Guarding Vertices versus Guarding Edges in a Simple Polygon

Olivier Devillers, Naji Mouawad

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Olivier Devillers[†] **Naji Mouawad**[‡]

Abstract: Let $\mathcal{P}$ be a simple polygon, V its set of vertices. A *minimal vertex cover* $\mathcal{C}$ of $\mathcal{P}$ is a minimal subset of V which covers V. The *extended cover* of $\mathcal{P}$ given $\mathcal{C}$ is the maximal subset of the *boundary of* $\mathcal{P}$ covered by $\mathcal{C}$. Let $\epsilon\mathcal{P}$ denotes the extended cover of $\mathcal{P}$ given $\mathcal{C}$, and $\bar{\epsilon}\mathcal{P}$ the complement of $\epsilon\mathcal{P}$ with respect to $\delta\mathcal{P}$. Denote by μ the cardinality of $\bar{\epsilon}\mathcal{P}$. In this paper we establish lower and upper bounds on μ as a function of n the cardinality of the edge set of $\mathcal{P}$ and k the cardinality of the covering set. In the restricted case where $k = 2$ we prove the bounds to be tight.

1 Introduction

Let $\mathcal{P}$ be a simple polygon, V its corresponding set of vertices and E its set of edges. Let $\delta\mathcal{P}$ denotes the *boundary* of $\mathcal{P}$. Two points of $\mathcal{P}$ are *visible* if the line segment joining them lies within the interior of $\mathcal{P}$. The 'interior' is assumed to include the boundary of $\mathcal{P}$. A subset $\mathcal{S}$ of $\mathcal{P}$ *covers* $\mathcal{P}$ if every point of $\mathcal{P}$ is visible from some point of $\mathcal{S}$. Chvatal [1] established a tight bound on the size of a *minimal* cover of $\mathcal{P}$. He showed that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient and sometimes necessary to cover a simple polygon. Furthermore the elements of the minimal cover can be restricted to the vertices of $\mathcal{P}$. The elements of the covering set $\mathcal{S}$ will be called *guards*. The original work of Chvatal was generalized in a number of directions which are beautifully presented in O'Rourke's monograph [5].

A related concept to that of polygon covering restricts the set that must be covered to that of the vertices of $\mathcal{P}$. A simple polygon is *vertex covered* by a subset $\mathcal{C}$ of $\mathcal{P}$ if every vertex of $\mathcal{P}$ is visible from some point of $\mathcal{C}$.

Finding a minimal cover for $\mathcal{P}$ is $\mathcal{NP}$-hard (see [4] [2]) even when the elements of the covering set are restricted to the vertices of $\mathcal{P}$. Finding a minimal vertex cover when the guards are restricted to the vertices of $\mathcal{P}$ is feasible in polynomial time by a simple greedy algorithm. A natural question to ask is whether the minimal vertex cover can act as a 'good approximation' of the minimal cover of $\mathcal{P}$.

In this paper we show that such is not the case. A simple polygon which is vertex covered by a set $\mathcal{C}$ may be 'poorly covered' by that same set $\mathcal{C}$. If n is the cardinality of the edge-set of $\mathcal{P}$ and k that of the covering set, then the number of distinct hidden 'portions' of $\delta\mathcal{P}$ is of the order of $O(kn)$. In order to properly formalize this idea we need to make precise the measurement criterion upon which this result is based. This is done in the following section . Section 3 establishes exact lower and upper bounds for the general case where the cardinality of the cover set is k. Section 4 establishes an exact tight bound for the restricted case where $k = 2$. Section 5 concludes the paper.

2 Notations and Definitions

Let $\mathcal{C}$ be a vertex cover of $\mathcal{P}$. In what follows the guards are restricted to the vertices of $\mathcal{P}$. The *extended Cover* of $\mathcal{P}$ given $\mathcal{C}$ is the *maximal* subset of the boundary of $\mathcal{P}$ which is covered by $\mathcal{C}$. Intuitively, the extended cover is a set of segments of the edges of $\mathcal{P}$ which is visible from $\mathcal{C}$. Let $\epsilon\mathcal{P}$ denotes the extended cover of $\mathcal{P}$ given $\mathcal{C}$. Clearly the polygon is covered by $\mathcal{C}$ if and only if $\epsilon\mathcal{P} = \delta\mathcal{P}$. In such a case the extended cover is said to be *complete*.

With respect to the extended cover, every edge of $\mathcal{P}$ may be sub-divided into *covered segments* and *hidden segments*. Covered segments are those which belong to the extended cover, while hidden segments are those which are invisible to all guards of the cover. Let $\bar{\epsilon}\mathcal{P}$ be the complementary set of $\epsilon\mathcal{P}$ with respect to δP. In other words $\bar{\epsilon}\mathcal{P}$ is the set of hidden segments of $\delta\mathcal{P}$ with respect to $\mathcal{C}$. Let $\mu = |\bar{\epsilon}\mathcal{P}|$. Clearly μ is finite since every hidden segment is determined by a pair of vertices.

We establish exact bounds on μ as a function of n and k. We show that $\frac{(n-k)(k-1)}{4} \leq \mu \leq \frac{(n-k+1)k(k-1)}{4k-2}$. In the specific case where $k = 2$ we prove that $\mu = \frac{2n-2}{3}$. Based on the 'μ-measurement' criterion one can see that there is a sizable difference between covering the entire polygon and covering its vertices.

In what follows, we say that a simple polygon $\mathcal{P}$ is k-vertex guardable if any minimal vertex cover of $\mathcal{P}$ has cardinality k.

Let $\mathcal{C}$ be a minimal vertex cover of $\mathcal{P}$, c a guard of $\mathcal{C}$.

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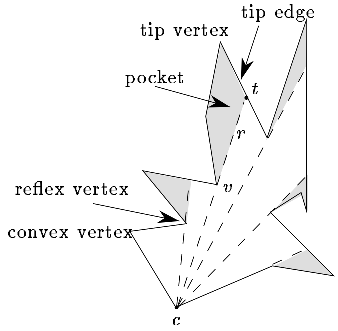

Figure 1: Pockets in the visibility polygon of g

By assumption all elements of $\mathcal{C}$ lie on vertices of $\mathcal{P}$. A vertex v is an *anchor* vertex if it hosts a guard c of $\mathcal{C}$. Alternatively we say that c is *positioned* at v.

The subset of $\mathcal{P}$ covered by c is called the *visibility polygon* of c with respect to $\mathcal{P}$ and is denoted by $vp(c|\mathcal{P})$ [3]. Let v be a reflex vertex in $vp(c|\mathcal{P})$, r the ray emanating from c and passing through v. The reflex vertex v is the *support* vertex of the ray r. Let t be the point of intersection of r and $\delta\mathcal{P}$ which is visible from c. The point t is called the *tip* of the ray r. Let T be the subset of $\delta\mathcal{P}$ delimited by v and t which does not include the anchor of c. Clearly T is hidden from c. As such it forms a *pocket* and the line segment (v, t) is called the *window* of T. The edge supporting the tip is called the *tip edge* of T and the vertex of the tip edge lying within T is called the *tip vertex* of T. In case $v = t$, the pocket is degenerate and consists of the singleton $\{v\}$.

Figure 1 illustrates many of these concepts.

Recall that no point of a hidden segment is visible from any guard of $\mathcal{C}$. All hidden segments are open intervals along the boundary of $\mathcal{P}$ and their limit-points are called *transition* points as they mark a transition from a hidden segment to a visible one. It should be clear that every transition point is the tip of a window defined by an anchor and a corresponding support vertex.

3 k guards

Lemma 1 *Let $\mathcal{P}$ be a simple polygon whose edge-set has cardinality n and which is k-vertex guardable. The cardinality μ of $\bar{\epsilon}\mathcal{P}$ is bounded above by $\frac{k(k-1)(n-k+1)}{4k-2}$.*

Proof: A simple geometric observation shows that every pair of distinct hidden segments is disjoint. Thus the number of transition points equals 2μ. Let t be a transition point, and let T be the pocket bounded by t. Because t is a transition point between a hidden and a visible segment, no guard of $\mathcal{C}$ may be placed at the tip or support vertices the pocket T. A pocket together with its window forms a simple polygon. As such it contains at least one convex vertex of $\mathcal{P}$.

Assume all convex vertices labeled 'unvisited' and let c_m be the element of $\mathcal{C}$ which determines the minimum number of transition points. Let such minimum be t_m. Clearly t_m is less than or equal to the average number of transition points per guard of $\mathcal{C}$. Thus, $t_m \leq \frac{2\mu}{k}$. There are t_m corresponding pockets within $vp(c_m|\mathcal{P})$. Each pocket contains at least one convex vertex. We mark each of these convex vertices as 'visited'. If there are more than two convex vertices in a pocket one is randomly marked 'visited'.

Let c' be an element of $\mathcal{C}$. Denote by t' the number of transition points defined by c' and by q' the number of already visited vertices belonging to $vp(c'|\mathcal{P})$. Each of the transition points determines a tip vertex (hidden from c') and a reflex vertex (visible from c'). These reflex vertices are distinct from the visited vertices which are all convex. Clearly, none of the remaining $k-1$ elements of $\mathcal{C}$ may be anchored at either the visible reflex vertex or the tip vertex. Thus $n \geq 2t' + q' + k - 1$.

The average value of $2t' + q' + k - 1$ when c' is any element of $\mathcal{C}$ different from c_m can be easily computed :

$$\begin{aligned}
&\sum_{c' \neq c_m} (2t' + q' + k - 1) \\
&\quad = 2\sum_{i\neq m} t_i + \sum_{i \neq m} q_i + (k-1)^2 \\
&\quad = 2(2\mu - t_m) + t_m + (k-1)^2 \\
&\quad = 4\mu - t_m + (k-1)^2
\end{aligned}$$

Since t_m is bounded by $2\mu/k$ and since the maximal value of the last quantity for at least one guard is greater than the average value,

$$n \geq \frac{4\mu}{k-1} - \frac{2\mu}{k(k-1)} + k - 1$$

. The lemma follows □

Lemma 2 *There exist an infinite family of polygons for which μ is greater than or equal to $\frac{(k-1)(n-k)}{4}$.*

Proof: Figure 2 exhibits a polygon achieving the bound. The polygon is obtained by expanding a convex k-gone in a specific manner. The elements of $\mathcal{C}$ are positioned at every vertex of the k-gone. Next, each edge is replaced by a 'dent' as shown in figure 2. Any two distinct elements of $\mathcal{C}$ determine an element of $\bar{\epsilon}\mathcal{P}$ in each dent. Thus the total number of hidden segments within any given dent is $k-1$.

Let p be the number of dents thus added. The total number of hidden segments within the newly formed polygon is $\mu = p(k-1)$. By construction , the size of

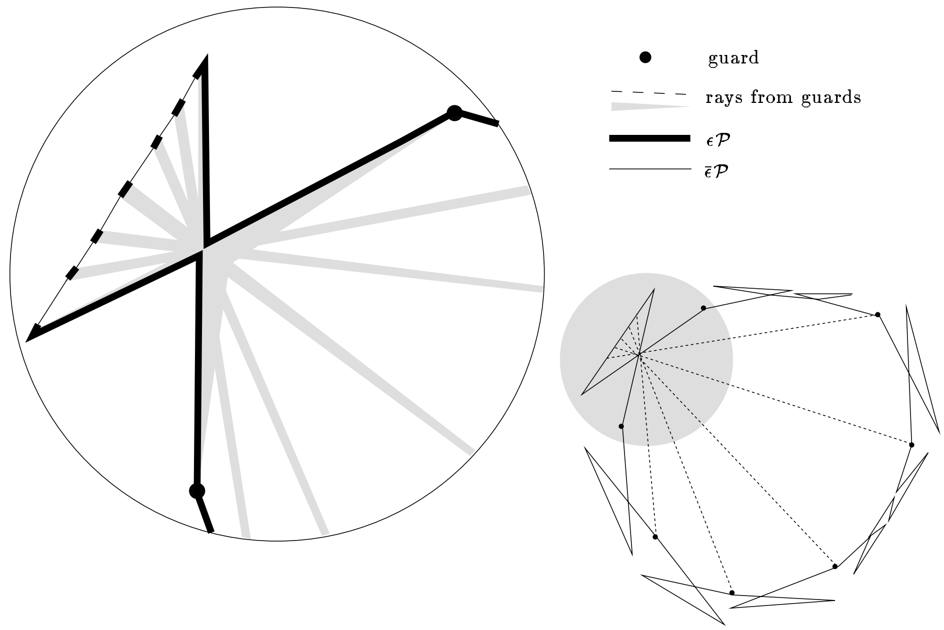

Figure 2: Tight example for lemma 2

the polygon is $n = k + 4p$. Back substituting we obtain

$$\mu = \frac{(n-k)(k-1)}{4}$$

$\square$

Theorem 3 *The cardinality of $\bar{\epsilon}\mathcal{P}$ does not exceed $\frac{k(k-1)(n-k+1)}{4}$ for all polygons and can equal $\frac{(n-k)(k-1)}{4}$ for an infinite family of polygons.*

Proof: Follows immediately from Lemma 1 and 2. $\square$

4 Two guards

Using Lemma 1 in the case of $k = 2$ yields $\mu \leq \frac{n-1}{3}$. This result can be slightly improved.

Lemma 4 *Let $\mathcal{P}$ be a simple polygon which is 2-vertex guardable, then μ is less than or equal to $\frac{n-2}{3}$.*

Proof: Let $\mathcal{C} = \{c_1, c_2\}$. Each of c_1 and c_2 defines exactly μ transition points. From the proof of Lemma 1 we know that $n \geq 4\mu - 2t_m + 1$. Without loss of generality, set $c_1 = c_m$. If c_2 is not positioned onto a visited (convex) vertex of the visibility polygon of c_1 with respect to $\mathcal{P}$ then the result follows by applying the counting argument of Lemma 1 onto the vertices of $vp(c_2|\mathcal{P})$. Assume c_2 positioned onto a visited convex vertex of the visibility polygon of c_1 with respect to $\mathcal{P}$. Such a vertex cannot be a tip vertex of a pocket of $vp(c_1|\mathcal{P})$. Thus there exist a

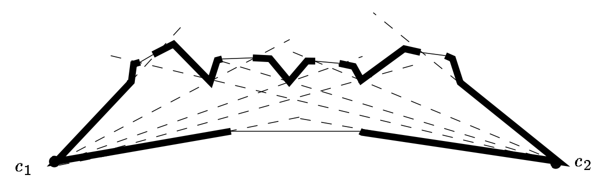

Figure 3: Tight example for lemma 5

pocket of $vp(c_1|\mathcal{P})$ comprising two vertices which are not covered by c_1. If such is the case, set $c_2 = c_m$ and apply the counting arguments of Lemma 1 onto the vertices of $vp(c_1|\mathcal{P})$. This yield $n \geq 4\mu - \mu + 2$. Thus $\mu \leq (n-2)/3$. $\square$

Lemma 5 *There exist an infinite family of simple polygons which are 2-vertex guardable such that the cardinality of the $\bar{\epsilon}\mathcal{P}$ set is $\mu = \frac{n-2}{3}$.*

Proof: The polygon in Figure 2 gives a lower bound of $\frac{n-2}{4}$. That of Figure 3 yields the desired bound. The improvement is due to the deletion of all edges linking consecutive dents and in the introduction of a reflex chain between the two vertices anchoring the guards of $\mathcal{C}$. Supposing there are $\mu - 1$ dents, the total number of vertices equals $3\mu + 2$. $\square$

Theorem 6 *The cardinality of $\bar{\epsilon}\mathcal{P}$ is less than or equal to $\frac{(n-2)}{3}$ for all polygons and there exist an infinite family of polygons for which this bound is tight.*

Proof: Follows immediately from Lemma 4 and 5. □

5 Conclusion

We have introduced the concept of extended cover, and showed that there is a sizable difference between the extended cover and the complete cover of a simple polygon $\mathcal{P}$. The measurement criterion used herein was the cardinality of the hidden set along the boundary of $\mathcal{P}$. Other criteria come to mind such as the ratio between the total length of the hidden set or the total surface of the hidden area with respect to the length of the boundary or the surface of the polygon.

Based on the μ-measurement criterion we were able to exhibit k-vertex guardable polygons with hidden sets of size $O(kn)$. A heuristic which would try to approximate a minimal cover of P by first computing a vertex cover and then adding extra-guards to account for the hidden segments would require $O(k+n)$ guards if it was applied to the polygons in Figures 2 and 3. A rather poor performance.

In the specific case of 2-vertex guardable polygons we were able to exhibit a tight bound on the size of the hidden set. The bound for the general case is still open.

The algorithmic version of this problem is interesting in its own right. Let $\mathcal{P}$ be a simple polygon which is k-guardable. Assuming the covering set positioned, compute its extended cover. Is it possible to do so in $O(kn \log n)$ time and linear space ? What about finding the maximum extended cover over all possible extended covers for a given simple polygon ?

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