Some Notes about Subshifts on Groups

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Abstract

In this note we prove the following results:

- If a finitely presented group G admits a strongly aperiodic SFT, then G has decidable word problem.

- For a large class of group G, $\mathbb{Z} \times G$ admits a strongly aperiodic SFT. In particular, this is true for the free group with 2 generators, Thompson's groups T and V, $PSL_2(\mathbb{Z})$ and any f.g. group of rational matrices which is bounded.

While Symbolic Dynamics [LM95] usually studies subshifts on $\mathbb{Z}$, there has been a lot of work generalizing these results to other groups, from dynamicians and computer scientists working in higher dimensions ($\mathbb{Z}^d$ [Lin04]) to group theorists interested in characterizing group properties in terms of topological or dynamical properties [CSC10].

In this note, we are interested in the existence of aperiodic SFTs, or more generally of aperiodic effective shifts.

There has been a lot of work proving how to build aperiodic SFTs in a large class of groups, and more generally tilings on manifolds. The most well known is probably Berger's construction [Ber64] of an aperiodic SFTs in the two-dimensional lattice $\mathbb{Z}^2$, but construction on wilder groups or symmetric spaces may be found [Moz97, Coh14].

It is an open question to characterize groups that admit strongly aperiodic SFTs.

Cohen[Coh14] showed that f.g. groups admitting strongly aperiodic SFTs are one ended and asked whether it is a sufficient condition. Our first result proves that it is not: If G is finitely presented, then it also must have decidable word problem. This is proven in section 2. This is true more generally for f.g. groups admitting strongly aperiodic effective subshifts, that is subshifts given by a list of forbidden patterns we can enumerate by a program.

In fact, we also do not need strongly aperiodic subshifts, but something weaker, that we call weakly strongly aperiodic subshifts: Strongly aperiodic

subshifts ask that the stabilizer of each point is finite. Here we ask that the stabilizer of each point does not contain a normal subgroup. The notation for this new object is quite unfortunate and better names are welcome.

This result may be generalized to any f.g. group, without any assumption about a finite or recursive presentation: the existence of a strongly aperiodic SFTs implies some structure on the word problem on G, namely that we can enumerate the non-identity elements of the group from the identity elements of the group. The latter property is for example true of any simple group. This generalization is the core of section 3, and might be omitted by any reader not familiar with recursion theory.

In the last section, we will remark how a variation on a technique by Kari gives aperiodic SFTs on $\mathbb{Z} \times G$ for a large class of group G. We do not know if there exists an easier proof of this statement.

1 Effective sets on Groups

We first give definitions of effective sets, which are some particular closed subsets of the Cantor Space A^G and $A^{\mathbb{F}_p}$. The reader fluent with symbolic dynamics should remark that the sets we consider are not supposed to be translation(shift)-invariant in this section.

1.1 Effective sets on the free group

Let $\mathbb{F}_p$ denote the free group on p generators. Let G be a finitely generated group with p generators that we see as a quotient of $\mathbb{F}_p$. Unless specified otherwise, the identity on G and on $\mathbb{F}_p$ will be denoted by λ, and the symbol 1 will be used only for denoting a number. Let ϕ be the natural map from $\mathbb{F}_p$ to G, and R the kernel of this map. Hence $G = \mathbb{F}_p/R = \langle x_1 \dots x_p | R \rangle$.

Let A be a finite alphabet.

A *pseudo-word* is a map w from a finite part of $\mathbb{F}_p$ to A. A pseudo-word is a *G-word* if $w_g = w_h$ whenever $\phi(g) = \phi(h)$ and both sides are defined.

A configuration $x \in A^{\mathbb{F}_p}$ *disagrees* with a pseudo-word w if there exists g so that $x_g \neq w_g$ and both sides are well defined. A configuration $x \in A^G$ *disagrees* with a pseudo-word w if there exists $g \in \mathbb{F}_p$ so that $x_{\phi(g)} \neq w_g$ and both sides are well defined. Note that a configuration in A^G always disagree with a pseudo-word which is not a G-word.

Let L be a list of pseudo-words. The set defined by L is the subset $S_L^{\mathbb{F}_p}$ of $A^{\mathbb{F}_p}$ of all configurations x that disagree with all words in L. The G-set defined by L is the subset S_L^G of A^G of all configurations x that disagree with all words in L.

It is easy to see that a set defined by L is a closed set for the prodiscrete topology on $A^{\mathbb{F}_p}$. Conversely, any closed set can be defined by some L.

If L has an effective enumeration (can be enumerated by a program, or a Turing machine), $S_L^{\mathbb{F}_p}$ and S_L^G are said to be *effective*.

It is important to note that our definition of effectivity differs from the notion of $\mathbb{Z}$-effectivity proposed in [AaS]. The definition are identical for finitely and recursively presented groups, but the definition we take here makes more sense for general groups, with presentation of arbitrary complexity.

We will use in the following a basic but important result:

Proposition 1.1. *There exists an algorithm that, given an effective enumeration L, halts iff $S_L^{\mathbb{F}_p}$ is empty.*

Proof. For a finite set L, it is easy to test if $S_L^{\mathbb{F}_p}$ is empty: just test all possible words of $A^{\mathbb{F}_p}$ defined on the union of the supports of all words in L.

Furthermore, by compactness, for an infinite L', $S_{L'}^{\mathbb{F}_p}$ is empty iff there exists a finite $L \subseteq L'$ so that $S_L^{\mathbb{F}_p}$ is empty.

Now, if L is effective, consider the following algorithm: enumerate all elements w_i in L, and test at each step if $S_{\{w_1,\ldots,w_n\}}^{\mathbb{F}_p}$ is empty. By the first remark, it is indeed an algorithm. By the second remark, this algorithm halts iff $S_L^{\mathbb{F}_p}$ is empty. $\qquad\square$

1.2 The Relation between $S_L^{\mathbb{F}_p}$ and S_L^G

Recall that $G = \mathbb{F}_p/R$ for some normal subgroup R.

Denote by Per_R the set of all configurations of $A^{\mathbb{F}_p}$ which are R-periodic, that is $x_{hg} = x_g$ for all h in R and all g in $\mathbb{F}_p$.

Note that if $x \in Per_R$, and $\phi(g) = \phi(g')$ then $x_g = x_{g'}$. Indeed, if $\phi(g) = \phi(g')$, then $g'g^{-1} \in R$, hence $x_{g'g^{-1}g} = x_g$ by definition of Per_R. Conversely, if for all g, g', $\phi(g) = \phi(g')$ implies that $x_g = x_{g'}$, then $x \in Per_R$.

Hence there is a natural map ψ from Per_R to G defined by $\psi(x) = y$ where $y_{\phi(g)} = x_g$. ψ is invertible with inverse defined by $\psi^{-1}(y) = x$ where $x_g = y_{\phi(g)}$.

Per_R can be given by a forbidden set of words: For two group elements g in $\mathbb{F}_p$ and h in R and two letter $a \neq b \in A$, denote by $w_{g,h,a,b}$ the word w defined over $\{g, gh\}$ by $w_g = a$ and $w_{gh} = b$. Then it is easy to see that $Per_R = S_{L_R}^{\mathbb{F}_p}$ where L_R is the set of all words of the form $w_{g,h,a,b}$ for $h \in R$ and $g \in \mathbb{F}_p$.

Fact 1.2. $\psi(S_L^{\mathbb{F}_p} \cap Per_R) = S_L^G$

(There is a natural bijection between configurations in $\mathbb{F}_p$ that are R-periodic and forbid the set L, and configurations in G that forbid the set L)

While $S_L^{\mathbb{F}_p}$ is always effective if L can be enumerated, it might be possible for $S_L^{\mathbb{F}_p} \cap Per_R$ to not be effective. In fact:

Proposition 1.3. *Let A be an alphabet of size at least 2.*
Per_R is effective iff G has a recognizable word problem.

A recognizable word problem means that there is an algorithm that, given a word w in $\mathbb{F}_p$, halts iff $w \in R$ (hence w codes the identity element of G). This is equivalent to saying that G is recursively presented.

Proof. If G has a recognizable word problem, we may enumerate all words $g \in \mathbb{F}_p$ that belong to R, and thus enumerate L_R, hence $Per_R = S_{L_R}^{\mathbb{F}_p}$ is effective.

Conversely, suppose that $Per_R = S_L^{\mathbb{F}_p}$ for some effective L.

Let $g \in G$. For any letter $a \in A$, consider the word w^a over $\{\lambda, g\}$ defined by $w_\lambda^a = a, w_g^a = a$. Let $L' = L \cup \{w^a, a \in A\}$.

Then $S_{L'}^{\mathbb{F}_p}$ is empty iff $g \in R$. Indeed, it is clear that if $g \in R$, then $S_{L'}^{\mathbb{F}_p}$ is empty. Conversely, suppose that $g \notin R$ and wlog $\{c, d\} \subseteq A$. Define x by $x_h = c$ if $h \in R$ and $x_h = d$ otherwise. Then $x \in Per_R$ and $x \in S_{L'}^{\mathbb{F}_p}$, hence $S_{L'}^{\mathbb{F}_p}$ is nonempty.

As emptyness of effective sets is recognizable in $\mathbb{F}_p$, this gives the result. $\square$

Corollary 1.4. *If G has a recognizable word problem and S_L^G is effective, then $S_L^{\mathbb{F}_p} \cap Per_R$ is effective.*

2   Effective subshifts on Groups

If $x \in A^G$, denote by gx the configuration of A^G defined by $(gx)_h = x_{g^{-1}h}$. This defines an action of G on A^G.

Definition 2.1. *A closed set X of A^G is said to be a subshift if $x \in X, g \in G$ implies that $gx \in X$.*

X is an effectively closed subshift if X is effective, and a subshift.

X is a SFT if there exists a finite set L so that $X = S_{\{g^{-1}w, g \in G, w \in L\}} = S_{\{g^{-1}w, g \in \mathbb{F}_p, w \in L\}}$. In particular a SFT is always effective.

Fact 2.2. *If X is a subshift, $\psi^{-1}(X) \cap Per_R$ is a subshift.*

Hence any subshift of A^G lifts up to a subshift of $A^{\mathbb{F}_p}$.

Let $X_{\leq 1}$ denote the subset of $\{0, 1\}^G$ of configurations that contains at most one symbol 1. It is easy to see that $X_{\leq 1}$ is closed, and a subshift.

Proposition 2.3. *Suppose that G has a recognizable word problem.*

If $X_{\leq 1}$ is effective then the word problem on G is decidable.

Proof. $X_{\leq 1}$ lifts up to a subshift Y on $\mathbb{F}_p$ with the property that (a) Y is effective (as G is recursively presented) (b) Y consists of all configurations so that $x_g = x_h = 1 \implies gh^{-1} \in R$.

Now let $g \in \mathbb{F}_p$. Let w be the word defined by $w_\lambda = 0$ and w' the word defined by $w_g = 0$, and consider $Y' = Y \cap S_{\{w, w'\}}^{\mathbb{F}_p}$. That is, any configuration x of Y' must have $x_\lambda = 1$ and $x_g = 1$. Thus Y' is empty iff $g \notin R$.

Emptyness is recognizable, hence the complement of the word problem is recognizable, therefore decidable. $\square$

Definition 2.4. *For $x \in A^G$ denote by $Stab(x) = \{g | gx = x\}$.*

A (nonempty) subshift X is strongly aperiodic iff for every $x \in X$, $Stab(x)$ is finite.

A (nonempty) subshift X is weakly strongly aperiodic iff for every $x \in X$, $\cap_{h \in G} Stab(hx)$ is finite.

Both properties are equivalent for commutative groups. $\cap_{h \in G} Stab(hx)$ will be called the *normal stabilizer* of x. It is indeed a normal subgroup of G, and the union of all normal subgroups of $Stab(x)$.

Our first result states that a strongly aperiodic effective subshift (and in particular a strongly aperiodic SFT) forces the group to have a decidable word problem in the class of torsion-free recursively presented group. The next proposition strenghtens the result by deleting the torsion-freeness requirement.

Proposition 2.5. *Let G be a torsion-free recursively presented group.*

If G admits a strongly aperiodic effective subshift, then G has decidable word problem.

Proof. Let X be the strongly aperiodic effective subshift. X lifts up to a subshift Y on $A^{\mathbb{F}_p}$.

Note that if $\psi(y) = x$, then $Stab(y) = Stab(x)R = RStab(x)$. Furthermore, if G is torsion-free, then $Stab(x) = \{\lambda\}$, hence for all $y \in Y$, $Stab(y) = R$.

Now let $g \in \mathbb{F}_p$. Let $Z = \{x | \forall t, x_{gt} = x_t\} = \{x \in A^{\mathbb{F}_p} | g \in Stab(x)\}$.

Z is effective. Furthermore $Y \cap Z = \emptyset$ iff $g \notin R$.

Emptyness is recognizable, hence the complement of the word problem is recognizable, hence decidable. $\square$

Proposition 2.6. *Let G be a recursively presented group.*

If G admits a weakly strongly aperiodic effective subshift, then it admits a weakly strongly aperiodic effective subshift X where for all $x \in X$, $\cap_{h \in G} Stab(hx) = \lambda$.

Proof. Let X be weakly strongly aperiodic.

The proof is in two steps. In the first step, we will prove that there exists a finite normal subgroup H of G and a nonempty effective subshift Y so that for all $x \in Y$, $\cap_{g \in G} Stab(gx) = H$.

Let $H_0 = \{\lambda\}$. Suppose that there exists $x \in X$ so that $H_0 \subsetneq \cap_{h \in G} Stab(hx)$. Then let's denote $H_1 \supset H_0$ the normal subgroup on the right.

We do the same for H_1, building progressively a chain of normal subgroups $H_1 \ldots H_n \ldots$.

It is impossible however to obtain an infinite chain this way. Indeed, as for all i, there exists x_i so that $\cap_{g \in G} Stab(gx_i) = H_i$, a limit point x of x_i would verify $\cap_{g \in G} Stab(gx) \supseteq \cup_i H_i$, hence x would be a configuration with an infinite normal stabilizer, impossible by definition.

Hence this process will stop, and we obtain some finite normal subgroup H of G and a point x_0 so that $\cap_{h \in G} Stab(hx) = H$ and no point x has a larger normal stabilizer.

Now let $Y = \{x \in X | \forall g \in G, h \in H, hgx = gx\}$. Y is nonempty, as it contains x_0. As H is finite, Y is clearly effective. As H is normal, it is a subshift. Furthermore, for all $x \in Y$, $\cap_{h \in G} Stab(hx) = H$.

5

Now the second step. Take $Z = \{x \in H^G | \forall g \in G, h \in H \setminus \{\lambda\}, x_{hg} \neq x_g\}$. Z is clearly effective. As H is normal, it is a subshift[1]. Z is also nonempty: Write $G = HI$ where I is a family of representatives of G/H. Then the point z defined by $z_g = h$ if $g \in hI$ is in Z, hence Z is nonempty[2]. Furthermore, if $z \in Z$, then $Stab(x) \cap H = \{\lambda\}$ [3]. As a consequence, $Z \times Y$ is a nonempty subshift for which for all $x \in Z \times Y$, $\cap_{h \in G} Stab(hx) = \{\lambda\}$. $\quad\square$

Corollary 2.7. *Let G be a recursively presented group.*

If G admits a weakly strongly aperiodic effective subshift, G has decidable word problem.

(As a strongly aperiodic effective subshift is also weakly strongly aperiodic, the result is also true for groups that admits strongly aperiodic effective subshifts, or groups that admits strongly aperiodic SFT)

Proof. This is more or less the same proof as before, with one slight difference.

Let X be the weakly strongly aperiodic effective subshift. We may suppose by the previous proposition that for all $x \in X$, $\cap_{h \in G} Stab(hx) = \{\lambda\}$.

X lifts up to a subshift Y on $A^{\mathbb{F}_p}$ with the following property: If $y \in Y$, then $\cap_{h \in \mathbb{F}_p} Stab(hy) = R$.

Now let $g \in \mathbb{F}_p$. Let $Z = \{y | \forall h \in \mathbb{F}_p, ghy = hy\} = \{y | g \in \cap_{h \in \mathbb{F}_p} Stab(hy)\}$. Z is effective. Furthermore $Y \cap Z = \emptyset$ iff $g \notin R$.

Emptyness is recognizable, hence the complement of the word problem is recognizable, hence decidable.

$\quad\square$

The converse of the previous corollary should be true. However we were not able to prove it for stupid reasons: We do now know how to prove that any group admit a strongly aperiodic subshift.

Open Problem 1. *Characterize groups admitting weakly strongly aperiodic subshifts.*

Open Problem 2. *Prove that any group admit a (weakly-?) strongly aperiodic subshift. From the proof, deduce that every group with decidable word problem admits a (weakly-?) strongly aperiodic effective subshift.*

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[1]Indeed, let $z \in Z$ and $t \in G$. Let $g \in G$ and $h \in H \setminus \{\lambda\}$. Then $(tz)_{hg} = z_{t^{-1}hg} = z_{t^{-1}htt^{-1}g} \neq z_{t^{-1}g} = (tz)_g$, hence $tz \in Z$

[2]Indeed, let $g \in G$ and $h \in H \setminus \{\lambda\}$. Then $z_g = k$ where $g \in kI$ for some $k \in H$. But $hg \in (hk)I$ hence $z_{hg} = hk \neq k = z_g$.

[3]Indeed, for $h \in H \setminus \{\lambda\}$, $(hx)_\lambda = x_{h^{-1}} \neq x_\lambda$, hence $h \notin Stab(x)$.

3 Enumeration degrees

In this section, we generalize the previous results to any finitely generated groups, whose presentation might be not recursive. We will prove in particular that if G admits a strongly aperiodic SFT, then the word problem of G is a total enumeration degree.

For this, we need to introduce enumeration degrees [FR59, Odi99].

Enumeration degrees, and enumeration reducibility, is a notion from computability theory that is quite natural in the context of presented groups and subshifts, as it captures (in computable terms) the fact that the only information we have about these objects are positive (or negative) information: In a subshift (effective or not), we usually have ways to describe patterns that do not appear, but no procedure to list patterns that appear. In a presented group, we have information about elements that correspond to the identity element of the group, but no easy way to prove that an element is different from the identity.

We are unaware of any previous use of this reduction in the context of symbolic dynamics. Note however that Aubrun and Sablik used a very similar reduction (strong enumeration reducibility) in the context of subactions [AS09].

3.1 Definitions

If A and B are two sets of numbers (or words in $\mathbb{F}_p$), we say that A is enumeration reducible to B if there exists an algorithm that produces an enumeration of A from any enumeration of B. Formally:

Definition 3.1. *A is enumeration reducible to B, written $A \leq_e B$, if there exists a computable function f that associates to each (n, i) a finite set $D_{n,i}$ s.t. $n \in A \iff \exists i, D_{n,i} \subseteq B$.*

We will first give here a few easy facts, and then examples relevant to group theory and symbolic dynamics.

Fact 3.2. *If A is recursively enumerable, then $A \leq_e B$. If A is recursively enumerable and $\overline{A} \leq_e A$ then A is computable*

(If we can enumerate $\overline{A}$ given an enumeration of A, and A is enumerable, then $\overline{A}$ is enumerable, hence computable)

Here are some examples relevant to group theory:

Fact 3.3. *(Formal version) Let $G = \langle X | R \rangle$ be a finitely generated group, with $R \subseteq \mathbb{F}_p$ and N be the normal subgroup of $\mathbb{F}_p$ generated by R. Then $N \leq_e R$. In particular, if R is finite, then N (hence the word problem over G) is recursively enumerable.*

(Informal version) From a presentation R of a group, we can list all elements that correspond to the identity element of the group (but in general we cannot list elements that are not identity of the group) In terms of reducibility, the set of all elements that correspond to the identity is the smallest possible presentation of a group.

Indeed $g \in N$ iff there exists $g_1 \ldots g_k \in \mathbb{F}_p, u_1 \ldots u_k \in R \cup R^{-1}$ so that $g = g_1 u_1 g_1^{-1} g_2 u_2 g_2^{-1} \ldots g_k u_k g_k^{-1}$. Given any enumeration of R (and as $\mathbb{F}_p$ is enumerable), we can therefore enumerate N.

Fact 3.4. *(Formal version) Let G be a finitely generated simple group, seen as a normal subgroup N of $\mathbb{F}_p$. Then $\overline{N} \leq_e N$ (the complement of the word problem is enumeration reducible to the word problem)*

In particular, if G is finitely presented, then G has a decidable word problem.

(Informal version) In a f.g. simple group, we may produce a list of elements that do not correspond to the identity element from a list of those that do.

This is well known when G is finitely presented, and can be extended as a necessary and sufficient condition [Tho80].

Proof. Fix $a \notin N$. Then by simplicity, $g \in \overline{N}$ iff a is in the normal subgroup generated by g and N iff there exists $g_1 \ldots g_k \in \mathbb{F}_p, u_1 \ldots u_k \in N \cup \{g, g^{-1}\}$ such that $a = g_1 u_1 g_1^{-1} g_2 u_2 g_2^{-1} \ldots g_k u_k g_k^{-1}$. Then with any enumeration of N, we can therefore enumerate $\overline{N}$

Formally, let $(D(g,i))_{i \in \mathbb{N}}$ be an enumeration of all finite sets $D \subseteq \mathbb{F}_p$ for which there exists $g_1 \ldots g_k \in \mathbb{F}_p, u_1 \ldots u_k \in D \cup \{g, g^{-1}\}$ such that we have $a = g_1 u_1 g_1^{-1} g_2 u_2 g_2^{-1} \ldots g_k u_k g_k^{-1}$. Then $g \in \overline{N} \iff \exists i, D(g,i) \subseteq N$. $\square$

Here are some examples relevant to symbolic dynamics or topology.

Fact 3.5. *(Formal version) Let $S = S_L$ be any closed set. Let $\mathcal{L}(S)$ be the set of words that disagree with every element of S (remark that $S = S_{\mathcal{L}(S)}$)*

Then $\mathcal{L}(S) \leq_e L$.

In particular if L is computable then $\mathcal{L}(S)$ is recursively enumerable.

(Informal version) From any description of a closed set in terms of some forbidden words, we may obtain a list of all words that do not appear (but usually not of patterns that appear). In terms of reducibility, the set of all words that do not appear is the smallest possible description of a closed set.

Subshifts are particular closed sets, so this is also true for subshifts. In particular the set of patterns that do not appear in a SFT (over $\mathbb{Z}$, or $\mathbb{F}_p$) is recursively enumerable.

Proof. Let w be any word, defined over a finite set B. For each position $g \in B$ and each letter $a \neq w_g$, consider the word $v_{g,a}$ defined only on position g, with value a, and take F_w the finite set of all such words. Then it is easy to see that S_{F_w} is exactly the set of all configurations that *agree* with w.

Hence $w \in \mathcal{L}(S)$ iff $S_{F_w} \cap S = \emptyset$. By compactness, $w \in \mathcal{L}(S)$ iff there exists a finite subset $L' \subseteq L$ such that $S_{F_w \cup L'} = \emptyset$

Thus, if $(F(n,w))_{n \in \mathbb{N}}$ is a computable enumeration of all finite sets of words so that $S_{F(n,w) \cup F_w} = \emptyset$, then $w \in \mathcal{L}(S)$ iff $\exists n, F(n,w) \subseteq L$. $\square$

<u>Fact</u> 3.6. *(Formal version) Let S be a minimal subshift of $A^{\mathbb{Z}}$. Then $\overline{\mathcal{L}(S)} \leq_e$ $\mathcal{L}(S)$. In particular, if S is effective, then $\mathcal{L}(S)$ is computable.*

(Informal version) In a minimal subshift, we may produce of list of patterns that do appear from a list of patterns that don't.

(The theorem also holds of course for subshifts over $\mathbb{F}_p$, or any group with a decidable word problem)

This result is well known in the effective case, see [Hoc09, Prop 9.6] or [BJ08, Cor 4.9].

Proof. In a minimal subshift S, a pattern p appears in S iff adding p to the list of forbidden patterns would result in an empty subshift. But by compactness, a finite part of the list would suffice to obtain an empty shift, thus providing the reduction.

Formally, for a pattern p, let $(F(n, p))_{n \in \mathbb{N}}$ be a computable enumeration of all finite sets of patterns so that $S_{F(n,p) \cup \{p\}} = \emptyset$, then $w \notin \mathcal{L}(S)$ iff $\exists n, F(n, p) \subseteq \mathcal{L}(S)$. □

Note that the result is assymmetric: This does not mean that we can produce a list of patterns that don't appear from a list of patterns that do, and it is indeed possible, using methods from [BJ10], to produce counterexamples. Intuitively, the list of forbidden patterns of a minimal subshift contain something more: We can compute (enumerate) from it the quasiperiodicity function of the minimal subshift. An exact theorem about this will be given in a subsequent paper.

3.2 Generalizations

Now we explain how this concept gives generalizations of the previous theorems.

First, we look at subsets of $A^{\mathbb{F}_p}$ that are effective given an enumeration of B. This definition is nonstandard:

Definition 3.7. *A set $S \subseteq A^{\mathbb{F}_p}$ is B-enumeration-effective if $S = S_L$ for some set of words L so that $L \leq_e B$.*

Here are a few examples:

- $\{x \in \{0, 1\}^{\mathbb{F}_p} | \forall h \in B, x_h = 1\}$ is B-enumeration effective

- $\{x \in \{0, 1\}^{\mathbb{F}_p} | \forall g \in \mathbb{F}_p, \forall h \in B, x_{gh} = x_h\}$ is B-enumeration effective

- $\{x \in \{0, 1\}^{\mathbb{F}_p} | \forall h \notin B, x_h = 1\}$ is usually not B-enumeration effective. It is B-enumeration effective iff the complement of B is enumeration reducible to B. It happens for example whenever the complement of B is enumerable, regardless of the status B.

Definition 3.8. *If $G = \mathbb{F}_p / R$ for a normal subgroup R of $\mathbb{F}_p$, we will say that $X \in A^{\mathbb{F}_p}$ is G-enumeration effective whenever X is R-enumeration effective.*

Proposition 3.9 (Analog of Prop 1.3). *Let A be an alphabet of size at least 2. Per_R is G-enumeration effective. Furthermore, if Per_R is B-enumeration effective for some set B, then $R \leq_e B$.*

This is obvious, as the set of words incompatible with Per_R is enumeration reducible to R, and R is enumeration reducible to the set of words incompatible with Per_R if A is of size at least 2.

Corollary 3.10 (Analog of Cor. 1.4). *If S_L^G is effective, then $S_L^{\mathbb{F}_p} \cap Per_R$ is G-enumeration effective.*

Proposition 3.11 (Analog of Prop. 2.3). *If $X_{\leq 1}$ is effective (in particular if $X_{\leq 1}$ is an SFT) then the complement of R is enumeration reducible to R.*

Proof. $X_{\leq 1}$ lifts up to a subshift X of $A^{\mathbb{F}_p}$ which is G-enumeration effective, i.e. the set of all words that disagrees with X is enumeration reducible to R. Write $X = S_L$ for some $L \leq_e R$.

By the proof of Prop. 2.3, there exists a uniform family w_g of words so that $g \notin R$ iff $X \cap S_{\{w_g\}} = \emptyset$.

Let $(F(n,g))_{n \in \mathbb{N}}$ be a computable enumeration of all finite sets F so that $S_{F \cup \{w_g\}} = \emptyset$.

Then $g \notin R$ iff $\exists n F(n,g) \subseteq L$, hence $\overline{R} \leq_e L \leq_e R$

$\square$

Proposition 3.12 (Analog of Cor. 2.7). *If G admits a weakly strongly aperiodic effective subshift, then the complement of R is enumeration reducible to R. (As a strongly aperiodic effective subshift is also weakly strongly aperiodic, the result is also true for groups that admits strongly aperiodic effective subshifts, or groups that admits strongly aperiodic SFT)*

Proof. From the proof of Cor. 2.7, there exists a G-enumeration effective subshift X on $\mathbb{F}_p$, and a family of effective subshifts X_g so that $X \cap X_g = \emptyset$ iff $g \notin R$. Write $X = S_L$, where $L \leq_e R$, and $X_g = S_{L_g}$.

Let $(F(n,g))_{n \in \mathbb{N}}$ be a computable enumeration of all finite sets F for which there exists $G \subseteq L_g$ so that $S_{F \cup G} = \emptyset$ (this can indeed be enumerated as L_g can be enumerated).

Then $g \notin R$ iff $\exists n F(n,g) \subseteq L$, hence $\overline{R} \leq_e L \leq_e R$

$\square$

4 On a Construction of Kari

4.1 Definitions

Kari provided a way in [Kar07] to convert a piecewise affine map into a tileset simulating it. We give here the relevant definitions. First, we introduce a formalism for Wang tiles that will be easier to deal with.

Definition 4.1. *Let G be a f.g. group with a set S of generators.*
A set of Wang tiles over G is a tuple $(C, (\phi_h)_{h \in S}, (\psi_h)_{h \in S})$ where, for each h, ϕ_h, ψ_h are maps from C to some finite set.
The subshift generated by C is

$$
\begin{aligned}
X_C &= \{x \in C^G | \forall g \in G, \forall h \in S, \phi_h(x_g) = \psi(x_{gh^{-1}})\} \\
&= \{x \in C^G | \forall g \in G, \forall h \in S, \phi_h((gx)_e) = \psi((gx)_{h^{-1}})\}
\end{aligned}
$$

(The last definition proves it is indeed a subshift, and in fact a subshift of finite type). If G has one generator (in particular if $G = \mathbb{Z} = \langle 1 \rangle$), we will write ϕ and ψ instead of ϕ_1 and ψ_1.

Definition 4.2. *Let $cont : \{0,1\}^{\mathbb{Z}} \to [0,1]$ defined by $cont(x) = \limsup_n \frac{\sum_{i \in [-n,n]} x_i}{2n+1}$ and $disc : [0,1] \to \{0,1\}^{\mathbb{Z}}$ defined by $disc(y)_n = \lfloor (n+1)x \rfloor - \lfloor nx \rfloor$.*
Remark that $cont(disc)(y) = y$.

Theorem 1 ([Kar07])**.** *Let a, b be rational numbers and $f(x) = ax + b$.*
Then there exists a set of Wang tiles (C, ϕ, ψ) over $\mathbb{Z}$ and two maps out, in from C to $\{0,1\}$ so that the two following properties hold

- *For any configuration x of X_C, $f(cont(in(x)) = cont(out(x))$*

- *For any $y \in [0,1]$ so that $f(y) \in [0,1]$, there exists a configuration x of C_G so that $in(x) = disc(y)$ and $out(x) = disc(f(y))$*

C is usually seen as a set of Wang tiles over $\mathbb{Z}^2$ rather than $\mathbb{Z}$ but this formalism is better for our purpose.
Two examples are given in Figure 1.

Corollary 4.3. *Let $f_1 \ldots f_k$ be a finite family of affine maps with rational coordinates.*
Then there exists a set of Wang tiles (C, ϕ, ψ) over $\mathbb{Z}$ and maps in et $(out_i)_{1 \leq i \leq k}$ from C to $\{0,1\}$ so that the two following properties hold

- *For any configuration x of X_C, $f_i(cont(in(x)) = cont(out_i(x))$*

- *For any $y \in [0,1]$ so that $f_i(y) \in [0,1]$ for all i, there exists a configuration x of C_G so that $in(x) = disc(y)$ and $out_i(x) = disc(f_i(y))$*
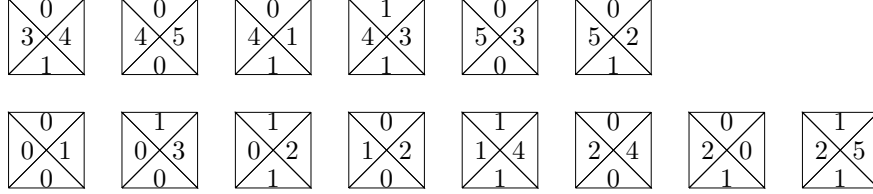
C_1 :
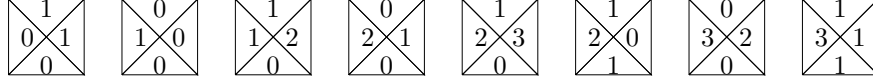
C_2 :

Figure 1: Two set of Wang tiles corresponding respectively to the maps $f(x) = (2x - 1)/3$ and $f(x) = (4x + 1)/3$. The colors on each tile $c \in C$ on east,west,north,south represent respectively $\phi(c), \psi(c), in(c), out(c)$.

Proof. let (C^i, ϕ^i, ψ^i) be the set of Wang tiles over $\mathbb{Z}$ corresponding to f_i, with maps out^i and in^i.

Let $C = \{y \in \prod C^i | \exists x \in \{0,1\}, \forall i, in^i(y^i) = x\}$. Let p^i denote the projection from C to C^i and define

$$\phi = \prod (\phi^i \circ p^i)$$

$$\psi = \prod (\psi^i \circ p^i)$$

$$in = in^1 \circ p^1 = in^2 \circ p^2 = \cdots = in^k \circ p^k$$

$$out_i = out^i \circ p^i$$

It is clear that C satisfies the desired properties. $\qquad\square$

Corollary 4.4. *Theorem 3 still holds when f is a piecewise affine rational homeomorphism from $[0,1]_{/0\sim1}$. As a consequence, the previous corollary also holds for a finite familiy of piecewise affine rational homeomorphisms.*

Let's define precisely what we mean by a piecewise affine rational homeomorphism from $[0,1]_{/0\sim1}$ to $[0,1]_{/0\sim1}$.

We first define a relation $\equiv$: $x \equiv y$ if $x = y$ or $\{x,y\} = \{0,1\}$. Then a piecewise affine homeomorphism from $[0,1]_{/0\sim1}$ to $[0,1]_{/0\sim1}$ is given by a finite family $[p_i, p_{i+1}]_{i<n}$ of n intervals with rational coordinates, with $p_0 = 0$ and $p_n = 1$, and a family of affine maps $f_i(x) = a_i x + b$ so that

- f_i and f_{i+1} agrees on their common boundary:

$$\forall i \in \mathbb{Z}/n\mathbb{Z}, f_i(p_{i+1}) \sim f_{(i+1)}(p_{i+1})$$

- $f = \cup_i f_i$ is injective: $f(x) \sim f(y) \implies x \sim y$

These properties imply that f is invertible, and its inverse is still piecewise affine

Proof. We give a proof of this easy result to prepare for another proof later on.

Let $\mathcal{F}$ be the class of relations on $[0,1] \times [0,1]$ for which the theorem is true. $\mathcal{F}$ contains all rational affine maps.

It is clear from the formalism that if f and g are in $\mathcal{F}$, then $f \cup g \in \mathcal{F}$, by taking $(C \cup C', \phi \cup \phi', \psi \cup \psi)$ once the range of ϕ and ϕ' have been made disjoint. In the same way, we may prove $f \circ g \in \mathcal{F}$ and $f; g \in \mathcal{F}$, where $f \circ g(x) = f(g(x))$ and $f; g(x) = g(f(x))$.

Let $i \in \{0 \dots n-1\}$. The function f_i is the composition of 5 functions in $\mathcal{F}$:

- $g_1 = id \cup (x \to x - 1) \cup (x \to x + 1)$ (g_1 is exactly the relation $\equiv$)

- $g_2 = (x \to p_{i+1} - x) \circ (x \to p_{i+1} - x)$ ($g_2(x) = x$, defined on $[0, p_{i+1}]$)

- $g_3 = (x \to x + p_i) \circ (x \to x - p_i)$ ($g_2(x) = x$, defined on $[p_i, 1]$)

- $g_4 = x \to a_i x + b_i$

- $g_5 = g_1$

Hence $f_i \in \mathcal{F}$, and $f \in \mathcal{F}$.

$\square$

Kari first used this construction [Kar96] to obtain a aperiodic SFT of $\mathbb{Z}^2$: Start with a piecewise rational homeomorphism f with no periodic points (f should be indeed over $[0, 1]_{/0 \sim 1}$ and not over $[0, 1]$ for this to work, as any continuous map from $[0, 1]$ to $[0, 1]$ has a fixed point by the intermediate value theorem). For example, take

$$f(x) = \begin{cases} (2x - 1)/3 & \text{if} \quad 1/2 \leq x \leq 1 \\ (4x + 1)/3 & \text{if} \quad 0 \leq x \leq 1/2 \end{cases}$$

Take the set of Wang tiles over $\mathbb{Z}$ given by the theorem, and consider it as a set of Wang tiles over $\mathbb{Z} \times \mathbb{Z}$ by mapping *in* and *out* to $\phi_{(0,1)}$ and $\psi_{(0,1)}$. For the specific function f above, we obtain the set of 22 tiles presented in Fig. 1 (where horizontal colors corresponding to different sets are supposed to be distinct). Then it is easy to see that this indeed gives a SFT over $\mathbb{Z}^2$ with no periodic points. The same construction can be refined [Kar96] to obtain aperiodic tilesets with fewer tiles, but that is not our purpose here.

The important point is that this construction may be easily generalized: There is no need to tile $\mathbb{Z} \times \mathbb{Z}$, we may use the exact same idea to obtain a SFT over $\mathbb{Z} \times G$ for some groups G.

Definition 4.5. *A f.g. group G is PA-recognizable iff there exists a finite set $\mathcal{F}$ of piecewise affine rational homeomorphisms of $[0,1]_{/0\sim 1}$ so that*

- *(A) The group generated by the homeomorphisms is isomorphic to G*

- *(B) For any $t \in [0,1]_{/0\sim 1}$, if $gf(t) = f(t)$ for all f, then $g = e$*

Note that by the property (A) every PA-recognizable group has decidable word problem.

Theorem 2. *If G is PA-recognizable and infinite, there exists a SFT over $\mathbb{Z} \times G$ which is strongly aperiodic.*

Proof. Let S be a set of generators for G. Let $(f_h)_{h \in S}$ be generators for G as a group of piecewise affine maps. And consider the set of Wang tiles (C, ϕ, ψ) and maps in, out_h, corresponding to them by Corollaries 4.3 and 4.4.

Now we look at the set of Wang tiles $(C', (\phi_i), (\psi_i))$ over $\mathbb{Z} \times G$ (with generators 1 and S) defined by $C' = C$ and:

$$\phi_1 = \phi$$

$$\psi_1 = \psi$$

$$\psi_h = in$$

$$\phi_h = out_h$$

We will prove that (a) $X_{C'}$ mimics the behaviour of the piecewise affine maps and (b) gives a strongly aperiodic SFT.

For an element $x \in X_{C'}$, $g \in G$, let $x_g : \mathbb{Z} \to C$ where $x_g(n) = x_{(n,g)}$. Now let $z_g = cont(in(x_g))$. Note that $x_g \in X_C$.

Note that by definition, for any h,

$$
\begin{aligned}
f_h(z_g) &= f_h(cont(in(x_g))) \\
&= cont(out_h(x_g)) \\
&= cont(\phi_h(x_g)) \\
&= cont(\psi_h(x_{gh^{-1}})) \\
&= cont(in(x_{gh^{-1}})) \\
&= z_{gh^{-1}}
\end{aligned}
$$

This implies that for any $g = g_1 \ldots g_k$, we have $z_{(g_1 \ldots g_k)^{-1}} = f_{g_1}(f_{g_2} \ldots f_{g_k}(z_\lambda) \ldots)$. That is, for all $g \in G$, $z_{g^{-1}} = f_g(z_\lambda)$.

Now, by the second part of theorem 3, for any collection $(z_g)_{g \in G}$ that satisfy $z_{g^{-1}} = f_g(z_\lambda)$, there exists a corresponding configuration in $X_{C'}$. This proves that the SFT is nonempty, by starting e.g. with $z_{g^{-1}} = f_g(0)$.

Now suppose that $x \in X_{C'}$ is weakly periodic, that is $Stab(x)$ is infinite. Let $(n, h) \in \mathbb{Z} \times G$ be in the stabilizer of x, that is for all $(m, g) \in \mathbb{Z} \times G$, we have $((n, h)x)_{(m,g)} = x_{(m,g)}$. This implies that $z_{gh^{-1}} = z_g$ for all g. Hence, $f_h f_g(z_\lambda) = f_g(z_\lambda)$ for all g. By PA-recognizability, this implies $h = \lambda$.

It remains to prove that $n = 0$. If $n \neq 0$, this means that for each g, the word x_g is periodic of period n. There are finitely many periodic words of length n, which means that z_g will take only finitely many values: $z_g \in Z$ for some finite set Z, which is closed under all maps f_h.

Then each element of G acts as a permutation on Z. Furthermore, by PA-recognizability, any element of G that acts like the identity on Z must be equal to the identity. This implies that any element of G is identified by the permutation of Z it induces, hence that G is finite. $\square$

4.2 Applications

Proposition 4.6. $\mathbb{Z}$ *is PA-recognizable. Hence* $\mathbb{Z} \times \mathbb{Z}$ *admits a strongly aperiodic subshift of finite type*

Proof. The function f seen previously provides a proof.

$$f(x) = \begin{cases} (2x - 1)/3 & \text{if} \quad 1/2 \leq x \leq 1 \\ (4x + 1)/3 & \text{if} \quad 0 \leq x \leq 1/2 \end{cases}$$

To understand better what f does, we will look at $f' = hfh^{-1}$ where $h(x) = x + 1$. Then it is easy to see that

$$f'(x) = \begin{cases} 2x/3 & \text{if} \quad 3/2 \leq x \leq 2 \\ 4x/3 & \text{if} \quad 1 \leq x \leq 3/2 \end{cases}$$

from which it is easy to see that the orbit of f is infinite, (hence the group generated by f is isomorphic to $\mathbb{Z}$), and that if $f^n(t) = t$ for some n and some t, then $n = 0$ (hence property (B)). Therefore G is PA-recognizable. $\square$

Proposition 4.7. *Thompson group T is PA-recognizable. Hence* $\mathbb{Z} \times T$ *admits a strongly aperiodic subshift of finite type*

Proof. T is the quintessential PA-recognizable group: It is formally the subgroup of all piecewise affine maps of $[0, 1]_{/0 \sim 1}$ where each affine map has dyadic coordinates and positive slope.

T is indeed finitely generated, more precisely it is generated by the three following functions [CFP96]:

$$a(x) = \begin{cases} x/2 & 0 \leq x \leq 1/2 \\ x - 1/4 & 1/2 \leq x \leq 3/4 \\ 2x - 1 & 3/4 \leq x \leq 1 \end{cases} \quad b(x) = \begin{cases} x & 0 \leq x \leq 1/2 \\ x/2 + 1/4 & 1/2 \leq x \leq 3/4 \\ x - 1/8 & 3/4 \leq x \leq 7/8 \\ 2x - 1 & 7/8 \leq x \leq 1 \end{cases}$$

$$c(x) = \begin{cases} x/2 + 3/4 & 0 \leq x \leq 1/2 \\ 2x - 1 & 1/2 \leq x \leq 3/4 \\ x - 1/4 & 3/4 \leq x \leq 1 \end{cases}$$

From the definition of T, it is easy to see that the orbit of any $z \in [0, 1]$ is dense, hence property (B) is true, and T is PA-recognizable. $\square$

Note that it is not clear if Thompson group F is PA-recognizable: F is the subgroup of T generated by a and b, and fixes 0: As a consequence, this particular representation does not satisfy property (B). Whether another representation of Thompson group F exists with this property is open.

Proposition 4.8. $PSL_2(\mathbb{Z})$ *is PA-recognizable. Hence* $\mathbb{Z} \times PSL_2(\mathbb{Z})$ *admits a strongly aperiodic subshift of finite type.*

Proof. $PSL_2(\mathbb{Z})$ is the subgroup of T generated by a and c, see for example [Fos11]. To see that this representation satisfies property (B), remark that this action is conjugated by the Minkowski question mark symbol (ibid.) to the action of $PSL_2(\mathbb{Z})$ over the projective line $\mathbb{R} \cup \{\infty\}$. From this point of view, it is clear that any element of $PSL_2(\mathbb{Z})$ different from the identity fixes at most two points. Hence if $gf(t) = f(t)$ for all t, then $g = \lambda$.

For this particular group, it is actually easy to work out all details and produce a concrete aperiodic set of Wang tiles, represented in Fig 2. It is obtained by taking $d = a$ and $e = ac$ as generators (rather than a and c) and looking at them as acting on $[0, 2]_{0 \sim 2}$ (rather than $[0, 1]_{0 \sim 1}$) by the formulas:

$$d(x) = \begin{cases} x/2 & 0 \le x \le 1 \\ x - 1/2 & 1 \le x \le 3/2 \\ 2x - 2 & 3/2 \le x \le 2 \end{cases} \quad e(x) = \begin{cases} x + 1 & 0 \le x \le 1 \\ x - 1 & 1 \le x \le 2 \end{cases}$$

(Of course, such details may also be provided for Thomspon group T. However the presence of the generator b produced an set of tiles too large to be depicted here.) $\square$

4.3 Generalizations

The construction of Kari works for more than piecewise affine homeomorphisms of $[0, 1]$. It works for any partial piecewise affine map from $[0, 1]^d$ to its image.

Theorem 3 ([Kar07])**.** *Let* $A \in M_{m \times n}(\mathbb{Q})$ *be a (possibly non square) matrix with rational coefficients,* $b \in \mathbb{Q}^m$ *a rational vector and* $f(x) = Ax + b$

Then there exists a set of Wang tiles (C, ϕ, ψ) *over* $\mathbb{Z}$ *(generated by 1) and two maps* out, in *from* C *to* $\{0, 1\}^m$ *and* $\{0, 1\}^n$ *so that the two following properties hold*

- *For any configuration x of X_C,* $f(cont_n(in(x)) = cont_m(out(x))$

- *For any* $y \in [0, 1]^n$ *so that* $f(y) \in [0, 1]^m$*, there exists a configuration x of C_G so that* $in(x) = disc_n(y)$ *and* $out(x) = disc_m(f(y))$

where $disc_i$ *and* $cont_i$ *are the natural i-dimensional analogues of* $disc$ *and* $cont$
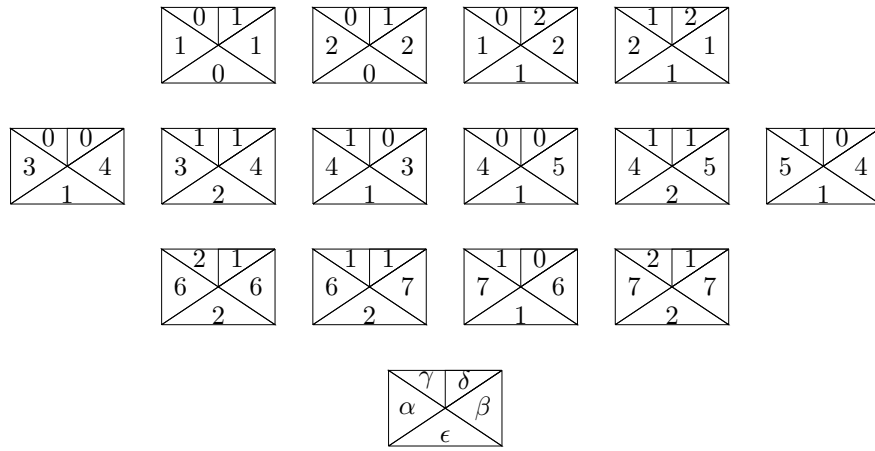
Figure 2: A strongly aperiodic set of 14 Wang tiles over $\mathbb{Z} \times PSL_2(\mathbb{Z})$, where $PSL_2(\mathbb{Z})$ is generated by $d = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The rules are as follows: Let x be the tile in position (n, g). Then the tile y in position $(n+1, g)$ must satisfy $y_\alpha = x_\beta$, the tile y in position (n, gd) must satisfy $y_\epsilon = x_\gamma$, the tile y in position (n, ge) must satisfy $y_\epsilon = x_\delta$.

Now we will be able to prove a theorem similar to the previous one for a larger class of maps (hence a larger class of groups). There are three directions in which we can go:

- Go to higher dimensions

- Look at piecewise affine maps defined on compact subsets of $\mathbb{R}^d$ different from $[0,1]^d$.

- Consider other identifications than $0 \sim 1$

In the following we will not use the full possible generalisation, and will not identify any points in our sets. This will be sufficient for the applications and already relatively painful to define. However, this means that the next definition will not encompass PA-recognizable groups.

Definition 4.9. *Let $\mathcal{F} = \{f_i : B_i \mapsto B_i', i = 1 \ldots k\}$ be a finite set of piecewise affine rational homeomorphisms, where each B_i and B_i' is a finite union of bounded rational polytopes of $\mathbb{R}^n$.*

Let $S_{\mathcal{F}}$ be the closure of the set f_i and f_i^{-1} under composition. Each element of $S_{\mathcal{F}}$ is a piecewise affine homeomorphism, whose domain is the union of finitely many bounded rational polytopes, and may possibly be empty.

Let $T_{\mathcal{F}}$ be the common domain of all functions in $S_{\mathcal{F}}$.

Then the group $G_{\mathcal{F}}$ *generated by $\mathcal{F}$ is the group $\{f_{|T_{\mathcal{F}}}, f \in S_{\mathcal{F}}\}$.*

Definition 4.10. *A f.g. group G is PA'-recognizable iff there exists a finite set $\mathcal{F}$ of piecewise affine rational homeomorphisms so that*

- *(A) G is isomorphic to $G_{\mathcal{F}}$.*

- *(B) For any $t \in T_{\mathcal{F}}$, if $gf(t) = f(t)$ for all f, then $g = \lambda$*

Note that $T_{\mathcal{F}}$ might not be computable in general. In particular, it is not clear that any PA'-recognizable has decidable word problem.

Theorem 4. *If G is PA'-recognizable, then the complement of the word problem on G is recognizable. In particular, if G is recursively presented, the word problem on G is decidable*

Proof. We assume that $G \neq \{\lambda\}$, hence $T_{\mathcal{F}} \neq \emptyset$.

Let g be an element of G, given by composition of some piecewise affine maps. Let $D = \{t | \forall f \in S_{\mathcal{F}}, f(t) \text{ is defined and } g(f(t)) = f(t)\}$

Note that $D \subseteq T_{\mathcal{F}}$. Furthermore, $g \neq \lambda$ iff $D = \emptyset$ by property (B).

This gives a semi algorithm to decide if $g \neq \lambda$. $\square$

Theorem 5. *If G is PA'-recognizable, $\mathbb{Z} \times G$ admits a strongly aperiodic subshift of finite type.*

Proof. Same proof as before. $\square$

Here a few applications:

Proposition 4.11. $\mathbb{Z}$ *is PA'-recognizable. Hence* $\mathbb{Z} \times \mathbb{Z}$ *admits a strongly aperiodic subshift of finite type.*

Proof. Let $A = \{(x, y) \in [-1, 1]^2, |x| + |y| \geq 1\}$. A is the union of four bounded polytopes.

Let

$$f: \quad \begin{array}{ccc} A & \to & f(A) \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

And let $\mathcal{F} = \{f\}$. f is clearly an homeomorphism. Note that f is a rotation of angle $\arccos 3/5$.

Now it is easy to see that $T_{\{f\}} = S_1 = \{(x, y) | x^2 + y^2 = 1\}$, and that $G_{\{f\}}$ is isomorphic to $\mathbb{Z}$. Furthermore, it is also clear that the orbit of any point of $T_{\{f\}}$ is dense in $T_{\{f\}}$, which implies property (B). Hence $\mathbb{Z}$ is PA'-recognizable. $\square$

Proposition 4.12. *Any finitely generated subgroup G of rational matrices of a compact matrix group is PA'-recognizable. Hence $\mathbb{Z} \times G$ admits a strongly aperiodic subshift of finite type.*

Proof. We assume familiary with representation theory of linear compact groups, see e.g. [OV90, Chap 3.4]. Let G be such a group, and let $M_1 \ldots M_n$ be the matrices of size $k \times k$ that generate G. Using elementary linear algebra we may suppose there exists a rational vector $y \in \mathbb{R}^k$ so that $gy = y$ implies g is the identity, and Gy spans $\mathbb{R}^k$.

Now, as G is a subgroup of a compact group, we can define a scalar product so that all matrices of G are unitary. Let $\mathbb{R}^k = V^1 \oplus V^2 \cdots \oplus V^p$ be a decomposition of $\mathbb{R}^k$ into orthogonal (for this scalar product) irreducible G-invariant vector spaces, that is $GV^i = V^i$ and no proper nonzero subspace of V^i is G-invariant. This is possible as G is a subgroup of a compact group hence completely reducible. Note that the vector spaces V^i might not have rational bases.

Let P^i be the orthogonal projection onto V^i. For a vector x, let $x^i = P^i x$, so that $x = \sum_i x^i$. For a matrix $g \in G$, let $g^i : V^i \to V^i$ be the restriction of g to V^i, so that $gx = \sum_i g^i x^i$.

Recall there is y so that $gy = y$ for $g \in G$ implies that $g = 1$. As Gy spans $\mathbb{R}^k$, $y^i \neq 0$ for all i.

Let $i \in \{1 \ldots p\}$. Let W^i be the topological closure of Gy^i. As y^i is nonzero and V^i is G-invariant, $W^i \subseteq V^i$ and is faraway from zero. That is, there exists constants $r^i, R^i > 0$ so that for all $y \in W^i$, $|y|_1 > r^i$ and $|y|_1 < R^i$.

Now let

$$T = \{y | \forall i, |P^i y|_1 > r^i \text{ and } |P^i y|_1 < R^i\}$$

and

$$T_0 = \{y | \forall i, |P^i y|_1 > r^i/2 \text{ and } |P^i y|_1 < 2R^i\}$$

Note that T is a polytope with real coordinates. Let T' be an approximation of T as a polytope with rational coordinates, so that $T \subseteq T' \subseteq T_0$.

Now define the maps f_i as restrictions of M_i from T' to $M_i T'$. Let $\mathcal{F}$ be the corresponding set of maps.

We cannot describe $T_\mathcal{F}$ exactly, but it is clear that it contains y, as T contains the G-orbit of y. As a consequence, $G_\mathcal{F}$ is isomorphic to G.

Now we prove property (B). Start from $t \in T_\mathcal{F}$ and $g \in G_\mathcal{F}$ so that $gf(t) = f(t)$ for all f.

Let $i \in \{1, \ldots, p\}$ and let $t^i = P^i t$ so that $t = \sum_i t^i$. As $t \in T_\mathcal{F} \subseteq T' \subset T_0$, we have $t^i \neq 0$. As a consequence, the orbit of $G t^i$ on W_i spans a nonzero G-invariant subspace of V_i, which is V_i by irreducibility. Now, as $gf(t) = f(t)$ for all f, we conclude that g is the identity on the orbit of $G t^i$, hence g is the identity on V^i. As this is true for all i, g is the identity matrix. $\square$

Corollary 4.13. *The free group $\mathbb{F}_2$ is PA'-recognizable. Every finite group is PA'-recognizable.*

Proposition 4.14. *Thompson's group V is PA'-recognizable.*

Proof. V is usually given [CFP96] as the generalization of T to discontinuous maps. However, our maps in the definition need to be continuous, so we will see V as acting on the "middle thirds" Cantor set (As a side note, V is therefore isomorphic to the group of all revertible generalized one-sided shifts [Moo91]).

Let
$$C_3 = \left\{ \sum_{i \geq 1} \frac{\alpha_i}{3^i}, \alpha \in \{0,2\}^{\mathbb{N}^+} \right\}$$

Let a, b, c, π_0 defined on C_3 by:

$$a(x) = \left\{ \begin{array}{ll} x/3 & 0 \leq x \leq 1/3 \\ x - 4/9 & 2/3 \leq x \leq 7/9 \\ 3x - 2 & 8/9 \leq x \leq 1 \end{array} \right. \quad b(x) = \left\{ \begin{array}{ll} x & 0 \leq x \leq 1/3 \\ x/3 + 4/9 & 2/3 \leq x \leq 7/9 \\ x - 4/27 & 8/9 \leq x \leq 25/27 \\ 3x - 2 & 26/27 \leq x \leq 1 \end{array} \right.$$

$$c(x) = \left\{ \begin{array}{ll} x/3 + 8/9 & 0 \leq x \leq 1/3 \\ 3x - 2 & 2/3 \leq x \leq 7/9 \\ x - 2/9 & 8/9 \leq x \leq 1 \end{array} \right. \quad \pi_0(x) = \left\{ \begin{array}{ll} x/3 + 2/3 & 0 \leq x \leq 1/3 \\ 3x - 2 & 2/3 \leq x \leq 7/9 \\ x & 8/9 \leq x \leq 1 \end{array} \right.$$

Now our definition does not permit to define a, b, c, π_0 on C_3, as the domain and range of each map should be a finite union of intervals with rational coordinates. So we will define them by the above formulas, but for $x \in [0,1]$ rather than $x \in C_3$. Note that they are already homeomorphisms onto their image.

Let $\mathcal{F} = \{a, b, c, \pi_0\}$. We claim that $T_\mathcal{F} = C_3$, which will prove that $G_\mathcal{F}$ is indeed isomorphic to V. As before, any orbit is dense, from which property (B) ensues and V will be PA'-recognizable.

It remains to prove that $T_\mathcal{F} = C_3$. Note that clearly $C_3 \subseteq T_\mathcal{F}$.

First note that

- $Dom(a) = [0, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

- $Range(a) = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 1]$

Which implies that $T_{\mathcal{F}} \subseteq [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

Now let $x \in T_{\mathcal{F}}$.

- If $0 \leq x \leq 1/9$, then $3x \in T_{\mathcal{F}}$ (apply a^{-1})

- if $2/9 \leq x \leq 1/3$, then $3x \in T_{\mathcal{F}}$ (apply a^{-1}, then c^{-1} then a)

- if $2/3 \leq x \leq 7/9$, then $3x - 2 \in T_{\mathcal{F}}$ (apply c)

- If $8/9 \leq x \leq 1$, then $3x - 2 \in T_{\mathcal{F}}$ (apply a)

This proves inductively that $x \in C_3$. $\qquad\square$

Open Problems

This is only one way of generalizing Kari's construction. There are many other ways to generalize it, one of which providing a (weakly) aperiodic SFT on the Baumslag Solitar group, see [AK13].

Here is an interesting open question: The construction uses representations of reals as words in $\{0,1\}^{\mathbb{Z}}$, can we use a representation in $\{0,1\}^{H}$, for some other group H ? This would possibly allow to prove that $H \times G$ has a strongly aperiodic SFT for G PA-recognizable.

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