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Certifying isolated singular points and their multiplicity structure

Jonathan D. Hauenstein*
Department of Applied and Computational Mathematics and Statistics,
University of Notre Dame
Notre Dame, IN, 46556, USA
hauenstein@nd.edu

Bernard Mourrain
Inria Sophia Antipolis
2004 route des Lucioles, B.P. 93,
06902 Sophia Antipolis, Cedex France
bernard.mourrain@inria.fr

Agnes Szanto†
Department of Mathematics,
North Carolina State University
Campus Box 8205,
Raleigh, NC, 27965, USA.
aszanto@ncsu.edu

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Abstract

This paper presents two new constructions related to singular solutions of polynomial systems. The first is a new deflation method for an isolated singular root. This construction uses a single linear differential form defined from the Jacobian matrix of the input, and defines the deflated system by applying this differential form to the original system. The advantages of this new deflation is that it does not introduce new variables and the increase in the number of equations is linear instead of the quadratic increase of previous methods. The second construction gives the coefficients of the so-called inverse system or dual basis, which defines the multiplicity structure at the singular root. We present a system of equations in the original variables plus a relatively small number of new variables. We show that the roots of this new system include the original singular root but now with multiplicity one, and the new variables uniquely determine the multiplicity structure. Both constructions are "exact" in that they permit one to treat all conjugate roots simultaneously and can be used in certification procedures for singular roots and their multiplicity structure with respect to an exact rational polynomial system.

1 Introduction

Our motivation for this work is twofold. On one hand, in a recent paper [1], two of the co-authors of the present paper studied a certification method for approximate roots of exact overdetermined and singular polynomial systems, and wanted to extend the method to certify the multiplicity structure at the root as well. Since all these problems are ill-posed, in [1] a hybrid symbolic-numeric approach was proposed, that included the exact computation of a square polynomial system that had the original root with multiplicity one. In certifying singular roots, this exact square system was obtain from a deflation technique that added subdeterminants of the Jacobian matrix to the system iteratively. However, the multiplicity structure is destroyed by this deflation technique, that is why it remained an open question how to certify the multiplicity structure of singular roots of exact polynomial systems.
Our second motivation was to find a method that simultaneously refines the accuracy of a singular root and the parameters describing the multiplicity structure at the root. In all previous numerical

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approaches that approximate these parameters, they apply numerical linear algebra to solve a linear system with coefficients depending on the approximation of the coordinates of the singular root. Thus the local convergence rate of the parameters was slowed from the quadratic convergence of Newton's iteration applied to the singular roots. We were interested if the parameters describing the multiplicity structure can be simultaniously approximated with the coordinates of the singular root using Newton's iteration.

In the present paper we first give a new improved version of the deflation method that can be used in the certification algorithm of [1], reducing the number of added equations at each deflation iteration from quadratic to linear. We prove that applying a single linear differential form to the input system, corresponding to a generic kernel element of the Jacobian matrix, already reduced both the multiplicity and the depth of the singular root. Secondly, we give a description of the multiplicity structure using a polynomial number of parameters, and express these parameters together with the coordinates of the singular point as the roots of a multivariate polynomial system. We prove that this new polynomial system has a root corresponding to the singular root but now with multiplicity one, and the new added coordinates describe the multiplicity structure. Thus this second approach completely deflates the system in one step. The number of equations and variables in the second construction depends polynomially on the number of variables and equations of the input system and the multiplicity of the singular root. Both constructions are exact in the sense that approximations of the coordinates of the singular point are only used to detect numerically non-singular submatrices, and not in the coefficients of the constructed polynomial systems.

Related work.

The treatment of singular roots is a critical issue for numerical analysis and there is a huge literature on methods which transform the problem into a new one for which Newton-type methods converge quadratically to the root.

Deflation techniques which add new equations in order to reduce the multiplicity have already been considered in [26], [25]: By triangulating the Jacobian matrix at the (approximate) root, new minors of the polynomial Jacobian matrix are added to the initial system in order to reduce the multiplicity of the singular solution.

A similar approach is used in [10] and [8], where a maximal invertible block of the Jacobian matrix at the (approximate) root is computed and minors of the polynomial Jacobian matrix are added to the initial system. In [8], an additional step is considered where the first derivatives of the input polynomials are added when the Jacobian matrix at the root vanishes.

These constructions are repeated until a system with a simple root is obtained.

In these methods, at each step, the number of added equations is $(n-r)\times(m-r)$, where n is number of variables, m is the number of equations and r is the rank of the Jacobian at the root.

In [12], a triangular presentation of the ideal in a good position and derivations with respect to the leading variables are used to iteratively reduce the multiplicity. This process is applied for p-adic lifting with exact computation.

In other approaches, new variables and new equations are introduced simultaneously. In [31], new variables are introduced to describe some perturbations of the initial equations and some differentials which vanish at the singular points. This approach is also used in [18], where it is shown that this iterated deflation process yields a system with a simple root.

In [20], perturbation variables are also introduced in relation with the inverse system of the singular point to obtain directly a deflated system with a simple root. The perturbation is constructed from a monomial basis of the local algebra at the multiple root.

In [13, 14], only variables for the differentials of the initial system are introduced. The analysis of this deflation is improved in [5], where it is shown that the number of steps is bounded by the order of the inverse system. This type of deflation is also used in [17], for the special case where the Jacobian matrix at the multiple root has rank $n-1$ (case of breath one).

In these methods, at each step, the number of variables is at least doubled and new equations are introduced, which are linear in these new variables.

The mentioned deflation techniques usually breaks the structure of the local ring at the singular

point. The first method to compute the inverse system describing this structure is due to F.S. Macaulay [19] and known as the dialytic method. More recent algorithms for the construction of inverse systems are described e.g. in [21], reducing the size of the intermediate linear systems (and exploited in [28]). In [32], the dialytic method is used, and they analyze the relationship of deflation some methods to the inverse system. It had been further improved in [23] and more recently in [20], using an integration method. This technique reduces significantly the cost of computing the inverse system, since it relies on the solution of linear system related to the inverse system truncated in some degree and not on the number of monomials in this degree. Multiplication matrices corresponding to systems with singular roots were studied in [22, 4].
The computation of inverse systems has been used to approximate a multiple root. In [27], a minimization approach is used to reduce the value of the equations and their derivatives at the approximate root, assuming a basis of the inverse system is known. In [30], the inverse system is constructed via Macaulay's method; tables of multiplications are deduced and their eigenvalues are used to improve the approximated root. They show that the convergence is quadratic at the multiple root. In [16] they show that in the breadth one case the parameters needed to describe the inverse system is small, and use it to compute the singular roots in [15]. In [20], the inverse system is used to transform the singular root into a simple root of an augmented system.

Contributions.

We propose a new deflation method for polynomial systems with isolated singular points, which does introduce new parameters. At each step, a single differential of the system is considered based on the analysis of the Jacobian at the singular point. A linear number of new equations is added instead of the quadratic increases of the previous deflations. The deflated system does not involved any approximate coefficients and can therefore be used in certification methods as in [1].
To approximate efficiently both the singular point and its inverse system, we propose a new deflation, which involves a small number of new variables compared to other approaches which rely on Macaulay matrices. It is based on a new characterization of the isolated singular point together with its multiplicity structure. The deflated polynomial system exploits the nilpotent and commutation properties of the multiplication matrices in the local algebra of the singular point. We prove that it has a simple root which yields the root and the coefficients of the inverse system at this singular point. Due to the upper triangular form of the multiplication matrices in a convenient basis of the local algebra, the number of new parameters introduced in this deflation is less than $\frac{1}{2}n(\delta-1)\delta$ where n is the number of variables and δ the multiplicity of the singular point. The parameters involved in the deflated system are determined from the analysis of an approximation of the singular point. Nevertheless, the deflated system does not involve any approximate coefficients and thus it can also be used in certification techniques as [1].
In this paper we present two new constructions. The first one is a new deflation method for a system of polynomials with an isolated singular root. The new construction uses a single linear differential form defined from the Jacobian matrix of the input, and defines the deflated system by applying this differential form to the original system. We prove that the resulting deflated system has strictly lower multiplicity and depth at the singular point than the original one. The advantage of this new deflation is that it does not introduce new variables, and the increase in the number of equations is linear, instead of the quadratic increase of previous deflation methods. The second construction gives the coefficients of the so called inverse system or dual basis, which defines the multiplicity structure at the singular root. The novelty of our construction is that we show that the nilpotent and commutation properties of the multiplication matrices define smoothly the singular points and its inverse system. We give a system of equations in the original variables plus a relatively small number of new variables, and prove that the roots of this new system correspond to the original multiple roots but now with multiplicity one, and they uniquely determine the multiplicity structure. The number of unknowns used to describe the multiplicity structure is significantly smaller, compared to the direct computation of the dual bases from the so called Macaulay matrices. Both constructions are "exact" in the sense that approximations of the coordinates of the singular point are only used to detect numerically non-singular submatrices,

and not in the rest of the construction. Thus these constructions would allow to treat all conjugate roots simultaneously, as well as to apply these constructions in the certification of the singular roots and the multiplicity structure of an exact rational polynomial system.

2 Preliminaries

Let $\mathbf{f} := (f_1, \ldots, f_N) \in \mathbb{K}[\mathbf{x}]^N$ with $\mathbf{x} = (x_1, \ldots, x_n)$ for some $\mathbb{K} \subset \mathbb{C}$ field. Let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ be an isolated multiple root of $\mathbf{f}$. Let $I = \langle f_1, \ldots, f_N \rangle$, $\mathfrak{m}_\xi$ be the maximal ideal at ξ and Q be the primary component of I at ξ so that $\sqrt{Q} = \mathfrak{m}_\xi$.

Consider the ring of power series $\mathbb{C}[[\boldsymbol{\partial}_\xi]] := \mathbb{C}[[\partial_{1,\xi}, \ldots, \partial_{n,\xi}]]$ and we use the notation for $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$

$$\boldsymbol{\partial}_\xi^\beta := \partial_{1,\xi}^{\beta_1} \cdots \partial_{n,\xi}^{\beta_n}.$$

We identify $\mathbb{C}[[\boldsymbol{\partial}_\xi]]$ with the dual space $\mathbb{C}[\mathbf{x}]^*$ by considering $\boldsymbol{\partial}_\xi^\beta$ as derivations and evaluations at ξ, defined by

$$\boldsymbol{\partial}_\xi^\beta(p) := \boldsymbol{\partial}^\beta(p)|_\xi := \frac{d^{|\beta|}p}{dx_1^{\beta_1} \cdots dx_n^{\beta_n}}(\xi) \quad \text{for } p \in \mathbb{C}[\mathbf{x}]. \tag{1}$$

Hereafter, the derivations "at $\mathbf{x}$" will be denoted $\boldsymbol{\partial}^\beta$ instead of $\boldsymbol{\partial}_\mathbf{x}^\beta$. The derivation with respect to the variable ∂_i is denoted d_{∂_i} $(i = 1, \ldots, n)$. Note that

$$\frac{1}{\beta!}\boldsymbol{\partial}_\xi^\beta((\mathbf{x} - \xi)^\alpha) = \begin{cases} 1 & \text{it } \alpha = \beta \\ 0 & \text{otherwise} \end{cases},$$

where we use the notation $\frac{1}{\beta!} = \frac{1}{\beta_1! \cdots \beta_n!}$.
For $p \in \mathbb{C}[\mathbf{x}]$ and $\Lambda \in \mathbb{C}[[\boldsymbol{\partial}_\xi]] = \mathbb{C}[\mathbf{x}]^*$, let

$$p \cdot \Lambda : q \mapsto \Lambda(p\, q).$$

We check that $p = (x_i - \xi_i)$ acts as a derivation on $\mathbb{C}[[\boldsymbol{\partial}_\xi]]$:

$$(x_i - \xi_i) \cdot \boldsymbol{\partial}_\xi^\beta = d_{\partial_{i,\xi}}(\boldsymbol{\partial}_\xi^\beta)$$

For an ideal $I \subset \mathbb{C}[\mathbf{x}]$, let $I^\perp = \{\Lambda \in \mathbb{C}[[\boldsymbol{\partial}_\xi]] \mid \forall p \in I, \Lambda(p) = 0\}$. The vector space $I^\perp$ is naturally identified with the dual space of $\mathbb{C}[\mathbf{x}]/I$. We check that $I^\perp$ is a vector subspace of $\mathbb{C}[[\boldsymbol{\partial}_\xi]]$, which is stable by the derivations $d_{\partial_{i,\xi}}$.

Lemma 2.1. *If Q is a $\mathfrak{m}_\xi$-primary component of I, then $Q^\perp = I^\perp \cap \mathbb{C}[\boldsymbol{\partial}_\xi]$.*

This lemma shows that to compute $Q^\perp$, it suffices to compute all polynomials of $\mathbb{C}[\boldsymbol{\partial}_\xi]$ which are in $I^\perp$. Let us denote this set $\mathscr{D} = I^\perp \cap \mathbb{C}[\boldsymbol{\partial}_\xi]$. It is a vector space stable under the derivations $d_{\partial_{i,\xi}}$. Its dimension is the dimension of $Q^\perp$ or $\mathbb{C}[\mathbf{x}]/Q$, that is the *multiplicity* of ξ, denote it by $\delta_\xi(I)$, or simply by δ if ξ and I is clear from the context.
For an element $\Lambda(\boldsymbol{\partial}_\xi) \in \mathbb{C}[\boldsymbol{\partial}_\xi]$ we define the *order* $\mathrm{ord}(\Lambda)$ to be the maximal $|\beta|$ such that $\boldsymbol{\partial}_\xi^\beta$ appears in $\Lambda(\boldsymbol{\partial}_\xi)$ with non-zero coefficient.
For $t \in \mathbb{N}$, let $\mathscr{D}_t$ be the elements of $\mathscr{D}$ of order $\leq t$. As $\mathscr{D}$ is of dimension d, there exists a smallest $t \geq 0$ such that $\mathscr{D}_{t+1} = \mathscr{D}_t$. Let us call this smallest t, the *nil-index* of $\mathscr{D}$ and denote it by $o_\xi(I)$, or simply by o. As $\mathscr{D}$ is stable by the derivations $d_{\partial_{i,\xi}}$, we easily check that for $t \geq o_\xi(I)$, $\mathscr{D}_t = \mathscr{D}$ and that $o_\xi(I)$ is the maximal degree of the elements in $\mathscr{D}$.

3 Deflation using first differentials

To improve the numerical approximation of a root, one usually applies a Newton-type method to converge quadratically from a nearby solution to the root of the system, provided it is simple. In the case of multiple roots, deflation techniques are employed to transform the system into another one which has an equivalent root with a smaller multiplicity or even with multiplicity one.
We describe here a construction, using differentials of order one, which leads to a system with a simple root. This construction improves the constructions in [13, 5] since no new variables are added. It also improves the constructions presented in [10] and the "kerneling" method of [8] by adding a smaller number of equations at each deflation step. Note that, in [8], there are smart preprocessing and postprocessing steps which could be utilized in combination with our method. In the preprocessor, one adds directly partial derivatives of polynomials which are zero at the root. The postprocessor extracts a square subsystem of the completely deflated system for which the Jacobian has full rank at the root.
Consider the Jacobian matrix $J_{\mathbf{f}}(\mathbf{x}) = [\partial_j f_i(\mathbf{x})]$ of the initial system $\mathbf{f}$. By reordering properly the rows and columns (i.e., polynomials and variables), it can be put in the form

$$J_{\mathbf{f}}(\mathbf{x}) := \begin{bmatrix} A(\mathbf{x}) & B(\mathbf{x}) \\ C(\mathbf{x}) & D(\mathbf{x}) \end{bmatrix} \tag{2}$$

where $A(\mathbf{x})$ is an $r \times r$ matrix with $r = \mathrm{rank} J_{\mathbf{f}}(\xi) = \mathrm{rank} A(\xi)$.
Suppose that $B(\mathbf{x})$ is an $r \times c$ matrix. The c columns

$$\det(A(\mathbf{x})) \begin{bmatrix} -A^{-1}(\mathbf{x})B(\mathbf{x}) \\ \mathrm{Id} \end{bmatrix}$$

(for $r = 0$ this is the identity matrix) yield the c elements

$$\Lambda_1^{\mathbf{x}} = \sum_{i=1}^{n} \lambda_{1,j}(\mathbf{x})\partial_j, \ \ \ldots, \ \Lambda_c^{\mathbf{x}} = \sum_{i=1}^{n} \lambda_{c,j}(\mathbf{x})\partial_j.$$

Their coefficients $\lambda_{i,j}(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$ are polynomial in the variables $\mathbf{x}$. Evaluated at $\mathbf{x} = \xi$, they generate the kernel of $J_{\mathbf{f}}(\xi)$ and form a basis of $\mathscr{D}_1$.

Definition 3.1. The family $D_1^{\mathbf{x}} = \{\Lambda_1^{\mathbf{x}}, \ldots, \Lambda_c^{\mathbf{x}}\}$ is the *formal* inverse system of order 1 at ξ. For $\boldsymbol{i} = \{i_1, \ldots, i_k\} \subset \{1, \ldots, c\}$ with $|\boldsymbol{i}| \neq 0$, the *$\boldsymbol{i}$-deflated system* of order 1 of $\mathbf{f}$ is

$$\{\mathbf{f}, \Lambda_{i_1}^{\mathbf{x}}(\mathbf{f}), \ldots, \Lambda_{i_k}^{\mathbf{x}}(\mathbf{f})\}.$$

By construction, for $i = 1, \ldots, c$,

$$\Lambda_i^{\mathbf{x}}(\mathbf{f}) = \sum_{j=1}^{n} \partial_j(\mathbf{f})\lambda_{i,j}(\mathbf{x}) = \det(A(\mathbf{x}))J_{\mathbf{f}}(\mathbf{x})[\lambda_{i,j}(\mathbf{x})]$$

has $n - c$ zero entries. Thus, the number of non-trivial new equations added in the $\boldsymbol{i}$-deflated system is $|\boldsymbol{i}| \cdot (N - n + c)$. The construction depends on the choice of the invertible block $A(\xi)$ in $J_{\mathbf{f}}(\xi)$. By a linear invertible transformation of the initial system and by computing a $\boldsymbol{i}$-deflated system, one obtains a deflated system constructed from any $|\boldsymbol{i}|$ linearly independent elements of the kernel of $J_{\mathbf{f}}(\xi)$.

Example 3.2. Consider the multiplicity 2 root $\xi = (0, 0)$ for the system $f_1(\mathbf{x}) = x_1 + x_2^2$ and $f_2(\mathbf{x}) = x_1^2 + x_2^2$. Then,

$$J_{\mathbf{f}}(\mathbf{x}) = \begin{bmatrix} A(\mathbf{x}) & B(\mathbf{x}) \\ C(\mathbf{x}) & D(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 1 & 2x_2 \\ 2x_1 & 2x_2 \end{bmatrix}.$$

The corresponding vector $[-2x_2 \;\; 1]^T$ yields the element

$$\Lambda_1^{\mathbf{x}} = -2x_2\partial_1 + \partial_2.$$

Since $\Lambda_1^{\mathbf{x}}(f_1) = 0$, the $\{1\}$-deflated system of order 1 of $\mathbf{f}$ is

$$\left\{x_1 + x_2^2, \;\; x_1^2 + x_2^2, \;\; -4x_1x_2 + 2x_2\right\}$$

which has a multiplicity 1 root at ξ.

We use the following to analyze this deflation procedure.

Lemma 3.3 (Leibniz rule)**.** *For $a, b \in \mathbb{K}[\boldsymbol{x}]$,*

$$\boldsymbol{\partial}^\alpha(a\,b) = \sum_{\beta\in\mathbb{N}^n} \frac{1}{\beta!}\boldsymbol{\partial}^\beta(a) d_\partial^{\boldsymbol{\beta}}(\boldsymbol{\partial}^\alpha)(b).$$

Proposition 3.4. *Let r be the rank of $J_{\mathbf{f}}(\xi)$. Assume that $r < n$. Let $\boldsymbol{i} \subset \{1,\dots,n\}$ with $0 < |\boldsymbol{i}| \leq n - r$ and $\mathbf{f}^{(1)}$ be the $\boldsymbol{i}$-*deflated system *of order 1 of $\mathbf{f}$. Then, $\delta_\xi(\mathbf{f}^{(1)}) \geq 1$ and $o_\xi(\mathbf{f}^{(1)}) < o_\xi(\mathbf{f})$, which also implies that $\delta_\xi(\mathbf{f}^{(1)}) < \delta_\xi(\mathbf{f})$.*

Proof. By construction, for $i \in \boldsymbol{i}$, the polynomials $\Lambda_i^{\mathbf{x}}(\mathbf{f})$ vanish at ξ, so that $\delta_\xi(\mathbf{f}^{(1)}) \geq 1$. By hypothesis, the Jacobian of $\mathbf{f}$ is not injective yielding $o_\xi(\mathbf{f}) > 0$. Let $\mathscr{D}^{(1)}$ be the inverse system of $\mathbf{f}^{(1)}$ at ξ. Since $(\mathbf{f}^{(1)}) \supset (\mathbf{f})$, we have $\mathscr{D}^{(1)} \subset \mathscr{D}$. In particular, for any non-zero element $\Lambda \in \mathscr{D}^{(1)} \subset \mathbb{K}[\boldsymbol{\partial}_\xi]$ and $i \in \boldsymbol{i}$, $\Lambda(\mathbf{f}) = 0$ and $\Lambda(\Lambda_i^{\mathbf{x}}(\mathbf{f})) = 0$.
Using Leibniz rule, for any $p \in \mathbb{K}[\mathbf{x}]$, we have

$$\begin{aligned}
\Lambda(\Lambda_i^{\mathbf{x}}(p)) &= \Lambda\left(\sum_{j=1}^n \lambda_{i,j}(\mathbf{x})\partial_j(p)\right)\\
&= \sum_{\beta\in\mathbb{N}^n}\sum_{j=1}^n \frac{1}{\beta!}\boldsymbol{\partial}_\xi^{\boldsymbol{\beta}}(\lambda_{i,j}(\mathbf{x})) d_{\partial_\xi}^{\boldsymbol{\beta}}(\Lambda)\partial_{j,\xi}(p)\\
&= \sum_{\beta\in\mathbb{N}^n}\sum_{j=1}^n \frac{1}{\beta!}\boldsymbol{\partial}_\xi^{\boldsymbol{\beta}}(\lambda_{i,j}(\mathbf{x}))\boldsymbol{\partial}_{j,\xi} d_{\partial_\xi}^{\boldsymbol{\beta}}(\Lambda)(p)\\
&= \sum_{\beta\in\mathbb{N}^n} \Delta_{i,\beta} d_{\partial_\xi}^{\boldsymbol{\beta}}(\Lambda)(p)
\end{aligned}$$

where

$$\Delta_{i,\boldsymbol{\beta}} = \sum_{j=1}^n \lambda_{i,j,\boldsymbol{\beta}}\partial_{j,\xi} \in \mathbb{K}[\boldsymbol{\partial}_\xi] \text{ and } \lambda_{i,j,\boldsymbol{\beta}} = \frac{1}{\boldsymbol{\beta}!}\partial_\xi^{\boldsymbol{\beta}}(\lambda_{i,j}(\mathbf{x})) \in \mathbb{K}.$$

The term $\Delta_{i,\mathbf{0}}$ is $\sum_{j=1}^n \lambda_{i,j}(\xi)\partial_{j,\xi}$ which has degree 1 in $\boldsymbol{\partial}_\xi$ since $[\lambda_{i,j}(\xi)]$ is a non-zero element of $\ker J_{\mathbf{f}}(\xi)$. For simplicity, let $\phi_i(\Lambda) := \sum_{\boldsymbol{\beta}\in\mathbb{N}^n} \Delta_{i,\boldsymbol{\beta}} d_\partial^{\boldsymbol{\beta}}(\Lambda)$.
For any $\Lambda \in \mathbb{C}[\boldsymbol{\partial}_\xi]$, we have

$$\begin{aligned}
d_{\partial_{j,\xi}}(\phi_i(\Lambda)) &= \sum_{\beta\in\mathbb{N}^n} \lambda_{i,j,\beta} d_\partial^{\boldsymbol{\beta}}(\Lambda) + \Delta_{i,\beta} d_\partial^{\boldsymbol{\beta}}(d_{\partial_{j,\xi}}(\Lambda))\\
&= \sum_{\beta\in\mathbb{N}^n} \lambda_{i,j,\beta} d_\partial^{\boldsymbol{\beta}}(\Lambda) + \phi_i(d_{\partial_{j,\xi}}(\Lambda)).
\end{aligned}$$

Moreover, if $\Lambda \in \mathscr{D}^{(1)}$, then by definition $\phi_i(\Lambda)(\mathbf{f}) = 0$. Since $\mathscr{D}$ and $\mathscr{D}^{(1)}$ are both stable by derivation, it follows that $\forall \Lambda \in \mathscr{D}^{(1)}$, $d_{\partial_{j,\xi}}(\phi_i(\Lambda)) \in \mathscr{D}^{(1)} + \phi_i(\mathscr{D}^{(1)})$. Since $\mathscr{D}^{(1)} \subset \mathscr{D}$, we know $\mathscr{D} + \phi_i(\mathscr{D}^{(1)})$ is stable by derivation. For any element Λ of $\mathscr{D} + \phi_i(\mathscr{D}^{(1)})$, $\Lambda(\mathbf{f}) = 0$. We deduce that $\mathscr{D} + \phi_i(\mathscr{D}^{(1)}) = \mathscr{D}$. Consequently, the order of the elements in $\phi_i(\mathscr{D}^{(1)})$ is at most $o_\xi(\mathbf{f})$. The statement follows since ϕ_i increases the order by 1, therefore $o_\xi(\mathbf{f}^{(1)}) < o_\xi(\mathbf{f})$. □

We consider now a sequence of deflations of the system $\mathbf{f}$. Let $\mathbf{f}^{(1)}$ be the $\boldsymbol{i}_1$-deflated system of $\mathbf{f}$. We construct inductively $\mathbf{f}^{(k+1)}$ as the $\boldsymbol{i}_{k+1}$-deflated system of $\mathbf{f}^{(k)}$ for some choices of $\boldsymbol{i}_j \subset \{1, \ldots, n\}$.

Proposition 3.5. *There exists $k \leq o_\xi(\mathbf{f})$ such that ξ is a simple root of $\mathbf{f}^{(k)}$.*

Proof. By Proposition 3.4, $\delta_\xi(\mathbf{f}^{(k)}) \geq 1$ and $o_\xi(\mathbf{f}^{(k)})$ is strictly decreasing with k until it reaches the value 0. Therefore, there exists $k \leq o_\xi(I)$ such that $o_{\boldsymbol{\xi}}(\mathbf{f}^{(k)}) = 0$ and $\delta_\xi(\mathbf{f}^{(k)}) \geq 1$. This implies that ξ is a simple root of $\mathbf{f}^{(k)}$. □

To minimize the number of equations added at each deflation step, we take $|\boldsymbol{i}| = 1$. Then, the number of non-trivial new equations added at each step is at most $N - n + c$.
We described this approach using first order differentials arising from the Jacobian, but this can be easily extended to use higher order differentials.

4 The multiplicity structure

Before describing our results, we start this section by recalling the definition of orthogonal primal-dual pairs of bases for the space $\mathbb{C}[\mathbf{x}]/Q$ and its dual. The following is a definition/lemma:

Lemma 4.1 (Orthogonal primal-dual basis pair)**.** *Let $\mathbf{f}$, ξ, Q, $\mathscr{D}$, $\delta = \delta_\xi(\mathbf{f})$ and $o = o_\xi(\mathbf{f})$ be as above. Then there exists a primal-dual basis pair of the local ring $\mathbb{C}[\mathbf{x}]/Q$ with the following properties:*

- *The* primal basis *of the local ring $\mathbb{C}[\mathbf{x}]/Q$ has the form*

$$B := \{(\mathbf{x} - \xi)^{\alpha_1}, (\mathbf{x} - \xi)^{\alpha_2}, \ldots, (\mathbf{x} - \xi)^{\alpha_\delta}\}. \tag{3}$$

  We can assume that $\alpha_1 = 0$ and that the monomials in B are connected to 1 *(c.f. [24]). Define the set of exponents in B*

$$E := \{\alpha_1, \ldots, \alpha_\delta\}.$$

- *There is a unique* dual basis $\boldsymbol{\Lambda} \subset \mathscr{D}$ *orthogonal to B, i.e. the elements of $\boldsymbol{\Lambda}$ are given in the following form:*

$$\begin{aligned} \Lambda_0 &= \boldsymbol{\partial}_\xi^{\alpha_1} = 1_\xi \\ \Lambda_1 &= \frac{1}{\alpha_1!}\boldsymbol{\partial}_\xi^{\alpha_1} + \sum_{\substack{|\beta| \leq o \\ \beta \notin E}} \nu_{\alpha_1,\beta}\, \boldsymbol{\partial}_\xi^{\beta} \\ &\vdots \\ \Lambda_{\delta-1} &= \frac{1}{\alpha_\delta!}\boldsymbol{\partial}_\xi^{\alpha_\delta} + \sum_{\substack{|\beta| \leq o \\ \beta \notin E}} \nu_{\alpha_\delta,\beta}\, \boldsymbol{\partial}_\xi^{\beta}, \end{aligned} \tag{4}$$

- *We have $0 = \mathrm{ord}(\Lambda_0) \leq \cdots \leq \mathrm{ord}(\Lambda_{\delta-1})$, and for all $0 \leq t \leq o$ we have*

$$\mathscr{D}_t = \mathrm{span}\{\Lambda_j \ : \ \mathrm{ord}(\Lambda_j) \leq t\},$$

  where $\mathscr{D}_t$ denotes the elements of $\mathscr{D}$ of order $\leq t$, as above.

Proof. Let $\succ$ be the graded reverse lexicographic ordering in $\mathbb{C}[\boldsymbol{\partial}]$ such that $\partial_1 \prec \partial_2 \prec \cdots \prec \partial_n$. We consider the initial $\mathrm{In}(\mathscr{D}) = \{\mathrm{In}(\Lambda) \mid \Lambda \in \mathscr{D}\}$ of $\mathscr{D}$ for the monomial ordering $\succ$. It is a finite set of increasing monomials $D := \{\boldsymbol{\partial}^{\alpha_0}, \boldsymbol{\partial}^{\alpha_1}, \ldots, \boldsymbol{\partial}^{\alpha_{\delta-1}}\}$, which are the leading monomials of the elements of $\boldsymbol{\Lambda} = \{\Lambda_0, \Lambda_1, \ldots, \Lambda_{\delta-1}\} \subset \mathscr{D}$. As $1 \in \mathscr{D}$ and is the lowest monomial $\succ$, we have $\Lambda_0 = 1$. As $\succ$

is refining the total degree in $\mathbb{C}[\boldsymbol{\partial}]$, we have $\mathrm{ord}(\Lambda_i) = |\alpha_i|$ and $0 = \mathrm{ord}(\Lambda_0) \leq \cdots \leq \mathrm{ord}(\Lambda_{\delta-1})$. Moreover, every element in $\mathscr{D}_t$ reduces to 0 by the elements in $\mathbf{\Lambda}$. As only the elements Λ_i of order $\leq t$ are involved in this reduction, we deduce that $\mathscr{D}_t$ is spanned by the elements Λ_i with $\mathrm{ord}(\Lambda_i) \leq t$.
Let $E = \{\alpha_0, \ldots, \alpha_{\delta-1}\}$. The elements Λ_i are of the form

$$\Lambda_i = \frac{1}{\alpha_i!}\boldsymbol{\partial}_\xi^{\alpha_i} + \sum_{|\beta| \prec |\alpha_i|} \nu_{\alpha_i,\beta}\, \boldsymbol{\partial}_\xi^{\beta}.$$

By auto-reduction of the elements Λ_i, we can even suppose that $\beta \notin E$ in the summation above, so that they are of the form (4).
Let $B(\xi) = \{(\mathbf{x}-\xi)^{\alpha_0}, \ldots, (\mathbf{x}-\xi)^{\alpha_{\delta-1}}\} \subset \mathbb{C}[\mathbf{x}]$. As $(\Lambda_i((\mathbf{x}-\xi)^{\alpha_j}))_{0\leq i,j\leq \delta-1}$ is the identity matrix, we deduce that B is a basis of $\mathbb{C}[\mathbf{x}]/Q$, which is dual to $\mathbf{\Lambda}$.
As $\mathscr{D}$ is stable by derivation, the leading term of $\frac{d}{d\partial_i}(\Lambda_j)$ is in D. If $\frac{d}{d\partial_i}(\boldsymbol{\partial}_\xi^{\alpha_j})$ is not zero, then it is the leading term of $\frac{d}{d\partial_i}(\Lambda_j)$, since the monomial ordering is compatible with the multiplication by a variable. This shows that D is stable by division by the variable ∂_i and that B is connected to 1. This ends the proof of the lemma. □

Such a basis of $\mathscr{D}$ can be obtained from any other basis of $\mathscr{D}$ by transforming first the coefficient matrix of the given dual basis into row echelon form and then reducing the elements above the pivot coefficients. The integration method described in [20] computes a primal-dual pair, such that the coefficient matrix has a block row-echelon form, each block being associated to an order. The computation of a basis as in Lemma 4.1 can be then performed order by order.

Example 4.2. Let

$$f_1 = x_1 - x_2 + x_1^2, f_2 = x_1 - x_2 + x_1^2,$$

which has a multiplicity 3 root at $\xi = (0,0)$. The integration method described in [20] computes a primal-dual pair

$$\tilde{B} = \{1, x_1, x_2\}\,,\ \tilde{\mathbf{\Lambda}} = \left\{1, \partial_1 + \partial_2, \partial_2 + \frac{1}{2}\partial_1^2 + \partial_1\partial_2 + \frac{1}{2}\partial_1^2\right\}.$$

This primal dual pair does not form an orthogonal pair, since $(\partial_1 + \partial_2)(x_2) \neq 0$. However, using let say the degree lexicographic ordering such that $x_1 > x_2$, we easily deduce the primal-dual pair of Lemma 4.1:

$$B = \left\{1, x_1, x_1^2\right\}, \quad \mathbf{\Lambda} = \tilde{\mathbf{\Lambda}}.$$

Throughout this section we assume that we are given a fixed primal basis B for $\mathbb{C}[\mathbf{x}]/Q$ such that a dual basis $\mathbf{\Lambda}$ of $\mathscr{D}$ satisfying the properties of Lemma 4.1 exists. Note that such a primal basis B can be computed numerically from an approximation of ξ and using a modification of the integration method of [20].
Given the primal basis B, a dual basis can be computed by Macaulay's dialytic method which can be used to deflate the root ξ as in [14]. This method would introduce $n + (\delta-1)\left(\binom{n+o}{n} - \delta\right)$ new variables, which is not polynomial in o. Below, we give a construction of a polynomial system that only depends on at most $n + n\delta(\delta-1)/2$ variables. These variables correspond to the entries of the *multiplication matrices* that we define next. Let

$$\begin{array}{rcl} M_i : \mathbb{C}[\mathbf{x}]/Q & \to & \mathbb{C}[\mathbf{x}]/Q \\ p & \mapsto & (x_i - \xi_i)\, p \end{array}$$

be the multiplication operator by $x_i - \xi_i$ in $\mathbb{C}[\mathbf{x}]/Q$. Its transpose operator is

$$\begin{array}{rcl} M_i^t : \mathscr{D} & \to & \mathscr{D} \\ \Lambda & \mapsto & \Lambda \circ M_i = (x_i - \xi_i)\cdot\Lambda = \dfrac{d}{d\partial_{i,\xi}}(\Lambda) = d_{\partial_{i,\xi}}(\Lambda) \end{array}$$

where $\mathscr{D} = Q^\perp \subset \mathbb{C}[\boldsymbol{\partial}]$. The matrix of M_i in the basis B of $\mathbb{C}[\mathbf{x}]/Q$ is denoted $\mathtt{M}_i$.
As B is a basis of $\mathbb{C}[\mathbf{x}]/Q$, we can identify the elements of $\mathbb{C}[\mathbf{x}]/Q$ with the elements of the vector space $\mathrm{span}_{\mathbb{C}}(B)$. We define the normal form $N(p)$ of a polynomial p in $\mathbb{C}[\mathbf{x}]$ as the unique element b of $\mathrm{span}_{\mathbb{C}}(B)$ such that $p - b \in Q$. Hereafter, we are going to identify the elements of $\mathbb{C}[\mathbf{x}]/Q$ with their normal form in $\mathrm{span}_{\mathbb{C}}(B)$.
For any polynomial $p(x_1, \ldots, x_n) \in \mathbb{C}[\mathbf{x}]$, let $p(\mathbf{M})$ be the operator of $\mathbb{C}[\mathbf{x}]/Q$ obtained by replacing $x_i - \xi_i$ by M_i.

Lemma 4.3. *For any $p \in \mathbb{C}[\mathbf{x}]$, the normal form of p is $N(p) = p(\mathbf{M})(1)$ and we have*

$$p(\mathbf{M})(1) = \Lambda_0(p)\, 1 + \Lambda_1(p)\, (\mathbf{x} - \xi)^{\alpha_1} + \cdots + \Lambda_{\delta-1}(p)\, (\mathbf{x} - \xi)^{\alpha_{\delta-1}}.$$

This shows that the coefficient vector $[p]$ of $N(p)$ in the basis B of is $[p] = (\Lambda_i(p))_{0 \le i \le \delta-1}$.
The following lemma is also well known, but we include it here with proof:

Lemma 4.4. *Let B as in (3) and denote the exponents in B by $E := \{\alpha_1, \ldots, \alpha_\delta\}$ as above. Let*

$$E^+ := \bigcup_{i=1}^{n} (E + \mathbf{e}_i)$$

with $E + \mathbf{e}_i = \{(\gamma_1, \ldots, \gamma_i + 1, \ldots, \gamma_n) : \gamma \in E\}$ and we denote $\partial(E) = E^+ \setminus E$. The values of the coefficients $\nu_{\alpha,\beta}$ for $(\alpha, \beta) \in E \times \partial(E)$ appearing in the dual basis (4) uniquely determine the system of pairwise commuting multiplication matrices $\mathtt{M}_i$, namely, for $i = 1, \ldots, n$

$$\mathtt{M}_i^t = \begin{array}{|ccccc|} \hline 0 & \nu_{\alpha_1,\mathbf{e}_i} & \nu_{\alpha_2,\mathbf{e}_i} & \cdots & \nu_{\alpha_{d-1},\mathbf{e}_i} \\ 0 & 0 & \nu_{\alpha_2,\alpha_1+\mathbf{e}_i} & \cdots & \nu_{\alpha_{d-1},\alpha_1+\mathbf{e}_i} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{\alpha_{d-1},\alpha_{d-2}+\mathbf{e}_i} \\ 0 & 0 & 0 & \cdots & 0 \\ \hline \end{array} \tag{5}$$

Moreover,

$$\nu_{\alpha_i,\alpha_k+\mathbf{e}_j} = \begin{cases} 1 & \text{if } \alpha_i = \alpha_k + \mathbf{e}_j \\ 0 & \text{if } \alpha_k + \mathbf{e}_j \in E,\ \alpha_i \neq \alpha_k + \mathbf{e}_j. \end{cases}$$

Proof. As M_i^t acts as a derivation on $\mathscr{D}$ and $\mathscr{D}$ is closed under derivation, so the third property in Lemma 4.1 implies that the matrix of M_i^t in the basis of $\mathbf{\Lambda}$ of $\mathscr{D}$ has an upper triangular form with zero (blocks) on the diagonal.
For an element Λ_j of order k, its image by M_i^t is

$$\begin{aligned} M_i^t(\Lambda_j) &= (x_i - \xi_i) \cdot \Lambda_j \\ &= \sum_{|\alpha_l| < k} \Lambda_j((x_i - \xi_i)(\mathbf{x} - \xi)^{\alpha_l})\Lambda_l \\ &= \sum_{|\alpha_l| < k} \Lambda_j((\mathbf{x} - \xi)^{\alpha_l + \mathbf{e}_i})\, \Lambda_l = \sum_{|\alpha_l| < k} \nu_{\alpha_j,\alpha_l+\mathbf{e}_i} \Lambda_l. \end{aligned}$$

This shows that the entries of $\mathtt{M}_i$ are the coefficients of the dual basis elements corresponding to exponents in $E \times \partial(E)$. The second claim is clear from the definition of $\mathtt{M}_i$. □

The previous lemma shows that the dual basis uniquely defines the system of multiplication ma-

trices for $i = 1, \ldots, n$

$$\mathtt{M}_i^t = \begin{bmatrix} \Lambda_0(x_i - \xi_i) & \cdots & \Lambda_{\delta-1}(x_i - \xi_i) \\ \Lambda_0\left((\mathbf{x}-\xi)^{\alpha_1+\mathbf{e}_i}\right) & \cdots & \Lambda_{\delta-1}\left((\mathbf{x}-\xi)^{\alpha_1+\mathbf{e}_i}\right) \\ \vdots & & \vdots \\ \Lambda_0\left((\mathbf{x}-\xi)^{\alpha_d+\mathbf{e}_i}\right) & \cdots & \Lambda_{\delta-1}\left((\mathbf{x}-\xi)^{\alpha_\delta+\mathbf{e}_i}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \nu_{\alpha_1,\mathbf{e}_i} & \nu_{\alpha_2,\mathbf{e}_i} & \cdots & \nu_{\alpha_{\delta-1},\mathbf{e}_i} \\ 0 & 0 & \nu_{\alpha_2,\alpha_1+\mathbf{e}_i} & \cdots & \nu_{\alpha_{\delta-1},\alpha_1+\mathbf{e}_i} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{\alpha_{\delta-1},\alpha_{\delta-2}+\mathbf{e}_i} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Note that these matrices are nilpotent by their upper triangular structure, and all 0 eigenvalues. As o is the maximal order of the elements of $\mathscr{D}$, we have $\mathtt{M}^\gamma = 0$ if $|\gamma| > o$.

Conversely, the system of multiplication matrices $\mathtt{M}_1, \ldots, \mathtt{M}_n$ uniquely defines the dual basis as follows. Consider $\nu_{\alpha_i,\gamma}$ for some (α_i, γ) such that $|\gamma| \leq o$ but $\gamma \notin E^+$. We can uniquely determine $\nu_{\alpha_i,\gamma}$ from the values of $\{\nu_{\alpha_j,\beta} \ : \ (\alpha_j, \beta) \in E \times \partial(E)\}$ from the following identities:

$$\nu_{\alpha_i,\gamma} = \Lambda_i((\mathbf{x}-\xi)^\gamma) = [\mathtt{M}_{(\mathbf{x}-\xi)^\gamma}]_{1,i} = [\mathtt{M}^\gamma]_{1,i}. \tag{6}$$

The next definition defines the *parametric multiplication matrices* that we use in our constriction.

Definition 4.5 (Parametric multiplication matrices)**.** Let B as in (3), and E, $\partial(E)$ as in Lemma 4.4. We define array μ of length $n\delta(\delta-1)/2$ consisting of 0's, 1's and the variables $\mu_{\alpha_i,\beta}$ as follows: for all $\alpha_i, \alpha_k \in E$ and $j \in \{1, \ldots, n\}$ the corresponding entry is

$$\mu_{\alpha_i,\alpha_k+\mathbf{e}_j} = \begin{cases} 1 & \text{if } \alpha_i = \alpha_k + \mathbf{e}_j \\ 0 & \text{if } \alpha_k + \mathbf{e}_j \in E,\ \alpha_i \neq \alpha_k + \mathbf{e}_j \\ \mu_{\alpha_i,\alpha_k+\mathbf{e}_j} & \text{if } \alpha_k + \mathbf{e}_j \notin E. \end{cases} \tag{7}$$

The *parametric multiplication matrices* are defined for $i = 1, \ldots, n$ by

$$\mathtt{M}_i^t(\mu) := \begin{bmatrix} 0 & \mu_{\alpha_1,\mathbf{e}_i} & \mu_{\alpha_2,\mathbf{e}_i} & \cdots & \mu_{\alpha_{\delta-1},\mathbf{e}_i} \\ 0 & 0 & \mu_{\alpha_2,\alpha_1+\mathbf{e}_i} & \cdots & \mu_{\alpha_{d-1},\alpha_1+\mathbf{e}_i} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{\alpha_{d-1},\alpha_{\delta-2}+\mathbf{e}_i} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{8}$$

We denote by

$$\mathtt{M}(\mu)^\gamma := \mathtt{M}_1(\mu)^{\gamma_1} \cdots \mathtt{M}_n(\mu)^{\gamma_n},$$

and note that for general parameters values μ, the matrices $\mathtt{M}_i(\mu)$ do not commute, so we fix their order by their indices in the above definition of $\mathtt{M}(\mu)^\gamma$.

Remark 4.6. Note that we can reduce the number of free parameters in the parametric multiplication matrices by exploiting the commutation rules of the multiplication matrices corresponding to a given primal basis B. For example, consider the breadth one case, where we can assume that $E = \{\mathbf{0}, \mathbf{e}_1, 2\mathbf{e}_1, \ldots, (\delta-1)\mathbf{e}_1\}$. In this case the only free parameters appear in the first columns of $\mathtt{M}_2(\mu), \ldots, \mathtt{M}_n(\mu)$, the other columns are shifts of these. Thus, it is enough to introduce $(n-1)(\delta-1)$ free parameters, similarly as in [17]. In Section 5 we present a modification of [17, Example 3.1] which has breadth two, but also uses at most $(n-1)(\delta-1)$ free parameters.

Definition 4.7 (Parametric normal form)**.** Let $\mathbb{K} \subset \mathbb{C}$ be a field. We define

$$\begin{aligned} \mathcal{N}_{\mathbf{z},\mu} : \mathbb{K}[\mathbf{x}] &\to \mathbb{K}[\mathbf{z},\mu]^\delta \\ p &\mapsto \mathcal{N}_{\mathbf{z},\mu}(p) := \sum_{\gamma\in\mathbb{N}^n} \frac{1}{\gamma!}\boldsymbol{\partial}^\gamma_{\mathbf{z}}(p)\,\mathtt{M}(\mu)^\gamma[1]. \end{aligned}$$

where $[1] = [1, 0, \ldots, 0]$ is the coefficient vector of 1 in the basis B. This sum is finite since for $|\gamma| \geq \delta$, $\mathtt{M}(\mu)^\gamma = 0$, so the entries of $\mathcal{N}_{\mathbf{z},\mu}(p)$ are polynomials in μ and $\mathbf{z}$.

Notice that this notation is not ambiguous, assuming that the matrices $\mathtt{M}_i(\mu)$ $(i = 1, \ldots, n)$ are commuting. The specialization at $(\mathbf{x}, \mu) = (\xi, \nu)$ is the vector

$$\mathcal{N}_{\xi,\nu}(p) = [\Lambda_0(p), \ldots, \Lambda_{\delta-1}(p)]^t \in \mathbb{C}^\delta.$$

4.1 The multiplicity structure equations of a singular point

We can now characterize the multiplicity structure by polynomial equations.

Theorem 4.8. *Let $\mathbb{K} \subset \mathbb{C}$ be any field, $\mathbf{f} \in \mathbb{K}[\mathbf{x}]^N$ and let $\xi \in \mathbb{C}^n$ be an isolated solution of $\mathbf{f}$. Let $\mathtt{M}_i(\mu)$ for $i = 1, \ldots n$ be the parametric multiplication matrices as in (8) and $\mathcal{N}_{\xi,\mu}$ be the parametric normal form as in Defn. 4.7 at $\mathbf{z} = \xi$. Then the ideal J_ξ of $\mathbb{C}[\mu]$ generated by the polynomial system*

$$\begin{cases} \mathcal{N}_{\xi,\mu}(f_k) \quad \text{for } k = 1, \ldots, N, \\ \mathtt{M}_i(\mu)\cdot\mathtt{M}_j(\mu) - \mathtt{M}_i(\mu)\cdot\mathtt{M}_i(\mu) \quad \text{for } i, j = 1, \ldots, n \end{cases} \tag{9}$$

is the maximal ideal

$$\mathfrak{m}_\nu = (\mu_{\alpha,\beta} - \nu_{\alpha,\beta}, (\alpha,\beta) \in E \times \partial(E))$$

where $\nu_{\alpha,\beta}$ are the coefficients of the dual basis defined in (4).

Proof. As before, the system (9) has a solution $\mu_{\alpha,\beta} = \nu_{\alpha,\beta}$ for $(\alpha,\beta) \in E \times \partial(E)$. Thus $J_\xi \subset \mathfrak{m}_\nu$. Conversely, let $C = \mathbb{C}[\mu]/J_\xi$ and consider the map

$$\Phi : C[\mathbf{x}] \to C^\delta, \;\; p \mapsto \mathcal{N}_{\xi,\mu}(p).$$

Let K be its kernel. Since the matrices $\mathtt{M}_i(\mu)$ are commuting modulo J_ξ, we can see that K is an ideal. As $f_k \in K$, we have $\mathcal{I} := (f_k) \subset K$.
Next we show that $Q \subset K$. By construction, for any $\alpha \in \mathbb{N}^n$ we have modulo J_ξ

$$\mathcal{N}_{\xi,\mu}((\mathbf{x}-\xi)^\alpha) = \sum_{\gamma\in\mathbb{N}^n} \frac{1}{\gamma!}\boldsymbol{\partial}^\gamma_\xi((\mathbf{x}-\xi)^\alpha)\,\mathtt{M}(\mu)^\gamma[1] = \mathtt{M}(\mu)^\alpha[1].$$

Using the previous relation, we check that $\forall p, q \in C[\mathbf{x}]$,

$$\Phi(pq) = p(\xi + \mathtt{M}(\mu))\Phi(q) \tag{10}$$

where $p(\xi + \mathtt{M}(\mu))$ is obtained by replacing $x_i - \xi_i$ by $\mathtt{M}_i(\mu)$. Let $q \in Q$. As Q is the $\mathfrak{m}_\xi$-primary component of $\mathcal{I}$, there exists $p \in \mathbb{C}[\mathbf{x}]$ such that $p(\xi) \neq 0$ and $p\,q \in \mathcal{I}$. By (10), we have

$$\Phi(p\,q) = p(\xi + \mathtt{M}(\mu))\Phi(q) = 0.$$

Since $p(\xi) \neq 0$ and $p(\xi + \mathtt{M}(\mu)) = p(\xi)Id + N$ with N lower triangular and nilpotent, $p(\xi + \mathtt{M}(\mu))$ is invertible. We deduce that $\Phi(q) = p(\xi + \mathtt{M}(\mu))^{-1}\Phi(pq) = 0$ and $q \in K$.
Let us show now that Φ is surjective and more precisely, that $\Phi((\mathbf{x}-\xi)^{\alpha_k}) = \mathbf{e}_k$ (abusing the notation as here $\mathbf{e}_k$ has length δ not n). Since B is connected to 1, either $\alpha_k = 0$ or there exists

$\alpha_j \in E$ such that $\alpha_k = \alpha_j + \mathbf{e}_i$ for some $i \in \{1, \ldots, n\}$. Thus the j^{th} column of $\mathtt{M}_i(\mu)$ is $\mathbf{e}_k$ by (7). As $\{\mathtt{M}_i(\mu) : i = 1, \ldots, n\}$ are pairwise commuting, we have $\mathtt{M}(\mu)^{\alpha_k} = \mathtt{M}_j(\mu)\mathtt{M}(\mu)^{\alpha_j}$, and if we assume by induction on $|\alpha_j|$ that the first column of $\mathtt{M}(\mu)^{\alpha_j}$ is $\mathbf{e}_j$, we obtain $\mathtt{M}(\mu)^{\alpha_k}[1] = \mathbf{e}_k$. Thus, for $k = 1, \ldots, \delta$, $\Phi((\mathbf{x} - \xi)^{\alpha_k}) = \mathbf{e}_k$.
We can now prove that $\mathfrak{m}_\nu \subset J_\xi$. As $M_i(\nu)$ is the multiplication by $(x_i - \xi_i)$ in $\mathbb{C}[\mathbf{x}]/Q$, for any $b \in B$ and $i = 1, \ldots, n$, we have $(x_i - \xi_i)\, b = M_i(\nu)(b) + q$ with $q \in Q \subset K$. We deduce that for $k = 0, \ldots, \delta - 1$,

$$\Phi((x_i - \xi_i)\,(\mathbf{x} - \xi)^{\alpha_k}) = \mathtt{M}_i(\mu)\Phi((\mathbf{x} - \xi)^{\alpha_k}) = \mathtt{M}_i(\mu)(\mathbf{e}_k) = \mathtt{M}_i(\nu)(\mathbf{e}_k).$$

This shows that $\mu_{\alpha,\beta} - \nu_{\alpha,\beta} \in J_\xi$ for $(\alpha, \beta) \in E \times \partial(E)$ and that $\mathfrak{m}_\nu = J_\xi$. □

In the proof of the next theorem we need to consider cases when the multiplication matrices do not commute. We introduce the following definition:

Definition 4.9. Let $\mathbb{K} \subset \mathbb{C}$ be any field. Let $\mathcal{C}$ be the ideal of $\mathbb{K}[\mathbf{z}, \mu]$ generated by entries of the commutation relations: $\mathtt{M}_i(\mu) \cdot \mathtt{M}_j(\mu) - \mathtt{M}_j(\mu) \cdot \mathtt{M}_i(\mu) = 0$, $i, j = 1, \ldots, n$. We call $\mathcal{C}$ the *commutator ideal*.

Lemma 4.10. *For any field $\mathbb{K} \subset \mathbb{C}$, $p \in \mathbb{K}[\mathbf{x}]$, and $i = 1, \ldots, n$, we have*

$$\mathcal{N}_{\mathbf{z},\mu}(x_i p) = x_i \mathcal{N}_{\mathbf{z},\mu}(p) + \mathtt{M}_i(\mu)\, \mathcal{N}_{\mathbf{z},\mu}(p) + O_{i,\mu}(p). \tag{11}$$

where $O_{i,\mu} : \mathbb{K}[\mathbf{x}] \to \mathbb{K}[\mathbf{z}, \mu]^\delta$ is linear with image in the commutator ideal $\mathcal{C}$.

Proof. $\mathcal{N}_{\mathbf{z},\mu}(x_i p) = \sum_\gamma \frac{1}{\gamma!}\, \partial^\gamma_{\mathbf{z}}(x_i p)\, \mathtt{M}(\mu)^\gamma[1]$

$$\begin{aligned}
&= x_i \sum_\gamma \frac{1}{\gamma!} \partial^\gamma_{\mathbf{z}}(p)\, \mathtt{M}(\mu)^\gamma[1] + \sum_\gamma \frac{1}{\gamma!} \gamma_i\, \partial^{\gamma - e_i}_{\mathbf{z}}(p)\, \mathtt{M}(\mu)^\gamma[1] \\
&= x_i \sum_\gamma \frac{1}{\gamma!} \partial^\gamma_{\mathbf{z}}(p)\, \mathtt{M}(\mu)^\gamma[1] + \sum_\gamma \frac{1}{\gamma!} \partial^\gamma_{\mathbf{z}}(p)\, \mathtt{M}(\mu)^{\gamma + e_i}[1] \\
&= x_i\, \mathcal{N}_{\mathbf{z},\mu}(p) + \mathtt{M}_i(\mu) \left(\sum_\gamma \frac{1}{\gamma!} \partial^\gamma_{\mathbf{z}}(p)\, \mathtt{M}(\mu)^\gamma[1] \right) \\
&\qquad + \sum_\gamma \frac{1}{\gamma!}\, \partial^\gamma_{\mathbf{z}}(p)\, O_{i,\gamma}(\mu)[1]
\end{aligned}$$

where $O_{i,\gamma} = \mathtt{M}_i(\mu)\mathtt{M}(\mu)^\gamma - \mathtt{M}(\mu)^{\gamma + e_i}$ is a $\delta \times \delta$ matrix with coefficients in $\mathcal{C}$. Therefore, $O_{i,\mu} : p \mapsto \sum_\gamma \frac{1}{\gamma!} \partial^\gamma_{\mathbf{z}}(p)\, O_{i,\gamma}(\mu)[1]$ is a linear functional of p with coefficients in $\mathcal{C}$. □

The next theorem proves that the system defined as in (9) for general $\mathbf{z}$ has (ξ, ν) as a simple root.

Theorem 4.11. *Let $\mathbf{f} \in \mathbb{K}[\mathbf{x}]^N$ and $\xi \in \mathbb{C}^n$ be as above. Let $\mathtt{M}_i(\mu)$ for $i = 1, \ldots n$ be the parametric multiplication matrices defined in (8) and $\mathcal{N}_{\mathbf{x},\mu}$ be the parametric normal form as in Defn. 4.7. Then $(\mathbf{z}, \mu) = (\xi, \nu)$ is an isolated root with multiplicity one of the polynomial system in $\mathbb{K}[\mathbf{z}, \mu]$:*

$$\begin{cases} \mathcal{N}_{\mathbf{z},\mu}(f_k) = 0 \ \text{ for } k = 1, \ldots, N, \\ \mathtt{M}_i(\mu) \cdot \mathtt{M}_j(\mu) - \mathtt{M}_j(\mu) \cdot \mathtt{M}_i(\mu) = 0 \ \text{ for } i, j = 1, \ldots, n. \end{cases} \tag{12}$$

Proof. For simplicity, let us denote the (non-zero) polynomials appearing in (12) by

$$P_1, \ldots, P_M \in \mathbb{K}[\mathbf{z}, \mu],$$

where $M \leq N\delta + n(n-1)(\delta - 1)(\delta - 2)/4$. To prove the theorem, it is sufficient to prove that the columns of the Jacobian matrix of the system $[P_1, \ldots, P_M]$ at $(\mathbf{z}, \mu) = (\xi, \nu)$ are linearly independent. The columns of this Jacobian matrix correspond to the elements in $\mathbb{C}[\mathbf{z}, \mu]^*$

$$\partial_{1,\xi}, \ldots, \partial_{n,\xi}, \text{ and } \partial_{\mu_{\alpha,\beta}} \ \text{ for } \ (\alpha, \beta) \in E \times \partial(E),$$

where $\partial_{i,\xi}$ defined in (1) for $\mathbf{z}$ replacing $\mathbf{x}$, and $\partial_{\mu_{\alpha,\beta}}$ is defined by

$$\partial_{\mu_{\alpha,\beta}}(q) = \frac{dq}{d\mu_{\alpha,\beta}}\,\big|_{(\mathbf{z},\mu)=(\xi,\nu)} \qquad \text{for } q \in \mathbb{C}[\mathbf{z},\mu].$$

Suppose there exist $a_1,\dots,a_n$, and $a_{\alpha,\beta} \in \mathbb{C}$ for $(\alpha,\beta) \in E \times \partial(E)$ not all zero such that

$$\Delta := a_1\partial_{1,\xi} + \cdots + a_n\partial_{n,\xi} + \sum_{\alpha,\beta} a_{\alpha,\beta}\partial_{\mu_{\alpha,\beta}} \in \mathbb{C}[\mathbf{z},\mu]^*$$

vanishes on all polynomials $P_1,\dots,P_M$ in (12). In particular, for an element $P_i(\mu)$ corresponding to the commutation relations and any polynomial $Q \in \mathbb{C}[\mathbf{x},\mu]$, using the product rule for the linear differential operator Δ we get

$$\Delta(P_iQ) = \Delta(P_i)Q(\xi,\nu) + P_i(\nu)\Delta(Q) = 0$$

since $\Delta(P_i) = 0$ and $P_i(\nu) = 0$. By the linearity of Δ, for any polynomial C in the commutator ideal $\mathcal{C}$, we have $\Delta(C) = 0$.
Furthermore, since $\Delta(\mathcal{N}_{\mathbf{z},\mu}(f_k)) = 0$ and

$$\mathcal{N}_{\xi,\nu}(f_k) = [\Lambda_0(f_k),\dots,\Lambda_{\delta-1}(f_k)]^t,$$

we get that

$$(a_1\partial_{1,\xi} + \cdots + a_n\partial_{n,\xi}) \cdot \Lambda_{\delta-1}(f_k) + \sum_{|\gamma|\le|\alpha_{\delta-1}|} p_\gamma(\nu)\,\boldsymbol{\partial}_\xi^\gamma(f_k) = 0 \tag{13}$$

where $p_\gamma \in \mathbb{C}[\mu]$ are some polynomials in the variables μ that do not depend on f_k. If $a_1,\dots,a_n$ are not all zero, we have an element $\tilde{\Lambda}$ of $\mathbb{C}[\boldsymbol{\partial}_\xi]$ of order strictly greater than $\mathrm{ord}(\Lambda_{\delta-1}) = o$ that vanishes on $f_1,\dots,f_N$.
Let us prove that this higher order differential also vanishes on all multiples of f_k for $k = 1,\dots,N$. Let $p \in \mathbb{C}[\mathbf{x}]$ such that $\mathcal{N}_{\xi,\nu}(p) = 0$, $\Delta(\mathcal{N}_{\mathbf{z},\mu}(p)) = 0$. By (11), we have

$$\begin{aligned}
&\mathcal{N}_{\xi,\nu}((x_i-\xi_i)p)\\
&\quad= \quad (x_i-\xi_i)\mathcal{N}_{\xi,\nu}(p) + \mathtt{M}_i(\nu)\mathcal{N}_{\xi,\nu}(p) + O_{i,\nu}(p) = 0
\end{aligned}$$

and $\Delta(\mathcal{N}_{\mathbf{z},\mu}((x_i-\xi_i)p))$

$$\begin{aligned}
&= \quad \Delta((x_i-\xi_i)\mathcal{N}_{\mathbf{z},\mu}(p)) + \Delta(\mathtt{M}_i(\mu)\mathcal{N}_{\mathbf{z},\mu}(p)) + \Delta(O_\mu(p))\\
&= \quad \Delta(x_i-\xi_i)\mathcal{N}_{\xi,\nu}(p) + (\xi_i-\xi_i)\Delta(\mathcal{N}_{\mathbf{z},\mu}(p))\\
&\qquad\quad + \Delta(\mathtt{M}_i(\mu))\mathcal{N}_{\xi,\mu}(p) + \mathtt{M}_i(\nu)\Delta(\mathcal{N}_{\mathbf{z},\mu}(p))\\
&\qquad\quad + \Delta(O_{i,\mu}(p))\\
&= \quad 0.
\end{aligned}$$

As $\mathcal{N}_{\xi,\nu}(f_k) = 0$, $\Delta(\mathcal{N}_{\mathbf{z},\mu}(f_k)) = 0$, $i = 1,\dots,N$, we deduce by induction on the degree of the multipliers and by linearity that for any element f in the ideal I generated by $f_1,\dots,f_N$, we have

$$\mathcal{N}_{\xi,\nu}(f) = 0 \quad \text{and} \quad \Delta(\mathcal{N}_{\mathbf{z},\mu}(f)) = 0,$$

which yields $\tilde{\Lambda} \in I^\perp$. Thus we have $\tilde{\Lambda} \in I^\perp \cap \mathbb{C}[\boldsymbol{\partial}_\xi] = Q^\perp$ (by Lemma 2.1). As there is no element of degree strictly bigger than o in $Q^\perp$, this implies that

$$a_1 = \cdots = a_n = 0.$$

Then, by specialization at $\mathbf{x} = \xi$, Δ yields an element of the kernel of the Jacobian matrix of the system (9). By Theorem 4.8, this Jacobian has a zero-kernel, since it defines the simple point ν. We deduce that $\Delta = 0$ and (ξ,ν) is an isolated and simple root of the system (12). □

The following corollary applies the polynomial system defined in (12) to refine the precision of an approximate multiple root together with the coefficients of its Macaulay dual basis. The advantage of using this, as opposed to using the Macaulay multiplicity matrix, is that the number of variables is much smaller, as was noted above.

Corollary 4.12. *Let $\mathbf{f} \in \mathbb{K}[\mathbf{x}]^N$ and $\xi \in \mathbb{C}^n$ be as above, and let $\Lambda_0(\nu), \ldots, \Lambda_{\delta-1}(\nu)$ be its dual basis as in (4). Let $E \subset \mathbb{N}^n$ be as above. Assume that we are given approximates for the singular roots and its inverse system as in (4)*

$$\tilde{\xi} \cong \xi \quad \text{and} \quad \tilde{\nu}_{\alpha_i,\beta} \cong \nu_{\alpha_i,\beta} \ \ \forall \alpha_i \in E,\ \beta \notin E,\ |\beta| \leq o.$$

Consider the overdetermined system in $\mathbb{K}[\mathbf{z}, \mu]$ from (12). Then a random square subsystem of (12) will have a simple root at $\mathbf{z} = \xi$, $\mu = \nu$ with high probability. Thus, we can apply Newton's method for this square subsystem to refine $\tilde{\xi}$ and $\tilde{\nu}_{\alpha_i,\beta}$ for $(\alpha_i, \beta) \in E \times \partial(E)$. For $\tilde{\nu}_{\alpha_i,\gamma}$ with $\gamma \notin E^+$ we can use (6) for the update.

Example 4.13. Reconsider the setup from Ex. 3.2 with primal basis $\{1, x_2\}$ and $E = \{(0,0), (0,1)\}$. We obtain

$$\mathtt{M}_1(\mu) = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix} \quad \text{and} \quad \mathtt{M}_2(\mu) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The resulting deflated system in (12) is

$$F(z_1, z_2, \mu) = \begin{bmatrix} z_1 + z_2^2 \\ \mu + 2z_2 \\ z_1^2 + z_2^2 \\ 2\mu z_1 + 2z_2 \end{bmatrix}$$

which has a nonsingular root at $(z_1, z_2, \mu) = (0, 0, 0)$ corresponding to the origin with multiplicity structure $\{1, \partial_2\}$.

5 Examples

Computations for the following examples, as well as several other systems, along with MATLAB code can be found at `www.nd.edu/~jhauenst/deflation/`.

5.1 A family of examples

We first consider a modification of [17, Example 3.1]. For any $n \geq 2$, the following system has n polynomials, each of degree at most 3, in n variables:

$$x_1^3 + x_1^2 - x_2^2,\ x_2^3 + x_2^2 - x_3, \ldots, x_{n-1}^3 + x_{n-1}^2 - x_n,\ x_n^2.$$

The origin is a multiplicity $\delta := 2^n$ root having breadth 2 (i.e., the corank of Jacobian at the origin is 2).

We apply our parametric normal form method described in § 4. Similarly as in Remark 4.6, we can reduce the number of free parameters to be at most $(n-1)(\delta-1)$ using the structure of the primal basis $B = \{x_1^a x_2^b : a < 2^{n-1},\ b < 2\}$.

The following table shows the multiplicity, number of variables and polynomials in the deflated system, and the time (in seconds) it took to compute this system (on a iMac, 3.4 GHz Intel Core i7 processor, 8GB 1600Mhz DDR3 memory). Note that when comparing our method to an approach using the null spaces of Macaulay multiplicity matrices (see for example [6, 14]), we found that for $n \geq 4$ the deflated system derived from the Macaulay multiplicity matrix was too large to compute.

This is because the nil-index at the origin is 2^{n-1}, so the size of the Macaulay multiplicity matrix is $n \cdot \binom{2^{n-1}+n-1}{n-1} \times \binom{2^{n-1}+n}{n}$.

		New approach			Null space		
n	mult	vars	poly	time	vars	poly	time
2	4	5	9	1.476	8	17	2.157
3	8	17	31	5.596	192	241	208
4	16	49	100	19.698	7189	19804	> 76000
5	32	129	296	73.168	N/A	N/A	N/A
6	64	321	819	659.59	N/A	N/A	N/A

5.2 Caprasse system

We consider the Caprasse system [3, 29]:

$$f(x_1,x_2,x_3,x_4) = \begin{bmatrix} {x_1}^3x_3 - 4\,x_1{x_2}^2x_3 - 4\,{x_1}^2x_2x_4 - 2\,{x_2}^3x_4 - 4\,{x_1}^2 + \\ \qquad 10\,{x_2}^2 - 4\,x_1x_3 + 10\,x_2x_4 - 2, \\ x_1{x_3}^3 - 4\,x_2{x_3}^2x_4 - 4\,x_1x_3{x_4}^2 - 2\,x_2{x_4}^3 - 4\,x_1x_3 + \\ \qquad 10\,x_2x_4 - 4\,{x_3}^2 + 10\,{x_4}^2 - 2, \\ {x_2}^2x_3 + 2\,x_1x_2x_4 - 2\,x_1 - x_3, \\ {x_4}^2x_1 + 2\,x_2x_3x_4 - 2\,x_3 - x_1 \end{bmatrix}$$

at the multiplicity 4 root $\xi = (2, -\sqrt{-3}, 2, \sqrt{-3})$.
We first consider simply deflating the root. Using the approaches of [6, 10, 13], one iteration suffices. For example, using an extrinsic and intrinsic version of [6, 13], the resulting system consists of 10 and 8 polynomials, respectively, and 8 and 6 variables, respectively. Following [10], using all minors results in a system of 20 polynomials in 4 variables which can be reduced to a system of 8 polynomials in 4 variables using the 3×3 minors containing a full rank 2×2 submatrix. The approach of § 3 using an $|\boldsymbol{i}| = 1$ step creates a deflated system consisting of 6 polynomials in 4 variables. In fact, since the null space of the Jacobian at the root is 2 dimensional, adding two polynomials is necessary and sufficient.
Next, we consider the computation of both the point and multiplicity structure. Using an intrinsic null space approach via a second order Macaulay matrix, the resulting system consists of 64 polynomials in 37 variables. In comparison, using the primal basis $\{1, x_1, x_2, x_1x_2\}$, the approach of § 4 constructs a system of 30 polynomials in 19 variables.

5.3 Examples with multiple iterations

In our last set of examples, we consider simply deflating a root of the last three systems from [6, § 7] and a system from [12, § 1], each of which required more than one iteration to deflate. These four systems and corresponding points are:

1: $\{x_1^4 - x_2x_3x_4, x_2^4 - x_1x_3x_4, x_3^4 - x_1x_2x_4, x_4^4 - x_1x_2x_3\}$ at $(0,0,0,0)$ with $\delta = 131$ and $o = 10$;

2: $\{x^4, x^2y + y^4, z + z^2 - 7x^3 - 8x^2\}$ at $(0,0,-1)$ with $\delta = 16$ and $o = 7$;

3: $\{14x + 33y - 3\sqrt{5}(x^2 + 4xy + 4y^2 + 2) + \sqrt{7} + x^3 + 6x^2y + 12xy^2 + 8y^3, 41x - 18y - \sqrt{5} + 8x^3 - 12x^2y + 6xy^2 - y^3 + 3\sqrt{7}(4xy - 4x^2 - y^2 - 2)\}$ at $Z_3 \approx (1.5055, 0.36528)$ with $\delta = 5$ and $o = 4$;

4: $\{2x_1 + 2x_1^2 + 2x_2 + 2x_2^2 + x_3^2 - 1, (x_1 + x_2 - x_3 - 1)^3 - x_1^3,$
$(2x_1^3 + 5x_2^2 + 10x_3 + 5x_3^2 + 5)^3 - 1000x_1^5\}$ at $(0,0,-1)$ with $\delta = 18$ and $o = 7$.

We compare using the following four methods: (A) intrinsic slicing version of [6, 13]; (B) isosingular deflation [10] via a maximal rank submatrix; (C) "kerneling" method in [8]; (D) approach of § 3 using an $|\boldsymbol{i}| = 1$ step. We performed these methods without the use of preprocessing and

postprocessing as mentioned in § 3 to directly compare the number of nonzero distinct polynomials, variables, and iterations for each of these four deflation methods.

	Method A			Method B			Method C			Method D		
	Poly	Var	It	Poly	Var	It	Poly	Var	It	Poly	Var	It
1	16	4	2	22	4	2	22	4	2	16	4	2
2	24	11	3	11	3	2	12	3	2	12	3	3
3	32	17	4	6	2	4	6	2	4	6	2	4
4	96	41	5	54	3	5	54	3	5	22	3	5

For breadth one singular points as in system 3, methods B, C, and D yield the same deflated system. Except for methods B and C on the second system, all four methods required the same number of iterations to deflate the root. For the first and third systems, our new approach matched the best of the other methods and resulted in a significantly smaller deflated system for the last one.

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