Helmholtz Equation in Highly Heterogeneous Media: a two Scales Analysis

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Heterogeneous Helmholtz equation

- the pressure u satisfies the equation

$$-\frac{\omega^2}{c^2}u - \Delta u = f$$

- where f is a source of excitation
- where ω is the angular frequency
- where c is the wave velocity

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Frequency

- the frequency is represented by the parameter $\omega \in \mathbb{R}_+^*$
- solving for high frequencies requires heavy computations
- high order methods may reduce the computational cost
- coarse mesh with high order elements

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Wave velocity

- the wave velocity is represented as a function $c : \Omega \to \mathbb{R}_+^*$
- in geophysical applications, c is piecewise constant
- in classical FEM, the mesh has to fit c
- c must be constant in each cell of the mesh
- for highly heterogenous media, we need a fine mesh

Existence and Uniqueness: Homogeneous case

- Fredholm Alternative
- Uniqueness (scaling argument)
- Existence
- Stability ($C(\omega)$)

Frequency explicit stability

- Use a special test function in the variational formulation
- Frequency explicit bounds:

$$|u|_{0,\Omega} \leq \frac{C}{\omega}|f|_{0,\Omega}, \quad |u|_{1,\Omega} \leq C|f|_{0,\Omega}, \quad |u|_{2,\Omega} \leq C\omega|f|_{0,\Omega},$$

with $C = C(\Omega)$

Sketch of the proof: Homogeneous case

- Varational form

$$B(u,v) = -\frac{\omega^2}{c^2}\int_\Omega u\bar{v} - i\frac{\omega}{c}\int_{\partial\Omega} u\bar{v}\int_\Omega \nabla u \cdot \nabla \bar{v}$$

- Test function $v = \nabla u \cdot \mathbf{x}$

$$2\mathrm{Re}\ -\frac{\omega^2}{c^2}\int_\Omega u\bar{v} = \frac{\omega^2}{c^2}\int_\Omega |u|^2 - \frac{\omega^2}{c^2}\int_{\partial\Omega} |u|^2\mathbf{x}\cdot\mathbf{n}$$

Sketch of the proof: Heterogeneous case

- Varational form

$$B(u,v) = -\omega^2 \int_\Omega \frac{1}{c^2} u\bar{v} - i\omega \int_{\partial\Omega} \frac{1}{c} u\bar{v} \int_\Omega \nabla u \cdot \nabla \bar{v}$$

- Test function $v = \nabla u \cdot \mathbf{x}$

$$\begin{aligned} 2\mathrm{Re} \; -\omega^2 \int_\Omega \frac{1}{c^2} u\bar{v} &= \omega^2 \int_\Omega \frac{1}{c^2} |u|^2 - \omega^2 \int_{\partial\Omega} \frac{1}{c^2} |u|^2 \mathbf{x}\cdot\mathbf{n} \\ &- \sum_{r,l} \int_{\Omega_r \cap \Omega_l} \left(\frac{1}{c_r^2} \mathbf{x}\cdot\mathbf{n}_r + \frac{1}{c_l^2} \mathbf{x}\cdot\mathbf{n}_l \right) |u|^2 \end{aligned}$$

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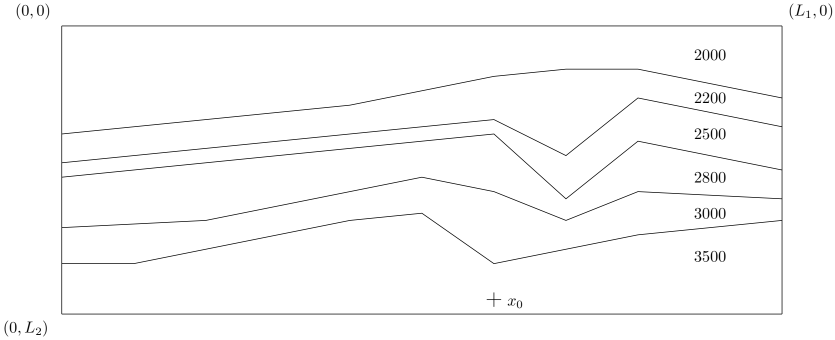
Additional Hypothesis

$$\frac{1}{c_r^2} \mathbf{x}\cdot\mathbf{n}_r + \frac{1}{c_l^2} \mathbf{x}\cdot\mathbf{n}_l \leq 0, \quad \forall x \in \Omega_r \cap \Omega_l, \quad \forall r,l$$

A Stratified medium



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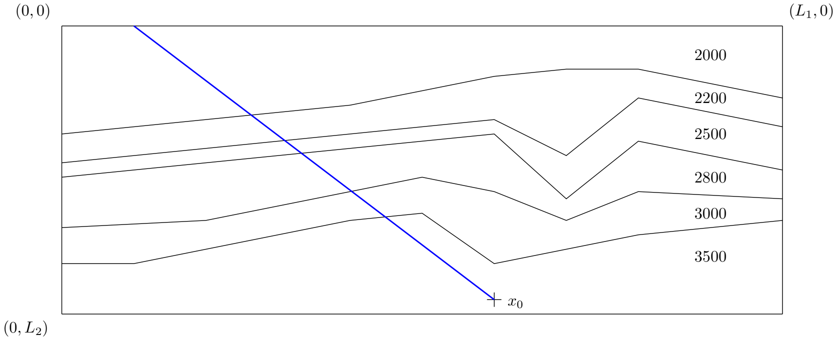

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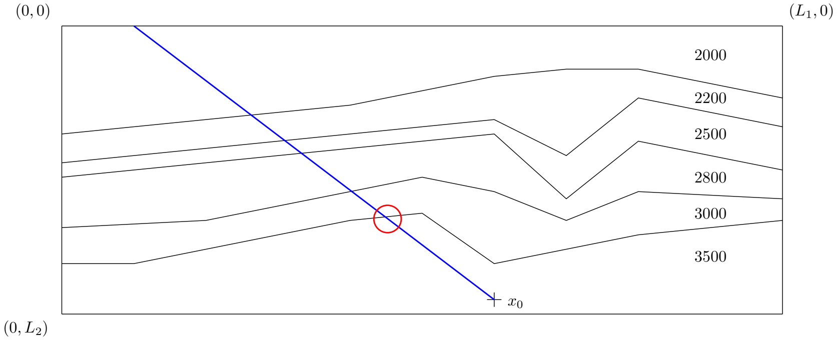

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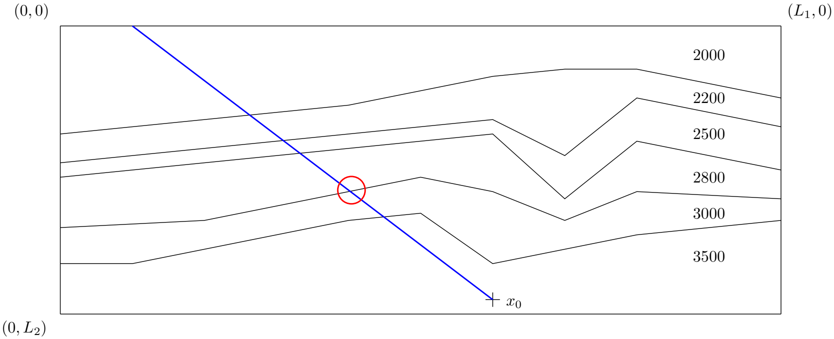

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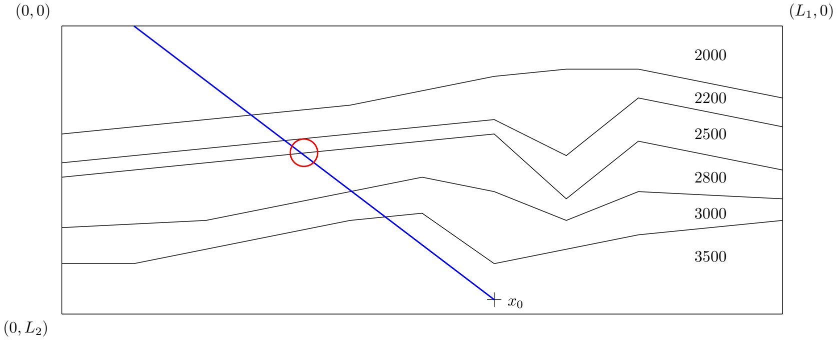

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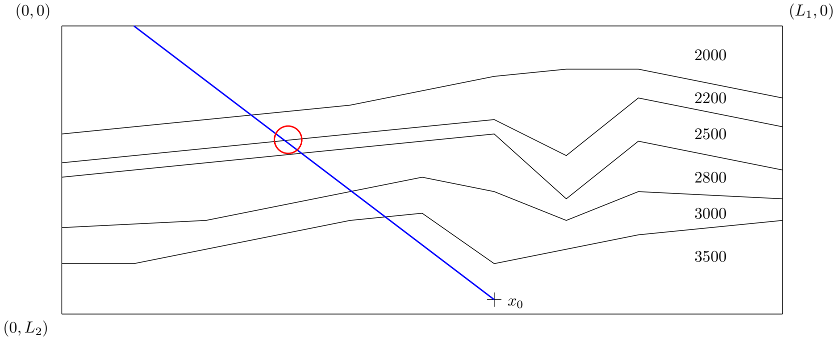

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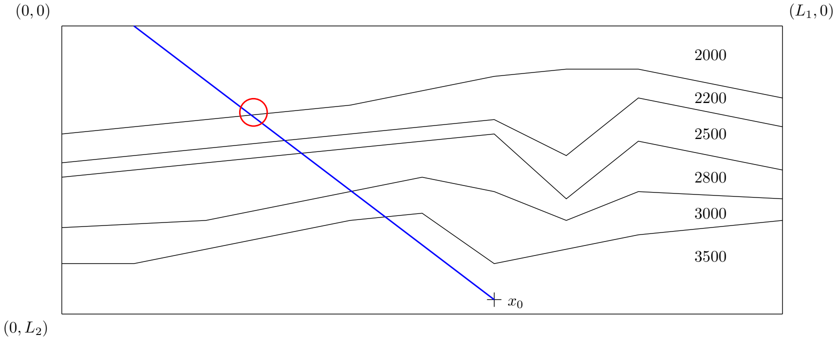
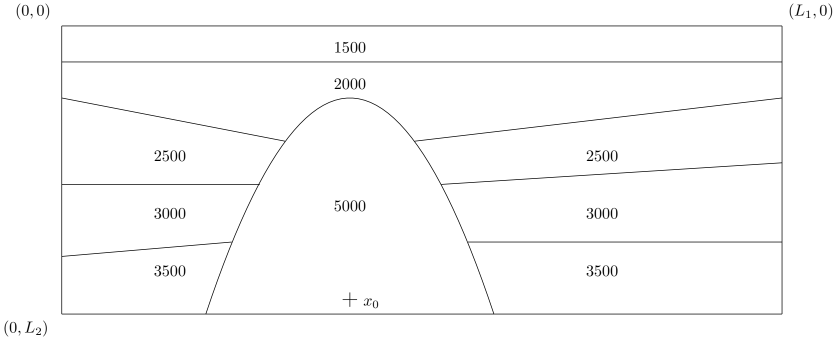


A Salt body



Stability (Heterogeneous case)

- Additional hypothesis on c
- Same bounds than the homogeneous case
- $C = C(\Omega, c_{\max}/c_{\min})$

Settings

- $\mathcal{T}_H$ mesh of Ω
- $\mathcal{T}_H$ conforming, regular
- H mesh step
- V_H^p discretisation space of order p

$$V_H^p = \left\{ v \in C^0(\bar{\Omega}) \quad | \quad v|_K \in \mathcal{P}_p(K) \quad \forall K \in \mathcal{T}_H \right\}$$

Homogeneous case

- Stability is proved under the condition that

$$\omega^{p+1} H^p \leq C,$$

- then we have

$$\omega |u - u_H|_{0,\Omega} + |u - u_H|_{1,\Omega} \leq C \omega^{p+2} H^{p+1},$$

for all $\omega > 0$.

Addional Constraint

- The parameter c must be taken into account

A solution

- Fix H and p to fit ω
- Use subquadrature schemes of step h
- Fix h to fit c

Numerical Integration

- One must integrate quantities

$$\int_K \frac{1}{c^2}\varphi^i\varphi^j, \quad \varphi^i, \varphi^j \in \mathcal{P}_p(K), \quad K \in \mathcal{T}_H$$

- Consider a submesh $\mathcal{T}_{H,h}^K$ of K
- Replace c by c_h
- c_h piecewise constant on $\mathcal{T}_{H,h}^K$

$$\int_K \frac{1}{c^2}\varphi^i\varphi^j \simeq \int_K \frac{1}{c_h^2}\varphi^i\varphi^j = \sum_{B\in\mathcal{T}_{H,h}^K} \frac{1}{c_h^2}\int_B \varphi^i\varphi^j$$

Numerical Integration

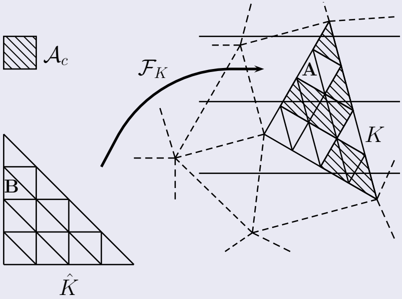
- Consider a mesh $\hat{\mathcal{T}}_h$ of $\hat{K}$
- Compute the reference integrals

$$\hat{I}_B^{i,j} = \int_{\hat{B}} \varphi^i \varphi^j, \quad \varphi^i, \varphi^j \in \mathcal{P}_p(\hat{K}), \quad \hat{B} \in \hat{\mathcal{T}}_h$$

- Map them using the Jacobian

$$\int_K \frac{1}{c_h^2} \varphi^i \varphi^j = \operatorname{jac} J_K \sum_{\hat{B} \in \hat{\mathcal{T}}_h} \frac{1}{c_h^2} \hat{I}_B^{i,j}$$

Mapping



Stability and convergence

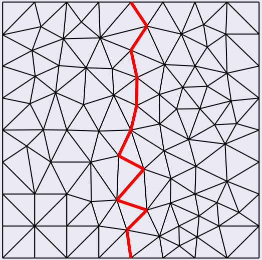
- Linear elements $p = 1$
- Stability is ensured if

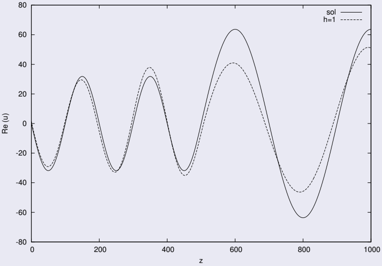
$$\omega^2 H + \omega h \leq C,$$

- and then

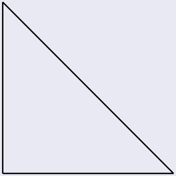
$$\omega |u - u_H|_{0,\Omega} + |u - u_H|_{1,\Omega} \leq C \left(\omega h + \omega^3 H^2 \right)$$

Two-layer medium ($\omega = 10\pi$, $c_1 = 1000$, $c_2 = 2000$, $\mathcal{P}_4$ elements)





Reference cell



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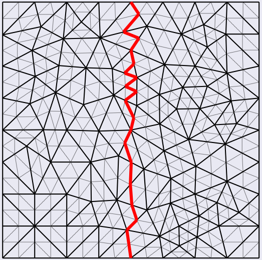

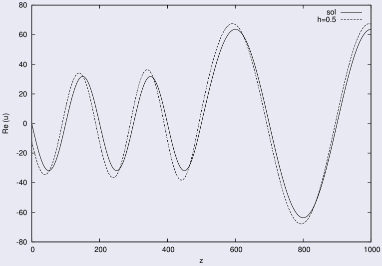

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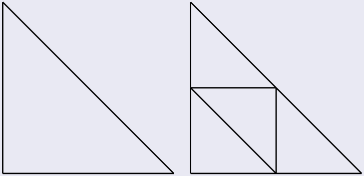

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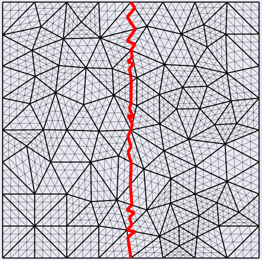

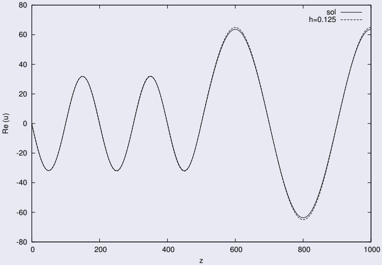

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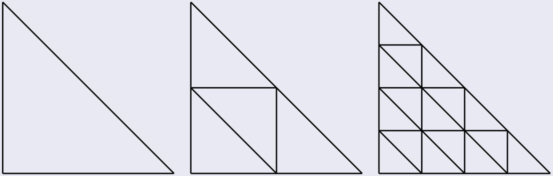

Multi-layer test case

- 1000 layers of 3 meters each
- $c_{min} = 500 m.s^{-1}$, $c_{max} = 5500 m.s^{-1}$
- $|c_j - c_{j+1}| \geq 1000$
- $\omega = 40\pi$ ($f = 20$ Hz)
- $\mathcal{P}_6$ elements

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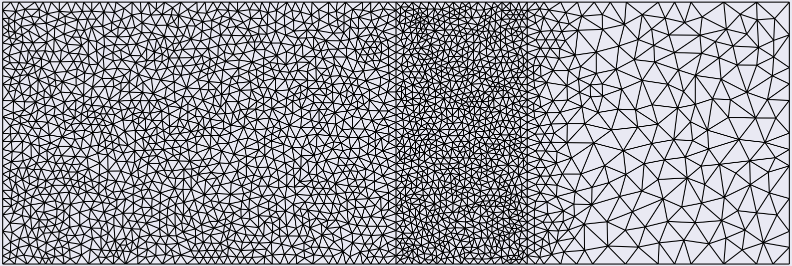
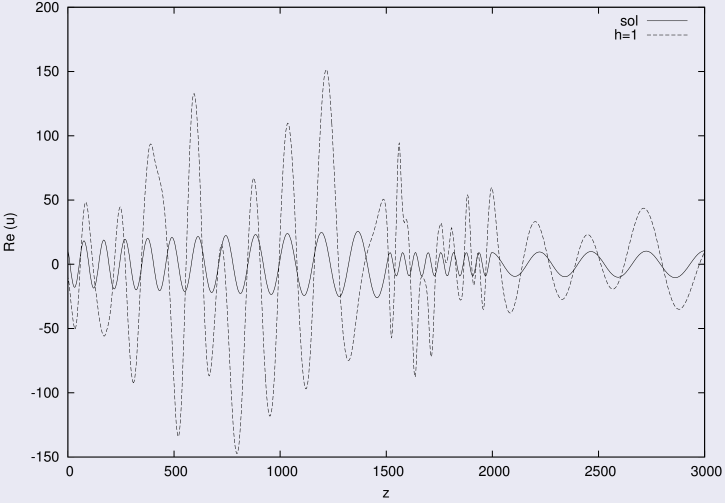
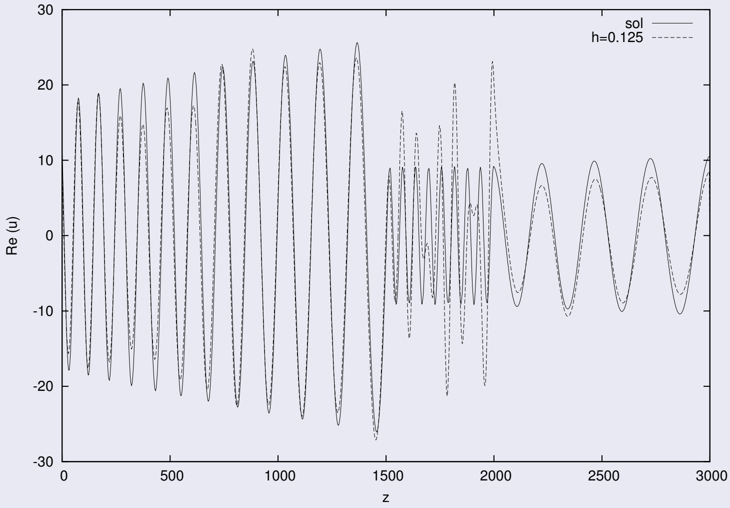
Mesh



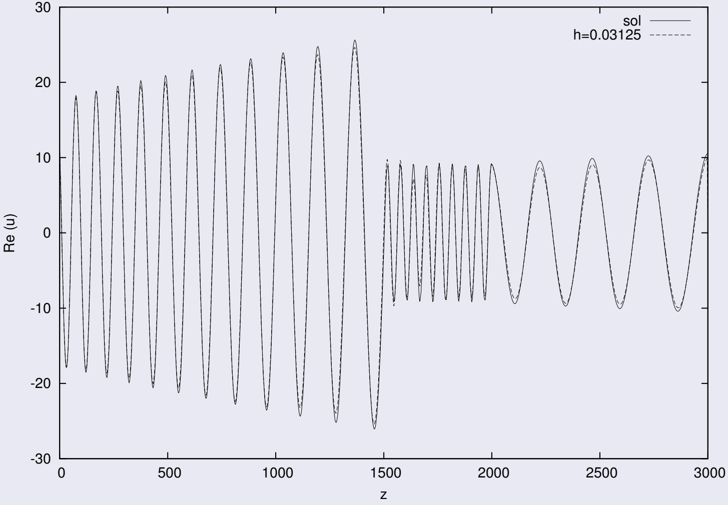
Without subquadrature (7.44 relative L^2 error)



64 subcells (4.45×10^{-1} relative L^2 error)



1024 subcells (7.00×10^{-2} relative L^2 error)



Conclusion

- subquadrature schemes capture fine scale heterogenities
- an arbitrary high order 2D solver has been implemented
- theoretical convergence issues have been adressed
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Perspective

- a 3D solver is on the way
- sharper estimates
- non-constant density

$$-\frac{\omega^2}{\kappa}u - \mathrm{div}\left(\frac{1}{\rho}\nabla u\right) = f$$