Piecewise Polynomial Reconstruction of Functions from Simplified Morse-Smale complex

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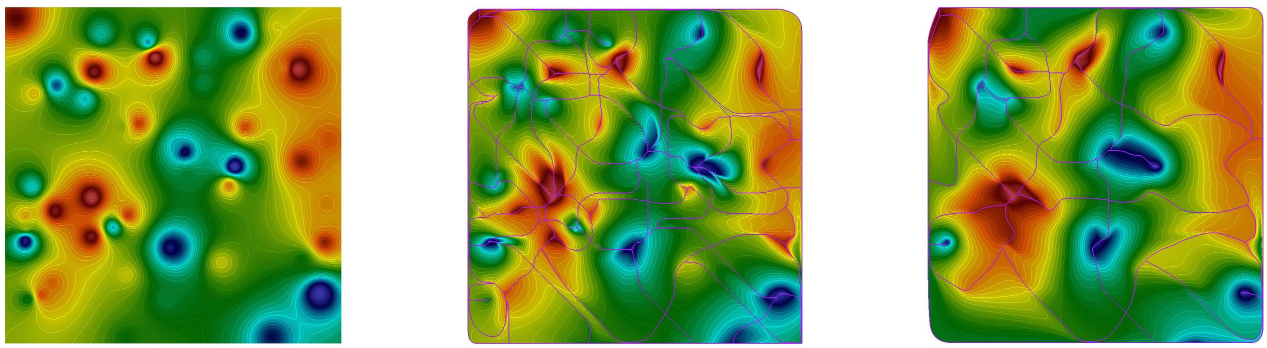

Figure 1: Left: the initial function. Middle: Our reconstruction of the Morse-Smale complex with 5% of simplification. The 1-cells of the Morse-Smale complex are the purple lines. Right: Our reconstruction of the Morse-Smale complex with 25% of simplification.

1 Introduction

Simulation of phenomena like climate often deals with large datasets. A process to extract the most salient features is needed to assist in the understanding of the dataset. The Morse-Smale (MS) complex is a topological structure defined on scalar functions which extracts critical points of the function and the links between them. Furthermore, it encodes a hierarchy between critical points, and less important critical points can be deleted in order to simplify the structure. Starting from an initial function $f : \mathbb{R}^2 \to \mathbb{R}$, the Morse-Smale complex of this function is computed, then simplified. From this simplified structure, we aim to construct a new function, which corresponds to this structure, closed to the initial function, thus approximating the initial data set by preserving the most salient features. The main difficulty we face, lies in the fact that both, the boundary curves (corresponding to the 1-cells of the MS complex) and the surface patches inside each 2-cell have to be monotonic functions. Furthermore, the geometry of the 2-cells may be very complex, see Fig. 1 (middle and right).

This poster proposes a novel approach for computing a scalar function coherent with a given simplified MS complex that privileges the use of piecewise polynomial functions. Based on techniques borrowed from Shape Preserving Design in Computer Aided Geometric Design, our method constructs the surface 2-cell by 2-cell using piecewise polynomials curves and surfaces. We first compute monotonic boundary curves as quartic B-splines. With the aid of a parametrization of the 2-cell's domain onto the unit square we then construct a piecewise polynomial monotonic surface inside the 2-cell interpolating the boundary curves and the critical points provided by the simplified MS complex.

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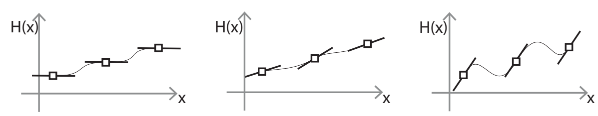

Figure 2: Cubic Hermite interpolation of monotone increasing data. Different derivatives at the data points give different curves. Right: The derivatives are equal to zero, the function is monotonic, but critical points are generated. Middle: These derivatives give a monotonic curve. Right: If the derivatives are too big, the curve is no longer monotonic.

2 Related Work

Several papers exist about computation and simplification of Morse-Smale complexes. Previous methods for reconstruction are mesh-based Laplacian techniques. [1] provides a C^0 reconstruction by first smoothing the height values of the 1-cells, then by Laplacian smoothing the surface mesh inside each 2-cell until they are monotonic. [6] proposes a C^1 reconstruction by first applying Laplacian smoothing to the 1-cells, then by optimizing a bi-Laplacian subject to constraints that ensure monotonicity. These methods are based on piecewise linear interpolants on the original mesh. In contrast we use polynomial curves and surfaces defined on a coarser mesh.

3 Piecewise polynomial monotonic interpolation of 2D gridded data

In this section, we define a new piecewise polynomial monotonic surface which interpolates data given on a grid. It will be used in Sect. 4.2 to compute the surface inside the 2-cells. [3] gives a method to get a monotonic surface which interpolates gridded data. The height values on the grid have to be increasing along all rows and columns. This axis aligned monotonicity is however too restrictive for our setting. We therefore generalize this idea to data being monotone increasing along the diagonals.

For each point $(i, j)_{1\geq i,j\geq n}$ of the grid, a height value $z_{i,j}$ is given. The data is supposed to be strictly increasing along the diagonal :

$$z_{i,j} < z_{i+1,j+1}$$

We interpolate this data set with a monotonic function using a modified Sibson-Split interpolant. For each grid cell it consists of four cubic C^1 triangular Bézier surfaces.

We develop an algorithm that computes the 25 control coefficients of the Bézier surfaces explicitly from the position and gradient values at the grid points by ensuring diagonal monotonicity. i.e. $\frac{\partial f}{\partial(x+y)} > 0$. It is based on a theorem where we derive sufficient conditions on the partial derivatives to ensure monotonicity of the interpolating function. Figure 2 illustrates in the analogue 1D case, why inappropriate derivatives lead to non monotonic interpolates. We then provide two algorithms which enable to compute these admissible gradient values at the grid vertices efficiently.

4 Our algorithm - MS reconstruction

Our algorithm takes as input the initial function defined on a triangulation and the simplified MS complex as computed by [5] in form of positions of singular points and links between them (1-cells) given as lists of 2D vertex positions and height values.

4.1 Monotonic smoothing of the 1-cells

Our algorithm begins by smoothing the 1-cells in the (x,y)-plane and in height values separately. Each 1-cell starts at a critical point (minimum, saddle) and ends at a critical point (saddle, maximum) with greater function value. After the simplification process of Morse-Smale complexes, the function values along the 1-cell path are no longer monotonic. So we smooth them by approximating the 1-cell data with a smooth quartic B-spline provided the curve is monotonic using [4] and then convert them in cubic Bézier curves.

4.2 Monotone interpolation inside the 2-cells

This part describes how we construct a monotonic surface inside each 2-cell. For each 2-cell, we first compute a piecewise polynomial monotonic surface defined on the unit square using our interpolant from Sect. 3. In this first step we only take into account the height of the function. We then reparametrize this monotonic interpolant to match the domain of the 2-cell in the plane.

4.2.1 Parametrization of the domain of a 2-cell

Let us denote where $\Omega \subset R^2$ the domain of the 2-cell. We define a parametrization $\Phi : \Omega \to [0,1]^2$, by first computing a constraint Delaunay triangulation of the domain and then applying Floater's Mean Value Coordinates [2] between this triangulation of the 2-cell and the unit square. To this end we associate the 4 critical points of a 2-cell to the domain corners. The minimum is mapped to the lower left corner, the maximum to the upper right corner and the two saddles to the remaining two corners.

4.2.2 Computation of a monotonic surface defined on $[0,1]^2$

Using the method described in Section 3, we can compute a piecewise polynomial monotonic surface inside a square as follows. We subdivide the boundary curves using Bézier subdivision until there is the same number of Bézier curves along all boundaries of $[0,1]^2$. This defines a subdivision of the parameter domain boundary and a tensor product grid inside $[0,1]^2$. The missing function values at the inner grid vertices are estimated (several strategies are possible here) so that they are diagonal increasing. Our modified Sibson-Split interpolant is then computed over the whole grid resulting in a C^1 piecewise cubic function f. An example of a surface inside the square is given in Figure 3 on the left. The isovalues show that the surface is monotonic and has no critical point inside its domain.

4.2.3 Construction of the surface inside the 2-cell

Using the parametrisation of Ω and the monotonic surfaces F, we obtain finally a monotonic function $F = f \circ \Phi : \Omega \to R$ by combining our Sibson interpolant with the parametrization. The resulting MS complex surface patch defined piecewise on all 2-cells is thus

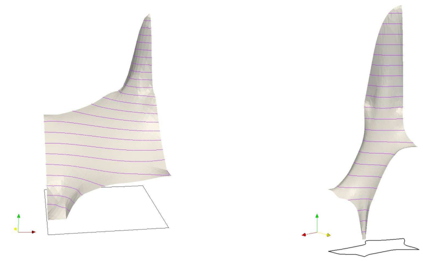

Figure 3: Left: Our computation of a monotonic surface inside the square $[0,1]^2$. Right: The same surface reparametrized inside the 2-cell. The purple lines are isovalues, they show that there is no critical points inside these surfaces.

monotonic inside each 2-cell, since Φ is a diffeomorphism and f is monotonic. This important monotonicity property guarantees us that the resulting surface corresponds to the given simplified MS complex and has no additional critical points. Figure 3 right shows an example of the reconstruction inside a 2-cell. The surface computed inside the square is on the left. As there is no critical point inside the surface in the square, there is no critical point after its reparametrization to the 2-cell.

5 Results

From an initial given function, a simplification of its Morse-Smale complex is computed and new function is computed using the results presented in this poster, which is coherent with the simplified Morse-Smale complex. Our method leads to a a C^0 reconstructed function. An example of a reconstruction is given at Figure 1. The reconstruction is piecewise polynomial inside each 2-cells. We can also see from the isocontour lines that the reconstructed function is C^0 across the boundary curves of the Morse-Smale complex, i.e. the 1-cells. Compared to other mesh-based methods, our reconstruction does not depend on the initial dense triangulation of the function. Furthermore, as we have an explicit piecewise polynomial representation of the reconstruction, our surface can be refined and evaluated explicitly everywhere.

6 Future Work

For future work, we plan to modify this technique using improved interpolants in order to have a C^1 reconstruction.

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