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Abstract Using the theory of non-commutative rings, the delay identification problem of nonlinear time-delay systems with unknown inputs is studied. Necessary and sufficient conditions are proposed to judge the identifiability of the delay, where two different cases are discussed for the dependent and independent outputs, respectively. After that, necessary and sufficient conditions are given to analyze the causal and non-causal observability for nonlinear time-delay systems with unknown inputs.

1 Introduction

Time-delay systems are widely used to model concrete systems in engineering sciences, such as biology, chemistry, mechanics and so on [16, 26, 31]. Many results have been reported for the purpose of stability and observability analysis, by assuming that the delay of the studied systems is known. It makes the delay identification be one of the most important topics in the field of time-delay systems. Up to now, various techniques have been proposed for the delay identification problem, such as identification by using variable structure observers [15, 35, 36], modified least squares techniques [37], neural network algorithms [45], convolution approach [5], algebraic fast identification technique [6,8] as initiated in [18], and so on (see [4,15] for additional references). Note that most of the papers on identification in presence of delays concern linear models. Another source of complexity comes from the presence of feedback loops involving the delays. Indeed, when the delay appears only on the inputs or outputs, the system has the finite dimension. When the delays are involved in a closed-loop manner, the resulting model has delayed states and become a functional differential equation, which has the infinite dimension [7, 38].

Gang Zheng and Jean-Pierre Richard
Non-A team, Inria Lille and LAGIS CNRS 8219, 40 Avenue Halley, 59650 Villeneuve d'Ascq, France , e-mail: gang.zheng@inria.fr, jean-pierre.richard@ec-lille.fr

Besides identifiability, the observability property has been exhaustively studied for nonlinear systems without delays. It has been characterized in [22, 27, 43] from a differential point of view, and in [14] from an algebraic point of view. However, when the system is subject to time delay, such analysis is more complicated (see the surveys [38] and [41]). For linear time-delay systems, various aspects of the observability have been studied in the literature, using different methods such as the functional analytic approach [9] or the algebraic approach [10, 17, 42].

The aim of this paper is to firstly identify the time delay of nonlinear time-delay system with unknown inputs and then study observability for this system with identified delay. The work of this paper is based on the theory of non-commutative rings, which was firstly proposed in [30] for the disturbance decoupling problem of nonlinear time-delay system. Then this method was applied to study observability of nonlinear time-delay systems with known inputs in [44], to analyze identifiability of parameter for nonlinear time-delay systems in [47], and to study state elimination and delay identification of nonlinear time-delay systems with known inputs in [1]. The motivation to study nonlinear time-delay system with unknown inputs is due to the fact that there exist some cases, such as observer design for time-delay systems, in which the inputs can be unknown [13, 25, 39, 46]. Moreover, some proposed unknown input observer design methods do depend on the known delay, which should be identified in advance. Motivated by this requirement, this paper investigates both the delay identification problem and observability problem for nonlinear time-delay systems with unknown inputs.

This paper is organized as follows. Section 2 recalls the algebraic framework proposed in [44]. Notations and preliminary result are given in Section 3. Necessary and sufficient conditions are discussed for identifying the delay in two different cases: dependent outputs over the non-commutative rings, and then independent ones. Section 5 deduces necessary and sufficient conditions of causal and non-causal observability for nonlinear time-delay systems with unknown inputs, and the proposed result is applied to identify the delay of a biological model in Section 6.

2 Algebraic framework

It is assumed that the delays are constant and commensurate, that is all of them are multiples of an *elementary unknown delay* τ. Under this assumption, the considered nonlinear time-delay system is described as follows:

$$\begin{cases} \dot{x} = f(x(t-i\tau)) + \sum_{j=0}^{s} g^j(x(t-i\tau))u(t-j\tau), \\ y = h(x(t-i\tau)) = [h_1(x(t-i\tau)), \ldots, h_p(x(t-i\tau))]^T, \\ x(t) = \psi(t),\ u(t) = \varphi(t),\ t \in [-s\tau, 0], \end{cases} \tag{1}$$

where the constant delays $i\tau$ are associated to the finite set of integers $i \in S_- = \{0, 1, \ldots, s\}$; $x \in W \subset R^n$ refers to the state variables; $u = [u_1, \ldots, u_m]^T \in R^m$ is the unknown input; $y \in R^p$ is the measurable output; f, g^j and h are meromorphic functions[1]; $f(x(t-i\tau)) = f(x, x(t-\tau), \ldots, x(t-s\tau))$; $\psi : [-s\tau, 0] \to R^n$ and $\varphi : [-s\tau, 0] \to R^m$ denote unknown continuous functions of initial conditions. Throughout this chapter, it is assumed that, for initial conditions ψ and φ, system (1) admits a unique solution.

Based on the algebraic framework introduced in [44], consider the field $\mathcal{K}$ of meromorphic functions of a finite number of the variables from $\{x_j(t-i\tau), j \in [1, n], i \in S_-\}$. For the sake of simplicity, we introduce the delay operator δ, which means, for $i \in Z^+$:

$$\delta^i \xi(t) = \xi(t-i\tau),\ \xi(t) \in \mathcal{K}, \tag{2}$$

$$\begin{aligned} \delta^i\left(a(t)\xi(t)\right) &= \delta^i a(t) \delta^i \xi(t) \\ &= a(t-i\tau)\xi(t-i\tau). \end{aligned} \tag{3}$$

Let $\mathcal{K}(\delta]$ denote the set of polynomials in δ over $\mathcal{K}$, of the form

$$a(\delta] = a_0(t) + a_1(t)\delta + \cdots + a_{r_a}(t)\delta^{r_a} \tag{4}$$

where $a_i(t) \in \mathcal{K}$ and $r_a \in Z^+$. The addition in $\mathcal{K}(\delta]$ is defined as usual, but the multiplication is given as:

$$a(\delta]b(\delta] = \sum_{k=0}^{r_a+r_b} \sum_{i+j=k}^{i\le r_a, j\le r_b} a_i(t) b_j(t-i\tau)\delta^k. \tag{5}$$

Considering (1) without input, differentiation of an output component $h_j(x(t-i\tau))$ with regard to time t is defined as follows:

$$\dot{h}_j(x(t-i\tau)) = \sum_{i=0}^{s} \frac{\partial h_j}{\partial x(t-i\tau)} \delta^i f.$$

Thanks to the definition of $\mathcal{K}(\delta]$, (1) can be rewritten in a more compact form:

$$\begin{cases} \dot{x} = f(x,\delta) + G(x,\delta)u = f(x,\delta) + \sum_{i=1}^{m} G_i(x,\delta)u_i(t) \\ y = h(x,\delta) \\ x(t) = \psi(t),\ u(t) = \varphi(t),\ t \in [-s\tau, 0], \end{cases} \tag{6}$$

where $f(x,\delta) = f(x(t-i\tau))$ and $h(x,\delta) = h(x(t-i\tau))$, with entries belonging to $\mathcal{K}$, $u = u(t)$, and $G(x,\delta) = [G_1, \cdots, G_m]$ with $G_i(x,\delta) = \sum_{l=0}^{s} g_i^l \delta^l$.

With the standard differential operator d, denote by $\mathcal{M}$ the left module over $\mathcal{K}(\delta]$:

$$\mathcal{M} = span_{\mathcal{K}(\delta]}\{d\xi,\ \xi \in \mathcal{K}\} \tag{7}$$

where $\mathcal{K}(\delta]$ acts on $d\xi$ according to (2) and (3). Note that $\mathcal{K}(\delta]$ is a non-commutative ring, however it is proved that it is a left Ore ring [24, 44], which enables to define the rank of a left module over $\mathcal{K}(\delta]$.

Define the vector space $\mathcal{E}$ over $\mathcal{K}$:

[1] means quotients of convergent power series with real coefficients [12, 44].

$$\mathcal{E} = span_{\mathcal{K}}\{d\xi : \xi \in \mathcal{K}\}$$

$\mathcal{E}$ is the set of linear combinations of a finite number of elements from $dx_j(t - i\tau)$ with row vector coefficients in $\mathcal{K}$. Since the delay operator δ and the standard differential operator are commutative, the one-form of $\omega \in \mathcal{M}$ can be written as: $\omega = \sum_{j=1}^{n} a_j(\delta]dx_j$, where $a(\delta] \in \mathcal{K}(\delta]$. For a given vector field $\beta = \sum_{j=1}^{n} b_j(\delta]\frac{\partial}{\partial x_j}$ with $b_j(\delta] \in \mathcal{K}(\delta]$, the inner product of ω and β is defined as follows:

$$\omega\beta = \sum_{j=1}^{n} a_j(\delta]b_j(\delta] \in \mathcal{K}(\delta].$$

3 Notations and preliminary result

Some efforts have been made to extend the Lie derivative [23] to nonlinear time-delay systems (see [11, 19–21, 32–34]) in the framework of commutative rings. In what follows, we define the derivative and Lie derivative for nonlinear time-delay systems from the non-commutative point of view.

For $0 \leq j \leq s$, let $f(x(t - j\tau))$ and $h(x(t - j\tau))$ respectively be an n and p dimensional vector with entries $f_r \in \mathcal{K}$ for $1 \leq r \leq n$ and $h_i \in \mathcal{K}$ for $1 \leq i \leq p$. Let

$$\frac{\partial h_i}{\partial x} = \left[\frac{\partial h_i}{\partial x_1}, \cdots, \frac{\partial h_i}{\partial x_n}\right] \in \mathcal{K}^{1\times n}(\delta], \tag{8}$$

where, for $1 \leq r \leq n$:

$$\frac{\partial h_i}{\partial x_r} = \sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t - j\tau)}\delta^j \in \mathcal{K}(\delta].$$

Then, the Lie derivative for nonlinear systems without delays can be extended to nonlinear time-delay systems in the framework of [44] as follows

$$L_f h_i = \frac{\partial h_i}{\partial x}(f) = \sum_{r=1}^{n}\sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t - j\tau)}\delta^j(f_r) \tag{9}$$

and in the same way one can define $L_{G_i}h_i$.

Based on the above notations, the relative degree can be defined in the following way.

Definition 1. (Relative degree) System (6) has the relative degree $(\nu_1, \cdots, \nu_p)$ in an open set $W \subseteq R^n$ if the following conditions are satisfied for $1 \leq i \leq p$:

1. for all $x \in W$, $L_{G_j}L_f^r h_i = 0$ for all $1 \leq j \leq m$ and $0 \leq r < \nu_i - 1$;
2. there exists $x \in W$ such that $\exists j \in \{1, \cdots, m\}$, $L_{G_j}L_f^{\nu_i - 1}h_i \neq 0$.

If the first condition is satisfied for all $r \geq 0$ and some $i \in \{1, \cdots, p\}$, we set $\nu_i = \infty$.

Moreover, for system (6), one can also define observability indices introduced in [27] over non-commutative rings. For $1 \leq k \leq n$, let $\mathcal{F}_k$ be the following left module over $\mathcal{K}(\delta]$:

$$\mathcal{F}_k := span_{\mathcal{K}(\delta]} \left\{ dh, dL_f h, \cdots, dL_f^{k-1} h \right\}.$$

It was shown that the filtration of $\mathcal{K}(\delta]$-module satisfies $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$, then define $d_1 = rank_{\mathcal{K}(\delta]} \mathcal{F}_1$, and $d_k = rank_{\mathcal{K}(\delta]} \mathcal{F}_k - rank_{\mathcal{K}(\delta]} \mathcal{F}_{k-1}$ for $2 \leq k \leq n$. Let $k_i = card \{d_k \geq i, 1 \leq k \leq n\}$, then $(k_1, \cdots, k_p)$ are the observability indices. Reorder, if necessary, the output components of (6) so that

$$\begin{aligned} & rank_{\mathcal{K}(\delta]} \{ \tfrac{\partial h_1}{\partial x}, \cdots, \tfrac{\partial L_f^{k_1-1} h_1}{\partial x}, \cdots, \tfrac{\partial h_p}{\partial x}, \cdots, \tfrac{\partial L_f^{k_p-1} h_p}{\partial x} \} \\ & = k_1 + \cdots + k_p. \end{aligned}$$

Based on the above definitions, let us define the following notations, which will be used in the sequel. For $1 \leq i \leq p$, denote by k_i the observability indices, ν_i the relative degree for y_i of (6), and

$$\rho_i = \min \{\nu_i, k_i\}.$$

Without loss of generality, suppose $\sum_{i=1}^{p} \rho_i = j$, thus $\{dh_1, \cdots, dL_f^{\rho_1 - 1} h_1, \cdots, dh_p, \cdots, dL_f^{\rho_p - 1} h_p\}$ are j linearly independent vectors over $\mathcal{K}(\delta]$. Then note:

$$\Phi = \{dh_1, \cdots, dL_f^{\rho_1 - 1} h_1, \cdots, dh_p, \cdots, dL_f^{\rho_p - 1} h_p\} \tag{10}$$

and

$$\pounds = span_{R[\delta]} \left\{ h_1, \cdots, L_f^{\rho_1 - 1} h_1, \cdots, h_p, \cdots, L_f^{\rho_p - 1} h_p \right\}, \tag{11}$$

where $R[\delta]$ is the commutative ring of polynomials in δ with coefficients belonging to the field R, and let $\pounds(\delta]$ be the set of polynomials in δ with coefficients over $\pounds$. The module spanned by element of Φ over $\pounds(\delta]$ is defined as follows:

$$\Omega = span_{\pounds(\delta]} \{\xi, \xi \in \Phi\}. \tag{12}$$

Define

$$\mathcal{G} = span_{R[\delta]} \{G_1, \ldots, G_m\},$$

where G_i is given in (6), and its *left* annihilator:

$$\mathcal{G}^{\perp} = span_{\pounds(\delta]} \{\omega \in \mathcal{M} \mid \omega\beta = 0, \forall \beta \in \mathcal{G}\}, \tag{13}$$

where $\mathcal{M}$ is defined in (7).

After having defined the relative degree and observability indices via the extended Lie derivative for nonlinear time-delay systems in the framework of noncommutative rings, now an observable canonical form can be derived.

Theorem 1. *Consider the system (6) with outputs $(y_1, \cdots, y_p)$ and the corresponding $(\rho_1, \cdots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where k_i and ν_i are the observability indices and the relative degree indices, respectively. There exists a change of coordinates $\phi(x, \delta) \in \mathcal{K}^{n\times 1}$, such that (6) is transformed into the following form:*

$$\dot{z}_{i,j} = z_{i,j+1} \tag{14}$$

$$\dot{z}_{i,\rho_i} = V_i(x, \delta) = L_f^{\rho_i} h_i(x, \delta) + \sum_{j=1}^{m} L_{G_j} L_f^{\rho_i - 1} h_i(x, \delta) u_j \tag{15}$$

$$y_i = C_i z_i = z_{i,1} \tag{16}$$

$$\dot{\xi} = \alpha(z, \xi, \delta) + \beta(z, \xi, \delta) u \tag{17}$$

where $z_i = (z_{i,1}, \cdots, z_{i,\rho_i})^T = \left(h_i, \cdots, L_f^{\rho_i - 1} h_i\right)^T \in \mathcal{K}^{\rho_i \times 1}$, $\alpha \in \mathcal{K}^{\mu\times 1}, \beta \in \mathcal{K}^{\mu\times 1}(\delta]$ with $\mu = n - \sum_{j=1}^{p} \rho_j$ and $C_i = (1, 0, \cdots, 0) \in R^{1\times \rho_i}$.

Moreover, if $k_i < \nu_i$, one has $V_i(x, \delta) = L_f^{\rho_i} h_i = L_f^{k_i} h_i$. $\square$

Proof. See [48].

Based on Theorem 1, noting $\rho_i = \min\{\nu_i, k_i\}$ for $1 \leq i \leq p$ where the k_i represent the observability indices and ν_i stands for the relative degree of y_i for (6), the following equality can be derived:

$$\mathcal{H}(x, \delta) = \Psi(x, \delta) + \Gamma(x, \delta) u, \tag{18}$$

with

$$\mathcal{H}(x, \delta) = \left(h_1^{(\rho_1)}, \cdots, h_p^{(\rho_p)}\right)^T,$$
$$\Psi(x, \delta) = \left(L_f^{\rho_1} h_1, \cdots, L_f^{\rho_p} h_p\right)^T,$$

and

$$\Gamma(x, \delta) = \begin{pmatrix} L_{G_1} L_f^{\rho_1 - 1} h_1 & \cdots & L_{G_m} L_f^{\rho_1 - 1} h_1 \\ \vdots & \ddots & \vdots \\ L_{G_1} L_f^{\rho_p - 1} h_p & \cdots & L_{G_m} L_f^{\rho_p - 1} h_p \end{pmatrix}, \tag{19}$$

where $H(x, \delta) \in \mathcal{K}^{p\times 1}$, $\Psi(x, \delta) \in \mathcal{K}^{p\times 1}$ and $\Gamma(x, \delta) \in \mathcal{K}^{p\times m}(\delta]$. Assume that $rank_{\mathcal{K}(\delta]} \Gamma = m$. Since $\Gamma \in \mathcal{K}^{p\times m}(\delta]$ with $m \leq p$, according to Lemma 4 in [29], there exists a matrix $\Xi \in \mathcal{K}^{p\times p}(\delta]$ such that:

$$\Xi \Gamma = \left[\bar{\Gamma}^T, \mathbf{0}\right]^T, \tag{20}$$

where $\bar{\Gamma} \in \mathcal{K}^{m\times m}(\delta]$ has full rank m. With the compact equation (18), identifiability and observability will be analyzed separately in Sections 4 and 5.

4 Identifiability

In order to study the delay identifiability of (6), let us firstly introduce the following definition of the identifiability of time delay, which is an adaptation of Definition 2 in [1].

Definition 2. For system (6), an equation with delays, containing only the output and a finite number of its derivatives:

$$\alpha(h, \dot{h} \ldots, h^{(k)}, \delta) = 0, k \in Z^{+}$$

is said to be an *output delay equation* (of order k). Moreover, this equation is said to be an *output delay-identifiable equation*[2] for (6) if it cannot be written as $\alpha(h, \dot{h} \ldots, h^{(k)}, \delta) = a(\delta]\tilde{\alpha}(h, \dot{h} \ldots, h^{(k)})$ with $a(\delta] \in \mathcal{K}(\delta]$.

As stated in [1], if there exists an output delay-identifiable equation for (6) (i.e. involving the delay in an essential way), then the delay can be identified for almost all[3] y by numerically finding zeros of such an equation. For this issue, the interested reader can refer to [3] and the references therein. Thus, delay identification for (6) boils down to the research of such an output delay equation.

4.1 Dependent outputs over $\mathcal{K}(\delta]$

Let us firstly consider the most simple case for identifying the delay for (6), i.e., from only the outputs of (6), which is stated in the following result.

Theorem 2. *There exists an output delay-identifiable equation (of order* 0*)* $\alpha(h, \delta)$ *for (6) if and only if*

$$rank_{\mathcal{K}(\delta]} \frac{\partial h}{\partial x} < rank_{\mathcal{K}} \frac{\partial h}{\partial x}. \tag{21}$$

□

Proof. see [48].

Example 1. Consider the following dynamical system:

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[2] in [1], this equation is stated to involve the delay *in an essential way*.

[3] i.e., singularity of the delay identification exists for a countable set of y, which case is excluded in this chapter.

$$\begin{cases} \dot{x} = f(x, u, \delta), \\ y_1 = x_1, \\ y_2 = x_1 \delta x_1 + x_1^2. \end{cases} \tag{22}$$

It can be seen that

$$\frac{\partial h}{\partial x} = \begin{pmatrix} 1, 0 \\ \delta x_1 + 2x_1 + x_1 \delta, 0 \end{pmatrix}$$

which yields $rank_{\mathcal{K}(\delta]} \frac{\partial h}{\partial x} = 1$ and $rank_{\mathcal{K}} \frac{\partial h}{\partial x} = 2$. Thus Theorem 2 is satisfied, and the delay of system (32) can be identified.

In fact, a straightforward calculation gives:

$$y_2 = y_1 \delta y_1 + y_1^2,$$

which permits to identify the delay δ by applying an algorithm to detect zero-crossing when varying δ. $\square$

Inequality (21) implies that the outputs of (6) are dependent over $\mathcal{K}(\delta]$. Theorem 2 can be seen as a special case of Theorem 2 in [1]. However, as it will be shown in the next section, this condition is not necessary for the case where the output of (6) is independent over $\mathcal{K}(\delta]$.

4.2 Independent outputs over $\mathcal{K}(\delta]$

Theorem 2 has analyzed the case where the outputs of (6) are dependent over $\mathcal{K}(\delta]$. In the contrary case (independence over $\mathcal{K}(\delta]$), the dynamics of system (6) have to be involved in order to deduce some output delay equations, which might be used to identify the delay. In the following, it will be firstly given the sufficient condition for the existence of a delay output equation for system (6) when the output is independent over $\mathcal{K}(\delta]$. Then a necessary and sufficient condition will be provided. Based on the deduction of (18), we can state the following result.

Theorem 3. *There exists an output delay equation for (6), if there exists a non zero* $\omega = \sum_{c=1}^{n} \sum_{j=1}^{p} q_j \frac{\partial L_f^{\rho_j - 1} h_j}{\partial x_c} dx_c$, *with* $q_j \in \mathcal{K}(\delta]$ *for* $1 \leq j \leq p$, *such that* $\omega \in \mathcal{G}^{\perp} \cap \Omega$ *and* $\omega f \in \pounds$, *where* $\mathcal{G}^{\perp}$ *is defined in (13),* Ω *in (12), and* $\pounds$ *in (11).* $\square$

Proof. Denote $Q = [q_1, \cdots, q_p]$ as $1 \times p$ vector with $q_j \in \mathcal{K}(\delta]$ for $1 \leq j \leq p$. Because of the associativity law over $\mathcal{K}(\delta]$, one has:

$$Q\Gamma = Q \begin{pmatrix} L_{G_1} L_f^{\rho_1 - 1} h_1 & \cdots & L_{G_m} L_f^{\rho_1 - 1} h_1 \\ \vdots & \ddots & \vdots \\ L_{G_1} L_f^{\rho_p - 1} h_p & \cdots & L_{G_m} L_f^{\rho_p - 1} h_p \end{pmatrix} = Q \begin{bmatrix} \frac{\partial L_f^{\rho_1 - 1} h_1}{\partial x} \\ \vdots \\ \frac{\partial L_f^{\rho_p - 1} h_p}{\partial x} \end{bmatrix} [G_1, \cdots, G_m]$$

Then, according to the definition in (8), one gets:

$$Q\Gamma = \omega\left[G_1, \cdots, G_m\right] = \omega G,$$

where $\omega = \sum_{c=1}^{n}\sum_{j=1}^{p} q_j \frac{\partial L_f^{\rho_j-1} h_j}{\partial x_c} dx_c$.

Moreover, one can check that:

$$\omega f = \left(Q \begin{bmatrix} \frac{\partial L_f^{\rho_1-1} h_1}{\partial x} \\ \vdots \\ \frac{\partial L_f^{\rho_p-1} h_p}{\partial x} \end{bmatrix} \right) f = Q \left(\begin{bmatrix} \frac{\partial L_f^{\rho_1-1} h_1}{\partial x} \\ \vdots \\ \frac{\partial L_f^{\rho_p-1} h_p}{\partial x} \end{bmatrix} f \right) = Q \begin{bmatrix} L_f^{\rho_1} h_1 \\ \vdots \\ L_f^{\rho_p} h_p \end{bmatrix} = Q\Psi.$$

According to (18), one has:

$$Q\mathcal{H} = Q(\Psi + \Gamma u) = \omega f + \omega G u, \tag{23}$$

where $\mathcal{H} = \left[y_1^{(\rho_1)}, \cdots, y_p^{(\rho_p)}\right]^T$. Thus, if $\omega \in \mathcal{G}^\perp \cap \Omega$ and $\omega f \in \pounds$, which implies there exists Q with entries belonging to $\pounds(\delta]$, one has

$$Q\Gamma = \omega G = 0$$

and

$$Q\mathcal{H} = \omega f \in \pounds.$$

Finally, one obtains the following relation:

$$Q(\mathcal{H} - \Psi) = 0, \tag{24}$$

which is exactly the output delay equation, since it contains only the output, its derivatives and delays. □

If, in addition, the deduced output delay equation (24) is an output delay-identifiable equation, i.e. containing the delay δ in an essential way, then the delay of (6) can be identified (at least locally) by detecting zero-crossing of (24). The following will give necessary and sufficient conditions guaranteeing the essential involvement of δ in (24). But before this, let us define:

$$\mathcal{Y} = \left(h_1, \ldots, L_f^{\rho_1-1} h_1, \ldots, h_p, \ldots, L_f^{\rho_p-1} h_p\right)^T,$$

and denote by $\mathcal{K}_0 \subset \mathcal{K}$ the field of meromorphic functions of x, which will be used in the following theorem (also involving Ψ defined in (18)).

Theorem 4. *The output delay equation (24) is an output delay-identifiable equation if and only if*

$$rank_{\mathcal{K}(\delta]} \frac{\partial \mathcal{Y}}{\partial x} < rank_{\mathcal{K}} \frac{\partial\{\mathcal{Y}, \Psi\}}{\partial x} \tag{25}$$

or for any element q_j of $Q \in \mathcal{K}^{1\times p}(\delta]$, $\nexists a(\delta] \in \mathcal{K}(\delta]$ such that

$$q_j = a(\delta]\bar{q}_j, \ \textit{with}\ \bar{q}_j \in \mathcal{K}_0, 1 \le j \le p \tag{26}$$

and

$$rank_{\mathcal{K}(\delta]} \frac{\partial \mathcal{Y}}{\partial x} = rank_{\mathcal{K}} \frac{\partial \{\mathcal{Y}, \Psi\}}{\partial x}. \tag{27}$$

□

Proof. See [48].

Remark 1. It is clear that Theorem 2 is a special case of Theorem 4, since the output delay-identifiable equation stated in Theorem 2 does not contain any derivative of the output.

In [1], a condition similar to (25) of Theorem 4 is stated as a necessary and sufficient condition for delay identification for nonlinear systems with *known* inputs. However, as we proved above, in the case of *unknown* inputs, this condition is sufficient, but not necessary.

5 Observability

Similarly to the observability definitions given in [22] and [14] for nonlinear delay-free systems, it has been given in [28] a definition of observability for nonlinear time-delay systems. The following gives a more generic definition of observability in the case of systems with unknown inputs.

Definition 3. System (6) is locally observable if the state $x(t)$ can be expressed as a function of the output and a finite number of its time derivatives with their backward and forward shifts. A locally observable system is locally causally observable if its state can be written as a function of the output and its derivatives with their backward shifts only. Otherwise, it is locally non-causally observable (and it depends also on the forward shifts).

In the same way, the following definition is given of systems with unknown inputs.

Definition 4. The unknown input $u(t)$ can be locally estimated if it can be written as a function of the output and a finite number of its time derivatives with backward and forward shifts. The input can be locally causally estimated if $u(t)$ can be expressed as a function of the output and its time derivatives with backward shifts only. Otherwise, it can be non causally estimated (and it depends also on the forward shifts).

Theorem 5. *Consider the system (6) with outputs* $(y_1, \cdots, y_p)$ *and their corresponding* $(\rho_1, \cdots, \rho_p)$ *with* $\rho_i = \min\{k_i, \nu_i\}$ *where* k_i *and* ν_i *are the observability indices and the relative degree indices, respectively. Consider* Φ *and* $\bar{\Gamma}$ *defined in (10) and* (20)*, respectively.*

If $rank_{\mathcal{K}(\delta]}\Phi = n$*,, then there exists a change of coordinates* $\phi(x, \delta)$ *such that (6) can be transformed into (14-17) with dim* $\xi = 0$.

Moreover, if the change of coordinates is locally bicausal over $\mathcal{K}$, then the state $x(t)$ of (6) is locally causally observable; if, in addition, $\bar{\Gamma} \in \mathcal{K}^{m\times m}(\delta]$ is unimodular over $\mathcal{K}(\delta]$, then the unknown input $u(t)$ of (6) can be locally causally estimated. □

Proof. According to Theorem 1, (6) can be transformed into (14-17) by using the change of coordinates $(z,\xi) = \phi(x,\delta)$. Hence, if $rank_{\mathcal{K}(\delta]}\Phi = n$, one has $\sum_{j=1}^{p} \rho_j = n$, which implies that (6) can be transformed into (14-17) with dim $\xi = 0$ and the change of coordinates is given by $z = \phi(x,\delta)$ where $z = (z_i^T, \cdots, z_p^T)^T$ and $z_i = (h_i, \cdots, L_f^{\rho_i-1}h_i)^T$.

Moreover, if $\phi(x,\delta) \in \mathcal{K}^{n\times 1}$ is locally bicausal over $\mathcal{K}$, one can write x as a function of y_i, its derivative and backward shift, which implies state x is locally causally observable.

Concerning the reconstruction of the unknown inputs, rewrite (18) as follows

$$\Gamma u = H(x,\delta) - \Psi(x,\delta) = \Upsilon(x,\delta). \tag{28}$$

Since $rank_{\mathcal{K}(\delta]}\Phi = n$ and x is causally observable, then $\Upsilon(x,\delta)$ is a vector of known meromorphic functions belonging to $\mathcal{K}$.

If $\bar{\Gamma} \in \mathcal{K}^{m\times m}(\delta]$ is unimodular over $\mathcal{K}(\delta]$, then there exists a matrix $\bar{\Gamma}^{-1} \in \mathcal{K}^{m\times m}(\delta]$ such that $\left[\, \bar{\Gamma}^{-1}\ \mathbf{0} \,\right] \Xi\Gamma = I_{m\times m}$ and $u = \left[\, \bar{\Gamma}^{-1}\ \mathbf{0} \,\right] \Xi\Upsilon$. Since $\bar{\Gamma}^{-1} \in \mathcal{K}^{m\times m}(\delta]$, $\Xi \in \mathcal{K}^{p\times p}$ and $\Upsilon \in \mathcal{K}^{p\times 1}$, then u is also causally observable. □

For the case where the condition $rank_{\mathcal{K}(\delta]}\Phi = n$ in Theorem 5 is not satisfied, a constructive algorithm was proposed in [2] to solve this problem for nonlinear systems without delays. In the following we are going to extend this idea to treat the observation problem for time-delay systems with unknown inputs. The objective is to generate additional variables from the available measurement and unaffected by the unknown input such that an extended canonical form similar to (14)-(15) can be obtained for the estimation of the remaining state ξ.

Theorem 6. *Consider the system (6) with outputs $y = (y_1, \cdots, y_p)^T$ and the corresponding $(\rho_1, \cdots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where k_i and ν_i are the observability indices and the relative degree indices, respectively. Suppose $rank_{\mathcal{K}(\delta]}\Phi < n$ where Φ is defined in (10). There exist l new independent outputs over $\mathcal{K}$ suitable to the causal estimation problem if and only if $rank_{\mathcal{K}}\mathcal{L} = l$ where*

$$\mathcal{L} = span_{R[\delta]}\{\omega \in \mathcal{G}^{\perp} \cap \Omega \mid \omega f \notin \pounds\} \tag{29}$$

with f defined in (6)*, $\pounds$ defined in* (11)*, Ω defined in* (12) *and $\mathcal{G}^{\perp}$ defined in* (13)*. Moreover, the l additional outputs, denoted as $\bar{y}_i$, $1 \le i \le l$, are given by:*

$$\bar{y}_i = \omega_i f \ mod\ \pounds$$

where $\omega_i \in \mathcal{L}$. □

Proof. See [49].

Remark 2. Theorem 6 gives a constructive way to treat the case where $rank_{\mathcal{K}(\delta]}\Phi < n$. Once additional new outputs are deduced according to Theorem 6, it enables to define a new Φ. If $rank_{\mathcal{K}(\delta]}\Phi = n$, Theorem 5 can then be applied. Otherwise, if $rank_{\mathcal{K}(\delta]}\Phi < n$ and if Theorem 6 is still valid, then one can still deduce new outputs for the studied system. Thus a "Check-Extend" procedure is iterated until $rank_{\mathcal{K}(\delta]}\Phi = n$ is obtained.

5.1 Non-causal observability

The previous results can be extended to the case of non-causal observations of the state and the unknown inputs, which can be very useful in some applications. For instance, some proposed delay feedback control methods can be applied for stabilizing nonlinear time-delay systems [40]. Furthermore, other applications, such as cryptography based on chaotic system, do not require real-time estimation, hence non-causal observations can still play an important role in those applications.

In order to treat the non-causal case, let us introduce the forward time-shift operator ∇, similarly to the backward time-shift operator δ defined in Section 2:

$$\nabla f(t) = f(t+\tau)$$

and, for $i, j \in Z^+$:

$$\nabla^i \delta^j f(t) = \delta^j \nabla^i f(t) = f\left(t-(j-i)\tau\right).$$

Following the principle of Section 2, denote by $\bar{\mathcal{K}}$ the field of meromorphic functions of a finite number of variables from $\{x_j(t-i\tau), j \in [1,n], i \in S\}$ where $S = \{-s, \cdots, 0, \cdots, s\}$ is a finite set of relative integers. One has $\mathcal{K} \subseteq \bar{\mathcal{K}}$. Denote by $\bar{\mathcal{K}}(\delta, \nabla]$ the set of polynomials of the form:

$$\begin{aligned} a(\delta,\nabla] = {} & \bar{a}_{r_{\bar{a}}}\nabla^{r_{\bar{a}}} + \cdots + \bar{a}_1\nabla \\ & + a_0(t) + a_1(t)\delta + \cdots + a_{r_a}(t)\delta^{r_a}, \end{aligned} \tag{30}$$

with $a_i(t)$ and $\bar{a}_i(t)$ belonging to $\bar{\mathcal{K}}$. Keep the usual definition of addition for $\bar{\mathcal{K}}(\delta, \nabla]$ and define the multiplication as follows:

$$\begin{aligned} a(\delta,\nabla]b(\delta,\nabla] = {} & \sum_{i=0}^{r_a}\sum_{j=0}^{r_b} a_i\delta^i b_j \delta^{i+j} + \sum_{i=0}^{r_a}\sum_{j=1}^{r_{\bar{b}}} a_i\delta^i \bar{b}_j\delta^i\nabla^j \\ & + \sum_{i=1}^{r_{\bar{a}}}\sum_{j=0}^{r_b} \bar{a}_i\nabla^i b_j\nabla^i\delta^j + \sum_{i=1}^{r_{\bar{a}}}\sum_{j=1}^{r_{\bar{b}}} \bar{a}_i\nabla^i \bar{b}_j\nabla^{i+j}. \end{aligned} \tag{31}$$

It is clear that $\mathcal{K}(\delta] \subseteq \bar{\mathcal{K}}(\delta, \nabla]$ and that the ring $\bar{\mathcal{K}}(\delta, \nabla]$ possesses the same properties as $\mathcal{K}(\delta]$. Thus, a module $\bar{\mathcal{M}}$ can be also defined over $\bar{\mathcal{K}}(\delta, \nabla]$, as follows: $\bar{\mathcal{M}} = span_{\bar{\mathcal{K}}(\delta,\nabla]}\{d\xi, \xi \in \bar{\mathcal{K}}\}$.

Given the above definitions, Theorem 5 is now extended so to deal with non-causal observability for nonlinear time-delay systems.

Theorem 7. *Consider the system (6) with outputs $(y_1, \cdots, y_p)$ and the corresponding $(\rho_1, \cdots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where k_i and ν_i are the observability indices and the relative degree indices, respectively. If $rank_{\mathcal{K}(\delta]}\Phi = n$, where Φ is defined in (10), then there exists a change of coordinates $\phi(x, \delta)$ such that (6) can be transformed into (14-17) with dim $\xi = 0$.*

Moreover, if the change of coordinates is locally bicausal over $\bar{\mathcal{K}}$, then the state $x(t)$ of (6) is at least locally non-causally observable; if, in addition, $\bar{\Gamma} \in \mathcal{K}^{m\times m}(\delta]$ is unimodular over $\bar{\mathcal{K}}(\delta, \nabla]$, then the unknown input $u(t)$ of (6) can be at least locally non-causally estimated. □

Proof. See [49].

6 Illustrative example

The following example aims at highlighting the proposed results in the case of delay identification and causal observability. Consider:

$$\begin{cases} \dot{x}_1 = -\delta x_1^2 + \delta x_4 u_1, \quad \dot{x}_2 = -x_1^2 \delta x_3 + x_2 + x_1 \delta x_4 u_1, \\ \dot{x}_3 = x_4 - x_1^2 \delta x_4 u_1, \quad \dot{x}_4 = x_5 + \delta x_1, \quad \dot{x}_5 = \delta x_1 \delta x_3 + u_2, \\ y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_1 \delta x_1 + x_3. \end{cases} \tag{32}$$

One can check that $\nu_1 = k_1 = \nu_2 = k_2 = 1, \nu_3 = 1, k_3 = 3$, yielding $\rho_1 = \rho_2 = \rho_3 = 1$ and $\Phi = \{dx_1, dx_2, (\delta x_1 + x_1\delta)dx_1 + dx_3\}$. One has $rank_{\mathcal{K}(\delta]}\Phi = 3 < n$.

Set $\mathcal{G} = span_{R[\delta]}\{G_1, \cdots, G_m\}$, then one has:

$$\mathcal{G}^{\perp} = span_{R[\delta]}\left\{x_1 dx_1 - dx_2, x_1^2 dx_1 + dx_3, dx_4\right\}.$$

Since $rank_{\mathcal{K}(\delta]}\Phi = 3$, thus $\pounds = span_{R[\delta]}\{x_1, x_2, x_1\delta x_1 + x_3\}$ and

$$\Omega = span_{\pounds(\delta]}\{dx_1, dx_2, dx_3\},$$

which yields:

$$\Omega \cap \mathcal{G}^{\perp} = span_{\pounds(\delta]}\left\{x_1 dx_1 - dx_2, x_1^2 dx_1 + dx_3\right\}.$$

In the following, identifiability and observability will be successively checked for (32).

Identifiability analysis:

Following Theorem 1, one has:

$$\mathcal{H} = [\dot{y}_1, \dot{y}_2, \dot{y}_3]^T,$$

$$\Psi = \left[-\delta x_1^2, -x_1^2\delta x_3 + x_2, x_4\right]^T,$$

and

$$\Gamma = \begin{bmatrix} \delta x_4, & 0 \\ x_1 \delta x_4, & 0 \\ -x_1^2 \delta x_4, & 0 \end{bmatrix}.$$

Thus, by choosing $Q = [x_1, -1, 0]$, a non zero one-form can be found, such as:

$$\omega = x_1 dx_1 - dx_2 \in \Omega \cap \mathcal{G}^{\perp},$$

satisfying

$$\omega f = -x_1 \delta x_1^2 + x_1^2 \delta x_3 - x_2 \in \pounds.$$

According to Theorem 3, the following equation is an output delay equation:

$$Q\left(\mathcal{H} - \Psi\right) = 0, \tag{33}$$

since it contains only the output, its derivatives and delays.

Since $\mathcal{Y} = (x_1, x_2, x_1 \delta x_1 + x_3)^T$, one has:

$$\frac{\partial \mathcal{Y}}{\partial x} = \begin{pmatrix} 1, & 0, 0, 0, \ 0 \\ 0, & 1, 0, 0, 0 \\ \delta x_1 + x_1 \delta, & 0, 1, 0, \ 0 \end{pmatrix},$$

and

$$\frac{\partial \Psi}{\partial x} = \begin{pmatrix} -2\delta x_1 \delta, & 0, & 0, & 0, 0 \\ -2x_1 \delta x_3, & 1, & -x_1^2 \delta, & 0, 0 \\ 0, & 0, & 0, & 1, 0 \end{pmatrix}.$$

Thus, one obtains:

$$rank_{\mathcal{K}(\delta]} \frac{\partial \mathcal{Y}}{\partial x} = 3 < rank_{\mathcal{K}} \frac{\partial \{\mathcal{Y}, \Psi\}}{\partial x} = 6.$$

Theorem 4 is satisfied and (33) involves δ in an essential way. A straightforward calculation gives:

$$y_1 \dot{y}_1 - \dot{y}_2 = -y_1 \delta y_1^2 + y_1^2 \delta y_3 - y_1^2 \delta y_1 \delta^2 y_1 - y_2,$$

which permits to identify the delay.

Observability analysis:

From the definition of $\mathcal{L}$ in (29), one can check that $rank_{\mathcal{K}} \mathcal{L} = 1$, which gives the one-form $\omega = x_1^2 dx_1 + dx_3$, satisfying $\omega \in \Omega \cap \mathcal{G}^{\perp}$ and $\omega f = -x_1^2 \delta x_1^2 + x_4 \notin \pounds$. Thus, according to Theorem 6, a new output $\bar{y}_1 = h_4$ is given by:

$$\bar{y}_1 = h_4 = \omega f \text{ mod } \pounds = x_4 = y_1^2 \dot{y}_1 + \dot{y}_3 + y_1^2 \delta y_1^2. \tag{34}$$

For the new output $\bar{y}_1$, one has $k_i = \nu_i = 1$ for $1 \le i \le 3$, $k_4 = \nu_4 = 2$, thus $\rho_i = 1$ for $1 \le i \le 3$ and $\rho_4 = 2$. Finally, one obtains the new Φ as follows:

$$\Phi = \{dx_1, dx_2, (\delta x_1 + x_1\delta)dx_1 + dx_3, dx_4, \delta dx_1 + dx_5\}.$$

It can be checked that $rank_{\mathcal{K}(\delta]}\Phi = 5 = n$, and the new $\pounds$ is:

$$\pounds = span_{R[\delta]}\{x_1, x_2, x_1\delta x_1 + x_3, x_4, x_5 + \delta x_1\}.$$

This gives the following change of coordinates:

$$z = \phi(x,\delta) = (x_1, x_2, x_1\delta x_1 + x_3, x_4, x_5 + \delta x_1)^T .$$

It is easy to check that it is bicausal over $\mathcal{K}(\delta]$, since:

$$x = \phi^{-1} = (z_1, z_2, z_3 - z_1\delta z_1, z_4, z_5 - \delta z_1)^T .$$

When $t \geq \tau$, one gets the following estimations of states:

$$\begin{cases} x_1 = y_1,\ x_2 = y_2,\ x_3 = y_3 - y_1\delta y_1, \\ x_4 = \bar{y}_1,\ x_5 = -\delta y_1 + \dot{\bar{y}}_1, \end{cases}$$

with $\bar{y}_1$ defined in (34).

Moreover, the matrix Γ with the new output $\bar{y}_1$ can be obtained as follows:

$$\Gamma = \begin{pmatrix} \delta x_4, & 0 \\ x_1\delta x_4, & 0 \\ x_1^2\delta x_4, & 0 \\ 0, & 1 \end{pmatrix},$$

with $rank_{\mathcal{K}(\delta]}\Gamma = 2$. One can find matrices $\Xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ x_1 & -1 & 0 & 0 \\ x_1^2 & 0 & 1 & 0 \end{pmatrix}$, $\bar{\Gamma} = \begin{pmatrix} \delta x_4 & 0 \\ 0 & 1 \end{pmatrix}$, and $\bar{\Gamma}^{-1} = \begin{pmatrix} \frac{1}{\delta x_4} & 0 \\ 0 & 1 \end{pmatrix}$ such that $\left[\, \bar{\Gamma}^{-1}\ \mathbf{0} \,\right] \Xi\Gamma = I_{2\times 2}$. Consequently, according to Theorem 5, u_1 and u_2 can be causally estimated. When $t \geq 3\tau$, a straightforward computation yields the following estimates for the unknown inputs:

$$\begin{cases} u_1 = \frac{\dot{y}_1 + \delta y_1^2}{\delta \bar{y}_1}, \\ u_2 = \ddot{\bar{y}}_1 - \delta \dot{y}_1 - \delta y_1 \delta y_3 + \delta y_1^2 \delta^2 y_1. \end{cases}$$

7 Conclusion

This chapter has studied identifiability and observability for nonlinear time-delay systems with unknown inputs. Concerning the identification of the delay, dependent and independent outputs over the non-commutative rings have been analyzed. Concerning the observability, necessary and sufficient conditions have been deduced for both causal and non-causal cases. The causal and non-causal estimations of unknown inputs of the studied systems have been analyzed as well.

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