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# Construction of Quasi-Cyclic Product Codes

Alexander Zeh

Computer Science Department  
Technion—Israel Institute of Technology  
Haifa, Israel  
alex@codingtheory.eu

San Ling

Division of Mathematical Sciences, School of Physical &  
Mathematical Sciences, Nanyang Technological University  
Singapore, Republic of Singapore  
lingsan@ntu.edu.sg

**Abstract**—Linear quasi-cyclic product codes over finite fields are investigated. Given the generating set in the form of a reduced Gröbner basis of a quasi-cyclic component code and the generator polynomial of a second cyclic component code, an explicit expression of the basis of the generating set of the quasi-cyclic product code is given. Furthermore, the reduced Gröbner basis of a one-level quasi-cyclic product code is derived.

**Index Terms**—Cyclic code, Gröbner basis, module minimization, product code, quasi-cyclic code, submodule

## I. INTRODUCTION

A linear block code of length  $\ell m$  over a finite field  $\mathbb{F}_q$  is a quasi-cyclic code if every cyclic shift of a codeword by  $\ell$  positions, for some integer  $\ell$  between one and  $\ell m$ , results in another codeword. Quasi-cyclic codes are a natural generalization of cyclic codes (where  $\ell = 1$ ), and have a closely linked algebraic structure. In contrast to cyclic codes, quasi-cyclic codes are known to be asymptotically good (see Chen–Peterson–Weldon [1]). Several such codes have been discovered with the highest minimum distance for a given length and dimension (see Gulliver–Bhargava [2] as well as Chen’s and Grassl’s databases [3], [4]). Several good LDPC codes are quasi-cyclic (see e.g. [5]) and the connection to convolutional codes was investigated among others in [6]–[8].

Recent papers of Barbier *et al.* [9], [10], Lally–Fitzpatrick [8], [11], [12], Ling–Solé [13]–[15], Semenov–Trifonov [16], Güneri–Özbudak [17] and ours [18] discuss different aspects of the algebraic structure of quasi-cyclic codes including lower bounds on the minimum Hamming distance and efficient decoding algorithms.

The focus of this paper is on a simple method to combine two given quasi-cyclic codes into a product code. More specifically, we give a description of a quasi-cyclic product code when one component code is quasi-cyclic and the second one is cyclic.

The work of Wasan [19] first considers quasi-cyclic product codes while investigating the mathematical properties of the wider class of quasi-abelian codes. Some more results were published in a short note by Wasan and Dass [20]. Koshy proposed a so-called “circle” quasi-cyclic product codes in [21].

Our work considers quasi-cyclic product codes that generalize the results of Burton–Weldon [22] and Lin–Weldon [23]

(see also [24, Chapter 18]) based on the reduced Gröbner basis representation of Lally–Fitzpatrick [11] of the quasi-cyclic component code. We derive a representation of the generating set of a quasi-cyclic product code, where one component code is quasi-cyclic and the other is cyclic (in Thm. 7) and we give a reduced Gröbner basis for the special class of one-level quasi-cyclic product codes (in Thm. 8).

The paper is structured as follows. In Section II, we give necessary preliminaries on quasi-cyclic codes over finite fields. We outline relevant basics of the reduced Gröbner basis representation of Lally–Fitzpatrick [11]. Furthermore, the special class of  $r$ -level quasi-cyclic codes is defined in this section. Section III contains the main result on quasi-cyclic product codes, where the row-code is quasi-cyclic and the column-code is cyclic. Moreover, an explicit expression of the basis of a 1-level quasi-cyclic product code is derived in Section III. For illustration, we explicitly give an example of a binary 2-quasi-cyclic product code in Section IV. Section V concludes this paper.

## II. PRELIMINARIES

Let  $\mathbb{F}_q$  denote the finite field of order  $q$  and  $\mathbb{F}_q[X]$  the polynomial ring over  $\mathbb{F}_q$  with indeterminate  $X$ . Let  $a, b$  with  $b > a$  be two positive integers and denote by  $[a, b)$  the set of integers  $\{a, a + 1, \dots, b - 1\}$  and by  $[b) = [0, b)$ . A vector of length  $n$  is denoted by a lowercase bold letter as  $\mathbf{v} = (v_0 \ v_1 \ \dots \ v_{n-1})$  and an  $m \times n$  matrix is denoted by a capital bold letter as  $\mathbf{M} = (m_{i,j})_{i \in [m], j \in [n]}$ .

A linear  $[\ell \cdot m, k, d]_q$  code  $\mathcal{C}$  of length  $\ell m$ , dimension  $k$  and minimum Hamming distance  $d$  over  $\mathbb{F}_q$  is  $\ell$ -quasi-cyclic if every cyclic shift by  $\ell$  of a codeword is again a codeword of  $\mathcal{C}$ , more explicitly if:

$$\begin{aligned} (c_{0,0} \cdots c_{\ell-1,0} \quad c_{0,1} \cdots c_{\ell-1,1} \quad \dots \quad c_{\ell-1,m-1}) &\in \mathcal{C} \\ \Rightarrow \\ (c_{0,m-1} \cdots c_{\ell-1,m-1} \quad c_{0,0} \cdots c_{\ell-1,0} \quad \dots \quad c_{\ell-1,m-2}) &\in \mathcal{C}. \end{aligned}$$

We can represent a codeword of an  $[\ell \cdot m, k, d]_q$   $\ell$ -quasi-cyclic code as  $\mathbf{c}(X) = (c_0(X) \ c_1(X) \ \dots \ c_{\ell-1}(X)) \in \mathbb{F}_q[X]^\ell$ , where

$$c_i(X) \stackrel{\text{def}}{=} \sum_{j=0}^{m-1} c_{i,j} X^j, \quad \forall i \in [\ell]. \quad (1)$$

Then, the defining property of  $\mathcal{C}$  is that each component  $c_i(X)$  of  $\mathbf{c}(X)$  is closed under multiplication by  $X$  and reduction modulo  $X^m - 1$ .

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**Lemma 1.** Let  $(c_0(X) \ c_1(X) \ \cdots \ c_{\ell-1}(X))$  be a codeword of an  $\ell$ -quasi-cyclic code  $\mathcal{C}$  of length  $m\ell$ , where the components are defined as in (1). Then a codeword in  $\mathcal{C}$  represented as one univariate polynomial of degree smaller than  $m\ell$  is

$$c(X) = \sum_{i=0}^{\ell-1} c_i(X^\ell)X^i. \quad (2)$$

*Proof.* Substitute (1) into (2):

$$c(X) = \sum_{i=0}^{\ell-1} c_i(X^\ell)X^i = \sum_{i=0}^{\ell-1} \sum_{j=0}^{m-1} c_{i,j}X^{j\ell+i}.$$

□

Lally and Fitzpatrick [11], [25] showed that this enables us to see a quasi-cyclic code as an  $R$ -submodule of the algebra  $R^\ell$ , where  $R = \mathbb{F}_q[X]/\langle X^m - 1 \rangle$ . The code  $\mathcal{C}$  is the image of an  $\mathbb{F}_q[X]$ -submodule  $\tilde{\mathcal{C}}$  of  $\mathbb{F}_q[X]^\ell$  containing  $\tilde{K} = \langle (X^m - 1)\mathbf{e}_j, j \in [\ell] \rangle$  (where  $\mathbf{e}_j$  is the standard basis vector with one in position  $j$  and zero elsewhere) under the natural homomorphism

$$\begin{aligned} \phi: \mathbb{F}_q[X]^\ell &\rightarrow R^\ell \\ (c_0(X) \ \cdots \ c_{\ell-1}(X)) &\mapsto (c_0(X) + \langle X^m - 1 \rangle \ \cdots \\ &\quad c_{\ell-1}(X) + \langle X^m - 1 \rangle). \end{aligned}$$

It has a generating set of the form  $\{\mathbf{a}_i, i \in [z], (X^m - 1)\mathbf{e}_j, j \in [\ell]\}$ , where  $\mathbf{a}_i \in \mathbb{F}_q[X]^\ell$  and  $z \leq \ell$  (see e.g. [26, Chapter 5] for further information). Therefore, its generating set can be represented as a matrix with entries in  $\mathbb{F}_q[X]$ :

$$\mathbf{M}(X) = \begin{pmatrix} a_{0,0}(X) & a_{0,1}(X) & \cdots & a_{0,\ell-1}(X) \\ a_{1,0}(X) & a_{1,1}(X) & \cdots & a_{1,\ell-1}(X) \\ \vdots & \vdots & \ddots & \vdots \\ a_{z-1,0}(X) & a_{z-1,1}(X) & \cdots & a_{z-1,\ell-1}(X) \\ X^m - 1 & & & \mathbf{0} \\ & X^m - 1 & & \mathbf{0} \\ & \mathbf{0} & \ddots & \\ & & & X^m - 1 \end{pmatrix}. \quad (3)$$

Every matrix  $\mathbf{M}(X)$  as in (3) of the preimage  $\tilde{\mathcal{C}}$  can be transformed into a reduced Gröbner basis (RGB) with respect to the position-over-term order (POT) in  $\mathbb{F}_q[X]^\ell$  (see [11], [25]). This basis can be represented in the form of an upper-triangular  $\ell \times \ell$  matrix with entries in  $\mathbb{F}_q[X]$  as follows:

$$\mathbf{G}(X) = \begin{pmatrix} g_{0,0}(X) & g_{0,1}(X) & \cdots & g_{0,\ell-1}(X) \\ & g_{1,1}(X) & \cdots & g_{1,\ell-1}(X) \\ & \mathbf{0} & \ddots & \vdots \\ & & & g_{\ell-1,\ell-1}(X) \end{pmatrix}, \quad (4)$$

where the following conditions must be fulfilled:

- 1)  $g_{i,j}(X) = 0, \quad \forall 0 \leq j < i < \ell,$
- 2)  $\deg g_{j,i}(X) < \deg g_{i,i}(X), \quad \forall j < i, i \in [\ell],$
- 3)  $g_{i,i}(X) \mid (X^m - 1), \quad \forall i \in [\ell],$
- 4) if  $g_{i,i}(X) = X^m - 1$  then  $g_{i,j}(X) = 0, \quad \forall j \in [i + 1, \ell].$

The rows of  $\mathbf{G}(X)$  with  $g_{i,i}(X) \neq X^m - 1$  (i.e., the rows that do not map to zero under  $\phi$ ) are called the reduced generating set of the quasi-cyclic code  $\mathcal{C}$ . A codeword of  $\mathcal{C}$  can be represented as  $\mathbf{c}(X) = \mathbf{i}(X)\mathbf{G}(X)$  and it follows that  $k = m\ell - \sum_{i=0}^{\ell-1} \deg g_{i,i}(X)$ . Let us recall the following definition (see also [25, Thm. 3.2]).

**Definition 2** ( $r$ -level Quasi-Cyclic Code). We call an  $\ell$ -quasi-cyclic code  $\mathcal{C}$  of length  $\ell m$  an  $r$ -level quasi-cyclic code if there is an index  $r \in [\ell]$  for which the RGB/POT matrix as defined in (4) is such that  $g_{r-1,r-1}(X) \neq X^m - 1$  and  $g_{r,r}(X) = \cdots = g_{\ell-1,\ell-1}(X) = X^m - 1$ .

We recall [25, Corollary 3.3] for the case of a 1-level quasi-cyclic code in the following.

**Corollary 3** (1-level Quasi-Cyclic Code). The generator matrix in RGB/POT form of a 1-level  $\ell$ -quasi-cyclic code  $\mathcal{C}$  of length  $\ell m$  is:

$$\mathbf{G}(X) = (g(X) \ g(X)f_1(X) \ \cdots \ g(X)f_{\ell-1}(X)),$$

where  $g(X) \mid (X^m - 1)$  and  $f_1(X), \dots, f_{\ell-1}(X) \in \mathbb{F}_q[X]$ .

To describe quasi-cyclic codes explicitly, we need to recall the following facts of cyclic codes. A  $q$ -cyclotomic coset  $M_m^{(i)}$  is defined as:  $M_m^{(i)} \stackrel{\text{def}}{=} \{iq^j \bmod m \mid j \in [a]\}$ , where  $a$  is the smallest positive integer such that  $iq^a \equiv i \pmod{m}$ . The minimal polynomial in  $\mathbb{F}_q[X]$  of the element  $\alpha^i \in \mathbb{F}_{q^r}$  is given by

$$m_m^{(i)}(X) = \prod_{j \in M_m^{(i)}} (X - \alpha^j). \quad (5)$$

The following fact is used in Section III.

**Fact 4.** Let four nonzero integers  $y, a, \ell, m$  be such that

$$y \equiv a\ell \pmod{m\ell}$$

holds. Then  $\ell \mid y$  and  $y/\ell \equiv a \pmod{m}$ .

### III. QUASI-CYCLIC PRODUCT CODE

Throughout this section we consider a linear product code  $\mathcal{A} \otimes \mathcal{B}$ , where  $\mathcal{A}$  is the row-code and  $\mathcal{B}$  the column-code, respectively. Furthermore, w.l.o.g. let  $\mathcal{A}$  be an  $[\ell \cdot m_A, k_A, d_A]_q$   $\ell$ -quasi-cyclic code with reduced Gröbner basis in POT form as defined in (4):

$$\mathbf{G}^A(X) = \begin{pmatrix} g_{0,0}^A(X) & g_{0,1}^A(X) & \cdots & g_{0,\ell-1}^A(X) \\ & g_{1,1}^A(X) & \cdots & g_{1,\ell-1}^A(X) \\ & \mathbf{0} & \ddots & \vdots \\ & & & g_{\ell-1,\ell-1}^A(X) \end{pmatrix}, \quad (6)$$

and let  $\mathcal{B}$  be an  $[m_B, k_B, d_B]_q$  cyclic code with generator polynomial  $g^B(X)$  of degree  $m_B - k_B$ .

Throughout the paper, we assume that  $\gcd(\ell m_A, m_B) = 1$  and we furthermore assume that the two integers  $a$  and  $b$  are such that

$$a\ell m_A + b m_B = 1. \quad (7)$$

We recall the lemma of Wasan [19], that generalizes the result of Burton–Weldon [22, Theorem I] for cyclic product codes to

the case of an  $\ell$ -quasi-cyclic product code of an  $\ell$ -quasi-cyclic code  $\mathcal{A}$  and a cyclic code  $\mathcal{B}$ . A codeword of  $\mathcal{A} \otimes \mathcal{B}$  represented as univariate polynomial  $c(X)$  can then be obtained from the matrix representation  $(m_{i,j})_{\substack{j \in [\ell m_A] \\ i \in [m_B]}}$  as follows:

$$c(X) \equiv \sum_{i=0}^{m_B-1} \sum_{j=0}^{\ell m_A-1} m_{i,j} X^{\mu(i,j)} \pmod{X^{\ell m_A m_B} - 1}, \quad (8)$$

where

$$\mu(i, j) \stackrel{\text{def}}{=} ialm_A + jbm_B \pmod{\ell m_A m_B}. \quad (9)$$

**Lemma 5** (Mapping to a Univariate Polynomial [19]). *Let  $\mathcal{A}$  be an  $\ell$ -quasi-cyclic code of length  $\ell m_A$  and let  $\mathcal{B}$  be a cyclic code of length  $m_B$ . The product code  $\mathcal{A} \otimes \mathcal{B}$  is an  $\ell$ -quasi-cyclic code of length  $\ell m_A m_B$  if  $\gcd(\ell m_A, m_B) = 1$ .*

*Proof.* Let  $(m_{i,j})_{\substack{j \in [\ell m_A] \\ i \in [m_B]}}$  be a codeword of the product code  $\mathcal{A} \otimes \mathcal{B}$ , where each row is a codeword of  $\mathcal{A}$  and each column is a codeword of  $\mathcal{B}$ . The entry  $m_{i,j}$  is the coefficient  $c_{\mu(i,j)}$  of the codeword  $\sum_i c_i X^i$  as in (8). In order to prove that  $\mathcal{A} \otimes \mathcal{B}$  is  $\ell$ -quasi-cyclic it is sufficient to show that a shift by  $\ell$  positions of a codeword serialized to a univariate polynomial by (9) of  $\mathcal{A} \otimes \mathcal{B}$  is again a codeword of  $\mathcal{A} \otimes \mathcal{B}$ .

A shift by  $\ell$  in each row and a shift by one each column clearly gives a codeword in  $\mathcal{A} \otimes \mathcal{B}$ , because  $\mathcal{A}$  is  $\ell$ -quasi-cyclic and  $\mathcal{B}$  is cyclic. With

$$\begin{aligned} & \mu(i+1, j+\ell) \\ & \equiv (i+1)alm_A + (j+\ell)bm_B \pmod{\ell m_A m_B} \\ & \equiv ialm_A + jbm_B + \ell(alm_A + bm_B) \pmod{\ell m_A m_B} \\ & \equiv \mu(i, j) + \ell \pmod{\ell m_A m_B}, \end{aligned}$$

we obtain an  $\ell$ -quasi-cyclic shift of the univariate codeword obtained by (8) and (9).  $\square$

Instead of representing a codeword of  $\mathcal{A} \otimes \mathcal{B}$  as one univariate polynomial as in (8), we want to represent it as  $\ell$  univariate polynomials as defined in (1).

**Lemma 6** (Mapping to  $\ell$  Univariate Polynomials). *Let  $\mathcal{A}$  be an  $\ell$ -quasi-cyclic code of length  $\ell m_A$  and let  $\mathcal{B}$  be a cyclic code of length  $m_B$ . Let the matrix  $(m_{i,j})_{\substack{j \in [\ell m_A] \\ i \in [m_B]}}$  be a codeword of  $\mathcal{A} \otimes \mathcal{B}$ , where each row is in  $\mathcal{A}$  and each column is in  $\mathcal{B}$ . The  $\ell$  univariate polynomials of the corresponding codeword  $(c_0(X) \ c_1(X) \ \cdots \ c_{\ell-1}(X))$ , where each component is defined as in (1), are given by:*

$$c_h(X) \equiv X^{h(-am_A)} \cdot \sum_{i=0}^{m_B-1} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^{\bar{\mu}(i,j)} \pmod{X^{\ell m_A m_B} - 1}, \quad \forall h \in [\ell], \quad (10)$$

where

$$\bar{\mu}(i, j) \equiv ialm_A + jbm_B \pmod{m_A m_B}. \quad (11)$$

*Proof.* From Fact 4 we have for the exponents in (10):

$$\begin{aligned} & \bar{\mu}(i, j) + h(-am_A) \equiv ialm_A + jbm_B \pmod{m_A m_B} \\ & \Leftrightarrow \\ & \ell(\bar{\mu}(i, j) + h(-am_A)) \\ & \equiv \ell(ialm_A + jbm_B + h(-am_A)) \pmod{\ell m_A m_B}. \end{aligned} \quad (12)$$

With  $-alm_A = bm_B - 1$ , we can rewrite (12):

$$\begin{aligned} \ell(\bar{\mu}(i, j) + h(-am_A)) &= \ell\bar{\mu}(i, j) + \ell h(-am_A) \\ &= \ell\bar{\mu}(i, j) + hbm_B - h, \end{aligned}$$

and this gives with  $\bar{\mu}(i, j)$  as in (11) and  $\mu(i, j)$  as in (9):

$$\begin{aligned} & \ell\bar{\mu}(i, j) + hbm_B - h \\ & \equiv \ell(ialm_A + jbm_B) + hbm_B - h \\ & \equiv \ell ialm_A + (j\ell + h)bm_B - h \pmod{\ell m_A m_B} \\ & = \mu(i, j\ell + h) - h. \end{aligned} \quad (13)$$

Inserting (13) in (2) of Lemma 1 leads to:

$$\begin{aligned} c(X) &= \sum_{h=0}^{\ell-1} c_h(X^\ell) X^h \\ &= \sum_{h=0}^{\ell-1} \sum_{i=0}^{m_B-1} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^{\mu(i,j\ell+h)} \\ &= \sum_{i=0}^{m_B-1} \sum_{j=0}^{\ell m_A-1} m_{i,j} X^{\mu(i,j)}, \end{aligned} \quad (14)$$

which equals (8).  $\square$

The mapping  $\bar{\mu}(i, j)$  from (11) of the  $\ell$  submatrices  $(m_{i,j\ell})_{\substack{j \in [m_A] \\ i \in [m_B]}}$ ,  $(m_{i,j\ell+1})_{\substack{j \in [m_A] \\ i \in [m_B]}}$ ,  $\dots$ ,  $(m_{i,j\ell+\ell-1})_{\substack{j \in [m_A] \\ i \in [m_B]}}$  to the  $\ell$  univariate polynomials  $c_0(X), c_1(X), \dots, c_{\ell-1}(X)$  is the same as the one used to map the codeword of a cyclic product code from its matrix representation to a polynomial representation (see [22, Thm. 1]).

In Fig. III, we illustrate the  $\mu(i, j)$  as in (9) for  $a = 1$ ,  $\ell = 2$ ,  $m_A = 17$  and  $b = -11$ ,  $m_B = 3$ . Subfigure 1(a) shows the values of  $\mu(i, j)$ . The two submatrices  $(m_{i,j_2})$  and  $(m_{i,j_2+1})$  for  $i \in [3]$  and  $j \in [17]$  are shown in Subfigure 1(b). Subfigure 1(c) contains the coefficients of the two univariate polynomials  $c_0(X)$  and  $c_1(X)$ , where  $(c_0(X) \ c_1(X))$  is a codeword of the 2-quasi-cyclic product code of length 102.

The following theorem gives the basis representation of a quasi-cyclic product code, where the row-code is quasi-cyclic and the column-code is cyclic.

**Theorem 7** (Quasi-Cyclic Product Code). *Let  $\mathcal{A}$  be an  $[\ell \cdot m_A, k_A, d_A]_q$   $\ell$ -quasi-cyclic code with generator matrix  $\mathbf{G}^A(X) \in \mathbb{F}_q[X]^{\ell \times \ell}$  as in (6) and let  $\mathcal{B}$  be an  $[m_B, k_B, d_B]_q$  cyclic code with generator polynomial  $g^B(X) \in \mathbb{F}_q[X]$ .*

*Then the  $\ell$ -quasi-cyclic product code  $\mathcal{A} \otimes \mathcal{B}$  has a generating matrix of the following (unreduced) form:*

$$\mathbf{G}(X) = \begin{pmatrix} \mathbf{G}^0(X) \\ \mathbf{G}^1(X) \end{pmatrix}, \quad (15)$$

0	69	36	3	72	39	6	75	42	9	78	45	12	81	48	15	84	51	18	87	54	21	90	57	24	93	60	27	96	63	30	99	66	33
68	35	2	71	38	5	74	41	8	77	44	11	80	47	14	83	50	17	86	53	20	89	56	23	92	59	26	95	62	29	98	65	32	101
34	1	70	37	4	73	40	7	76	43	10	79	46	13	82	49	16	85	52	19	88	55	22	91	58	25	94	61	28	97	64	31	100	67

(a) The  $3 \times (2 \cdot 17)$  codeword matrix  $(m_{i,j})$  of the 2-quasi-cyclic product code  $\mathcal{A} \otimes \mathcal{B}$ . Each entry contains the index of the coefficient  $c_i$  of the univariate polynomial  $c(X) = \sum_{i=0}^{101} c_i X^i \in \mathcal{A} \otimes \mathcal{B}$ .

0	36	72	6	42	78	12	48	84	18	54	90	24	60	96	30	66
68	2	38	74	8	44	80	14	50	86	20	56	92	26	62	98	32
34	70	4	40	76	10	46	82	16	52	88	22	58	94	28	64	100

69	3	39	75	9	45	81	15	51	87	21	57	93	27	63	99	33
35	71	5	41	77	11	47	83	17	53	89	23	59	95	29	65	101
1	37	73	7	43	79	13	49	85	19	55	91	25	61	97	31	67

(b) The two submatrices  $(m_{i,2j})_{i \in [3]}^{j \in [17]}$  and  $(m_{i,2j+1})_{i \in [3]}^{j \in [17]}$  with entries that are the coefficients of  $c_0(X^2)$  and  $Xc_1(X^2)$ .

0	18	36	3	21	39	6	24	42	9	27	45	12	30	48	15	33
34	1	19	37	4	22	40	7	25	43	10	28	46	13	31	49	16
17	35	2	20	38	5	23	41	8	26	44	11	29	47	14	32	50

34	1	19	37	4	22	40	7	25	43	10	28	46	13	31	49	16
17	35	2	20	38	5	23	41	8	26	44	11	29	47	14	32	50
0	18	36	3	21	39	6	24	42	9	27	45	12	30	48	15	33

(c) The left submatrix contains the coefficients  $c_{0,i}$  of the univariate polynomials  $c_0(X)$  (the right one contains  $c_{1,i}$  of  $c_1(X)$ , respectively).

Figure 1. Illustration of the mapping  $\mu(i, j)$  (as defined in (9)) from a codeword of a quasi-cyclic product code represented as matrix to a polynomial representation. The product code  $\mathcal{A} \otimes \mathcal{B}$  is 2-quasi-cyclic. The row-code  $\mathcal{A}$  is 2-quasi-cyclic and has length  $\ell m_A = 2 \cdot 17$  and the column-code  $\mathcal{B}$  is cyclic and has length  $m_B = 3$  (Subfigure 1(a), here  $a = 1$  and  $b = -11$ ). The mapping  $\bar{\mu}(i, j)$  (as in (11)) to two univariate polynomials is illustrated in Subfigure 1(b) and Subfigure 1(c).

where

$$\mathbf{G}^0(X) = g^B(X^{a\ell m_A}).$$

$$\begin{pmatrix} g_{0,0}^A(X^{bm_B}) & g_{0,1}^A(X^{bm_B}) & \cdots & g_{0,\ell-1}^A(X^{bm_B}) \\ & g_{1,1}^A(X^{bm_B}) & \cdots & g_{1,\ell-1}^A(X^{bm_B}) \\ & & \ddots & \vdots \\ & & & g_{\ell-1,\ell-1}^A(X^{bm_B}) \end{pmatrix},$$

$$\cdot \text{diag}(1, X^{-am_A}, X^{-2am_A}, \dots, X^{-(\ell-1)am_A})$$

(16)

and

$$\mathbf{G}^1(X) = (X^{m_A m_B} - 1) \mathbf{I}_\ell, \quad (17)$$

where  $\mathbf{I}_\ell$  is the  $\ell \times \ell$  identity matrix.

*Proof.* We first give an explicit expression for each component of a codeword  $(c_0(X) \ c_1(X) \ \cdots \ c_{\ell-1}(X))$  in  $\mathcal{A} \otimes \mathcal{B}$  depending on the components of a codeword  $(a_0(X) \ a_1(X) \ \cdots \ a_{\ell-1}(X))$  of the row-code  $\mathcal{A}$  and depending the column-code  $\mathcal{B}$  based on the expression of Lemma 6. Let the  $m_B \times \ell m_A$  matrix  $(m_{i,j})$  be a codeword of the  $\ell$ -quasi-cyclic product code  $\mathcal{A} \otimes \mathcal{B}$  and let the polynomial

$$a_{i,h}(X) \stackrel{\text{def}}{=} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^j, \quad \forall h \in [\ell], i \in [m_B] \quad (18)$$

denote the  $h$ th component of a codeword  $(a_{i,0}(X) \ a_{i,1}(X) \ \cdots \ a_{i,\ell-1}(X))$  in  $\mathcal{A}$  in the  $i$ th row of the matrix  $(m_{i,j})$ . Denote a codeword  $b_j(X)$  of  $\mathcal{B}$  in the  $j$ th column by

$$b_j(X) = \sum_{i=0}^{m_B-1} m_{i,j} X^i, \quad \forall j \in [\ell m_A], \quad (19)$$

respectively. From (10), we have for the  $h$ th component of a codeword of the product code  $\mathcal{A} \otimes \mathcal{B}$ :

$$c_h(X) \equiv X^{h(-am_A)} \sum_{i=0}^{m_B-1} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^{\bar{\mu}(i,j)} \pmod{X^{m_A m_B} - 1}, \quad \forall h \in [\ell], \quad (20)$$

and with  $\bar{\mu}(i, j)$  as in (11) of Lemma 6 we can write (20) explicitly:

$$c_h(X) \equiv X^{h(-am_A)} \sum_{i=0}^{m_B-1} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^{ia\ell m_A + jbm_B} \pmod{X^{m_A m_B} - 1}, \quad \forall h \in [\ell]. \quad (21)$$

We define a shifted component:

$$\tilde{c}_h(X) \equiv c_h(X) X^{h(am_A)} \pmod{X^{m_A m_B} - 1}, \quad \forall h \in [\ell]. \quad (22)$$

Since

$$\begin{aligned} & \sum_{i=0}^{m_B-1} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^{ia\ell m_A + jbm_B} \\ &= \sum_{i=0}^{m_B-1} X^{ia\ell m_A} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^{jbm_B} \\ &= \sum_{i=0}^{m_B-1} X^{ia\ell m_A} a_{i,h}(X^{bm_B}), \quad \forall h \in [\ell], \end{aligned}$$

and from (22) and in terms of the components of the row-code as defined in (18), we obtain:

$$\tilde{c}_h(X) = q_h(X) (X^{m_A m_B} - 1) + \sum_{i=0}^{m_B-1} X^{ia\ell m_A} a_{i,h}(X^{bm_B}), \quad \forall h \in [\ell], \quad (23)$$

for some  $q_h(X) \in \mathbb{F}_q[X]$ . Therefore  $\tilde{c}_h(X)$  is a multiple of  $\sum_{i=0}^h \epsilon_i(X)g_{i,h}^A(X^{bm_B})$  for some  $\epsilon_i(X) \in \mathbb{F}_q[X]$ . A codeword  $b_j(X)$  in  $\mathcal{B}$  in the  $j$ th column of  $(m_{i,j})$  is a multiple of  $g^B(X)$  and we obtain:

$$\begin{aligned} & \sum_{i=0}^{m_B-1} \sum_{j=0}^{\ell m_A-1} m_{i,j} X^{ialm_A+jbm_B} \\ &= \sum_{j=0}^{\ell m_A-1} X^{jbm_B} \sum_{i=0}^{m_B-1} m_{i,j} X^{ialm_A} \\ &= \sum_{j=0}^{\ell m_A-1} X^{jbm_B} b_j(X^{al m_A}), \end{aligned}$$

and therefore  $\tilde{c}_h(X)$  is a multiple of  $g^B(X^{al m_A})$  modulo  $X^{m_A m_B} - 1$ .

Similar to the proof of [22, Thm. III], it can be shown that every shifted component  $\tilde{c}_h(X)$  is a multiple of the product of  $g^B(X^{al m_A})$  and  $\sum_{i=0}^h \epsilon_i g_{i,h}^A(X^{bm_B})$  modulo  $(X^{m_A m_B} - 1)$ . Therefore, we can represent each codeword in  $\mathcal{A} \otimes \mathcal{B}$  as:

$$\begin{aligned} & (c_0(X) \ c_1(X) \ \cdots \ c_{\ell-1}(X)) \\ &= (i_0(X) \ i_1(X) \ \cdots \ i_{\ell-1}(X)) \mathbf{G}(X), \end{aligned}$$

where  $\mathbf{G}(X)$  is as in (15).  $\square$

The following theorem gives the reduced Gröbner basis (as defined in (4)) representation of the quasi-cyclic product code from Thm. 7, where the row-code is a 1-level quasi-cyclic code.

**Theorem 8 (1-Level Quasi-Cyclic Product Code).** *Let  $\mathcal{A}$  be an  $[\ell \cdot m_A, k_A, d_A]_q$  1-level  $\ell$ -quasi-cyclic code with generator matrix in RGB/POT form:*

$$\begin{aligned} & \mathbf{G}^A(X) \\ &= (g_{0,0}^A(X) \ g_{0,1}^A(X) \ \cdots \ g_{0,\ell-1}^A(X)) \\ &= (g^A(X) \ g^A(X)f_1^A(X) \ \cdots \ g^A(X)f_{\ell-1}^A(X)) \end{aligned} \quad (24)$$

as shown in Corollary 3. Let  $\mathcal{B}$  be an  $[m_B, k_B, d_B]_q$  cyclic code with generator polynomial  $g^B(X) \in \mathbb{F}_q[X]$ .

Then a generator matrix of the 1-level  $\ell$ -quasi-cyclic product code in RGB/POT form is:

$$\begin{aligned} \mathbf{G}(X) &= (g(X) \ g(X)f_1^A(X^{bm_B}) \ \cdots \ g(X)f_{\ell-1}^A(X^{bm_B})) \\ &\quad \cdot \text{diag}(1, X^{-am_A}, X^{-2am_A}, \dots, X^{-(\ell-1)am_A}), \end{aligned}$$

where

$$g(X) = \gcd(X^{m_A m_B} - 1, g^A(X^{bm_B})g^B(X^{al m_A})). \quad (25)$$

*Proof.* Let two polynomials  $u_0(X), v_0(X) \in \mathbb{F}_q[X]$  be such that:

$$\begin{aligned} g(X) &= u_0(X)g^A(X^{bm_B})g^B(X^{al m_A}) \\ &\quad + v_0(X)(X^{m_A m_B} - 1). \end{aligned} \quad (26)$$

We show now how to reduce the basis representation to the RGB/POT form. We denote a new Row  $i$  by  $R[i]'$ . For ease of notation, we omit the term  $\text{diag}(1, X^{-am_A}, X^{-2am_A}, \dots, X^{-(\ell-1)am_A})$  and denote by  $Y = X^{bm_B}$  and  $Z = X^{al m_A}$ .

We write the basis of the submodule in unreduced form (as in (15)):

$$\begin{pmatrix} g^A(Y)g^B(Z) & g^A(Y)f_1^A(Y)g^B(Z) & \cdots \\ X^{m_A m_B} - 1 & X^{m_A m_B} - 1 & \\ \mathbf{0} & & \ddots \end{pmatrix} \quad (27)$$

$$\begin{aligned} \rightarrow R[0]' &= u_0(X)R[0] + v_0(X)R[1] + v_0(X)f_1^A(Y)R[2] \\ &\quad + \cdots + v_0(X)f_{\ell-1}^A(Y)R[\ell] \end{aligned}$$

$$\begin{pmatrix} g(X) & g(X)f_1^A(Y) & \cdots \\ g^A(Y)g^B(Z) & g^A(Y)f_1^A(Y)g^B(Z) & \cdots \\ X^{m_A m_B} - 1 & X^{m_A m_B} - 1 & \\ \mathbf{0} & & \ddots \end{pmatrix}, \quad (28)$$

where the  $i$ th entry in new row 0 was obtained using:

$$\begin{aligned} & u_0(X)g^A(Y)f_i^A(Y)g^B(Z) + v_0(X)f_i^A(Y)(X^{m_A m_B} - 1) \\ &= f_i^A(Y)(u_0(X)g^A(Y)g^B(Z) \\ &\quad + v_0(X)(X^{m_A m_B} - 1)), \end{aligned} \quad (29)$$

and with (26) we obtain from (29)

$$\begin{aligned} & f_i^A(Y)(u_0(X)g^A(Y)g^B(Z) + v_0(X)(X^{m_A m_B} - 1)) \\ &= f_i^A(Y)g(X). \end{aligned}$$

Clearly,  $g(X)$  divides  $g^A(Y)g^B(Z)$  and it is easy to check that Row 1 of the matrix in (28) can be obtained from Row 0 by multiplying by  $g^A(Y)g^B(Z)/g(X)$ . Therefore, we can omit the linearly dependent Row 1 in (28) and write the reduced basis as:

$$(g(X) \ g(X)f_1^A(X^{bm_B}) \ \cdots \ g(X)f_{\ell-1}^A(X^{bm_B})),$$

where we omitted the matrix  $\text{diag}(1, X^{-am_A}, X^{-2am_A}, \dots, X^{-(\ell-1)am_A})$  for the first row during the proof, but it will only influence the row-operations by a factor.  $\square$

Note that (25) is exactly the generator polynomial of a cyclic product code. A 1-level  $\ell$ -quasi-cyclic product has rate greater than  $(\ell - 1)/\ell$  and is therefore of high practical relevance. The explicit RGB/POT form of the 1-level quasi-cyclic product code as in Thm. 8 allows statements on the minimum distance and to develop decoding algorithms.

#### IV. EXAMPLE

We consider a 2-quasi-cyclic product code with the same parameters as the one illustrated in Fig. III. In this section we investigate a more explicit example to be able to calculate the basis as given in Thm. 8.

Let  $\mathcal{A}$  be a binary 2-quasi-cyclic code of length  $\ell m_A = 2 \cdot 17 = 34$  and let  $\mathcal{B}$  be a cyclic code of length  $m_B = 3$ . We have  $X^{17} - 1 = m_0^{(17)}(X)m_1^{(17)}(X)m_3^{(17)}(X)$ , where the minimal polynomials are as defined in (5). Let the generator

matrix of  $\mathcal{A}$  in RGB/POT form as in (4) be  $\mathbf{G}^A(X) = (g_{0,0}^A(X) \ g_{0,1}^A(X))$  where

$$\begin{aligned} g_{0,0}^A(X) &= m_1^{\langle 17 \rangle}(X) \\ &= X^8 + X^7 + X^6 + X^4 + X^2 + X + 1, \\ g_{0,1}^A(X) &= m_1^{\langle 17 \rangle}(X) \cdot m_0(X)^3 \cdot (X^3 + X^2 + 1) \\ &= X^{14} + X^{13} + X^{12} + X^{11} + X^8 + 1, \end{aligned}$$

and  $\mathcal{A}$  is a  $[17 \cdot 2, 9, 11]_2$  2-quasi-cyclic code. Let  $\alpha$  be a 17th root of unity in  $\mathbb{F}_{2^8}[X] \cong \mathbb{F}_2[X]/(X^8 + X^4 + X^3 + X^2 + 1)$ . Let  $g^B(X) = m_0^{\langle 3 \rangle}(X) = X + 1$  be the generator polynomial of the  $[3, 2, 2]_2$  cyclic code  $\mathcal{B}$  and let  $a = 1$  and  $b = -11$  be such that (7) holds. We have

$$\begin{aligned} X^{51} - 1 &= m_0^{\langle 51 \rangle}(X) m_1^{\langle 51 \rangle}(X) m_3^{\langle 51 \rangle}(X) m_5^{\langle 51 \rangle}(X) m_9^{\langle 51 \rangle}(X) \\ &\quad m_{11}^{\langle 51 \rangle}(X) m_{17}^{\langle 51 \rangle}(X) m_{19}^{\langle 51 \rangle}(X). \end{aligned}$$

According to Thm. 8, we calculate

$$\begin{aligned} f_1^A(X^{-11 \cdot 3}) &\equiv f_{0,1}^A(X^{18}) = m_0(X^{18})^3 \cdot (X^{54} + X^{36} + 1) \\ &= (X^{18} + 1)^3 \cdot (X^{54} + X^{36} + 1) \\ &= X^{108} + X^{54} + X^{18} + 1 \\ &\equiv X^{18} + X^6 + X^3 + 1 \pmod{(X^{51} + 1)}, \end{aligned}$$

and we obtain the generator matrix  $\mathbf{G}(X) = (g_{0,0}(X) \ g_{0,1}(X))$  of  $\mathcal{A} \otimes \mathcal{B}$ , where:

$$\begin{aligned} g_{0,0}(X) &= m_0^{\langle 51 \rangle}(X) m_1^{\langle 51 \rangle}(X) m_3^{\langle 51 \rangle}(X) m_9^{\langle 51 \rangle}(X) m_{19}^{\langle 51 \rangle}(X) \\ &= X^{33} + X^{32} + X^{30} + X^{27} + X^{25} + X^{23} + X^{20} \\ &\quad + X^{18} + X^{17} + X^{16} + X^{15} + X^{13} + X^{10} + X^8 \\ &\quad + X^6 + X^3 + X + 1. \end{aligned}$$

With Thm. 8, we obtain:

$$\begin{aligned} g_{0,1}(X) &\equiv a_1^A(X^{-11 \cdot 3}) g_{0,0}(X) \\ &\equiv X^{50} + X^{48} + X^{45} + X^{43} + X^{41} + X^{39} + X^{36} \\ &\quad + X^{34} + X^{32} + X^{29} + X^{27} + X^{26} + X^{25} + X^{23} \\ &\quad + X^{22} + X^{21} + X^{19} + X^{18} + X^{17} + X^{16} + X^{15} \\ &\quad + X^{14} + X^{12} + X^{11} + X^{10} + X^8 + X^7 + X^6 \\ &\quad + X^4 + X \pmod{(X^{51} + 1)}. \end{aligned}$$

## V. CONCLUSION AND OUTLOOK

Based on the RGB/POT representation of an  $\ell$ -quasi-cyclic code  $\mathcal{A}$  and the generator polynomial of a cyclic code  $\mathcal{B}$ , a basis representation of the  $\ell$ -quasi-cyclic product code  $\mathcal{A} \otimes \mathcal{B}$  was proven. The reduced basis representation of the special case of a 1-generator quasi-cyclic product code was derived.

The general case of the basis representation of an  $\ell_A \ell_B$ -quasi cyclic product code from an  $\ell_A$ -quasi-cyclic code  $\mathcal{A}$  and an  $\ell_B$ -quasi-cyclic code  $\mathcal{B}$  as well as the reduction of the basis remains an open future work. Furthermore, a technique to bound the minimum distance of a given quasi-cyclic code by embedding it into a product code similar to [27] seems to be realizable.

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