Construction of Quasi-Cyclic Product Codes

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***Abstract*—Linear quasi-cyclic product codes over finite fields are investigated. Given the generating set in the form of a reduced Gröbner basis of a quasi-cyclic component code and the generator polynomial of a second cyclic component code, an explicit expression of the basis of the generating set of the quasi-cyclic product code is given. Furthermore, the reduced Gröbner basis of a one-level quasi-cyclic product code is derived.**

***Index Terms*—Cyclic code, Gröbner basis, module minimization, product code, quasi-cyclic code, submodule**

I. Introduction

A linear block code of length ℓm over a finite field $\mathbb{F}_q$ is a quasi-cyclic code if every cyclic shift of a codeword by ℓ positions, for some integer ℓ between one and ℓm, results in another codeword. Quasi-cyclic codes are a natural generalization of cyclic codes (where $\ell = 1$), and have a closely linked algebraic structure. In contrast to cyclic codes, quasi-cyclic codes are known to be asymptotically good (see Chen–Peterson–Weldon [1]). Several such codes have been discovered with the highest minimum distance for a given length and dimension (see Gulliver–Bhargava [2] as well as Chen's and Grassl's databases [3], [4]). Several good LDPC codes are quasi-cyclic (see e.g. [5]) and the connection to convolutional codes was investigated among others in [6]–[8].

Recent papers of Barbier *et al.* [9], [10], Lally–Fitzpatrick [8], [11], [12], Ling–Solé [13]–[15], Semenov–Trifonov [16], Güneri–Özbudak [17] and ours [18] discuss different aspects of the algebraic structure of quasi-cyclic codes including lower bounds on the minimum Hamming distance and efficient decoding algorithms.

The focus of this paper is on a simple method to combine two given quasi-cyclic codes into a product code. More specifically, we give a description of a quasi-cyclic product code when one component code is quasi-cyclic and the second one is cyclic.

The work of Wasan [19] first considers quasi-cyclic product codes while investigating the mathematical properties of the wider class of quasi-abelian codes. Some more results were published in a short note by Wasan and Dass [20]. Koshy proposed a so-called "circle" quasi-cyclic product codes in [21].

Our work considers quasi-cyclic product codes that generalize the results of Burton–Weldon [22] and Lin–Weldon [23] (see also [24, Chapter 18]) based on the reduced Gröbner basis representation of Lally–Fitzpatrick [11] of the quasi-cyclic component code. We derive a representation of the generating set of a quasi-cyclic product code, where one component code is quasi-cyclic and the other is cyclic (in Thm. 7) and we give a reduced Gröbner basis for the special class of one-level quasi-cyclic product codes (in Thm. 8).

The paper is structured as follows. In Section II, we give necessary preliminaries on quasi-cyclic codes over finite fields. We outline relevant basics of the reduced Gröbner basis representation of Lally–Fitzpatrick [11]. Furthermore, the special class of r-level quasi-cyclic codes is defined in this section. Section III contains the main result on quasi-cyclic product codes, where the row-code is quasi-cyclic and the column-code is cyclic. Moreover, an explicit expression of the basis of a 1-level quasi-cyclic product code is derived in Section III. For illustration, we explicitly give an example of a binary 2-quasi-cyclic product code in Section IV. Section V concludes this paper.

II. Preliminaries

Let $\mathbb{F}_q$ denote the finite field of order q and $\mathbb{F}_q[X]$ the polynomial ring over $\mathbb{F}_q$ with indeterminate X. Let a, b with $b > a$ be two positive integers and denote by $[a, b)$ the set of integers $\{a, a+1, \dots, b-1\}$ and by $[b) = [0, b)$. A vector of length n is denoted by a lowercase bold letter as $\mathbf{v} = (v_0 \ v_1 \ \cdots \ v_{n-1})$ and an $m \times n$ matrix is denoted by a capital bold letter as $\mathbf{M} = (m_{i,j})_{i \in [m)}^{j \in [n)}$.

A linear $[\ell \cdot m, k, d]_q$ code $\mathcal{C}$ of length ℓm, dimension k and minimum Hamming distance d over $\mathbb{F}_q$ is ℓ-quasi-cyclic if every cyclic shift by ℓ of a codeword is again a codeword of $\mathcal{C}$, more explicitly if:

$$
\begin{array}{llll}
(c_{0,0} \cdots c_{\ell-1,0} & c_{0,1} \cdots c_{\ell-1,1} & \dots & c_{\ell-1,m-1}) \in \mathcal{C} \\
& \Rightarrow & & \\
(c_{0,m-1} \cdots c_{\ell-1,m-1} & c_{0,0} \cdots c_{\ell-1,0} & \dots & c_{\ell-1,m-2}) \in \mathcal{C}.
\end{array}
$$

We can represent a codeword of an $[\ell \cdot m, k, d]_q$ ℓ-quasi-cyclic code as $\mathbf{c}(X) = (c_0(X) \ c_1(X) \ \cdots \ c_{\ell-1}(X)) \in \mathbb{F}_q[X]^\ell$, where

$$
c_i(X) \stackrel{\text{def}}{=} \sum_{j=0}^{m-1} c_{i,j} X^j, \quad \forall i \in [\ell). \tag{1}
$$

Then, the defining property of $\mathcal{C}$ is that each component $c_i(X)$ of $\mathbf{c}(X)$ is closed under multiplication by X and reduction modulo $X^m - 1$.

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Lemma 1. *Let $(c_0(X)\ c_1(X)\ \cdots\ c_{\ell-1}(X))$ be a codeword of an ℓ-quasi-cyclic code $\mathcal{C}$ of length $m\ell$, where the components are defined as in* (1). *Then a codeword in $\mathcal{C}$ represented as one univariate polynomial of degree smaller than $m\ell$ is*

$$c(X) = \sum_{i=0}^{\ell-1} c_i(X^\ell) X^i. \tag{2}$$

Proof. Substitute (1) into (2):

$$c(X) = \sum_{i=0}^{\ell-1} c_i(X^\ell) X^i = \sum_{i=0}^{\ell-1} \sum_{j=0}^{m-1} c_{i,j} X^{j\ell+i}.$$

□

Lally and Fitzpatrick [11], [25] showed that this enables us to see a quasi-cyclic code as an R-submodule of the algebra R^ℓ, where $R = \mathbb{F}_q[X]/\langle X^m - 1\rangle$. The code $\mathcal{C}$ is the image of an $\mathbb{F}_q[X]$-submodule $\widetilde{\mathcal{C}}$ of $\mathbb{F}_q[X]^\ell$ containing $\widetilde{K} = \langle (X^m-1)\mathbf{e}_j, j \in [\ell)\rangle$ (where $\mathbf{e}_j$ is the standard basis vector with one in position j and zero elsewhere) under the natural homomorphism

$$\begin{aligned} \phi:\ \mathbb{F}_q[X]^\ell &\to R^\ell \\ (c_0(X)\ \cdots\ c_{\ell-1}(X)) &\mapsto (c_0(X) + \langle X^m\text{-}1\rangle\ \cdots \\ &\qquad c_{\ell-1}(X) + \langle X^m\text{-}1\rangle). \end{aligned}$$

It has a generating set of the form $\{\mathbf{a}_i, i \in [z), (X^m-1)\mathbf{e}_j, j \in [\ell)\}$, where $\mathbf{a}_i \in \mathbb{F}_q[X]^\ell$ and $z \leq \ell$ (see e.g. [26, Chapter 5] for further information). Therefore, its generating set can be represented as a matrix with entries in $\mathbb{F}_q[X]$:

$$\mathbf{M}(X) = \begin{pmatrix} a_{0,0}(X) & a_{0,1}(X) & \cdots & a_{0,\ell-1}(X) \\ a_{1,0}(X) & a_{1,1}(X) & \cdots & a_{1,\ell-1}(X) \\ \vdots & \vdots & \ddots & \vdots \\ a_{z-1,0}(X) & a_{z-1,1}(X) & \cdots & a_{z-1,\ell-1}(X) \\ X^m-1 & & & \\ & X^m-1 & & \mathbf{0} \\ & \mathbf{0} & \ddots & \\ & & & X^m-1 \end{pmatrix}. \tag{3}$$

Every matrix $\mathbf{M}(X)$ as in (3) of the preimage $\widetilde{\mathcal{C}}$ can be transformed into a reduced Gröbner basis (RGB) with respect to the position-over-term order (POT) in $\mathbb{F}_q[X]^\ell$ (see [11], [25]). This basis can be represented in the form of an upper-triangular $\ell \times \ell$ matrix with entries in $\mathbb{F}_q[X]$ as follows:

$$\mathbf{G}(X) = \begin{pmatrix} g_{0,0}(X) & g_{0,1}(X) & \cdots & g_{0,\ell-1}(X) \\ & g_{1,1}(X) & \cdots & g_{1,\ell-1}(X) \\ & \mathbf{0} & \ddots & \vdots \\ & & & g_{\ell-1,\ell-1}(X) \end{pmatrix}, \tag{4}$$

where the following conditions must be fulfilled:

1) $g_{i,j}(X) = 0, \quad \forall 0 \leq j < i < \ell,$
2) $\deg g_{j,i}(X) < \deg g_{i,i}(X), \quad \forall j < i, i \in [\ell),$
3) $g_{i,i}(X) \mid (X^m-1), \quad \forall i \in [\ell),$
4) if $g_{i,i}(X) = X^m - 1$ then $g_{i,j}(X) = 0, \quad \forall j \in [i+1, \ell).$

The rows of $\mathbf{G}(X)$ with $g_{i,i}(X) \neq X^m - 1$ (i.e., the rows that do not map to zero under ϕ) are called the reduced generating set of the quasi-cyclic code $\mathcal{C}$. A codeword of $\mathcal{C}$ can be represented as $\mathbf{c}(X) = \mathbf{i}(X)\mathbf{G}(X)$ and it follows that $k = m\ell - \sum_{i=0}^{\ell-1} \deg g_{i,i}(X)$. Let us recall the following definition (see also [25, Thm. 3.2]).

Definition 2 (r-level Quasi-Cyclic Code)**.** *We call an ℓ-quasi-cyclic code $\mathcal{C}$ of length ℓm an r-level quasi-cyclic code if there is an index $r \in [\ell)$ for which the RGB/POT matrix as defined in* (4) *is such that $g_{r-1,r-1}(X) \neq X^m - 1$ and $g_{r,r}(X) = \cdots = g_{\ell-1,\ell-1}(X) = X^m - 1$.*

We recall [25, Corollary 3.3] for the case of a 1-level quasi-cyclic code in the following.

Corollary 3 (1-level Quasi-Cyclic Code)**.** *The generator matrix in RGB/POT form of a 1-level ℓ-quasi-cyclic code $\mathcal{C}$ of length ℓm is:*

$$\mathbf{G}(X) = \begin{pmatrix} g(X) & g(X)f_1(X) & \cdots & g(X)f_{\ell-1}(X) \end{pmatrix},$$

where $g(X) | (X^m-1)$ and $f_1(X), \dots, f_{\ell-1}(X) \in \mathbb{F}_q[X]$.

To describe quasi-cyclic codes explicitly, we need to recall the following facts of *cyclic* codes. A q-cyclotomic coset $M_m^{\langle i\rangle}$ is defined as: $M_m^{\langle i\rangle} \overset{\text{def}}{=} \{iq^j \mod m \,|\, j \in [a)\}$, where a is the smallest positive integer such that $iq^a \equiv i \bmod m$. The minimal polynomial in $\mathbb{F}_q[X]$ of the element $\alpha^i \in \mathbb{F}_{q^r}$ is given by

$$m_m^{\langle i\rangle}(X) = \prod_{j \in M_m^{\langle i\rangle}} (X - \alpha^j). \tag{5}$$

The following fact is used in Section III.

Fact 4. *Let four nonzero integers y, a, ℓ, m be such that*

$$y \equiv a\ell \mod m\ell$$

holds. Then $\ell \mid y$ and $y/\ell \equiv a \mod m$.

III. Quasi-Cyclic Product Code

Throughout this section we consider a linear product code $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{A}$ is the row-code and $\mathcal{B}$ the column-code, respectively. Furthermore, w.l.o.g. let $\mathcal{A}$ be an $[\ell \cdot m_A, k_A, d_A]_q$ ℓ-quasi-cyclic code with reduced Gröbner basis in POT form as defined in (4):

$$\mathbf{G}^A(X) = \begin{pmatrix} g^A_{0,0}(X) & g^A_{0,1}(X) & \cdots & g^A_{0,\ell-1}(X) \\ & g^A_{1,1}(X) & \cdots & g^A_{1,\ell-1}(X) \\ & \mathbf{0} & \ddots & \vdots \\ & & & g^A_{\ell-1,\ell-1}(X) \end{pmatrix}, \tag{6}$$

and let $\mathcal{B}$ be an $[m_B, k_B, d_B]_q$ cyclic code with generator polynomial $g^B(X)$ of degree $m_B - k_B$.

Throughout the paper, we assume that $\gcd(\ell m_A, m_B) = 1$ and we furthermore assume that the two integers a and b are such that

$$a\ell m_A + b m_B = 1. \tag{7}$$

We recall the lemma of Wasan [19], that generalizes the result of Burton–Weldon [22, Theorem I] for cyclic product codes to the case of an ℓ-quasi-cyclic product code of an ℓ-quasi-cyclic code $\mathcal{A}$ and a cyclic code $\mathcal{B}$. A codeword of $\mathcal{A}\otimes\mathcal{B}$ represented as univariate polynomial $c(X)$ can then be obtained from the matrix representation $(m_{i,j})_{i\in[m_B)}^{j\in[\ell m_A)}$ as follows:

$$c(X) \equiv \sum_{i=0}^{m_B-1}\sum_{j=0}^{\ell m_A-1} m_{i,j}X^{\mu(i,j)} \mod X^{\ell m_A m_B}-1, \quad (8)$$

where

$$\mu(i,j) \overset{\text{def}}{=} ia\ell m_A\ell + jbm_B \mod \ell m_A m_B. \quad (9)$$

Lemma 5 (Mapping to a Univariate Polynomial [19])**.** *Let $\mathcal{A}$ be an ℓ-quasi-cyclic code of length ℓm_A and let $\mathcal{B}$ be a cyclic code of length m_B. The product code $\mathcal{A}\otimes\mathcal{B}$ is an ℓ-quasi-cyclic code of length $\ell m_A m_B$ if* $\gcd(\ell m_A, m_B)=1$.

Proof. Let $(m_{i,j})_{i\in[m_B)}^{j\in[\ell m_A)}$ be a codeword of the product code $\mathcal{A}\otimes\mathcal{B}$, where each row is a codeword of $\mathcal{A}$ and each column is a codeword of $\mathcal{B}$. The entry $m_{i,j}$ is the coefficient $c_{\mu(i,j)}$ of the codeword $\sum_i c_iX^i$ as in (8). In order to prove that $\mathcal{A}\otimes\mathcal{B}$ is ℓ-quasi-cyclic it is sufficient to show that a shift by ℓ positions of a codeword serialized to a univariate polynomial by (9) of $\mathcal{A}\otimes\mathcal{B}$ is again a codeword of $\mathcal{A}\otimes\mathcal{B}$.

A shift by ℓ in each row and a shift by one each column clearly gives a codeword in $\mathcal{A}\otimes\mathcal{B}$, because $\mathcal{A}$ is ℓ-quasi-cyclic and $\mathcal{B}$ is cyclic. With

$$\begin{aligned}&\mu(i+1,j+\ell)\\ &\quad\equiv (i+1)a\ell m_A\ell + (j+\ell)bm_B \mod \ell m_A m_B\\ &\quad\equiv ia\ell m_A\ell + jbm_B + \ell(a\ell m_A + bm_B) \mod \ell m_A m_B\\ &\quad\equiv \mu(i,j)+\ell \mod \ell m_A m_B,\end{aligned}$$

we obtain an ℓ-quasi-cyclic shift of the univariate codeword obtained by (8) and (9). □

Instead of representing a codeword of $\mathcal{A}\otimes\mathcal{B}$ as one univariate polynomial as in (8), we want to represent it as ℓ univariate polynomials as defined in (1).

Lemma 6 (Mapping to ℓ Univariate Polynomials)**.** *Let $\mathcal{A}$ be an ℓ-quasi-cyclic code of length ℓm_A and let $\mathcal{B}$ be a cyclic code of length m_B. Let the matrix $(m_{i,j})_{i\in[m_B)}^{j\in[\ell m_A)}$ be a codeword of $\mathcal{A}\otimes\mathcal{B}$, where each row is in $\mathcal{A}$ and each column is in $\mathcal{B}$. The ℓ univariate polynomials of the corresponding codeword $(c_0(X)\ c_1(X)\ \cdots\ c_{\ell-1}(X))$, where each component is defined as in* (1)*, are given by:*

$$c_h(X) \equiv X^{h(-am_A)}\cdot\sum_{i=0}^{m_B-1}\sum_{j=0}^{m_A-1} m_{i,j\ell+h}X^{\overline{\mu}(i,j)} \mod X^{m_A m_B}-1, \quad \forall h\in[\ell), \quad (10)$$

where

$$\overline{\mu}(i,j) \equiv ia\ell m_A + jbm_B \mod m_A m_B. \quad (11)$$

Proof. From Fact 4 we have for the exponents in (10):

$$\begin{aligned}&\overline{\mu}(i,j) + h(-am_A) \equiv ia\ell m_A + jbm_B \mod m_A m_B\\ &\Leftrightarrow\\ &\ell\big(\overline{\mu}(i,j)+h(-am_A)\big)\\ &\quad\equiv \ell(ia\ell m_A + jbm_B + h(-am_A)) \mod \ell m_A m_B. \quad (12)\end{aligned}$$

With $-a\ell m_A = bm_B - 1$, we can rewrite (12):

$$\begin{aligned}\ell\big(\overline{\mu}(i,j)+h(-am_A)\big) &= \ell\overline{\mu}(i,j) + \ell h(-am_A)\\ &= \ell\overline{\mu}(i,j) + hbm_B - h,\end{aligned}$$

and this gives with $\overline{\mu}(i,j)$ as in (11) and $\mu(i,j)$ as in (9):

$$\begin{aligned}&\ell\overline{\mu}(i,j) + hbm_B - h\\ &\quad\equiv \ell(ia\ell m_A + jbm_B) + hbm_B - h\\ &\quad\equiv \ell ia\ell m_A + (j\ell+h)bm_B - h \mod \ell m_A m_B\\ &\quad= \mu(i,j\ell+h) - h. \quad (13)\end{aligned}$$

Inserting (13) in (2) of Lemma 1 leads to:

$$\begin{aligned}c(X) &= \sum_{h=0}^{\ell-1} c_h(X^\ell)X^h\\ &= \sum_{h=0}^{\ell-1}\sum_{i=0}^{m_B-1}\sum_{j=0}^{m_A-1} m_{i,j\ell+h}X^{\mu(i,j\ell+h)}\\ &= \sum_{i=0}^{m_B-1}\sum_{j=0}^{\ell m_A-1} m_{i,j}X^{\mu(i,j)}, \quad (14)\end{aligned}$$

which equals (8). □

The mapping $\overline{\mu}(i,j)$ from (11) of the ℓ submatrices $(m_{i,j\ell})_{i\in[m_B)}^{j\in[m_A)}, (m_{i,j\ell+1})_{i\in[m_B)}^{j\in[m_A)}, \dots, (m_{i,j\ell+\ell-1})_{i\in[m_B)}^{j\in[m_A)}$ to the ℓ univariate polynomials $c_0(X), c_1(X), \dots, c_{\ell-1}(X)$ is the same as the one used to map the codeword of a cyclic product code from its matrix representation to a polynomial representation (see [22, Thm. 1]).

In Fig. III, we illustrate the $\mu(i,j)$ as in (9) for $a=1$, $\ell=2$, $m_A=17$ and $b=-11$, $m_B=3$. Subfigure 1(a) shows the values of $\mu(i,j)$. The two submatrices $(m_{i,j2})$ and $(m_{i,j2+1})$ for $i\in[3)$ and $j\in[17)$ are shown in Subfigure 1(b). Subfigure 1(c) contains the coefficients of the two univariate polynomials $c_0(X)$ and $c_1(X)$, where $(c_0(X)\ c_1(X))$ is a codeword of the 2-quasi-cyclic product code of length 102.

The following theorem gives the basis representation of a quasi-cyclic product code, where the row-code is quasi-cyclic and the column-code is cyclic.

Theorem 7 (Quasi-Cyclic Product Code)**.** *Let $\mathcal{A}$ be an $[\ell\cdot m_A, k_A, d_A]_q$ ℓ-quasi-cyclic code with generator matrix $\mathbf{G}^A(X)\in\mathbb{F}_q[X]^{\ell\times\ell}$ as in* (6) *and let $\mathcal{B}$ be an $[m_B,k_B,d_B]_q$ cyclic code with generator polynomial $g^B(X)\in\mathbb{F}_q[X]$.*

Then the ℓ-quasi-cyclic product code $\mathcal{A}\otimes\mathcal{B}$ has a generating matrix of the following (unreduced) form:

$$\mathbf{G}(X) = \begin{pmatrix}\mathbf{G}^0(X)\\ \mathbf{G}^1(X)\end{pmatrix}, \quad (15)$$

0	69	36	3	72	39	6	75	42	9	78	45	12	81	48	15	84	51	18	87	54	21	90	57	24	93	60	27	96	63	30	99	66	33
68	35	2	71	38	5	74	41	8	77	44	11	80	47	14	83	50	17	86	53	20	89	56	23	92	59	26	95	62	29	98	65	32	101
34	1	70	37	4	73	40	7	76	43	10	79	46	13	82	49	16	85	52	19	88	55	22	91	58	25	94	61	28	97	64	31	100	67

(a) The $3 \times (2 \cdot 17)$ codeword matrix $(m_{i,j})$ of the 2-quasi-cyclic product code $\mathcal{A} \otimes \mathcal{B}$. Each entry contains the index of the coefficient c_i of the univariate polynomial $c(X) = \sum_{i=0}^{101} c_i X^i \in \mathcal{A} \otimes \mathcal{B}$.

0	36	72	6	42	78	12	48	84	18	54	90	24	60	96	30	66
68	2	38	74	8	44	80	14	50	86	20	56	92	26	62	98	32
34	70	4	40	76	10	46	82	16	52	88	22	58	94	28	64	100

69	3	39	75	9	45	81	15	51	87	21	57	93	27	63	99	33
35	71	5	41	77	11	47	83	17	53	89	23	59	95	29	65	101
1	37	73	7	43	79	13	49	85	19	55	91	25	61	97	31	67

(b) The two submatrices $(m_{i,2j})_{i\in[3]}^{i\in[17]}$ and $(m_{i,2j+1})_{i\in[3]}^{i\in[17]}$ with entries that are the coefficients of $c_0(X^2)$ and $Xc_1(X^2)$.

0	18	36	3	21	39	6	24	42	9	27	45	12	30	48	15	33
34	1	19	37	4	22	40	7	25	43	10	28	46	13	31	49	16
17	35	2	20	38	5	23	41	8	26	44	11	29	47	14	32	50

34	1	19	37	4	22	40	7	25	43	10	28	46	13	31	49	16
17	35	2	20	38	5	23	41	8	26	44	11	29	47	14	32	50
0	18	36	3	21	39	6	24	42	9	27	45	12	30	48	15	33

(c) The left submatrix contains the coefficients $c_{0,i}$ of the univariate polynomials $c_0(X)$ (the right one contains $c_{1,i}$ of $c_1(X)$, respectively).

Figure 1. Illustration of the mapping $\mu(i,j)$ (as defined in (9)) from a codeword of a quasi-cyclic product code represented as matrix to a polynomial representation. The product code $\mathcal{A} \otimes \mathcal{B}$ is 2-quasi-cyclic. The row-code $\mathcal{A}$ is 2-quasi-cyclic and has length $\ell m_A = 2 \cdot 17$ and the column-code $\mathcal{B}$ is cyclic and has length $m_B = 3$ (Subfigure 1(a), here $a = 1$ and $b = -11$). The mapping $\overline{\mu}(i,j)$ (as in (11)) to two univariate polynomials is illustrated in Subfigure 1(b) and Subfigure 1(c).

where

$$\mathbf{G}^0(X) = g^B(X^{a\ell m_A}) \cdot \begin{pmatrix} g^A_{0,0}(X^{bm_B}) & g^A_{0,1}(X^{bm_B}) & \cdots & g^A_{0,\ell-1}(X^{bm_B}) \\ & g^A_{1,1}(X^{bm_B}) & \cdots & g^A_{1,\ell-1}(X^{bm_B}) \\ \mathbf{0} & & \ddots & \vdots \\ & & & g^A_{\ell-1,\ell-1}(X^{bm_B}) \end{pmatrix}, \cdot \operatorname{diag}\left(1, X^{-am_A}, X^{-2am_A}, \dots, X^{-(\ell-1)am_A}\right) \tag{16}$$

and

$$\mathbf{G}^1(X) = (X^{m_A m_B} - 1)\mathbf{I}_\ell, \tag{17}$$

where $\mathbf{I}_\ell$ *is the* $\ell \times \ell$ *identity matrix.*

Proof. We first give an explicit expression for each component of a codeword $(c_0(X) \; c_1(X) \; \cdots \; c_{\ell-1}(X))$ in $\mathcal{A} \otimes \mathcal{B}$ depending on the components of a codeword $(a_0(X) \; a_1(X) \; \cdots \; a_{\ell-1}(X))$ of the row-code $\mathcal{A}$ and depending the column-code $\mathcal{B}$ based on the expression of Lemma 6. Let the $m_B \times \ell m_A$ matrix $(m_{i,j})$ be a codeword of the ℓ-quasi-cyclic product code $\mathcal{A} \otimes \mathcal{B}$ and let the polynomial

$$a_{i,h}(X) \overset{\text{def}}{=} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^j, \quad \forall h \in [\ell), i \in [m_B) \tag{18}$$

denote the hth component of a codeword $(a_{i,0}(X) \; a_{i,1}(X) \; \cdots \; a_{i,\ell-1}(X))$ in $\mathcal{A}$ in the ith row of the matrix $(m_{i,j})$. Denote a codeword $b_j(X)$ of $\mathcal{B}$ in the jth column by

$$b_j(X) = \sum_{i=0}^{m_B-1} m_{i,j} X^i, \quad \forall j \in [\ell m_A), \tag{19}$$

respectively. From (10), we have for the hth component of a codeword of the product code $\mathcal{A} \otimes \mathcal{B}$:

$$c_h(X) \equiv X^{h(-am_A)} \sum_{i=0}^{m_B-1} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^{\overline{\mu}(i,j)} \mod X^{m_A m_B} - 1, \quad \forall h \in [\ell), \tag{20}$$

and with $\overline{\mu}(i,j)$ as in (11) of Lemma 6 we can write (20) explicitly:

$$c_h(X) \equiv X^{h(-am_A)} \sum_{i=0}^{m_B-1} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^{ia\ell m_A + jbm_B} \mod X^{m_A m_B} - 1, \quad \forall h \in [\ell). \tag{21}$$

We define a shifted component:

$$\widetilde{c}_h(X) \equiv c_h(X) X^{h(am_A)} \mod X^{m_A m_B}\text{-}1, \; \forall h \in [\ell). \tag{22}$$

Since

$$\begin{aligned} &\sum_{i=0}^{m_B-1} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^{ia\ell m_A + jbm_B} \\ &= \sum_{i=0}^{m_B-1} X^{ia\ell m_A} \sum_{j=0}^{m_A-1} m_{i,j\ell+h} X^{jbm_B} \\ &= \sum_{i=0}^{m_B-1} X^{ia\ell m_A} a_{i,h}(X^{bm_B}), \quad \forall h \in [\ell), \end{aligned}$$

and from (22) and in terms of the components of the row-code as defined in (18), we obtain:

$$\widetilde{c}_h(X) = q_h(X)(X^{m_A m_B} - 1) + \sum_{i=0}^{m_B-1} X^{ia\ell m_A} a_{i,h}(X^{bm_B}), \quad \forall h \in [\ell), \tag{23}$$

for some $q_h(X) \in \mathbb{F}_q[X]$. Therefore $\widetilde{c}_h(X)$ is a multiple of $\sum_{i=0}^{h} \epsilon_i(X) g^A_{i,h}(X^{bm_B})$ for some $\epsilon_i(X) \in \mathbb{F}_q[X]$. A codeword $b_j(X)$ in $\mathcal{B}$ in the jth column of $(m_{i,j})$ is a multiple of $g^B(X)$ and we obtain:

$$\begin{aligned}
&\sum_{i=0}^{m_B-1} \sum_{j=0}^{\ell m_A-1} m_{i,j} X^{ia\ell m_A + jbm_B} \\
&\qquad = \sum_{j=0}^{\ell m_A-1} X^{jbm_B} \sum_{i=0}^{m_B-1} m_{i,j} X^{ia\ell m_A} \\
&\qquad = \sum_{j=0}^{\ell m_A-1} X^{jbm_B} b_j(X^{a\ell m_A}),
\end{aligned}$$

and therefore $\widetilde{c}_h(X)$ is a multiple of $g^B(X^{a\ell m_A})$ modulo $X^{m_A m_B} - 1$.

Similar to the proof of [22, Thm. III], it can be shown that every shifted component $\widetilde{c}_h(X)$ is a multiple of the product of $g^B(X^{a\ell m_A})$ and $\sum_{i=0}^{h} \epsilon_i g^A_{i,h}(X^{bm_B})$ modulo $(X^{m_A m_B} - 1)$. Therefore, we can represent each codeword in $\mathcal{A} \otimes \mathcal{B}$ as:

$$\begin{aligned}
&\big(c_0(X) \ c_1(X) \ \cdots \ c_{\ell-1}(X)\big) \\
&\qquad = \big(i_0(X) \ i_1(X) \ \cdots \ i_{\ell-1}(X)\big) \mathbf{G}(X),
\end{aligned}$$

where $\mathbf{G}(X)$ is as in (15). □

The following theorem gives the reduced Gröbner basis (as defined in (4)) representation of the quasi-cyclic product code from Thm. 7, where the row-code is a 1-level quasi-cyclic code.

Theorem 8 (1-Level Quasi-Cyclic Product Code)**.** *Let $\mathcal{A}$ be an $[\ell \cdot m_A, k_A, d_A]_q$ 1-level ℓ-quasi-cyclic code with generator matrix in RGB/POT form:*

$$\begin{aligned}
&\mathbf{G}^A(X) \\
&= \big(g^A_{0,0}(X) \quad g^A_{0,1}(X) \quad \cdots \quad g^A_{0,\ell-1}(X)\big) \\
&= \big(g^A(X) \quad g^A(X) f^A_1(X) \quad \cdots \quad g^A(X) f^A_{\ell-1}(X)\big) \qquad (24)
\end{aligned}$$

as shown in Corollary 3. Let $\mathcal{B}$ be an $[m_B, k_B, d_B]_q$ cyclic code with generator polynomial $g^B(X) \in \mathbb{F}_q[X]$.

Then a generator matrix of the 1-level ℓ-quasi-cyclic product code in RGB/POT form is:

$$\begin{aligned}
\mathbf{G}(X) = &\big(g(X) \quad g(X) f^A_1(X^{bm_B}) \quad \cdots \quad g(X) f^A_{\ell-1}(X^{bm_B})\big) \\
&\cdot \operatorname{diag}\left(1, X^{-am_A}, X^{-2am_A}, \ldots, X^{-(\ell-1)am_A}\right),
\end{aligned}$$

where

$$g(X) = \gcd\big(X^{m_A m_B} - 1, g^A(X^{bm_B}) g^B(X^{a\ell m_A})\big). \qquad (25)$$

Proof. Let two polynomials $u_0(X), v_0(X) \in \mathbb{F}_q[X]$ be such that:

$$\begin{aligned}
g(X) = u_0(X) g^A(X^{bm_B}) g^B(X^{a\ell m_A}) \\
+ v_0(X)(X^{m_A m_B} - 1).
\end{aligned} \qquad (26)$$

We show now how to reduce the basis representation to the RGB/POT form. We denote a new Row i by $\mathsf{R}[i]'$. For ease of notation, we omit the term $\operatorname{diag}(1, X^{-am_A}, X^{-2am_A}, \ldots, X^{-(\ell-1)am_A})$ and denote by $Y = X^{bm_B}$ and $Z = X^{a\ell m_A}$.

We write the basis of the submodule in unreduced form (as in (15)):

$$\begin{pmatrix}
g^A(Y) g^B(Z) & g^A(Y) f^A_1(Y) g^B(Z) & \cdots \\
X^{m_A m_B} - 1 & & \\
& X^{m_A m_B} - 1 & \\
\mathbf{0} & & \ddots
\end{pmatrix} \qquad (27)$$

$$\begin{aligned}
\to \mathsf{R}[0]' = u_0(X)\mathsf{R}[0] + v_0(X)\mathsf{R}[1] + v_0(X) f^A_1(Y)\mathsf{R}[2] \\
+ \cdots + v_0(X) f^A_{\ell-1}(Y)\mathsf{R}[\ell]
\end{aligned}$$

$$\begin{pmatrix}
g(X) & g(X) f^A_1(Y) & \cdots \\
g^A(Y) g^B(Z) & g^A(Y) f^A_1(Y) g^B(Z) & \cdots \\
X^{m_A m_B} - 1 & & \\
& X^{m_A m_B} - 1 & \\
\mathbf{0} & & \ddots
\end{pmatrix}, \qquad (28)$$

where the ith entry in new row 0 was obtained using:

$$\begin{aligned}
&u_0(X) g^A(Y) f^A_i(Y) g^B(Z) + v_0(X) f^A_i(Y)(X^{m_A m_B} - 1) \\
&= f^A_i(Y)\big(u_0(X) g^A(Y) g^B(Z) \\
&\qquad + v_0(X)(X^{m_A m_B} - 1)\big), \qquad (29)
\end{aligned}$$

and with (26) we obtain from (29)

$$\begin{aligned}
&f^A_i(Y)\big(u_0(X) g^A(Y) g^B(Z) + v_0(X)(X^{m_A m_B} - 1)\big) \\
&= f^A_i(Y) g(X).
\end{aligned}$$

Clearly, $g(X)$ divides $g^A(Y) g^B(Z)$ and it is easy to check that Row 1 of the matrix in (28) can be obtained from Row 0 by multiplying by $g^A(Y) g^B(Z)/g(X)$. Therefore, we can omit the linearly dependent Row 1 in (28) and write the reduced basis as:

$$\big(g(X) \quad g(X) f^A_1(X^{bm_B}) \quad \cdots \quad g(X) f^A_{\ell-1}(X^{bm_B})\big),$$

where we omitted the matrix $\operatorname{diag}(1, X^{-am_A}, X^{-2am_A}, \ldots, X^{-(\ell-1)am_A})$ for the first row during the proof, but it will only influence the row-operations by a factor. □

Note that (25) is exactly the generator polynomial of a cyclic product code. A 1-level ℓ-quasi-cyclic product has rate greater than $(\ell - 1)/\ell$ and is therefore of high practical relevance. The explicit RGB/POT form of the 1-level quasi-cyclic product code as in Thm. 8 allows statements on the minimum distance and to develop decoding algorithms.

IV. Example

We consider a 2-quasi-cyclic product code with the same parameters as the one illustrated in Fig. III. In this section we investigate a more explicit example to be able to calculate the basis as given in Thm. 8.

Let $\mathcal{A}$ be a binary 2-quasi-cyclic code of length $\ell m_A = 2 \cdot 17 = 34$ and let $\mathcal{B}$ be a cyclic code of length $m_B = 3$. We have $X^{17} - 1 = m_0^{\langle 17 \rangle}(X) m_1^{\langle 17 \rangle}(X) m_3^{\langle 17 \rangle}(X)$, where the minimal polynomials are as defined in (5). Let the generator matrix of $\mathcal{A}$ in RGB/POT form as in (4) be $\mathbf{G}^A(X) = \begin{pmatrix} g_{0,0}^A(X) & g_{0,1}^A(X) \end{pmatrix}$ where

$$\begin{aligned} g_{0,0}^A(X) &= m_1^{\langle 17 \rangle}(X) \\ &= X^8 + X^7 + X^6 + X^4 + X^2 + X + 1, \\ g_{0,1}^A(X) &= m_1^{\langle 17 \rangle}(X) \cdot m_0(X)^3 \cdot (X^3 + X^2 + 1) \\ &= X^{14} + X^{13} + X^{12} + X^{11} + X^8 + 1, \end{aligned}$$

and $\mathcal{A}$ is a $[17 \cdot 2, 9, 11]_2$ 2-quasi-cyclic code. Let α be a 17th root of unity in $\mathbb{F}_{2^8}[X] \cong \mathbb{F}_2[X]/(X^8+X^4+X^3+X^2+1)$. Let $g^B(X) = m_0^{\langle 3 \rangle}(X) = X + 1$ be the generator polynomial of the $[3,2,2]_2$ cyclic code $\mathcal{B}$ and let $a = 1$ and $b = -11$ be such that (7) holds. We have

$$\begin{aligned} X^{51} - 1 = m_0^{\langle 51 \rangle}(X) m_1^{\langle 51 \rangle}(X) m_3^{\langle 51 \rangle}(X) m_5^{\langle 51 \rangle}(X) m_9^{\langle 51 \rangle}(X) \\ m_{11}^{\langle 51 \rangle}(X) m_{17}^{\langle 51 \rangle}(X) m_{19}^{\langle 51 \rangle}(X). \end{aligned}$$

According to Thm. 8, we calculate

$$\begin{aligned} f_1^A(X^{-11 \cdot 3}) &\equiv f_{0,1}^A(X^{18}) = m_0(X^{18})^3 \cdot (X^{54} + X^{36} + 1) \\ &= (X^{18} + 1)^3 \cdot (X^{54} + X^{36} + 1) \\ &= X^{108} + X^{54} + X^{18} + 1 \\ &\equiv X^{18} + X^6 + X^3 + 1 \mod (X^{51} + 1), \end{aligned}$$

and we obtain the generator matrix $\mathbf{G}(X) = (g_{0,0}(X) \; g_{0,1}(X))$ of $\mathcal{A} \otimes \mathcal{B}$, where:

$$\begin{aligned} g_{0,0}(X) &= m_0^{\langle 51 \rangle}(X) m_1^{\langle 51 \rangle}(X) m_3^{\langle 51 \rangle}(X) m_9^{\langle 51 \rangle}(X) m_{19}^{\langle 51 \rangle}(X) \\ &= X^{33} + X^{32} + X^{30} + X^{27} + X^{25} + X^{23} + X^{20} \\ &\quad + X^{18} + X^{17} + X^{16} + X^{15} + X^{13} + X^{10} + X^8 \\ &\quad + X^6 + X^3 + X + 1. \end{aligned}$$

With Thm. 8, we obtain:

$$\begin{aligned} g_{0,1}(X) &\equiv a_1^A(X^{-11 \cdot 3}) g_{0,0}(X) \\ &\equiv X^{50} + X^{48} + X^{45} + X^{43} + X^{41} + X^{39} + X^{36} \\ &\quad + X^{34} + X^{32} + X^{29} + X^{27} + X^{26} + X^{25} + X^{23} \\ &\quad + X^{22} + X^{21} + X^{19} + X^{18} + X^{17} + X^{16} + X^{15} \\ &\quad + X^{14} + X^{12} + X^{11} + X^{10} + X^8 + X^7 + X^6 \\ &\quad + X^4 + X \mod (X^{51} + 1). \end{aligned}$$

V. Conclusion and Outlook

Based on the RGB/POT representation of an ℓ-quasi-cyclic code $\mathcal{A}$ and the generator polynomial of a cyclic code $\mathcal{B}$, a basis representation of the ℓ-quasi-cyclic product code $\mathcal{A} \otimes \mathcal{B}$ was proven. The reduced basis representation of the special case of a 1-generator quasi-cyclic product code was derived.

The general case of the basis representation of an $\ell_A \ell_B$-quasi cyclic product code from an ℓ_A-quasi-cyclic code $\mathcal{A}$ and an ℓ_B-quasi-cyclic code $\mathcal{B}$ as well as the reduction of the basis remains an open future work. Furthermore, a technique to bound the minimum distance of a given quasi-cyclic code by embedding it into a product code similar to [27] seems to be realizable.

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