u_n \in X_n\} = \prod_{n \geq 0} h_S(X_n)$. □

Lemma 3.5 *Suppose that (S, V) is a continuous Kleene ω -algebra satisfying $1^\omega = \perp$. Let (h_S, h_V) be a continuous homomorphism $(P(A^*), P(A^\omega)) \rightarrow (S, V)$. Then (h_S, h_V) factors through (φ_S, φ_V) .*

Proof. Define $h'_S = h_S$ and $h'_V : P(A^\omega) \rightarrow V$ by $h'_V(Y) = h_V(Y)$, for all $Y \subseteq A^\omega$. Then clearly $h_S = \varphi_S h'_S$. Moreover, $h_V = \varphi_V h'_V$, since for all $Y \subseteq A^\omega$, $Y \varphi_V h'_V = (Y \cap A^\omega) h_V = (Y \cap A^\omega) h_V \vee (Y \cap A^*) h_S 1^\omega = (Y \cap A^\omega) h_V \vee ((Y \cap A^*) 1^\omega) h_V = ((Y \cap A^\omega) \cup (Y \cap A^*) 1^\omega) h_V = Y h_V$.

Since (φ_S, φ_V) and (h_S, h_V) are homomorphisms. so is (h'_S, h'_V) . It is clear that h'_V preserves all suprema. \square

Corollary 3.6 *For each set A , $(P(A^*), P(A^\omega))$ is the free continuous Kleene ω -algebra satisfying $1^\omega = \perp$, freely generated by A .*

Proof. Suppose that (S, V) is a continuous Kleene ω -algebra satisfying $1^\omega = \perp$. Let $h : A \rightarrow S$. By Theorem 3.3, there is a unique continuous homomorphism $(h_S, h_V) : (P(A^*), P(A^\omega)) \rightarrow (S, V)$ extending h . By Lemma 3.5, h_S and h_V factor as $h_S = \varphi_S h'_S$ and $h_V = \varphi_V h'_V$, where (h'_S, h'_V) is a continuous homomorphism $(P(A^*), P(A^\omega)) \rightarrow (S, V)$. This homomorphism (h'_S, h'_V) is the required extension of h to a continuous homomorphism $(P(A^*), P(A^\omega)) \rightarrow (S, V)$. Since the factorization is unique, so is this extension. \square

4 *-Continuous Kleene ω -algebras

In this section, we define **-continuous Kleene ω -algebras* and *finitary *-continuous Kleene ω -algebras* as an extension of the **-continuous Kleene algebras* of [13]. We establish several basic properties of these structures, including the existence of the supremum of certain subsets including those corresponding to regular ω -languages.

We define a **-continuous Kleene ω -algebra* (S, V) as a **-continuous Kleene algebra*

$$(S, \vee, \cdot, \perp, 1, *)$$

acting on a (necessarily idempotent) semimodule $V = (V, \vee, \perp)$ subject to the usual laws of unitary action as well as the following axiom **Ax0**:

$$xy^*v = \bigvee_{n \geq 0} xy^n v,$$

for all $x, y \in S$ and $v \in V$. Moreover, there is an infinite product operation mapping an ω -sequence (x_0, x_1, \dots) (or ω -word $x_0 x_1 \dots$) over S to an element $\prod_{n \geq 0} x_n$ of V . Thus, infinite product is a function $S^\omega \rightarrow V$, where S^ω denotes the set of all ω -sequences over S .

Infinite product is subject to the following axioms relating the infinite product to the other operations of Kleene ω -algebras and operations on ω -sequences. The first two axioms are the same as for continuous Kleene ω -algebras. The last two ones are weaker forms of the complete additivity of the infinite product of continuous Kleene ω -algebras.

Ax1: For all $x_0, x_1, \dots \in S$, $\prod_{n \geq 0} x_n = x_0 \prod_{n \geq 0} y_n$, where $y_n = x_{n+1}$ for all $n \geq 0$.

Ax2: Suppose that $x_0, x_1, \dots \in S$ and $0 = n_0 \leq n_1 \leq \dots$ is a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \geq 0$. Then $\prod_{n \geq 0} x_n = \prod_{k \geq 0} y_k$.

Ax3: For all x_0, x_1, \dots and y, z in S ,

$$\prod_{n \geq 0} x_n (y \vee z) = \bigvee_{x'_n \in \{y, z\}} \prod_{n \geq 0} x_n x'_n.$$

Ax4: Suppose that x, y_0, y_1, \dots are in S . Then

$$\prod_{n \geq 0} x^* y_n = \bigvee_{k_n \geq 0} \prod_{n \geq 0} x^{k_n} y_n.$$

It is clear that every continuous Kleene ω -algebra is $*$ -continuous.

Some of our results will also hold for weaker structures. We define a *finitary* $*$ -continuous Kleene ω -algebra as a structure (S, V) as above, equipped with a star operation and an infinite product $\prod_{n \geq 0} x_n$ restricted to *finitary ω -sequences* (or finitary ω -words) over S , i.e., to sequences x_0, x_1, \dots such that there is a finite subset F of S such that each x_n is a finite product of elements of F . (Note that F is not fixed and may depend on the sequence x_0, x_1, \dots .) It is required that Ax1, Ax2 and Ax3 hold whenever the sequence x_0, x_1, \dots is finitary, and that Ax4 holds whenever the sequence y_0, y_1, \dots is finitary.

Finally, a *generalized $*$ -continuous Kleene algebra* (S, V) is defined as a $*$ -continuous Kleene ω -algebra, but without the infinite product (and without Ax1–Ax4). However, it is assumed that Ax0 holds.

The above axioms have a number of consequences. For example, if $x_0, x_1, \dots \in S$ and $x_i = \perp$ for some i , then $\prod_{n \geq 0} x_n = \perp$. Indeed, if $x_i = \perp$, then $\prod_{n \geq 0} x_n = x_0 \cdots x_i \prod_{n \geq i+1} x_n = \perp \prod_{n \geq i+1} x_n = \perp$. By Ax1 and Ax2, each $*$ -continuous Kleene ω -algebra is an ω -semigroup, and equipped with the natural order relation, an ordered ω -semigroup, cf. [16, pp. 92f].

Similarly, when (S, V) is a finitary $*$ -continuous Kleene algebra and x_0, x_1, \dots is a finitary sequence over S , and if some x_i is \perp , then $\prod_{n \geq 0} x_n = \perp$.

Suppose that (S, V) is a $*$ -continuous Kleene ω -algebra. To each word $w \in S^*$ there is a corresponding element \overline{w} of S which is the product of the letters of w in the semiring S . Similarly, when $w \in S^*V$, there is an element \overline{w} of V corresponding to w . Suppose now that a language $X \subseteq S^*$ or $X \subseteq S^*V$. Then we can associate with X the set $\overline{X} = \{\overline{w} : w \in X\}$, which is a subset of S or V . The reader will probably be relieved that below we will denote \overline{w} and \overline{X} by just w and X , respectively,

The following facts are well-known (and follow from the result that the semirings of regular languages are the free $*$ -continuous Kleene algebras [13] (essentially shown also in [4, 17]), and the free Kleene algebras [14]).

Lemma 4.1 *Suppose that S is a $*$ -continuous Kleene algebra. If $R \subseteq S^*$ is regular, then $\bigvee R$ exists. Moreover, for all $x, y \in S$, $x(\bigvee R)y = \bigvee xRy$.*

Lemma 4.2 *Let S be a $*$ -continuous Kleene-algebra. Suppose that R, R_1 and R_2 are regular*

subsets of S^* . Then

$$\begin{aligned}\bigvee(R_1 \cup R_2) &= \bigvee R_1 \vee \bigvee R_2 \\ \bigvee(R_1 R_2) &= (\bigvee R_1)(\bigvee R_2) \\ \bigvee(R^*) &= (\bigvee R)^*.\end{aligned}$$

In a similar way, we can prove:

Lemma 4.3 *Let (S, V) be a generalized $*$ -continuous Kleene-algebra. If $R \subseteq S^*$ is regular, $x \in S$ and $v \in V$, then $x(\bigvee R)v = \bigvee xRv$.*

Proof. Suppose that $R = \emptyset$. Then $x(\bigvee R)v = \perp = \bigvee xRv$. If R is a singleton set $\{y\}$, then $x(\bigvee R)v = xyv = \bigvee xRv$. Suppose now that $R = R_1 \cup R_2$ or $R = R_1 R_2$, where R_1, R_2 are regular, and suppose that our claim holds for R_1 and R_2 . Then, if $R = R_1 \cup R_2$,

$$\begin{aligned}x(\bigvee R)v &= x(\bigvee R_1 \vee \bigvee R_2)v \quad (\text{by Lemma 4.2}) \\ &= x(\bigvee R_1)v \vee x(\bigvee R_2)v \\ &= \bigvee xR_1v \vee \bigvee xR_2v \\ &= \bigvee x(R_1 \cup R_2)v \\ &= \bigvee xRv,\end{aligned}$$

where the third equality uses the induction hypothesis. If $R = R_1 R_2$, then

$$\begin{aligned}x(\bigvee R)v &= x(\bigvee R_1)(\bigvee R_2)v \quad (\text{by Lemma 4.2}) \\ &= \bigvee (xR_1(\bigvee R_2)v) \\ &= \bigvee \{y(\bigvee R_2)v : y \in xR_1\} \\ &= \bigvee \{\bigvee yR_2v : y \in xR_1\} \\ &= \bigvee xR_1R_2v \\ &= \bigvee xRv,\end{aligned}$$

where the second equality uses the induction hypothesis for R_1 and the fourth the one for R_2 . Suppose last that $R = R_0^*$, where R_0 is regular and our claim holds for R_0 . Then, using the previous case, it follows by induction that

$$x(\bigvee R_0^n)v = \bigvee xR_0^n v \quad (1)$$

for all $n \geq 0$. Using this and Ax0, it follows now that

$$\begin{aligned}
x(\bigvee R)v &= x(\bigvee R_0^*)y \\
&= x(\bigvee_{n \geq 0} \bigvee R_0^n)v \\
&= x(\bigvee_{n \geq 0} (\bigvee R_0)^n)v \quad (\text{by Lemma 4.2}) \\
&= \bigvee_{n \geq 0} x(\bigvee R_0)^n v \quad (\text{by Ax0}) \\
&= \bigvee_{n \geq 0} x(\bigvee R_0^n)v \quad (\text{by Lemma 4.2}) \\
&= \bigvee_{n \geq 0} \bigvee xR_0^n v \quad (\text{by (1)}) \\
&= \bigvee xR_0^*v \\
&= \bigvee xRv.
\end{aligned}$$

The proof is complete. □

We can also prove:

Lemma 4.4 *Let (S, V) be a $*$ -continuous Kleene ω -algebra. Suppose that the languages $R_0, R_1, \dots \subseteq S^*$ are regular and form a finite set \mathcal{R} of languages. Moreover, let $x_0, x_1, \dots \in S$. Then*

$$\prod_{n \geq 0} x_n(\bigvee R_n) = \bigvee \prod_{n \geq 0} x_n R_n.$$

Proof. If one of the R_i is empty, our claim is clear since both sides are equal to \perp , so we suppose they are all nonempty.

Below we will suppose that each regular language comes with a fixed decomposition having a minimal number of operations needed to obtain the language from the empty set and singleton sets. For a regular language R , let $|R|$ denote the minimum number of operations needed to construct it. When \mathcal{R} is a finite set of regular languages, let \mathcal{R}' denote the set of non-singleton languages in it. Let $|\mathcal{R}| = \sum_{R \in \mathcal{R}'} 2^{|R|}$. Our definition ensures that if $\mathcal{R} = \{R, R_1, \dots, R_n\}$ and $R = R' \cup R''$ or $R = R'R''$ according to the fixed minimal decomposition of R , and if $\mathcal{R}' = \{R', R'', R_1, \dots, R_n\}$, then $|\mathcal{R}'| < |\mathcal{R}|$. Similarly, if $R = R_0^*$ by the fixed minimal decomposition and $\mathcal{R}' = \{R_0, R_1, \dots, R_n\}$, then $|\mathcal{R}'| < |\mathcal{R}|$.

We will argue by induction on $|\mathcal{R}|$.

When $|\mathcal{R}| = 0$, then \mathcal{R} consists of singleton languages and our claim follows from Ax3. Suppose that $|\mathcal{R}| > 0$. Let R be a non-singleton language appearing in \mathcal{R} . If R appears only a finite number of times among the R_n , then there is some m such that R_n is different from R for all

$n \geq m$. Then,

$$\begin{aligned}
\prod_{n \geq 0} x_n(\bigvee R_n) &= \prod_{i < m} x_i(\bigvee R_i) \prod_{n \geq m} x_n(\bigvee R_n) \quad (\text{by Ax1}) \\
&= (\bigvee x_0 R_0 \cdots x_{n-1} R_{n-1}) \prod_{n \geq m} x_n(\bigvee R_n) \quad (\text{by Lemma 4.2}) \\
&= \bigvee (x_0 R_0 \cdots x_{n-1} R_{n-1} \prod_{n \geq m} x_n(\bigvee R_n)) \quad (\text{by Lemma 4.3}) \\
&= \bigvee \{y \prod_{n \geq m} x_n(\bigvee R_n) : y \in x_0 R_0 \cdots x_{n-1} R_{n-1}\} \\
&= \bigvee \{ \bigvee y \prod_{n \geq m} x_n R_n : y \in x_0 R_0 \cdots x_{n-1} R_{n-1} \} \\
&= \bigvee \prod_{n \geq 0} x_n R_n,
\end{aligned}$$

where the passage from the 4th line to the 5th uses induction hypothesis and Ax1.

Suppose now that R appears an infinite number of times among the R_n . Let R_{i_1}, R_{i_2}, \dots be all the occurrences of R among the R_n . Define

$$\begin{aligned}
y_0 &= x_0(\bigvee R_0) \cdots (\bigvee R_{i_1-1}) x_{i_1} \\
y_j &= x_{i_j+1}(\bigvee R_{i_j+1}) \cdots (\bigvee R_{i_{j+1}-1}) x_{i_{j+1}},
\end{aligned}$$

for $j \geq 1$. Similarly, define

$$\begin{aligned}
Y_0 &= x_0 R_0 \cdots R_{i_1-1} x_{i_1} \\
Y_j &= x_{i_j+1} R_{i_j+1} \cdots R_{i_{j+1}-1} x_{i_{j+1}},
\end{aligned}$$

for all $j \geq 1$. It follows from Lemma 4.2 that

$$y_j = \bigvee Y_j$$

for all $j \geq 0$. Then

$$\prod_{n \geq 0} x_n(\bigvee R_n) = \prod_{j \geq 0} y_j(\bigvee R), \quad (2)$$

by Ax2, and

$$\prod_{n \geq 0} x_n R_n = \prod_{j \geq 0} Y_j R.$$

If $R = R' \cup R''$, then:

$$\begin{aligned}
\prod_{n \geq 0} x_n(\bigvee R_n) &= \prod_{j \geq 0} y_j(\bigvee (R' \cup R'')) \quad (\text{by (2)}) \\
&= \prod_{j \geq 0} y_j(\bigvee R' \vee \bigvee R'') \quad (\text{by Lemma 4.2}) \\
&= \bigvee_{z_j \in \{\bigvee R', \bigvee R''\}} \prod_{j \geq 0} y_j z_j \quad (\text{by Ax3}) \\
&= \bigvee_{z_j \in \{\bigvee R', \bigvee R''\}} \bigvee \prod_{j \geq 0} Y_j z_j \\
&= \bigvee_{Z_j \in \{R', R''\}} \bigvee \prod_{j \geq 0} Y_j Z_j \\
&= \bigvee \prod_{n \geq 0} x_n(R' \cup R'') \\
&= \bigvee \prod_{n \geq 0} x_n R,
\end{aligned}$$

where the 4th and 5th equalities hold by the induction hypothesis and Ax2.

Suppose now that $R = R' R''$. Then applying the induction hypothesis almost directly we have

$$\begin{aligned}
\prod_{n \geq 0} x_n(\bigvee R_n) &= \prod_{j \geq 0} y_j(\bigvee R' R'') \\
&= \prod_{j \geq 0} y_j(\bigvee R')(\bigvee R'') \quad (\text{by Lemma 4.2}) \\
&= \bigvee \prod_{j \geq 0} Y_j(\bigvee R')(\bigvee R'') \\
&= \bigvee \prod_{j \geq 0} Y_j R' R'' \\
&= \bigvee \prod_{n \geq 0} x_n R' R'' \\
&= \bigvee \prod_{n \geq 0} x_n R,
\end{aligned}$$

where the third and fourth equalities come from the induction hypothesis and Ax2.

The last case to consider is when $R = T^*$, where T is regular. We argue as follows:

$$\begin{aligned}
\prod_{n \geq 0} x_n(\bigvee R_n) &= \prod_{j \geq 0} y_j(\bigvee T^*) \\
&= \prod_{j \geq 0} y_j(\bigvee T)^* \quad (\text{by Lemma 4.2}) \\
&= \bigvee_{k_0, k_1, \dots, j \geq 0} \prod y_j(\bigvee T)^{k_j} \quad (\text{by Ax1 and Ax4}) \\
&= \bigvee_{k_0, k_1, \dots} \bigvee_{j \geq 0} \prod Y_j(\bigvee T)^{k_j} \\
&= \bigvee_{k_0, k_1, \dots} \bigvee_{j \geq 0} \prod Y_j T^{k_j} \\
&= \bigvee_{j \geq 0} Y_j T^* \\
&= \bigvee_{j \geq 0} Y_j R_j \\
&= \bigvee_{n \geq 0} x_n R_n,
\end{aligned}$$

where the 4th and 5th equalities follow from the induction hypothesis and Ax2. The proof is complete. \square

By the same proof, we have the following version of Lemma 4.4 for the finitary case:

Lemma 4.5 *Let (S, V) be a finitary $*$ -continuous Kleene ω -algebra. Suppose that the languages $R_0, R_1, \dots \subseteq S^*$ are regular and form a finite set \mathcal{R} of languages. Moreover, let x_0, x_1, \dots is a finitary sequence of elements of S . Then*

$$\prod_{n \geq 0} x_n(\bigvee R_n) = \bigvee \prod_{n \geq 0} x_n R_n.$$

Note that each sequence $x_0, y_0, x_1, y_1, \dots$ with $y_n \in R_n$ is finitary.

As an immediate consequence of Lemma 4.5, we get the following result:

Corollary 4.6 *Let (S, V) be a finitary $*$ -continuous Kleene ω -algebra. Suppose that $R_0, R_1, \dots \subseteq S^*$ are regular and form a finite set of languages. Then $\bigvee \prod_{n \geq 0} R_n$ exists and is equal to $\prod_{n \geq 0} \bigvee R_n$.*

Suppose that (S, V) is a $*$ -continuous Kleene ω -algebra. When $v = x_0 x_1 \dots \in S^\omega$ is an ω -word over S , it naturally determines the element $\prod_{n \geq 0} x_n$ of V . Thus, any subset X of S^ω determines a subset of V . Using this convention, Lemma 4.4 may be rephrased as follows. For any $*$ -continuous Kleene ω -algebra (S, V) , $x_0, x_1, \dots \in S$ and regular sets $R_0, R_1, \dots \subseteq S^*$ that form a finite set, it holds that $\prod_{n \geq 0} x_n(\bigvee R_n) = \bigvee X$ where $X \subseteq S^\omega$ is the set of all ω -words $x_0 y_0 x_1 y_1 \dots$ with $y_i \in R_i$ for all $i \geq 0$, i.e., $X = x_0 R_0 x_1 R_1 \dots$. Similarly, Corollary 4.6 asserts that if a subset of V corresponds to an infinite product over a finite collection of ordinary regular languages in S^* , then the supremum of this set exists.

In any (finitary or non-finitary) $*$ -continuous Kleene ω -algebra (S, V) , we define an ω -power operation $S \rightarrow V$ by $x^\omega = \prod_{n \geq 0} x$ for all $x \in S$. From the axioms we immediately have:

Corollary 4.7 *Suppose that (S, V) is a $*$ -continuous Kleene ω -algebra or a finitary $*$ -continuous Kleene ω -algebra. Then the following hold for all $x, y \in S$:*

$$\begin{aligned} x^\omega &= xx^\omega \\ (xy)^\omega &= x(yx)^\omega \\ x^\omega &= (x^n)^\omega, \quad n \geq 2. \end{aligned}$$

Thus, each $*$ -continuous Kleene ω -algebra gives rise to a Wilke algebra, even an ordered Wilke algebra [16, 18].

Lemma 4.8 *Let (S, V) be a $*$ -continuous Kleene ω -algebra or a finitary $*$ -continuous Kleene ω -algebra. Suppose that $R \subseteq S^\omega$ is ω -regular. Then $\bigvee R$ exists in V .*

Proof. It is well-known that R can be written as a finite union of sets of the form $R_0(R_1)^\omega$ where $R_0, R_1 \subseteq S^*$ are regular, moreover, R_1 does not contain the empty word. It suffices to show that $\bigvee R_0(R_1)^\omega$ exists. But this holds by Corollary 4.6. \square

Lemma 4.9 *Let (S, V) be a $*$ -continuous Kleene ω -algebra. For all ω -regular sets $R_1, R_2 \subseteq S^\omega$ and regular sets $R \subseteq S^*$ it holds that*

$$\begin{aligned} \bigvee (R_1 \cup R_2) &= \bigvee R_1 \vee \bigvee R_2 \\ \bigvee (RR_1) &= (\bigvee R)(\bigvee R_1). \end{aligned}$$

And if R does not contain the empty word, then

$$\bigvee R^\omega = (\bigvee R)^\omega.$$

Proof. The first claim is clear. The second follows from Lemma 4.3. For the last, see the proof of Lemma 4.8. \square

5 Free finitary $*$ -continuous Kleene ω -algebras

Recall that for a set A , $R(A^*)$ denotes the collection of all regular languages in A^* . It is well-known that $R(A^*)$, equipped with the usual operations, is a $*$ -continuous Kleene algebra on A . Actually, $R(A^*)$ is characterized up to isomorphism by the following universal property.

Call a function $f : S \rightarrow S'$ between $*$ -continuous Kleene algebras a $*$ -continuous homomorphism if it preserves all operations including star, so that it preserves the suprema of subsets of S of the form $\{x^n : n \geq 0\}$, where $x \in S$.

Theorem 5.1 [14] *For each set A , $R(A^*)$ is the free $*$ -continuous Kleene algebra, freely generated by A .*

Thus, if S is any $*$ -continuous Kleene algebra and h is any mapping of A into S , then h has a unique extension to a $*$ -continuous Kleene algebra homomorphism $h^\sharp : R(A^*) \rightarrow S$.

Now let $R'(A^\infty)$ denote the collection of all subsets of A^∞ which are finite unions of sets of the form $\prod_{n \geq 0} R_n$, where each $R_n \subseteq A^*$ is regular, and the set $\{R_0, R_1, \dots\}$ is finite. Note that $R'(A^\infty)$ contains the empty set and is closed under finite unions. Moreover, when $Y \in R'(A^\infty)$ and $u = a_0 a_1 \dots \in Y \cap A^\omega$, then the alphabet of u is finite, i.e., the set $\{a_n : n \geq 0\}$ is finite.

Also, $R'(A^\infty)$ is closed under the action of $R(A^*)$ inherited from $(P(A^*), P(A^\infty))$. The infinite product of a sequence of regular languages in $R(A^*)$ is not necessarily contained in $R'(A^\infty)$, but by definition it contains all infinite products of finitary sequences over $R(A^*)$.

Example 5.2 Let $A = \{a, b\}$ and consider the set $\{aba^2b \dots a^n b \dots\}$ containing a single ω -word. It can be written as an infinite product of singleton subsets of A^* , but it cannot be written as an infinite product $R_0 R_1 \dots$ of regular languages in A^* such that the set $\{R_0, R_1, \dots\}$ is finite.

Proposition 5.3 $(R(A^*), R'(A^\infty))$ is a finitary $*$ -continuous Kleene ω -algebra.

Proof. This is clear from the fact that $(P(A^*), P(A^\infty))$ is a continuous Kleene ω -algebra and that $R(A^*)$ is a $*$ -continuous semiring. \square

Theorem 5.4 $(R(A^*), R'(A^\infty))$ is the free finitary $*$ -continuous Kleene ω -algebra, freely generated by A .

Proof. Our proof is modeled after the proof of Theorem 3.3.

Suppose that (S, V) is any finitary $*$ -continuous Kleene ω -algebra and let $h : A \rightarrow S$ be a mapping. For each $u = a_0 \dots a_{n-1}$ in A^* , define $h_S(u) = h(a_0) \dots h(a_{n-1})$ and $h_V(u) = h(a_0) \dots h(a_{n-1}) 1^\omega = \prod_{k \geq 0} b_k$, where $b_k = a_k$ for all $k < n$ and $b_k = 1$ for all $k \geq n$. When $u = a_0 a_1 \dots \in A^\omega$ whose alphabet is finite, define $h_V(u) = \prod_{k \geq 0} h(a_k)$. This infinite product exists in $R'(A^\infty)$.

Note that we have $h_S(uv) = h_S(u)h_S(v)$ for all $u, v \in A^*$, and $h_S(\epsilon) = 1$. And if $u \in A^*$ and $v \in A^\infty$ such that the alphabet of v is finite, then $h_V(uv) = h_S(u)h_V(v)$. Also, $h_V(XY) = h_S(X)h_V(Y)$ for all $X \subseteq A^*$ in $R(A^*)$ and $Y \subseteq A^\infty$ in $R'(A^\infty)$.

Moreover, for all u_0, u_1, \dots in A^* , if $u_i \neq \epsilon$ for infinitely many i , such that the alphabet of $u_0 u_1 \dots$ is finite, then $h_V(u_0 u_1 \dots) = \prod_{k \geq 0} h_S(u_k)$. If on the other hand, $u_k = \epsilon$ for all $k \geq n$, then $h_V(u_0 u_1 \dots) = h_S(u_0) \dots h_S(u_{n-1}) 1^\omega$. In either case, if $X_0, X_1, \dots \subseteq A^*$ are regular and form a finitary sequence, then the sequence $h_S(X_0), h_S(X_1), \dots$ is also finitary as is each infinite word in $\prod_{n \geq 0} X_n$, and $h_V(\prod_{n \geq 0} X_n) = \prod_{n \geq 0} h_S(X_n)$.

Suppose now that $X \subseteq A^*$ is regular and $Y \subseteq A^\infty$ is in $R'(A^\infty)$. We define $h_S^\sharp(X) = \bigvee h_S(X)$ and $h_V^\sharp(Y) = \bigvee h_V(Y)$. It is well-known that h_S^\sharp is a $*$ -continuous semiring morphism $R(A^*) \rightarrow S$. Also, h_V^\sharp preserves finite suprema, since when I is finite, $h_V^\sharp(\bigcup_{i \in I} Y_i) = \bigvee h_V(\bigcup_{i \in I} Y_i) = \bigvee \bigcup_{i \in I} h_V(Y_i) = \bigvee_{i \in I} \bigvee h_V(Y_i) = \bigvee_{i \in I} h_V^\sharp(Y_i)$.

We prove that the action is preserved. Let $X \in R(A^*)$ and $Y \in R'(A^\infty)$. Then $h_V^\sharp(XY) = \bigvee h_V(XY) = \bigvee h_S(X)h_V(Y) = \bigvee h_S(X) \bigvee h_V(Y) = h_S^\sharp(X)h_V^\sharp(Y)$.

Finally, we prove that infinite product of finitary sequences is preserved. Let X_0, X_1, \dots be a finitary sequence of regular languages in $R(A^*)$. Then, using Corollary 4.6, $h_V^\#(\prod_{n \geq 0} X_n) = \bigvee h_V(\prod_{n \geq 0} X_n) = \bigvee \prod_{n \geq 0} h_S(X_n) = \prod_{n \geq 0} \bigvee h_S(X_n) = \prod_{n \geq 0} h_S^\#(X_n)$.

It is clear that h_S extends h , and that (h_S, h_V) is unique. \square

Consider now $(R(A^*), R'(A^\omega))$ equipped with the infinite product operation $\prod_{n \geq 0} X_n = \{u_0 u_1 \in A^\omega : u_n \in X_n, n \geq 0\}$, defined on finitary sequences X_0, X_1, \dots of languages in $R'(A^*)$.

Proposition 5.5 $(R(A^*), R'(A^\omega))$ is a $*$ -continuous Kleene ω -algebra. Moreover, it satisfies $1^\omega = \perp$.

Lemma 5.6 $(R(A^*), R'(A^\omega))$ is a quotient of $(R(A^*), R'(A^\infty))$ under the $*$ -continuous homomorphism (φ_S, φ_V) such that φ_S is the identity function on $P(A^*)$ and φ_V maps $Y \in R'(A^\omega)$ to $Y \cap A^\omega$.

Lemma 5.7 Suppose that (S, V) is a finitary $*$ -continuous Kleene ω -algebra satisfying $1^\omega = \perp$. Let (h_S, h_V) be a $*$ -continuous homomorphism $(R(A^*), R'(A^\infty)) \rightarrow (S, V)$. Then (h_S, h_V) factors through (φ_S, φ_V) .

Corollary 5.8 For each set A , $(R(A^*), R'(A^\omega))$ is the free finitary $*$ -continuous Kleene ω -algebra satisfying $1^\omega = \perp$, freely generated by A .

Proof. This follows from Theorem 5.4 using Lemma 5.7. \square

6 $*$ -continuous Kleene ω -algebras and iteration semiring-semimodule pairs

In this section, our aim is to relate $*$ -continuous Kleene ω -algebras to iteration semiring-semimodule pairs. Our main result will show that every (finitary or non-finitary) $*$ -continuous Kleene ω -algebra is an iteration semiring-semimodule pair.

Some definitions are in order. Suppose that S is a semiring. Following [2], we call S a *Conway semiring* if S is equipped with a star operation $*$: $S \rightarrow S$ satisfying

$$\begin{aligned} (x + y)^* &= (x^* y)^* x^* \\ (xy)^* &= 1 + x(yx)^* y \end{aligned}$$

for all $x, y \in S$.

It is known that if S is a Conway semiring, then for each $n \geq 1$, so is the semiring $S^{n \times n}$ of all $n \times n$ -matrices over S with the usual sum and product operations and the star operation defined by induction on n so that if $n > 1$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a and d are square matrices of dimension $< n$, then

$$M^* = \begin{pmatrix} (a + bd^*c)^* & (a + bd^*c)^* bd^* \\ (d + ca^*b)^* ca^* & (d + ca^*b)^* \end{pmatrix}.$$

It is known that the above inductive definition does not depend on how M is split into 4 submatrices.

Suppose that S is a Conway semiring and $G = \{g_1, \dots, g_n\}$ is a finite group of order n . For each $x_{g_1}, \dots, x_{g_n} \in S$, consider the $n \times n$ matrix $M_G = M_G(x_{g_1}, \dots, x_{g_n})$ whose i th row is $(x_{g_i^{-1}g_1}, \dots, x_{g_i^{-1}g_n})$, for $i = 1, \dots, n$, so that each row (and column) is a permutation of the first row. We say that the group identity [4] associated with G holds in S if for each x_{g_1}, \dots, x_{g_n} , the first (and then any) row sum of M_G^* is $(x_{g_1} + \dots + x_{g_n})^*$. Finally, we call S an *iteration semiring* [2,5] if all group identities hold in S .

Classes of examples of (idempotent) iteration semirings are given by the continuous Kleene algebras and the $*$ -continuous Kleene algebras defined in the introduction. As mentioned above, the language semirings $P(A^*)$ and the semirings $P(A \times A)$ of binary relations are continuous and hence also $*$ -continuous Kleene algebras, and the semirings $R(A^*)$ of regular languages are $*$ -continuous Kleene algebras.

When S is a $*$ -continuous Kleene algebra and n is a nonnegative integer, then the matrix semiring $S^{n \times n}$ is also a $*$ -continuous Kleene algebra and hence an iteration semiring, cf. [13]. The star operation is defined by

$$M_{i,j}^* = \bigvee_{m \geq 0, 1 \leq k_1, \dots, k_m \leq n} M_{i,k_1} M_{k_1,k_2} \cdots M_{k_m,j},$$

for all $M \in S^{n \times n}$ and $1 \leq i, j \leq n$. It is not immediately obvious to prove that the above supremum exists. The fact that M^* is well-defined can be established by induction on n together with the well-known matrix star formula mentioned above: Suppose that $n \geq 2$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a and d are square matrices of dimension $< n$, then

$$M^* = \begin{pmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^*bd^* \\ (d \vee ca^*b)^*ca^* & (d \vee ca^*b)^* \end{pmatrix}.$$

A semiring-semimodule pair (S, V) is a *Conway semiring-semimodule pair* if it is equipped with a star operation $*$: $S \rightarrow S$ and an omega operation ω : $S \rightarrow V$ such that S is a Conway semiring and the following hold for all $x, y \in S$:

$$\begin{aligned} (x + y)^\omega &= (x^*y)^*x^\omega + (x^*y)^\omega \\ (xy)^\omega &= x(yx)^\omega. \end{aligned}$$

It is known that when (S, V) is a Conway semiring-semimodule pair, then so is $(S^{n \times n}, V^n)$ for each n , where V^n denotes the $S^{n \times n}$ -semimodule of all n -dimensional (column) vectors over V with the action of $S^{n \times n}$ defined similarly to matrix-vector product, and where the omega operation is defined by induction so that when $n > 1$ and M is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a and d are square matrices of dimension $< n$, then

$$M^\omega = \begin{pmatrix} (a + bd^*c)^\omega + (a + bd^*c)^*bd^\omega \\ (d + ca^*b)^\omega + (d + ca^*b)^*ca^\omega \end{pmatrix}.$$

We also define *iteration semiring-semimodule pairs* [2,10] as those Conway semiring-semimodule pairs such that S is an iteration semiring and the omega operation satisfies the following condition: let $M_G = M_G(x_{g_1}, \dots, x_{g_n})$ with $x_{g_1}, \dots, x_{g_n} \in S$ for a finite group $G = \{g_1, \dots, g_n\}$ of order n , then the first (and hence any) entry of M_G^ω is equal to $(x_{g_1} + \dots + x_{g_n})^\omega$.

Examples of (idempotent) iteration semiring-semimodule pairs include the semiring-semimodule pairs $(P(A^*), P(A^\omega))$ of languages and ω -languages over an alphabet A , mentioned in the introduction. The omega operation is defined by $X^\omega = \prod_{n \geq 0} X_n$, where $X_n = X$ for all $n \geq 0$. More generally, it is known that every continuous Kleene ω -algebra gives rise to an iteration semiring-semimodule pair. The omega operation is defined as for languages: $x^\omega = \prod_{n \geq 0} x_n$ with $x_n = x$ for all $n \geq 0$.

Other not necessarily idempotent examples include the *complete* and the (*symmetric*) *bi-inductive semiring-semimodule pairs* of [9, 10].

Suppose now that (S, V) is a $*$ -continuous Kleene ω -algebra. Then for each $n \geq 1$, $(S^{n \times n}, V^n)$ is a semiring-semimodule pair. The action of $S^{n \times n}$ on V^n is defined by a formula similar to matrix multiplication (viewing the elements of V^n as column matrices). It is easy to see that $(S^{n \times n}, V^n)$ is a generalized $*$ -continuous Kleene algebra for each $n \geq 1$.

Suppose that $n \geq 2$. We would like to define an infinite product operation $(S^{n \times n})^\omega \rightarrow V^n$ on matrices in $S^{n \times n}$ by

$$\left(\prod_{m \geq 0} M_m \right)_i = \bigvee_{1 \leq i_1, i_2, \dots \leq n} (M_0)_{i, i_1} (M_1)_{i_1, i_2} \cdots$$

for all $1 \leq i \leq n$. However, unlike in the case of complete semiring-semimodule pairs [10], the supremum on the right-hand side may not exist. Nevertheless it is possible to define an omega operation $S^{n \times n} \rightarrow V^n$ and to turn $(S^{n \times n}, V^n)$ into an iteration semiring-semimodule pair.

Lemma 6.1 *Let (S, V) be a (finitary) $*$ -continuous Kleene ω -algebra. Suppose that $M \in S^{n \times n}$, where $n \geq 2$. Then for every $1 \leq i \leq n$,*

$$\left(\prod_{m \geq 0} M \right)_i = \bigvee_{1 \leq i_1, i_2, \dots \leq n} M_{i, i_1} M_{i_1, i_2} \cdots$$

exists, so that we define M^ω by the above equality.

Moreover, when $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a and d are square matrices of dimension $< n$, then

$$M^\omega = \begin{pmatrix} (a \vee bd^*c)^\omega \vee (a \vee bd^*c)^*bd^\omega \\ (d \vee ca^*b)^\omega \vee (d \vee ca^*b)^*ca^\omega \end{pmatrix}. \quad (3)$$

Proof. Suppose that $n = 2$. Then by Corollary 4.6, $(a \vee bd^*c)^\omega$ is the supremum of the set of all infinite products $A_{1, i_1} A_{i_1, i_2} \cdots$ containing a or c infinitely often, and $(a \vee bd^*c)^*bd^\omega$ is the supremum of the set of all infinite products $A_{1, i_1} A_{i_1, i_2} \cdots$ containing a and c only finitely often. Thus, $(a \vee bd^*c)^\omega \vee (a \vee bd^*c)^*bd^\omega$ is the supremum of the set of all infinite products $A_{1, i_1} A_{i_1, i_2} \cdots$. Similarly, $(d \vee ca^*b)^\omega \vee (d \vee ca^*b)^*ca^\omega$ is the supremum of the set of all infinite products $A_{2, i_1} A_{i_1, i_2} \cdots$.

The proof of the induction step is similar. Suppose that $n > 2$, and let a be $k \times k$. Then by induction hypothesis, for every i with $1 \leq i \leq k$, the i th component of $(a \vee bd^*c)^\omega$ is the supremum of the set of all infinite products $A_{i, i_1} A_{i_1, i_2} \cdots$ containing an entry of a or c infinitely often, whereas the i th component of $(a \vee bd^*c)^*bd^\omega$ is the supremum of all infinite products $A_{i, i_1} A_{i_1, i_2} \cdots$ containing entries of a and c only finitely often. Thus, the i th component of $(a \vee bd^*c)^\omega \vee (a \vee bd^*c)^*bd^\omega$ is the supremum of the set of all infinite products $A_{i, i_1} A_{i_1, i_2} \cdots$. A similar fact holds for $(d \vee ca^*b)^\omega \vee (d \vee ca^*b)^*ca^\omega$. The proof is complete. \square

Theorem 6.2 *Every (finitary) *-continuous Kleene ω -algebra is an iteration semiring-semimodule pair.*

Proof. Suppose that (S, V) is a finitary *-continuous Kleene ω -algebra. Then

$$(x \vee y)^\omega = (x^*y)^\omega \vee (x^*y)^*x^\omega,$$

since by Lemma 4.2 and Lemma 4.9, $(x^*y)^\omega$ is the supremum of the set of all infinite products over $\{x, y\}$ containing y infinitely often, and $(x^*y)^*x^\omega$ is the supremum of the set of infinite products over $\{x, y\}$ containing y finitely often. Thus, $(x^*y)^\omega \vee (x^*y)^*x^\omega$ is equal to $(x \vee y)^\omega$, which by Ax3 is the supremum of all infinite products over $\{x, y\}$. As noted above, also

$$(xy)^\omega = x(yx)^\omega$$

for all $x, y \in S$. Thus, (S, V) is a Conway semiring-semimodule pair and hence so is each $(S^{n \times n}, V^n)$.

To complete the proof of the fact that (S, V) is an iteration semiring-semimodule pair, suppose that $x_1, \dots, x_n \in S$, and let $x = x_1 \vee \dots \vee x_n$. Let A be an $n \times n$ matrix whose rows are permutations of the x_1, \dots, x_n . We need to prove that each component of A^ω is x^ω . We use Lemma 6.1 and Ax3 to show that both are equal to the supremum of the set of all infinite products over the set $X = \{x_1, \dots, x_n\}$.

By Lemma 6.1, for each $i_0 = 1, \dots, n$, the i_0 th row of A^ω is $\bigvee_{i_1, i_2, \dots} a_{i_0, i_1} a_{i_1, i_2} \dots$. It is clear that each infinite product $a_{i_0, i_1} a_{i_1, i_2} \dots$ is an infinite product over X . Suppose now that $x_{j_0} x_{j_1} \dots$ is an infinite product over X . We define by induction on $k \geq 0$ an index i_{k+1} such that $a_{i_k, i_{k+1}} = x_{j_k}$. Suppose that $k = 0$. Then let i_1 be such that $a_{i_0, i_1} = x_{j_0}$. Since x_{j_0} appears in the i_0 th row, there is such an i_1 . Suppose that $k > 0$ and that i_k has already been defined. Since x_{j_k} appears in the i_k th row, there is some i_{k+1} with $a_{i_k, i_{k+1}} = x_{j_k}$. We have completed the proof of the fact that the i_0 th entry of A^ω is the supremum of the set of all infinite products over the set $X = \{x_1, \dots, x_n\}$.

Consider now $x^\omega = xx \dots$. We use induction on n to prove that x^ω is also the supremum of the set of all infinite products over the set $X = \{x_1, \dots, x_n\}$. When $n = 1$ this is clear. Suppose now that $n > 0$ and that the claim is true for $n - 1$. Let $y = x_1 \vee \dots \vee x_{n-1}$ so that $x = y \vee x_n$. We have:

$$\begin{aligned} x^\omega &= (y \vee x_n)^\omega \\ &= (x_n^*y)^*x_n^\omega \vee (x_n^*y)^\omega \\ &= (x_n^*y)^*x_n^\omega \vee (x_n^*x_1 \vee \dots \vee x_n^*x_{n-1})^\omega. \end{aligned}$$

Now

$$(x_n^*y)^*x_n^\omega = \bigvee_{k, m_1, \dots, m_k \geq 0} x_n^{m_1} y \dots x_n^{m_k} y x_n^\omega,$$

by Lemma 4.3, which is the supremum of all infinite products over X containing x_1, \dots, x_{n-1} only a finite number of times.

Also, using the induction hypothesis and Ax4,

$$\begin{aligned} (x_n^*x_1 \vee \dots \vee x_n^*x_{n-1})^\omega &= \bigvee_{1 \leq i_1, i_2, \dots \leq n-1} x_n^* x_{i_1} x_n^* x_{i_2} \dots \\ &= \bigvee_{1 \leq i_1, i_2, \dots \leq n-1} \bigvee_{k_0, k_1, \dots} x_n^{k_0} x_{i_1} x_n^{k_1} x_{i_2} \dots \end{aligned}$$

which is the supremum of all infinite products over X containing one of x_1, \dots, x_{n-1} an infinite number of times. Thus, x^ω is the supremum of all infinite products over X as claimed. The proof is complete. \square

6.1 Relation to bi-inductive semiring-semimodule pairs

Recall that when P is a partially ordered set and f is a function $P \rightarrow P$, then a *pre-fixed point* of f is an element x of P with $xf \leq x$. Similarly, $x \in P$ is a *post-fixed point* of f if $x \leq xf$. Suppose that f is monotone and has x as its least pre-fixed point. Then x is a *fixed point*, i.e., $xf = x$, and thus the least fixed point of f . Similarly, when f is monotone, then the greatest post-fixed point of f , whenever it exists, is the greatest fixed point of f .

When S is a $*$ -continuous Kleene algebra, then S is a Kleene algebra as defined in [13]. Thus, for all $x, y \in S$, x^*y is the least pre-fixed point (and thus the least fixed point) of the function $S \rightarrow S$ defined by $z \mapsto xz \vee y$ for all $z \in S$. Moreover, yx^* is the least pre-fixed point and the least fixed point of the function $S \rightarrow S$ defined by $z \mapsto zx \vee y$, for all $z \in S$.

Similarly, when (S, V) is a generalized $*$ -continuous Kleene algebra, then for all $x \in S$ and $v \in V$, x^*v is the least pre-fixed point and the least fixed point of the function $V \rightarrow V$ defined by $z \mapsto xz \vee v$, where z ranges over V .

A *bi-inductive semiring-semimodule pair* is defined as a semiring-semimodule pair (S, V) for which both S and V are partially ordered by the natural order \leq such that the semiring and semimodule operations and the action are monotone, and which is equipped with a star operation $*$: $S \rightarrow S$ and an omega operation ω : $S \rightarrow V$ such that the following hold for all $x, y \in S$ and $v \in V$:

- x^*y is the least pre-fixed point of the function $S \rightarrow S$ mapping each $z \in S$ to $xz \vee y$,
- x^*v is the least pre-fixed point of the function $V \rightarrow V$ mapping each $z \in V$ to $xz \vee v$,
- $x^\omega \vee x^*v$ is the greatest post-fixed point of the function $V \rightarrow V$ mapping each $z \in V$ to $xz \vee v$.

A bi-inductive semiring-semimodule pair is symmetric if for all $x, y \in S$, yx^* is the least pre-fixed point of the functions $S \rightarrow S$ defined by $z \mapsto zx + y$ for all $z \in S$.

By the above remarks we have:

Proposition 6.3 *Suppose that (S, V) is a finitary $*$ -continuous Kleene ω -algebra. When for all $x \in S$ and $v \in V$, $x^\omega \vee x^*v$ is the greatest post-fixed point of the function $V \rightarrow V$ defined by $z \mapsto xz \vee v$, then (S, V) is a symmetric bi-inductive semiring-semimodule pair.*

(It is known that every bi-inductive semiring-semimodule pair is an iteration semiring-semimodule pair, see [10].)

7 Büchi automata in $*$ -continuous Kleene ω -algebras

A generic definition of Büchi automata in Conway semiring-semimodule pairs was given in [2,9]. In this section, we recall this general definition and apply it to (finitary) $*$ -continuous Kleene

ω -algebras. We give two different definitions of the behavior of a Büchi automaton, an algebraic and a combinatorial, and show that these two definitions are equivalent.

Suppose that (S, V) is a Conway semiring-semimodule pair, S_0 is a subsemiring of S closed under star, and A is a subset of S . We write $S_0\langle A \rangle$ to denote the set of all finite sums $s_1a_1 + \cdots + s_ma_m$ with $s_i \in S_0$ and $a_i \in A$, for each $i = 1, \dots, m$.

We define a (weighted) Büchi automaton over (S_0, A) of dimension $n \geq 1$ in (S, V) as a system $\mathbf{A} = (\alpha, M, k)$ where $\alpha \in S_0^{1 \times n}$ is the initial vector, $M \in S_0\langle A \rangle^{n \times n}$ is the transition matrix, and k is an integer $0 \leq k \leq n$. In order to define the behavior $|\mathbf{A}|$ of \mathbf{A} , let us split M into 4 parts as above, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a \in S_0\langle A \rangle^{k \times k}$ the top-left k -by- k submatrix. Then we define

$$|\mathbf{A}| = \alpha \begin{pmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{pmatrix}.$$

We give another more combinatorial definition. A Büchi automaton $\mathbf{A} = (\alpha, M, k)$ of dimension n may be represented as a transition system whose set of states is $\{1, \dots, n\}$. For any pair of states i, j , the transitions from i to j are determined by the (i, j) th entry $M_{i,j}$ of the transition matrix. Let $M_{i,j} = s_1a_1 + \cdots + s_ma_m$, say. Then either there are m transitions from i to j , respectively labeled s_1a_1, \dots, s_ma_m , or there is just one transition, whose label is $M_{i,j}$. A run of the Büchi automaton starting in state i is an infinite path starting in state i which infinitely often visits at least one of the first k states, and the weight of such a run is the infinite product of the path labels. The behavior of the automaton in state i is the supremum of the weights of all runs starting in state i . Finally, the behavior of the automaton is $\alpha_1w_1 + \cdots + \alpha_nw_n$, where for each i , α_i is the i th component of α and w_i is the behavior in state i . Let $|\mathbf{A}'|$ denote the behavior of \mathbf{A} according to this second definition.

Theorem 7.1 *For every Büchi automaton \mathbf{A} over (S_0, A) in a finitary $*$ -continuous Kleene ω -algebra, it holds that $|\mathbf{A}| = |\mathbf{A}'|$.*

Proof. This holds by Lemma 6.1 and the fact that any matrix semiring over a $*$ -continuous Kleene algebra is itself a $*$ -continuous Kleene algebra. \square

For completeness we also mention a Kleene theorem for the Büchi automata introduced above, which is a direct consequence of the Kleene theorem for Conway semiring-semimodule pairs, cf. [9, 11].

Theorem 7.2 *Suppose that (S, V) is a $*$ -continuous Kleene ω -algebra, S_0 is a subsemiring of S closed under star, and $A \subseteq S$. Then an element of V is the behavior of a Büchi automaton over (S_0, V) iff it is regular (or rational) over (S_0, A) , i.e., when it can be generated from the elements of $S_0 \cup A$ by the semiring and semimodule operations, the action, and the star and omega operations.*

It is a routine matter to show that an element of V is rational over (S_0, A) iff it can be written as $\bigvee_{i=1}^n x_i y_i^\omega$, where each x_i and y_i can be generated from $S_0 \cup A$ by \bigvee, \cdot and $*$.

8 *-continuous (generalized) Kleene algebras of locally finite and \top -continuous functions

In the following two sections our aim is to provide additional examples of *-continuous Kleene ω -algebras. These examples are motivated by energy problems for hybrid systems and stem from [6]. We will deal with a generalization of the notion of *energy function* used in [6].

In this section, we define locally finite and \top -continuous functions over complete lattices and show that they give rise to (generalized) *-continuous Kleene algebras.

Suppose that $L = (L, \leq)$ is a complete lattice with bottom and top elements \perp and \top . Then a *finitely additive function* over L is a function $f : L \rightarrow L$ with $\perp f = \perp$ and $(x \vee y)f = xf \vee yf$ for all $x, y \in L$. (We write function application and composition in the diagrammatic order, from left to write.) In the same way, one may define finitely additive functions $L \rightarrow L'$ for complete lattices L, L' . Note that when $f : L \rightarrow L'$ is finitely additive, then $(\bigvee X)f = \bigvee Xf$ for all finite sets $X \subseteq L$.

Consider the collection $\mathbf{FinAdd}_{L,L'}$ of all finitely additive functions $f : L \rightarrow L'$, ordered pointwise. Since the (pointwise) supremum of any set of finitely additive functions is finitely additive, $\mathbf{FinAdd}_{L,L'}$ is also a complete lattice, in which the supremum of any set of functions can be constructed pointwise. The least and greatest elements are the constant functions with value \perp and \top , respectively. By an abuse of notation, we will denote these functions by \perp and \top as well.

Suppose now that $L = L'$, so that we just write \mathbf{FinAdd}_L . Since the composition of finitely additive functions is finitely additive and the identity function id over L is finitely additive, and since composition of finitely additive functions distributes over finite suprema, \mathbf{FinAdd}_L , equipped with the operation \vee (binary supremum), $;$ (composition), and the constant function \perp and the identity function id as 1, is an idempotent semiring. It follows that when f is finitely additive, then so is $f^* = \bigvee_{n \geq 0} f^n$. Moreover, $f \leq f^*$ and $f^* \leq g^*$ whenever $f \leq g$. Below we will usually write just fg for the composition $f;g$.

All functions between complete lattices considered in this paper will be (at least) finitely additive.

Lemma 8.1 *Let L be a complete lattice and S be any subsemiring of \mathbf{FinAdd}_L closed under the star operation. Then S is a *-continuous Kleene algebra iff for all $g, h \in S$, $g^*h = \bigvee_{n \geq 0} g^n h$.*

Proof. Suppose that the above condition holds. We need to show that

$$f\left(\bigvee_{n \geq 0} g^n\right)h = \bigvee_{n \geq 0} fg^n h$$

for all $f, g, h \in S$. But $f\left(\bigvee_{n \geq 0} g^n\right)h = f\left(\bigvee_{n \geq 0} g^n h\right)$ by assumption, and we conclude that $f\left(\bigvee_{n \geq 0} g^n h\right) = \bigvee_{n \geq 0} fg^n h$ since the supremum is pointwise. \square

Definition 8.2 *Let L be a complete lattice and $f : L \rightarrow L$ a finitely additive function. We call f locally finite if for each $x \in L$, either $xf^* = \top$ or $xf^* = x \vee \dots \vee xf^n$ for some $n \geq 0$. We call f \top -continuous if either f is the function \perp , or whenever $X \subseteq L$ with $\bigvee X = \top$, then $\bigvee Xf = \top$.*

Note that if $f : L \rightarrow L$ is a \top -continuous function which is different from the function \perp , then $\top f = \top$.

Example 8.3 Suppose that L is a complete lattice and $f : L \rightarrow L$ is finitely additive and satisfies that for each $x \in L$, either $xf^n = \perp$ for some $n \geq 0$, or $xf = x$, or $\bigvee_{n \geq 0} xf^n = \top$. Then f is locally finite. The functions id and \perp are locally finite and \top -continuous.

When L has at least two elements and \top is the supremum of all elements different from \top , then the locally finite function $f : L \rightarrow L$ with $xf = \perp$ for all $x < \top$ and $\top f = \top$ is not \top -continuous.

Lemma 8.4 Suppose that L is a complete lattice and $f : L \rightarrow L$ is finitely additive. Then the following conditions are equivalent.

- f is locally finite.
- For each $x \in L$, either $\bigvee_{n \geq 0} xf^n = \top$, or there is some n with $x \vee \dots \vee xf^n = x \vee \dots \vee xf^{n+1}$.
- For each $x \in L$, either $\bigvee_{n \geq 0} xf^n = \top$, or there is some n such that $x \vee \dots \vee xf^n = x \vee \dots \vee xf^m$ for all $m \geq n$.
- For each $x \in L$, either $\bigvee_{n \geq 0} xf^n = \top$, or there is some n such that $x \vee \dots \vee xf^n = x \vee \dots \vee xf^n \vee xf^m$ for all $m \geq n$.

Proof. Suppose that $x \in L$ and $xf^* < \top$, i.e., $\bigvee_{n \geq 0} xf^n < \top$. If f is locally finite then there is some n with $xf^* = x \vee \dots \vee xf^n$. Thus, for all $m \geq n$,

$$\begin{aligned} x \vee \dots \vee xf^n &\leq x \vee \dots \vee xf^n \vee xf^m \\ &\leq x \vee \dots \vee xf^m \\ &\leq xf^* \\ &= x \vee \dots \vee xf^n. \end{aligned}$$

Also, if $x \vee \dots \vee xf^n = x \vee \dots \vee xf^{n+1}$, then it follows by induction that $x \vee \dots \vee xf^n = x \vee \dots \vee xf^m$ for all $m \geq n$ and thus $xf^* = x \vee \dots \vee xf^n$. Indeed, this is clear when $m = n, n + 1$. Suppose that $m > n + 1$. Then, using the induction hypothesis and the fact that f is finitely additive,

$$\begin{aligned} x \vee \dots \vee xf^m &= x \vee (x \vee \dots \vee xf^{m-1})f \\ &= x \vee (x \vee \dots \vee xf^n)f \\ &= x \vee \dots \vee xf^{n+1} \\ &= x \vee \dots \vee xf^n. \end{aligned}$$

□

It is clear that the composition of \top -continuous functions is also \top -continuous. However, as the next example demonstrates, the composition of locally finite (and \top -continuous) functions is not necessarily locally finite.

Example 8.5 Let L be the following complete lattice (the linear sum of three infinite chains):

$$\perp < x_0 < x_1 < \cdots < y_0 < y_1 < \cdots < z_0 < z_1 < \cdots \top$$

Since L is a chain, a function $L \rightarrow L$ is finitely additive iff it is monotone and preserves \perp .

Let $f, g : L \rightarrow L$ be the following functions. First, $\perp f = \perp g = \perp$ and $\top f = \top g = \top$. Moreover, $x_i f = y_i$, $y_i f = z_i g = \top$ and $x_i g = \perp$, $y_i g = x_{i+1}$, $z_i g = \top$ for all i . Then f, g are monotone, $uf^* = u \vee uf \vee uf^2$ and $ug^* = u \vee ug$ for all $u \in L$. Also, f and g are \top -continuous, since if $\bigvee X = \top$ then either $\top \in X$ or $X \cap \{z_0, z_1, \dots\}$ is infinite, but then $\bigvee X f = \bigvee X g = \top$. However, fg is not locally finite, since $x_0(fg)^* = x_0 \vee x_0(fg) \vee x_0(fg)^2 \cdots = x_0 \vee x_1 \vee x_2 \vee \cdots = y_0$.

We now show that if a set of locally finite and \top -continuous functions enjoys appropriate closure properties, then it is necessarily a $*$ -continuous Kleene algebra.

Proposition 8.6 Suppose that $S \subseteq \mathbf{FinAdd}_L$ is closed under the operations \vee , composition and star and contains the functions \perp and id . Moreover, suppose that each $f \in S$ is locally finite and \top -continuous. Then S is a $*$ -continuous Kleene algebra.

Proof. Suppose that $g, h \in S$. By Lemma 8.1, it suffices to show that $g^*h = \bigvee_{n \geq 0} g^n h$. Since this is clear when $h = \perp$, we assume that h is a function different from \perp . Since $g^n h \leq g^*h$ for all $n \geq 0$, it holds that $\bigvee_{n \geq 0} g^n h \leq g^*h$. To prove the opposite inequality, suppose that $x \in L$. If $xg^* = \top$, then $\bigvee_{n \geq 0} xg^n = \top$, so that $\bigvee_{n \geq 0} xg^n h = \top$ by \top -continuity. Thus, $xg^*h = \top = \bigvee_{n \geq 0} xg^n h$.

Suppose that $xg^* < \top$. Then there is some $m \geq 0$ with

$$\begin{aligned} xg^*h &= (x \vee \cdots \vee xg^m)h \\ &= xh \vee \cdots \vee xg^m h \\ &\leq \bigvee_{n \geq 0} xg^n h \\ &= x \left(\bigvee_{n \geq 0} g^n h \right). \end{aligned}$$

The proof is complete. □

Next we study the closure properties of finitely additive \top -continuous functions in more detail. We will give a sufficient condition to the effect that the supremum of two such functions be also locally finite and \top -continuous. We will also prove that if f is locally finite and \top -continuous, then so is f^* .

The following two propositions are also of independent interest.

Proposition 8.7 Suppose that L is a complete lattice and $f, g_1, \dots, g_k : L \rightarrow L$, $k \geq 0$ are locally finite and \top -continuous functions. Then

$$f^* g_1 \cdots g_k = \bigvee_{n \geq 0} f^n g_1 \cdots g_k.$$

Proof. When $k = 0$ this is obvious by definition of star, so we assume that $k > 0$. It is clear that $\bigvee_{n \geq 0} f^n g_1 \cdots g_k \leq f^* g_1 \cdots g_k$ in the pointwise order. To complete the proof, we show that $x f^* g_1 \cdots g_k \leq \bigvee_{n \geq 0} x f^n g_1 \cdots g_k$ for all $x \in L$. We may assume that the g_i are different from the function \perp since in the opposite case $x f^* g_1 \cdots g_k = \perp$.

Suppose first that $x f^* < \top$. Then there is some n such that $x f^* = x \vee \cdots \vee x f^n$. Since the g_i are finitely additive, it follows that $x f^* g_1 \cdots g_k = \bigvee_{i=0}^n x f^i g_1 \cdots g_k \leq \bigvee_{n \geq 0} x f^n g_1 \cdots g_k$.

Suppose next that $x f^* = \top$, so that $x f^* g_1 \cdots g_k = \top$. Our task is to prove that $\bigvee_{n \geq 0} x f^n g_1 \cdots g_k = \top$. To this end, let h denote the function $h = g_1 \cdots g_k$. Since each g_i is \top -continuous, so is h . Moreover, h is not the function \perp . Thus, since $\bigvee \{x f^n : n \geq 0\} = \top$ and h is \top -continuous,

$$\bigvee_{n \geq 0} x f^n g_1 \cdots g_k = \bigvee_{n \geq 0} x f^n h = \top. \quad \square$$

Proposition 8.8 *Suppose that L is a complete lattice and $f, g \in \mathbf{FinAdd}_L$ such that f and f^*g are locally finite and \top -continuous functions. Then*

$$(f \vee g)^* = (f^*g)^* f^*.$$

Proof. It is clear that $(f \vee g)^n \leq \bigvee_{i \leq n} (f^*g)^i f^*$ for all $n \geq 0$, hence $(f \vee g)^* \leq \bigvee_{n \geq 0} (f^*g)^n f^* \leq (f^*g)^* f^*$. Below we will prove that

$$(f^*g)^* f^* \leq \bigvee_{n \geq 0} \bigvee_{k_i \geq 0} f^{k_0} g \cdots g f^{k_n} = (f \vee g)^*.$$

In our argument, we will make use of two observations.

Claim 1. Suppose that $x(f^*g)^n < \top$ for some $n \geq 0$. Then there exist integers $k_1, \dots, k_n \geq 0$ with

$$x(f^*g)^n \leq \bigvee \{x f^{i_1} g \cdots f^{i_n} g : i_1 \leq k_1, \dots, i_n \leq k_n\}.$$

Claim 2. Suppose that $x(f^*g)^n f^* < \top$ for some $n \geq 0$. Then there exist integers $k_1, \dots, k_n, k_{n+1} \geq 0$ with

$$x(f^*g)^n f^* \leq \bigvee \{x f^{i_1} g \cdots f^{i_n} g f^{i_{n+1}} : i_1 \leq k_1, \dots, i_{n+1} \leq k_{n+1}\}.$$

Since the proofs of these claims are similar, we only prove the first one. We argue by induction on n . When $n = 0$, our claim is clear. Suppose now that $n > 0$. Since $x(f^*g)^n < \top$ and f^*g preserves \top , $x(f^*g)^{n-1} = y < \top$. Since f is locally finite, $y f^* = y \vee \cdots \vee y f^m$ for some $m \geq 0$. Since g is finitely additive, $y f^* g = y g \vee \cdots \vee y f^m g$. By the induction hypothesis, $y \leq \bigvee \{x f^{i_1} g \cdots f^{i_{n-1}} g : i_1 \leq k_1, \dots, i_{n-1} \leq k_{n-1}\}$ for some $k_1, \dots, k_{n-1} \geq 0$. Thus, using the fact that f and g are finitely additive, it follows that

$$x(f^*g)^n \leq \bigvee \{x f^{i_1} g \cdots f^{i_n} g : i_1 \leq k_1, \dots, i_n \leq k_n\}$$

where $k_n = m$.

We now return to the proof of the Proposition. Let $x \in L$. We want to show that $x(f^*g)^* f^* \leq x(f \vee g)^*$, which will follow if we can prove that $x(f^*g)^* f^*$ is the supremum of a finite or infinite number of elements of the form $x f^{k_0} g f^{k_1} \cdots g f^{k_n}$ where $n \geq 0$ and $k_0, \dots, k_n \geq 0$. Let us denote by Δ_x the set of all such elements.

When $x = \top$, then our claim is clear. So we may assume that $x < \top$. We consider several cases.

Case 1. $x(f^*g)^*f^* < \top$. Then, since f^*g is locally finite and f^* is finitely additive,

$$\begin{aligned} x(f^*g)^*f^* &= (x \vee \cdots \vee x(f^*g)^n)f^* \\ &= xf^* \vee \cdots \vee x(f^*g)^nf^* \end{aligned}$$

for some $n \geq 0$. Clearly, $x(f^*g)^i < \top$ and $x(f^*g)^if^* < \top$ for all $i \leq n$. Now using Claim 2 and the fact that f is finitely additive, it follows that each $x(f^*g)^if^*$ is the supremum of a finite number of terms of Δ_x .

Case 2. $x(f^*g)^*f^* = \top$.

Within this case, we consider sub-cases.

Sub-case 2.1. $x(f^*g)^* < \top$. In this case, it follows as in Case 1 using Claim 1 that $y = x(f^*g)^*$ is the supremum of a finite number of elements of Δ_x . Since $yf^* = y \vee yf \vee \cdots = \top$ and for each $k \geq 0$, yf^k is also a finite supremum of elements of Δ_x , since f is finitely additive, it follows that $x(f^*g)^*f^* = yf^*$ is a possibly infinite supremum of elements of Δ_x .

Sub-case 2.2. $x(f^*g)^* = \bigvee_{n \geq 0} x(f^*g)^n = \top$. Suppose that $\bigvee_{i \leq n} (f^*g)^i < \top$ for all $n \geq 0$. Then, as shown above, for each $n \geq 0$ we can write $\bigvee_{i \leq n} (f^*g)^i$ as a finite supremum of elements of Δ_x . Thus, $x(f^*g)^*f^* = \top = x(f^*g)^*$ is a possibly infinite supremum of elements of Δ_x . Suppose now that there is some n with $\bigvee_{i \leq n} (f^*g)^i = \top$. Then there is a least such n . Let us denote it by n_0 . Since $x \neq \top$, it holds $n_0 > 0$. Then $x(f^*g)^i < \top$ for all $i < n_0$. If also $x(f^*g)^{n_0} < \top$, then $x(f^*g)^*f^* = \top = \bigvee_{i \leq n_0} (f^*g)^i$ is a finite supremum of elements of Δ_x . Last, assume that $x(f^*g)^{n_0} = \top$. Let us denote $x(f^*g)^{n_0-1}$ by y , so that $y < \top$ and $yf^*g = \top$. Note that y is a finite supremum of elements of Δ_x .

Suppose that $yf^* = y \vee yf \vee \cdots = \top$. Since f is finitely additive, each yf^i is a finite supremum over Δ_x , and we conclude that so is $x(f^*g)^*f^* = \top = yf^*$ is a possibly infinite supremum over Δ_x . If $yf^* < \top$, then $yf^* = y \vee \cdots \vee yf^n$, and by finite additivity, we conclude that yf^*g is a finite supremum of elements of Δ_x . \square

Next we prove that the star of a locally finite \top -continuous function also has these properties.

Lemma 8.9 *Suppose that L is a complete lattice and $f \in \mathbf{FinAdd}_L$ is locally finite. Then f^* is also locally finite. And if f is additionally \top -continuous, then so is f^* .*

Proof. Suppose that f is locally finite. We prove that $xf^{**} = x \vee xf^* = xf^*$ for all $x \in L$. Indeed, this is clear when $xf^* = \top$, since $f^* \leq f^{**}$. Otherwise $xf^* = \bigvee_{k \leq n} xf^k$ for some $n \geq 0$.

By finite additivity, it follows that $xf^*f^* = \bigvee_{k \leq n} xf^kf^*$. But for each k , $xf^kf^* = xf^k \vee xf^{k+1} \vee \cdots \leq xf^*$ and thus $xf^*f^* = xf^*$ and $xf^* = xf^{**}$. It follows by Lemma 8.4 that f^* is locally finite.

Suppose now that f is additionally \top -continuous. We need to show is that f^* is also \top -continuous. To this end, let $X \subseteq L$ with $\bigvee X = \top$. Since $x \leq xf^*$, for all $x \in X$, it holds that $\bigvee Xf^* \geq \bigvee X = \top$. Thus $\bigvee Xf^* = \top$. \square

Lemma 8.10 *Suppose that L is a complete lattice and $f, g, f^*g : L \rightarrow L$ are locally finite \top -continuous functions. Then $f \vee g$ is also locally finite and \top -continuous.*

Proof. We know that $f \vee g$ is finitely additive. By Proposition 8.8, $(f \vee g)^* = (f^*g)^*f^*$.

Suppose that there is $x \in L$ with $x(f \vee g)^* = x(f^*g)^*f^* < \top$. Then, since f^*g and f are locally finite and \top -continuous, we have that $x(f^*g)^* < \top$, $y = x(f^*g)^* = x \vee \dots \vee x(f^*g)^n$, and $yf^* = y \vee \dots \vee yf^m$ for some $n, m \geq 0$. Moreover, as shown in the proof of Proposition 8.8, for each $i \leq n$ there exist $n_1, \dots, n_i \geq 0$ with $x(f^*g)^i = \bigvee_{j_1 \leq n_1, \dots, j_i \leq n_i} x f^{j_1} g \dots f^{j_i} g$. It follows that $x(f \vee g)^* = x \vee \dots \vee x(f \vee g)^k$ for some $k \geq 0$. We have proved that $f \vee g$ is locally finite.

Since f and g are \top -continuous, $f \vee g$ is clearly also \top -continuous. \square

8.1 Generalized $*$ -continuous Kleene algebras of finitely additive functions

By the previous results, locally finite and \top -continuous functions provide examples of $*$ -continuous Kleene algebras. Next we show that they also yield generalized $*$ -continuous Kleene algebras.

Suppose that L, L' are complete lattices, and consider the collection $\mathbf{FinAdd}_{L,L'}$ of all finitely additive functions $L \rightarrow L'$, ordered pointwise. Then, as noted above, $\mathbf{FinAdd}_{L,L'}$ is a complete lattice. Define a left action of \mathbf{FinAdd}_L on $\mathbf{FinAdd}_{L,L'}$ by $fv = f;v$, for all $f \in \mathbf{FinAdd}_L$ and $v \in \mathbf{FinAdd}_{L,L'}$. It is a routine matter to check that $\mathbf{FinAdd}_{L,L'}$, equipped with the above action, the binary supremum operation \vee and the constant \perp is an (idempotent) left \mathbf{FinAdd}_L -semimodule.

We extend the notion of \top -continuous functions $L \rightarrow L$ over a complete lattice to a more general situation in a natural way. Suppose that L and L' are complete lattices and let $f : L \rightarrow L'$. We call f \top -continuous if f is finitely additive and $\bigvee Xf = \top$ whenever $X \subseteq L$ with $\bigvee X = \top$.

We will make use of a general fact which is similar to Lemma 8.1.

Lemma 8.11 *Suppose that L, L' are complete lattices, S is a $*$ -continuous Kleene algebra in \mathbf{FinAdd}_L , and $V \subseteq \mathbf{FinAdd}_{L,L'}$ is an S -semimodule. Then (S, V) is a generalized $*$ -continuous Kleene algebra iff for all $g \in S$ and $v \in V$, $g^*v = \bigvee_{n \geq 0} g^n v$ (and thus $\bigvee_{n \geq 0} g^n v \in V$).*

Proof. The above condition yields that for all $f, g \in S$ and $v \in V$, $fg^*v = \bigvee_{n \geq 0} fg^n v$, so that $Ax0$ holds. \square

Proposition 8.12 *Let S be a $*$ -continuous Kleene algebra of locally finite and \top -continuous functions in \mathbf{FinAdd}_L , and let V be a subsemimodule of $\mathbf{FinAdd}_{L,L'}$ closed under the induced S -action, which contains only \top -continuous functions. Then (S, V) is a generalized $*$ -continuous Kleene algebra.*

Proof. The proof is similar to that of Proposition 8.6. \square

Remark 8.13 *Actually it suffices to require in the above proposition the following. For all $v \in V$, $v \neq \perp$, and for all $f \in S$ and $x \in L$, if $xf^* = \top$ then $\bigvee_{n \geq 0} xf^n v = \top$. Then (S, V) is a generalized $*$ -continuous Kleene algebra. Indeed, suppose that $f \in S$ and $v \in V$. In order to prove that $f^*v = \bigvee_{n \geq 0} f^n v$, it suffices to show that $xf^*v \leq \bigvee_{n \geq 0} xf^n v$ for all x .*

If $xf^* < \top$, then $xf^* = x \vee \cdots \vee xf^n$ for some n . Since v is finitely additive, $xf^*v = xv \vee \cdots \vee xf^nv \leq \bigvee_{n \geq 0} xf^nv$. If $xf^* = \top$, then $xf^*v = \top$ unless v is \perp . It follows that $f^*v = \bigvee_{n \geq 0} f^nv$.

9 *-continuous Kleene ω -algebras of locally finite and \top -continuous functions

In this section, our aim is to study *-continuous Kleene ω -algebras of finitely additive functions. For this reason, we define an infinite product mapping ω -sequences of finitely additive functions over a complete lattice into the 2-element lattice $\mathbf{2}$.

In this section, let L be an arbitrary complete lattice and let $L' = \mathbf{2}$, the 2-element lattice $\{\perp, \top\}$. Our aim is to study generalized *-continuous Kleene ω -algebras (S, V) , where S consists of locally finite \top -continuous functions $L \rightarrow L$ and V consists of \top -continuous functions $L \rightarrow \mathbf{2}$.

For each sequence $f_0, f_1, \dots \in \mathbf{FinAdd}_L$, define $g = \prod_{n \geq 0} f_n : L \rightarrow \mathbf{2}$ by $xg = \perp$ iff there is an index $n \geq 0$ for which $xf_0 \cdots f_n = \perp$. Thus, if x, xf_0, xf_0f_1, \dots are all different from \perp , then $x \prod_{n \geq 0} f_n = \top$, otherwise $x \prod_{n \geq 0} f_n = \perp$. Note that in the latter case the sequence x, xf_0, xf_0f_1, \dots is eventually \perp .

It is easy to see that $\prod_{n \geq 0} f_n$ is also finitely additive. Indeed, $\perp \prod_{n \geq 0} f_n = \perp$ clearly holds, and for all $x, y \in L$, if $x \leq y$ in L , then $x \prod_{n \geq 0} f_n \leq y \prod_{n \geq 0} f_n$. Thus, to prove that for all $x, y \in L$, it holds that $(x \vee y) \prod_{n \geq 0} f_n = x \prod_{n \geq 0} f_n \vee y \prod_{n \geq 0} f_n$, it suffices to show that if $x \prod_{n \geq 0} f_n = y \prod_{n \geq 0} f_n = \perp$, then $(x \vee y) \prod_{n \geq 0} f_n = \perp$. But if $x \prod_{n \geq 0} f_n = y \prod_{n \geq 0} f_n = \perp$, then there exist m, k with $xf_0 \cdots f_m = yf_0 \cdots f_k = \perp$. Let $\ell = \max\{m, k\}$. We have that $(x \vee y)f_0 \cdots f_\ell = xf_0 \cdots f_\ell \vee yf_0 \cdots f_\ell = \perp$, and thus $(x \vee y) \prod_{n \geq 0} f_n = \perp$.

It is clear that Ax1 and Ax2 hold.

Proposition 9.1 *Suppose that f_0, f_1, \dots are finitely additive functions $L \rightarrow L$ and $0 = n_0 \leq n_1 \leq \dots$ is a sequence of integers which increases without a bound. Let $g_k = f_{n_k} \cdots f_{n_{k+1}-1}$ for all $k \geq 0$, so that each g_k is also finitely additive. Then*

$$\begin{aligned} \prod_{n \geq 0} f_n &= f_0 \prod_{n \geq 0} f_{n+1} \\ \prod_{n \geq 0} f_n &= \prod_{k \geq 0} g_k. \end{aligned}$$

Proposition 9.2 *The following property holds for all finitely additive functions f_0, f_1, \dots and $g_0, g_1, \dots : L \rightarrow L$:*

$$\prod_{n \geq 0} (f_n \vee g_n) = \bigvee_{h_n \in \{f_n, g_n\}} \prod_{n \geq 0} h_n. \quad (4)$$

In particular, Ax3 holds.

Proof. Suppose that $f_0, g_0, f_1, g_1, \dots$ are in \mathbf{FinAdd}_L . Since infinite product is monotone, the term on the right-hand side of (4) is less than or equal to the term on the left-hand side. To prove that equality holds, let $x \in L$ and suppose that $x \prod_{n \geq 0} (f_n \vee g_n) = \top$. It suffices to

show that there is a choice of the functions $h_n \in \{f_n, g_n\}$ such that $x \prod_{n \geq 0} h_n = \top$. Consider the full (ordered) binary tree where each node at level $n \geq 0$ is the source of an edge labeled f_n and an edge labeled g_n , ordered as indicated. We can assign to each node u the composition h_u of the functions that occur as the labels of the edges along the unique path from the root to that node. Let us mark a node u if $xh_u \neq \perp$. Since $x \prod_{n \geq 0} (f_n \vee g_n) = \top$, each level contains a marked node. Moreover, whenever a node is marked and has a predecessor, its predecessor is also marked. By König's lemma there is an infinite path going through marked nodes. This infinite path gives rise to the sequence h_0, h_1, \dots with $x \prod_{n \geq 0} h_n = \top$. \square

Proposition 9.3 *Suppose that f, g_0, g_1, \dots are locally finite and \top -continuous functions $L \rightarrow L$. Then $\prod_{n \geq 0} f^* g_n = \bigvee_{k_0, k_1, \dots \geq 0} \prod_{n \geq 0} f^{k_n} g_n$, i.e., Ax4 holds.*

Proof. Since f is locally finite and \top -continuous, so is f^* , by Lemma 8.9. Let $x \in L$. Suppose that $x \prod_{n \geq 0} f^* g_n = \top$. We want to prove that there exist integers k_0, k_1, \dots such that $x \prod_{n \geq 0} f^{k_n} g_n = \top$. Consider the ordered infinite binary tree and label the left out-edge of each vertex v by f and the right out-edge by g_n iff the unique path from the root to v contains exactly n right out-edges. To each vertex z we can canonically associate the function h_z which is the composition of the edge labels that occur along the path from the root to z . Let us mark a vertex z if following condition holds: if z_1, \dots, z_k are some ancestors of z such that each z_{i+1} is the left successor of z_i , for $i < k$, then it is not the case that $xh_{z_k} \leq xh_{z_1} \vee \dots \vee xh_{z_{k-1}}$.

Since $x \prod_{n \geq 0} f^* g_n = \top$, for each n there is a marked vertex which is the target of an edge labeled g_n . Moreover, any ancestor of a marked vertex is marked. By König's lemma, there is an infinite path going through marked vertices. If this path contains infinitely many right edges, we are done, since this infinite path determines the integers k_n , $n \geq 0$. Indeed, the edge labels of the path form an infinite word $f^{k_0} g_0 f^{k_1} g_1 \dots$.

Suppose the path contains only a finite number of right edges. For simplicity, we may as well assume that the path contains only left edges. This means that x, xf, xf^2, \dots satisfy that for all n , $xf^n \not\leq \bigvee_{i < n} xf^i$ and thus $xf^* = \top$, since f is locally finite and \top -continuous. Since none of the functions g_0, g_1, \dots is \perp , $v = \prod_{n \geq 0} g_n$ is not \perp , since $\top v = \top$. Now $\bigvee_{n \geq 0} f^n v = f^* v$ and since $xf^* v = \top$, also $\bigvee_{n \geq 0} xf^n v = \top$. Thus, $xf^m v \neq \perp$ for some m . We may thus choose $k_0 = m$ and $k_n = 0$ for all $n > 0$. \square

Corollary 9.4 *Suppose that L is a complete lattice and (S, V) is a generalized $*$ -continuous Kleene algebra of locally finite and \top -continuous functions $L \rightarrow L$ and \top -continuous functions $L \rightarrow \mathbf{2}$. Suppose that $\prod_{n \geq 0} f_n \in V$ for all sequences f_0, f_1, \dots of functions in S . Then (S, V) is a $*$ -continuous Kleene ω -algebra.*

Corollary 9.5 *Suppose that L is a complete lattice and (S, V) is a generalized $*$ -continuous Kleene algebra of locally finite and \top -continuous functions $L \rightarrow L$ and \top -continuous functions $L \rightarrow \mathbf{2}$. Suppose that $\prod_{n \geq 0} f_n \in V$ for all finitary sequences f_0, f_1, \dots of functions in S . Then (S, V) is a finitary $*$ -continuous Kleene ω -algebra.*

Suppose that L is a complete lattice and f_0, f_1, \dots is a sequence of locally finite \top -continuous functions $L \rightarrow L$, and consider the function $v = \prod_{n \geq 0} f_n : L \rightarrow \mathbf{2}$ defined above. While v is finitely additive, it may not be \top -continuous. Below we provide a sufficient condition under which this function is also \top -continuous.

Lemma 9.6 *Suppose that L is a complete lattice for which it holds that whenever $\bigvee X = \top$ for some $X \subseteq L$, then for all $x < \top$ in L there is $y \in X$ with $x \leq y$. If f_0, f_1, \dots is a sequence of locally finite \top -continuous functions $L \rightarrow L$, then $\prod_{n \geq 0} f_n$ is \top -continuous.*

Proof. Let $v = \prod_{n \geq 0} f_n$. We already know that v is finitely additive. We need to show that if $v \neq \perp$, then v is \top -continuous. But if $v \neq \perp$, then there is some $x < \top$ with $xv = \top$, i.e., such that $xf_0 \cdots f_n > \perp$ for all n . By assumption, there is some $y \in X$ with $x \leq y$. It follows that $yf_0 \cdots f_n \geq xf_0 \cdots f_n > \perp$ for all n and thus $\bigvee Xv = \top$. \square

9.1 Fixed points

In this short section, we relate $*$ -continuous Kleene ω -algebras of locally finite and \top -continuous functions to bi-inductive semiring-semimodule pairs.

Lemma 9.7 *Suppose that L is a complete lattice and (S, V) is a generalized $*$ -continuous Kleene algebra of locally finite \top -continuous functions $L \rightarrow L$ and \top -continuous functions $L \rightarrow \mathbf{2}$. Suppose that for each $f \in S$, $f^\omega = \prod_{n \geq 0} f \in V$. Then for all $f \in S$ and $v \in V$, $f^\omega \vee f^*v$ is the greatest post-fixed point of the map $V \rightarrow V$ given by $z \mapsto fz \vee v$.*

Proof. Since $ff^\omega = f^\omega$, we have that $f(f^\omega \vee f^*v) \vee v = ff^\omega \vee ff^*v \vee v = f^\omega \vee f^*v$.

Suppose that $u \in V$ with $u \leq fu \vee v$. Then $u \leq f^n u \vee f^{n-1}v \vee \cdots \vee v$ for all $n \geq 0$. We want to show that $xu \leq xf^\omega \vee xf^*v$ for all $x \in L$. This is clear when $xf^\omega = \top$. So suppose that $xf^\omega = \perp$. Then $xf^n = \perp$ for some n . Thus, $xu \leq xf^{n-1}v \vee \cdots \vee xv \leq xf^*v = xf^\omega \vee xf^*v$. \square

Corollary 9.8 *Suppose that L is a complete lattice and (S, V) is a generalized $*$ -continuous Kleene algebra of locally finite \top -continuous functions $L \rightarrow L$ and \top -continuous functions $L \rightarrow \mathbf{2}$. Suppose that for each $f \in S$, $f^\omega = \prod_{n \geq 0} f \in V$. Then (S, V) is a symmetric bi-inductive semiring-semimodule pair.*

10 Application: Energy Problems

We finish the paper by showing how the setting developed in the last two sections can be applied to solve so-called energy problems. We refer to [6] for a more detailed account of, and motivation for, energy problems in the context of formal modeling and verification.

Let $L = [0, \top]_\perp$ be the complete lattice of nonnegative real numbers together with $\top = \infty$ and an extra bottom element \perp , and extend the usual order and operations on real numbers to $[0, \top]_\perp$ by declaring that $\perp < x < \top$, $\perp - x = \perp$ and $\top + x = \top$ for all nonnegative reals x .

Note that L satisfies the condition required in Lemma 9.6: If $X \subseteq [0, \top]_\perp$ is such that $\bigvee X = \top$, then for all $x \in L$ with $x < \top$ there is $y \in X$ for which $x \leq y$.

An *energy function* is a mapping $f : [0, \top]_\perp \rightarrow [0, \top]_\perp$ for which $\perp f = \perp$, $\top f = \perp$ if $xf = \perp$ for all $x < \top$ and $\top f = \top$ otherwise, and $yf \geq xf + y - x$ whenever $\perp < x < y < \top$. The set of such functions is denoted \mathcal{E} .

The above entails that for all $f \in \mathcal{E}$ and all $x < y \in [0, \top]_{\perp}$, $xf = \top$ implies $yf = \top$ and $yf = \perp$ implies $xf = \perp$. Note that energy functions are monotone, hence finitely additive, and that \mathcal{E} is closed under (pointwise) binary supremum \vee and composition.

Lemma 10.1 *Energy functions are \top -continuous.*

Proof. Let $X \subseteq [0, \top]_{\perp}$ such that $\bigvee X = \top$ and $f \in \mathcal{E}$, $f \neq \perp$. We have $X \neq \{\perp\}$, so let $x_0 \in X \setminus \{\perp\}$ and, for all $n \geq 0$, $x_n = x_0 + n$. Let $y_n = x_n f$. If $y_n = \perp$ for all $n \geq 0$, then also $n f = \perp$ for all $n \geq 0$ (as $x_n \geq n$), hence $f = \perp$. We must thus have an index N for which $y_N > \perp$. But then $y_{N+k} \geq y_N + k$ for all $k \geq 0$, hence $\bigvee X f = \top$. \square

Lemma 10.2 *For $f \in \mathcal{E}$, f^* is given by $xf^* = x$ if $xf \leq x$ and $xf^* = \top$ if $xf > x$. Hence f is locally finite and $f^* \in \mathcal{E}$.*

Proof. We have $\perp f^* = \perp$ and $\top f^* = \top$. Let $x \neq \perp, \top$. If $xf \leq x$, then $xf^n \leq x$ for all $n \geq 0$, so that $x \leq \bigvee_{n \geq 0} xf^n \leq x$, whence $xf^* = x$. If $xf > x$, then let $a = xf - x > 0$. We have $xf \geq x + a$, hence $xf^n \geq x + na$ for all $n \geq 0$, so that $xf^* = \bigvee_{n \geq 0} xf^n = \top$. \square

We can now apply Proposition 8.6:

Corollary 10.3 *\mathcal{E} is a $*$ -continuous Kleene algebra.*

Remark 10.4 *It is not true that \mathcal{E} is a continuous Kleene algebra: Let $f_n, g \in \mathcal{E}$ be defined by $xf_n = x + 1 - \frac{1}{n+1}$ for $x \geq 0$, $n \geq 0$ and $xg = x$ for $x \geq 1$, $xg = \perp$ for $x < 1$. Then $0(\bigvee_{n \geq 0} f_n)g = (\bigvee_{n \geq 0} 0f_n)g = 1g = 1$, whereas $0\bigvee_{n \geq 0}(f_n g) = \bigvee_{n \geq 0}(0f_n g) = \bigvee_{n \geq 0}((1 - \frac{1}{n+1})g) = \perp$.*

Also note that not all locally finite functions $f : [0, \top]_{\perp} \rightarrow [0, \top]_{\perp}$ are energy functions: the function f defined by $xf = 1$ for $x < 1$ and $xf = x$ for $x \geq 1$ is locally finite, but $f \notin \mathcal{E}$.

Now let $L' = \mathbf{2}$, and let \mathcal{V} denote the \mathcal{E} -semimodule of all \top -continuous functions $[0, \top]_{\perp} \rightarrow \mathbf{2}$. For $f_0, f_1, \dots \in \mathcal{E}$, define an infinite product $f = \prod_{n \geq 0} f_n : [0, \top]_{\perp} \rightarrow \mathbf{2}$ by $xf = \perp$ if there is an index n for which $xf_0 \cdots f_n = \perp$ and $xf = \top$ otherwise. By Lemma 9.6, $\prod_{n \geq 0} f_n$ is \top -continuous, i.e., $\prod_{n \geq 0} f_n \in \mathcal{V}$.

By Proposition 8.12, $(\mathcal{E}, \mathcal{V})$ is a generalized $*$ -continuous Kleene algebra. We can now apply Corollaries 9.4 and 9.8:

Corollary 10.5 *$(\mathcal{E}, \mathcal{V})$ is a $*$ -continuous Kleene ω -algebra and a symmetric bi-inductive semiring-semimodule pair.*

The paper [6] is concerned with reachability and Büchi acceptance in so-called *energy automata*, which are finite automata labeled with energy functions. This is useful for solving certain energy and resource management problems in real-time and hybrid systems, see [6] for details.

Let $\mathbf{A} = (\alpha, M, k)$ be an energy automaton of dimension $n \geq 1$, with $\alpha \in \{\perp, \text{id}\}^{1 \times n} \subseteq \mathcal{E}^{1 \times n}$, $M \in \mathcal{E}(\mathcal{E})^{n \times n}$ and $0 \leq k \leq n$ as in Section 7 (hence $S_0 = A = \mathcal{E}$ in the notation of that section). Define $\zeta \in \{\perp, \text{id}\}^{n \times 1}$ by $\zeta_i = \text{id}$ if $i \leq k$, $\zeta_i = \perp$ for $i > k$. We can now apply the results of this paper to conclude the following:

Theorem 10.6 *A final state in \mathbf{A} is reachable with initial energy $x \in [0, \top]_{\perp}$ iff $x\alpha M^*\zeta > \perp$. There is an infinite run in \mathbf{A} from initial energy x which visits an accepting state infinitely often iff $x\alpha|\mathbf{A}| = \top$.*

11 Conclusion

We have introduced continuous and (finitary and non-finitary) $*$ -continuous Kleene ω -algebras and exposed some of their basic properties. Continuous Kleene ω -algebras are idempotent complete semiring-semimodule pairs, and conceptually, $*$ -continuous Kleene ω -algebras are a generalization of continuous Kleene ω -algebras in much the same way as $*$ -continuous Kleene algebras are of continuous Kleene algebras: In $*$ -continuous Kleene algebras, suprema of finite sets and of sets of powers are required to exist and to be preserved by the product; in $*$ -continuous Kleene ω -algebras these suprema are also required to be preserved by the infinite product.

It is known that in a Kleene algebra, $*$ -continuity is precisely what is required to be able to compute the reachability value of a weighted automaton (or its power series) using the matrix star operation. Similarly, we have shown that the Büchi values of automata over $*$ -continuous ω -algebras can be computed using the matrix omega operation.

We have seen that the sets of finite and infinite languages over an alphabet are the free continuous Kleene ω -algebras, and that the free finitary $*$ -continuous Kleene ω -algebras are given by the sets of regular languages and of finite unions of finitary infinite products of regular languages. A characterization of the free (non-finitary) $*$ -continuous Kleene ω -algebras (and whether they even exist) is left open.

We have shown that other examples of $*$ -continuous Kleene ω -algebras are given by locally finite and \top -continuous functions over complete lattices. We have seen that every $*$ -continuous Kleene ω -algebra is an iteration semiring-semimodule pair, which permits to compute the behavior of Büchi automata with weights in a $*$ -continuous Kleene ω -algebra using ω -powers of matrices. Hence the algebraic setting developed here can be employed to solve energy problems for hybrid systems.

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