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A Lyapunov function for economic MPC without terminal conditions

Lars Grüne[1] and Marleen Stieler[2]

***Abstract*— We consider nonlinear economic model predictive control (MPC) without terminal constraints or costs. We show that under suitable conditions, consisting of strict dissipativity, the turnpike property and appropriate continuity properties, a practical Lyapunov function exists for the MPC closed loop. This Lyapunov function is given by the optimal value function of the optimal control problem with rotated stage costs originating from the dissipativity condition. Alternative sufficient conditions in terms of suitable controllability properties and finite time optimality of the closed loop trajectories during the transient phase are also discussed.**

I. Introduction

One of the major accomplishments in economic Model Predictive Control (MPC) was the observation that under a strict dissipativity condition the existence of an optimal equilibrium follows which is asymptotically stable for the MPC closed loop. The proof of this result relies on the fact that the optimal value function of an optimal control problem with a rotated stage cost provides a Lyapunov function for the closed loop. This was first proved in [4] under a linear variant of strict dissipativity — which basically translates to strong duality of linear programs — and then extended to the general notion of strict dissipativity in [2]. See also [9] for an extension of the former approach to periodic orbits.

All these results (and also stability results for alternative economic MPC approaches approaches like, e.g., [5], [8]) have in common that terminal conditions — i.e., terminal constraints and/or costs — on the optimal control problem solved in each step of the MPC scheme are imposed and crucially exploited in the stability proof. In practice, however, terminal costs are often omitted in order to simplify the design. Moreover, terminal constraints may restrict the operating region of the resulting controller. For these reasons, it is of interest to analyze the behavior of economic MPC schemes without any terminal conditions. In this paper, we show that the same Lyapunov function construction as in [2], [4] also works without terminal conditions if one relaxes the stability notion to practical asymptotic stability. Essentially, the terminal conditions are replaced by the turnpike property, which states that optimal trajectories pass by near the optimal equilibrium even without enforcing this by additional state constraints. Since the turnpike property only ensures that the trajectories pass by near the optimal equilibrium but do not necessarily reach this point exactly, as a second ingredient we need a uniform continuity assumption on the optimal value functions which ensures that the difference between exact and approximately reaching the equilibrium has only small effects on the optimal value along the resulting trajectories. Our main result in this paper, Theorem 3.6, makes this precise. Since both, the turnpike property and the uniform continuity cannot be checked directly in terms of the problem data, we also derive alternative (sufficient) conditions in terms of suitable controllability properties in Theorems 4.4 and 4.9. Moreover, using the practical Lyapunov function we will be able to derive finite horizon approximate optimality properties of the MPC closed loop trajectories during their transient phase, as stated in Theorem 5.1.

In the technical parts of this paper, we heavily rely upon preliminary results from [7]. Indeed, the proof of the main Theorem 3.6 essentially consists in cleverly re-arranging inequalities from this reference in order to verify the practical Lyapunov function property. By doing so, we improve the results from [7] by obtaining significantly stronger properties while removing the requirement of exponential (or at least superlinear) convergence of the error terms induced by the turnpike property. While exponential turnpike is still discussed in this paper as a special case, it does no longer belong to the conditions for practical stability and approximately optimal transient performance.

The organization of this paper is as follows. After formulating the problem, introducing the concept of practical Lyapunov functions and explaining their relevance in Section II, we formulate and prove the main practical stability theorem in Section III. Alternative sufficient conditions are discussed in Section IV and transient optimality in Section V. Section VI illustrates our results by means of a numerical example and Section VII concludes our paper.

II. Problem Formulation

We consider nonlinear discrete time control systems given by

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

for $f : X \times U \to X$, with normed spaces X and U denoting the state and control space, respectively. The solution of system (1) for a control sequence $u = (u(0), u(1), \dots, u(K-1)) \in U^K$ emanating from the initial value x is denoted by $x_u(k, x),\ k = 0, \dots, K-1$. The sets $\mathbb{X}$ and $\mathbb{U}$ denote the admissible states and controls. For a given initial value $x \in \mathbb{X}$, a control sequence $u \in \mathbb{U}^K$ is called *admissible*

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if $x_u(k,x) \in \mathbb{X}$ holds for all time instants $k = 0, \dots, K$. The set of all admissible control sequences is denoted by $\mathbb{U}^K(x)$. For the infinite case $u = (u(0), u(1), \dots) \in U^\infty$ we define the sets $\mathbb{U}^\infty$ and $\mathbb{U}^\infty(x)$ similarly. In order to avoid feasibility issues we assume $\mathbb{U}^K(x) \neq \emptyset$ for all $x \in \mathbb{X}$ and all $K \in \mathbb{N} \cup \{\infty\}$.

For a given stage cost $\ell : X \times U \to \mathbb{R}$ we define the finite horizon cost functional

$$J_N(x,u) := \sum_{k=0}^{N-1} \ell(x_u(k,x), u(k)), \tag{2}$$

and the corresponding optimal value function

$$V_N(x) := \inf_{u \in \mathbb{U}^N(x)} J_N(x,u). \tag{3}$$

In the sequel we assume that for all $x \in \mathbb{X}$ and all $N \in \mathbb{N}$ there is a control sequence $u^\star_{N,x} \in \mathbb{U}^N(x)$, such that the equality $V_N(x) = J_N(x, u^\star_{N,x})$ holds, i.e. $u^\star_{N,x}$ solves the *optimal control problem* of minimizing $J_N(x,u)$ with respect to $u \in \mathbb{U}^N(x)$. We remark that optimal control sequences need not be unique; in this case $u^\star_{N,x}$ denotes one of the possible optimal control sequences.

The optimal control problem just defined can be used in order to define a feedback law using the following iterative *model predictive control (MPC)* scheme. Fixing an optimization horizon $N \in \mathbb{N}$, at each time instant n we perform the following steps:

1) Measure the current state $x = x(n)$ of the system.
2) Solve the optimization problem of minimizing $J_N(x,u)$ with respect to $u \in \mathbb{U}^N(x)$ subject to $x_u(0,x) = x$ and $x_u(k+1,x) = f(x_u(k,x), u(k))$. Denote the resulting optimal control sequence by $u^\star_{N,x}$.
3) Apply the first element of $u^\star_{N,x}$ as a feedback control value until the next time instant, i.e., define the feedback law $\mu_N(x) := u^\star_{N,x}(0)$.

The resulting *MPC closed loop system* is given by $x(n+1) = f(x(n), \mu_N(x(n)))$. Trajectories of this system with initial value $x \in \mathbb{X}$ will be denoted by $x_{\mu_N}(n,x)$

As the MPC feedback law is derived from minimizing (2), questions about the optimality properties of the closed loop naturally arise. Here, we will investigate the values

$$J^{cl}_K(x, \mu_N) := \sum_{n=0}^{K-1} \ell(x_{\mu_N}(n,x), \mu_N(x_{\mu_N}(n,x))),$$

for arbitrary $K \in \mathbb{N}$. Moreover, stability properties of the closed loop are of interest and — as we will see — form an important prerequisite for approximate optimality estimates. In this respect, the key contribution of this paper is the proof that essentially the same Lyapunov function which can be used in economic MPC with terminal conditions [4], [2] can also be used in our setting.

For the definition of stability we will make use of the following classes of comparison functions

$$\mathcal{L} := \left\{ \delta : \mathbb{R}^+_0 \to \mathbb{R}^+_0 \;\middle|\; \begin{array}{l} \delta \text{ continuous and decreasing} \\ \text{with } \lim_{k\to\infty} \delta(k) = 0 \end{array} \right\},$$
$$\mathcal{K} := \left\{ \alpha : \mathbb{R}^+_0 \to \mathbb{R}^+_0 \;\middle|\; \begin{array}{l} \alpha \text{ continuous and strictly} \\ \text{increasing with } \alpha(0) = 0 \end{array} \right\},$$
$$\mathcal{K}_\infty := \{\alpha \in \mathcal{K} \,|\, \alpha \text{ unbounded}\},$$
$$\mathcal{KL} := \left\{ \beta : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}^+_0 \;\middle|\; \begin{array}{l} \beta \text{ continuous,} \\ \beta(\cdot,t) \in \mathcal{K},\ \beta(r,\cdot) \in \mathcal{L} \end{array} \right\}.$$

Stability will be considered for optimal steady states defined as follows.

Definition 2.1: A pair $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ that satisfies the condition $f(x^e, u^e) = x^e$ is called *steady state* or *equilibrium* for the control system (1). A steady state is *optimal*, if it solves the optimization problem

$$\min_{x \in \mathbb{X}, u \in \mathbb{U}} \ell(x,u) \text{ s.t. } f(x,u) - x = 0. \tag{4}$$

Definition 2.2: Let $x^e \in \mathbb{X}$ be an equilibrium for the closed loop system, i.e. $x^e = f(x^e, \mu(x^e))$. The equilibrium is called *practically asymptotically stable* w.r.t. $\varepsilon \geq 0$ on a set $S \subseteq \mathbb{X}$ with $x^e \in S$ if there exists $\beta \in \mathcal{KL}$ such that

$$\|x_\mu(k,x) - x^e\| \leq \max\{\beta(\|x - x^e\|, k),\, \varepsilon\} \tag{5}$$

holds for all $x \in S$ and all $k \in \mathbb{N}$. The equilibrium is *globally practically asymptotically stable* w.r.t. $\varepsilon \geq 0$ if (5) holds on $S = \mathbb{X}$.

A sufficient condition for this stability property is the existence of a practical Lyapunov function in the following sense.

Definition 2.3: A function $V : \mathbb{X} \to \mathbb{R}$ is a *practical Lyapunov function* w.r.t. $\delta > 0$ for the closed loop system on a set $S \subseteq \mathbb{X}$ with $x^e \in S$, if there are $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{K}$ such that

$$\alpha_1(\|x - x^e\|) \leq V(x) \leq \alpha_2(\|x - x^e\|) \tag{6}$$

holds for all $x \in \mathbb{X}$ and

$$V(f(x, \mu(x))) \leq V(x) - \alpha_3(\|x - x^e\|) + \delta \tag{7}$$

holds for all $x \in S$.

The relevance of the existence of a practical Lyapunov function follows from the following theorem, which is standard and can be found in similar forms in various references. The particular form given here is proved in [6].

Theorem 2.4: Let V be a practical Lyapunov function w.r.t. some $\delta > 0$ on a set $S \subseteq \mathbb{X}$. Assume that either $S = \mathbb{X}$ or $S = V^{-1}[0, L] := \{x \in \mathbb{X} \,|\, V(x) \leq L\}$ for some $L > \alpha_2(\alpha_3^{-1}(\delta)) + \delta$. Then x^e is practically asymptotically stable on S w.r.t. $\varepsilon = \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(\delta)) + \delta)$.

III. The basic stability result

In this section we show how to re-arrange the inequalities proved in [7] in order to verify that $\widetilde{V}_N$ is a practical Lyapunov function for the economic MPC closed loop. The conditions we impose for this result are

- strict dissipativity, equal to that used in [2]

- continuity and local Lipschitz or uniform continuity of all data near x^e and u^e
- uniform continuity of the optimal value functions in x^e
- the turnpike property

and are given in rigorous form in the following four assumptions.

Assumption 3.1 (Strict dissipativity): The optimal control problem of minimizing (2) is *strictly dissipative*, i.e., there is an equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$, a function $\alpha_\ell \in \mathcal{K}_\infty$ and a *storage function* $\lambda : X \to \mathbb{R}$ such that

$$\min_{u \in \mathbb{U}} \tilde{\ell}(x, u) \geq \alpha_\ell(\|x - x^e\|) \tag{8}$$

holds for all $x \in \mathbb{X}$, where $\tilde{\ell}$ denotes the *rotated* stage costs

$$\tilde{\ell}(x, u) := \ell(x, u) + \lambda(x) - \lambda(f(x, u)) - \ell(x^e, u^e). \tag{9}$$

In the next assumptions we use the balls $\mathcal{B}_\delta(x^e) := \{x \in \mathbb{X} \,|\, \|x - x^e\| < \delta\}$ for $\delta > 0$.

Assumption 3.2 (Continuity of data): The functions f, ℓ and λ are continuous and Lipschitz continuous with constants L_f, L_ℓ and L_λ on balls $\mathcal{B}_\delta(x^e)$ and $\mathcal{B}_\delta(u^e)$ around x^e and u^e, respectively, and $\tilde{\ell}$ satisfies the inequality

$$\tilde{\ell}(x, u) \leq \alpha(\|x - x^e\|) + \alpha(\|u - u^e\|) \tag{10}$$

for all $x \in \mathbb{X},\ u \in \mathbb{U}$ and a suitable $\alpha \in \mathcal{K}_\infty$.

We remark that under Assumption 3.1 the function $\tilde{\ell}$ is zero in (x^e, u^e). Hence, in the finite dimensional case with $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ inequality (10) follows from continuity of $\tilde{\ell}$.

In order to formulate the next assumptions we need the following additional definition.

Definition 3.3: For the rotated stage cost $\tilde{\ell}$ from Assumption 3.1, we define $\widetilde{J}_N(x, u)$ and $\widetilde{V}_N(x)$ similar to (2) and (3) with $\tilde{\ell}$ in place of ℓ. The corresponding optimal control sequences are denoted by $\tilde{u}^\star_{N,x}$.
We remark that in general the optimal trajectories for the original and the rotated stage cost do not coincide.

Assumption 3.4 (Uniform continuity of V_N and $\widetilde{V}_N$): There exist γ_V, $\tilde{\gamma}_V \in \mathcal{K}$ such that

$$|V_N(x) - V_N(x^e)| \leq \gamma_V(\|x - x^e\|) \quad \text{and}$$
$$|\widetilde{V}_N(x) - \widetilde{V}_N(x^e)| \leq \tilde{\gamma}_V(\|x - x^e\|)$$

holds for all $x \in \mathbb{X}$ and all $N \in \mathbb{N}$.

Assumption 3.5 (Turnpike property): There exists $c \in (7/8, 1)$ and $\sigma \in \mathcal{L}$ such that for each $x \in \mathbb{X}$ and each $N \in \mathbb{N}$ the number $Q_N := \#P_N$ for

$$P_N := \{k \in \{0, \ldots, N-1\} : \|x_{u^\star_{N,x}}(k, x) - x^e\| \leq \sigma(N)\}$$

satisfies $Q_N \geq cN$. The same estimate holds for the optimal trajectories $x_{\tilde{u}^\star_{N,x}}$ of the rotated problem.

The following theorem shows that under these conditions the function $\widetilde{V}_N$ is a practical Lyapunov function for the economic MPC closed loop.

Theorem 3.6: Consider an economic MPC problem without terminal constraints satisfying Assumptions 3.1, 3.2, 3.4 and 3.5. Then there exists $N_0 \in \mathbb{N}$ and functions $\delta \in \mathcal{L}$ and $\alpha_V \in \mathcal{K}_\infty$ such that the inequalities

$$\alpha_\ell(\|x - x^e\|) \leq \widetilde{V}_N(x) \leq \alpha_V(\|x - x^e\|) \tag{11}$$

and

$$\begin{aligned} \widetilde{V}_N(f(x, \mu_N(x))) &\leq \widetilde{V}_N(x) - \tilde{\ell}(x, \mu_N(x)) + \delta(N) \\ &\leq \widetilde{V}_N(x) - \alpha_\ell(\|x - x^e\|) + \delta(N) \end{aligned} \tag{12}$$

hold for all $N \geq N_0$ and $x \in \mathbb{X}$. In particular, the functions $\widetilde{V}_N$ are practical Lyapunov functions for the economic MPC closed loop system and the closed loop is practically asymptotically stable w.r.t. $\varepsilon \to 0$ as $N \to \infty$.

Proof: The lower bound on $\widetilde{V}_N$ follows directly from Assumption 3.1 and the fact that $\widetilde{V}_N(x) \geq \min_{u \in \mathbb{U}} \tilde{\ell}(x, u)$. The upper bound follows from Assumption 3.4 with $\alpha_V = \tilde{\gamma}_V$. From [7, Theorem 4.2] applied with $K = 1$ we get

$$\ell(x, \mu_N(x)) \leq V_N(x) - V_N(f(x, \mu_N(x))) + \varepsilon(N-1)$$

with $\varepsilon(N) = \gamma_V(\sigma(N)) + \gamma_V(L_f(\sigma(N))) + L_\ell(\sigma(N))$, implying that [7, Eq. (18)] holds with $\varepsilon(N-1)$ in place of $\delta(N)$.

The fact that the set P_N in Assumption 3.5 contains more than $7N/8$ elements implies that the intersection of eight such sets contains at least one element $P \in \{0, \ldots, N-1\}$. Hence, we can proceed as in the proof of Theorem 7.6 in [7] with $K = 1$ in order to conclude

$$\tilde{\ell}(x, \mu_N(x)) \leq \widetilde{V}_N(x) - \widetilde{V}_N(f(x, \mu_N(x))) + \varepsilon(N-1) + R(N),$$

where the remainder term $R(N)$ is a sum of six terms of the form $\gamma_V(\sigma(N))$, $\tilde{\gamma}_V(\sigma(N))$, $L_\lambda \sigma(N)$.

Hence, (12) follows with $\delta(N) = \varepsilon(N-1) + R(N)$ which is an the $\mathcal{L}_\mathbb{N}$-function. The last inequality follows from Assumption 3.1. □

IV. Alternative sufficient conditions

While Assumptions 3.1 und 3.2 are easy to check once the data is available (and in case λ is not available there are at least sufficient conditions guaranteeing the existence of λ, see, e.g., [3]), Assumptions 3.4 and 3.5 involve the optimal value functions and trajectories whose a priori computation we would like to avoid. To this end, in this section we present sufficient controllability and stabilizability conditions under which these two assumptions can be concluded.

The first set of conditions applies to nonlinear systems with compact state and control constraints.

Assumption 4.1 (Compactness): The state and control constraint set $\mathbb{X}$ and $\mathbb{U}$ are compact.

Assumption 4.2 (Local controllability on $\mathcal{B}_\varepsilon(x^e)$): There is $\varepsilon > 0$, $M' \in \mathbb{N}, C > 0$ such that $\forall x \in \mathcal{B}_\varepsilon(x^e)\ \exists u_1 \in \mathbb{U}^{M'}(x), u_2 \in \mathbb{U}^{M'}(x^e)$ with

$$x_{u_1}(M', x) = x^e, x_{u_2}(M', x^e) = x$$

and

$$\begin{aligned} \max\{&\|x_{u_1}(k, x) - x^e\|, \|x_{u_2}(k, x^e) - x^e\|, \\ &\|u_1(k) - u^e\|, \|u_2(k) - u^e\|\} \leq C\|x - x^e\| \end{aligned}$$

for $k = 0, 1, \cdots, M' - 1$.

Assumption 4.3 (Finite time controllability into $\mathcal{B}_\varepsilon(x^e)$): For $\varepsilon > 0$ from Assumption 4.2 there is $K \in \mathbb{N}$ such that for each $x \in \mathbb{X}$ there is $k \leq K$ and $u \in \mathbb{U}^k(x)$ with $x_u(k,x) \in \mathcal{B}_\varepsilon(x^e)$.

The following theorem shows that these assumptions can be used in order to replace Assumptions 3.4 and 3.5.

Theorem 4.4: Consider an economic MPC problem without terminal constraints satisfying Assumptions 3.1, 3.2 and 4.1 – 4.3. Then there exists $N_0 \in \mathbb{N}$ and functions $\delta \in \mathcal{L}$ and $\alpha_V \in \mathcal{K}_\infty$ such that the inequalities (11) and (12) hold for all $N \geq N_0$ and $x \in \mathbb{X}$. In particular, the functions $\widetilde{V}_N$ are practical Lyapunov functions for the economic MPC closed loop system and the closed loop is practically asymptotically stable w.r.t. $\varepsilon \to 0$ as $N \to \infty$.

Sketch of proof *(for details see [6, Proof of Theorem 3.7]):* We prove the theorem by showing that Assumptions 3.4 and 3.5 and thus all assumptions of Theorem 3.6 are satisfied.

Assumptions 4.2 and 4.3 imply that that every initial state x can be steered to x^e in a (globally bounded) finite number of steps with control effort linear in $x - x^e$ if x is sufficiently close to x^e. Together with (10) this implies Assumption 3.4 for $\widetilde{V}_N$. The proof of Assumption 3.4 for V_N is more involved. It follows by [7, Theorem 6.4] from Assumptions 3.1 and 4.2. Similar to [7, Theorem 5.3] one sees that Assumption 3.1 and Assumption 4.3 imply the turnpike property from Assumption 3.5 with

$$\sigma(N) = \alpha_\ell^{-1}\left(\frac{C'}{1-c}N\right)$$

with c from Assumption 3.5 and $C' = \max_{x\in\mathbb{X}} 2|\lambda(x)| + \max_{x\in\mathbb{X}} \gamma_V(\|x - x^e\|)$. □

Remark 4.5: Note that the assumptions of Theorem 4.4 are not much more restrictive than those needed in [2] for proving stability for terminal constrained economic MPC. Strict dissipativity[1] and continuity are also assumed in this reference, Assumption 4.2 is slightly stronger but conceptually similar to [2, Assumption 2] and Assumption 4.3 will hold if we restrict $\mathbb{X}$ to the feasible set $\mathcal{X}_N$ from [2].

Remark 4.6: If we additionally assume the following polynomial growth condition: *There are constants $C_1, C_2, p, \eta > 0$ such that*

$$C_1(\|x-x^e\|^p) \leq \tilde{\ell}(x,u) \leq C_2(\|x-x^e\|^p + \|u-u^e\|^p) \quad (13)$$

holds for all $x \in \mathcal{B}_\eta(x^e), u \in \mathcal{B}_\eta(u^e)$ with x^e, u^e and $\tilde{\ell}$ from Assumption 3.1. Then, it follows from [3, Theorem 6.5] that $\sigma(N)$ and thus also $\delta(N)$ converge to 0 exponentially fast, i.e., there are $C > 0$ and $\theta \in (0,1)$ with $\delta(N) \leq C\theta^N$.

Our second set of conditions covers unconstrained linear quadratic problems. In this setting, we make the following assumptions.

Assumption 4.7 (Linear quadratic problem): The dynamics and the cost functions are given by

$$f(x,u) = Ax + Bu + c \qquad \text{and}$$

$$\ell(x,u) = x^T Rx + u^T Qu + s^T x + v^T u$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A, B, R, Q are matrices and s, v are vectors of appropriate dimensions with R and Q symmetric and positive definite.

Assumption 4.8 (No constraints): There are no state and control constraints, i.e., $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{U} = \mathbb{R}^m$.

Note that in this setting there exists a unique optimal steady state x^e in the sense of Definition 2.1. Moreover, [3, Proposition 4.5] shows that x^e is strictly dissipative with $\tilde{\ell}$ satisfying (13) .

Theorem 4.9: Consider an economic MPC problem without terminal constraints satisfying Assumptions 4.7 and 4.8 and let x^e be the optimal steady state. Then x^e is practically asymptotically stable on each compact subset $S \subset \mathbb{R}^n$ w.r.t. $\varepsilon \to 0$ as $N \to \infty$ if and only if the pair (A, B) is stabilizable. In this case, the problem is strictly dissipative and the functions $\widetilde{V}_N$ are practical Lyapunov functions for the closed loop satisfying (11) and (12), and ε converges to 0 exponentially fast in N.

Sketch of proof *(for details see [6, Proof of Theorem 3.11]):* "⇒": Clearly, practical asymptotic stability implies stabilizability of (A, B).

"⇐": If (A, B) is stabilizable, then strict dissipativity from Assumption 3.1 follows from [3, Proposition 4.5], the turnpike property in Assumption 3.5 was proved in [3, Theorem 6.2] and the uniform continuity of V_N and $\widetilde{V}_N$ in Assumption 3.4 follows from the explicit representation of these functions via the corresponding Riccati equations. Since Assumption 3.2 is obviously satisfied, practical asymptotic stability and the fact that $\widetilde{V}_N$ are Lyapunov functions follow from Theorem 3.6.

Since ε in Theorem 2.4 depends on $\delta = \delta(N)$ in a polynomial way, exponential convergence follows by Remark 4.6, noting that [3, Theorem 6.2] yields exponential turnpike and the quadratic stage cost ℓ satisfies the polynomial bounds from Remark 4.6 with $p = 2$. □

V. Transient performance

The fact that $\widetilde{V}_N$ can be used as a practical Lyapunov function enables us to prove an approximate finite horizon optimality property of economic MPC without terminal constraints. Since the performance on finite horizons is essentially determined by the transient behavior of the closed loop trajectories, we use the notion of "transient performance". In order to formulate this concept in detail, assume that the MPC closed loop is practically asymptotically stable, implying $x_{\mu_N}(K,x) \to x^e$ as $N \to \infty$ and $K \to \infty$. Then, *transient optimality* means that among all trajectories $x_u(k,x)$ satisfying $\|x_u(K,x) - x^e\| \leq \|x_{\mu_N}(K,x) - x^e\|$, the MPC closed loop trajectories are those with the smallest cost $J_K(x,u)$ — up to an error term which vanishes as $N \to \infty$ and $\|x_{\mu_N}(K,x) - x^e\| \to 0$. We define

$$\mathbb{U}^K_\varepsilon(x) := \{u \in \mathbb{U}^K(x) \,|\, x_u(K,x) \in \mathcal{B}_\varepsilon(x)\}.$$

We remark that for arbitrary $u \in \mathbb{U}^K(x)$ in general $J_K(x,u)$ can be much smaller than $J^{cl}_K(x,\mu_N)$, since even under the assumption of strict dissipativity finite horizon optimal

[1] The counterpart to the function α_ℓ in [2] is only assumed to be positive definite and not of class $\mathcal{K}_\infty$ as in our Assumption 3.1; however, for compact $\mathbb{X}$ this does not make a difference.

trajectories need not end up near x^e, cf., e.g., the examples in [3].

Theorem 5.1: Assume that x^e is practically asymptotically stable on a set $S \subseteq \mathbb{X}$ w.r.t. $\varepsilon = \varepsilon(N)$ for the economic MPC closed loop with Lyapunov function $\widetilde{V}_N$ satisfying (11), (12). Assume that there exists $\alpha_\lambda \in \mathcal{K}_\infty$ with $|\lambda(x)| \leq \alpha_\lambda(\|x - x^e\|)$ for all $x \in \mathbb{X}$. Let $\varepsilon_{K,N} := \|x_{\mu_N}(K,x) - x^e\| \leq \max\{\beta(\|x - x^e\|, K), \varepsilon(N)\}$. Then the inequality

$$J_K^{cl}(x, \mu_N(x)) \leq \inf_{u \in \mathbb{U}^K_{\varepsilon_{K,N}}(x)} J_K(x,u) + \alpha_V(\varepsilon_{K,N}) + 2\alpha_\lambda(\varepsilon_{K,N}) + K\delta(N) \quad (14)$$

holds for all $K, N \in \mathbb{N}$ and all $x \in S$.

Proof: First, by induction from (12) we obtain

$$\sum_{k=0}^{K-1} \tilde{\ell}(x_{\mu_N}(k,x), \mu_N(x_{\mu_N}(k,x))) \leq \widetilde{V}_N(x) - \widetilde{V}_N(x_{\mu_N}(K)) + K\delta(N). \quad (15)$$

Second, from the dynamic programming principle

$\widetilde{V}_N(x) = \inf_{u \in U^K(x)}\{\widetilde{J}_K(x,u) + \widetilde{V}_{N-K}(x_u(K,x))\}$

and (11) we obtain for all $K \in \{1, \ldots, N\}$ and $u \in \mathbb{U}^K_\varepsilon(x)$

$$\begin{aligned}\widetilde{J}_K(x,u) &= \underbrace{\widetilde{J}_K(x,u) + \widetilde{V}_{N-K}(x_u(K,x))}_{\geq \widetilde{V}_N(x)} - \underbrace{\widetilde{V}_{N-K}(x_u(K,x))}_{\leq \alpha_V(\varepsilon)} \\ &\geq \widetilde{V}_N(x) - \alpha_V(\varepsilon)\end{aligned} \quad (16)$$

and we note that for $K \geq N$ non-negativity of $\tilde{\ell}$ implies the inequality $\widetilde{J}_K(x,u) \geq \widetilde{V}_N(x)$ for all $u \in \mathbb{U}^K(x)$, implying again (16). Third, we have

$$\sum_{k=0}^{K-1} \tilde{\ell}(x_u(k,x), u(k)) = \widetilde{J}_K(x,u) = \lambda(x) + J_K(x,u) - \lambda(x_u(K,x)) \quad (17)$$

and $\widetilde{V}_N \geq 0$. Using these inequalities for all $u \in \mathbb{U}^K_{\varepsilon_{K,N}}(x)$ we obtain

$$\begin{aligned}
&J_K^{cl}(x, \mu_N(x)) \\
&\overset{(17)}{=} \sum_{k=0}^{K-1} \tilde{\ell}(x_{\mu_N}(k,x), \mu_N(x_{\mu_N}(k,x))) - \lambda(x) + \lambda(x_{\mu_N}(K,x)) \\
&\overset{(15)}{\leq} \widetilde{V}_N(x) - \widetilde{V}(x_{\mu_N}(K,x)) + K\delta(N) - \lambda(x) + \lambda(x_{\mu_N}(K,x)) \\
&\overset{(16)}{\leq} \widetilde{J}_K(x,u) + \alpha_V(\varepsilon_{K,N}) - \widetilde{V}(x_{\mu_N}(K,x)) + K\delta(N) - \lambda(x) + \lambda(x_{\mu_N}(K,x)) \\
&\overset{(17)}{=} J_K(x,u) + \alpha_V(\varepsilon_{K,N}) - \widetilde{V}(x_{\mu_N}(K,x)) + K\delta(N) - \lambda(x_u(K,x)) + \lambda(x_{\mu_N}(K,x)) \\
&\leq J_K(x,u) + \alpha_V(\varepsilon_{K,N}) + K\delta(N) + 2\alpha_\lambda(\varepsilon_{K,N})
\end{aligned}$$

implying the desired inequality. □

Remark 5.2: Note that all assumptions of Theorem 5.1 are satisfied under the assumptions of one of the Theorems 3.6, 4.4 or 4.9. In the linear quadratic case of Theorem 4.9, the existence of α_λ follows because in this setting λ is either a linear or a quadratic function, cf. [3]. Moreover, if the condition from Remark 4.6 holds then $\delta(N)$ converges to 0 exponentially fast as $N \to \infty$, implying that the error terms on the right hand side of (14) converge to 0 if $K, N \to \infty$ with $K \leq cN$ for some $c > 0$. In addition, in this case $\tilde{\ell}$ and $\widetilde{V}$ have identical polynomial growth near x^e, implying that the convergences $\beta(r,k) \to 0$ as $k \to \infty$ and $\varepsilon(N) \to 0$ as $N \to \infty$ are exponentially fast and thus all error terms in (14) converge to 0 exponentially fast as $K, N \to \infty$ with $K \leq cN$ for some $c > 0$.

VI. NUMERICAL EXAMPLE

We illustrate our findings by means of the example in [1], [4] that models a chemical reaction in an isothermal continuously stirred tank reactor (CSTR) of two reactants. The state space description of the continuous model is given by dynamics

$$\begin{aligned}\dot{x}_1(t) &= u(t)(c_1 - x_1(t))/V_R - k_r x_1(t) \\ \dot{x}_2(t) &= u(t)(c_2 - x_2(t))/V_R + k_r x_1(t),\end{aligned}$$

where the states $x_1,\ x_2$ denote the respective concentration of the reactants, the control u the steerable flow through the CSTR and the constants $c_1,\ c_2$ the feed concentration of the chemicals, k_r the rate of the reaction (chemical 1 $\to$ chemical 2). V_R denotes the volume of the reactor.

The economic stage cost is given by

$$\ell(x,u) = -2ux_2 + 0.5u + 0.1(u-4)^2. \quad (18)$$

We note that the stage cost has been regularized in order to render the problem strictly dissipative, cf. [1]. The optimal steady state is given by $(x_1^e, x_2^e, u^e) = (0.5, 0.5, 4)$.

For our analysis we use the parameters and constraints from [1], namely $V_R = 10,\ c_1 = 1,\ c_2 = 0,\ k_r = 0.4$ and $\mathbb{X} = [0,1]^2,\ \mathbb{U} = [0,20]$. Moreover, we use the sampling rate $T = 0.5$ in order to obtain a model in discrete time that fits our setting.

In [1] and [4] terminal costs or terminal equality constraints were used in the optimization problem in the MPC algorithm for ensuring convergence to the optimal steady state. Here, we waive all additional constraints or penalties in the cost functional and expect practical asymptotic stability. Indeed, Figure 1 shows that the MPC closed loop trajectories converge into a neighborhood of x^e, which is shrinking as N increases. In order to analyse the speed of the observed convergence, we measure the distance of the closed loop trajectory to x^e at time $k = 5$ in the 2-norm with respect to different optimization horizon N. Figure 2 shows that the normed distance of the endpoint of the closed loop to the optimal steady state decreases exponentially fast in N.

Now, we aim to compare the performance of closed loop trajectory to other trajectories, that converge into a neighborhood of the optimal steady state (cf. Theorem 5.1).

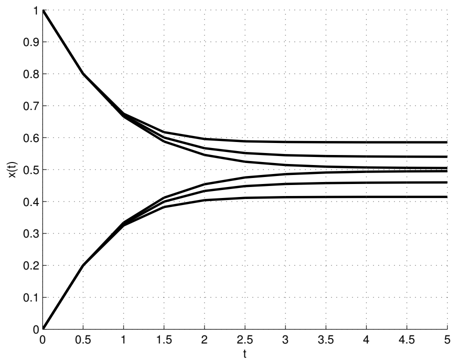

Fig. 1. Closed loop trajectories with respect to optimization horizon $N = 1,\ 2$ and 5 (from outside to inside) and $x_0 = (1, 0)^T$

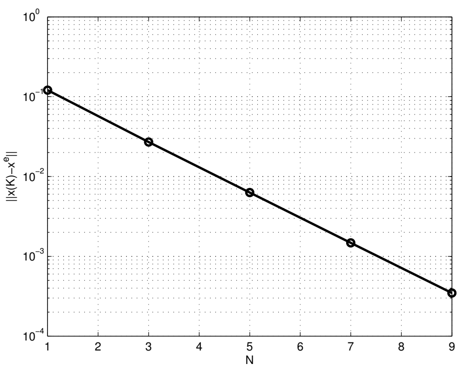

Fig. 2. Distance of the closed loop trajectories to x^e at time $k = 10$ for $N = 1, \dots, 10$ on a semi-logarithmic scale.

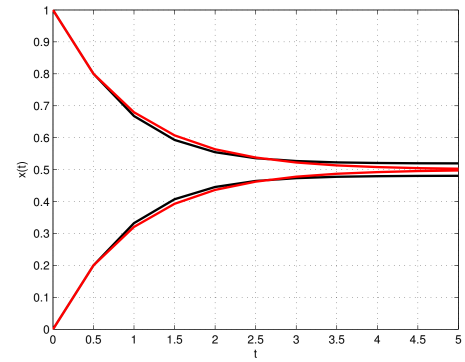

Fig. 3. Closed loop trajectory w.r.t. original (black) and stabilizing (red) stage costs and $N = 3$.

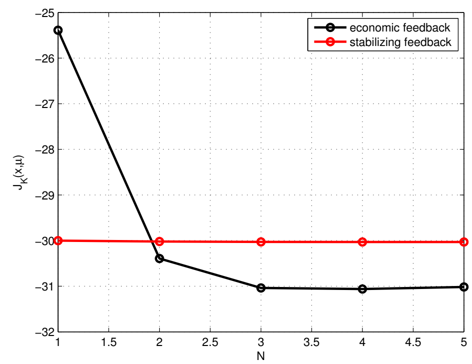

Fig. 4. Performance of the economic (black) and the stabilizing (red) feedback for varying N and fixed $K = 20$.

If we use a stabilizing stage cost instead of (18), i.e., $\ell(x, u) = \|x - (x_1^e, x_2^e)^T\|_2^2 + 0.1(u - u^e)^2$, we see in Figure 3, that the trajectory with respect to stabilizing stage costs converges exactly. We are interested in the performance of both, the feedback that stems from the original stage costs and the "stabilizing feedback" μ_N^{stab} in terms of the original cost criterion. This means, we compare $J_K^{cl}(x, \mu_N)$ to $J_K^{cl}(x, \mu_N^{\text{stab}})$. Figure 4 illustrates the statement of Theorem 5.1.

VII. Conclusion

In this paper, we have shown that the existence of a Lyapunov function for economic MPC does not necessarily rely on appropriate terminal conditions. Indeed, we have shown that under appropriate conditions a practical Lyapunov function exists also without including additional terminal constraints or costs to the MPC scheme. Like in the terminal constrained case, the Lyapunov function is given by the value function for the rotated stage cost obtained from a strict dissipativity condition. The particular form of the Lyapunov function moreover allows to prove an approximate optimality estimate for the MPC closed loop trajectories during the transient phase.

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