A Logic of Quantum Measurement

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Abstract

We present a formulation of quantum mechanics based on a logic representing some aspects of the behaviour of the measurement process. With such an approach, we make no direct mention of quantum states, and thus avoid the problems associated to this rather evasive notion. We then study some properties of the models of this logic, and deduce some characteristics that any model (and hence any formulation of quantum mechanics compatible with its prediction and relying on a notion of measurement) should verify. The main results we obtain are that in the case of a Hilbert space of dimension at least 3, no model can lead to the prediction with certainty of more than one atomic outcome. Moreover, if the Hilbert space is finite dimensional, then we are able to precisely describe the structure of the predictions of any model of our logic. As a consequence, we finally show that all the models of our logic make exactly the same predictions regarding whether a given sequence of outcomes is possible or not.

As Jaynes puts it so vividly, "*our present [quantum mechanical] formalism is not purely epistemological; it is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble*" [13].

One origin for this difficulty is, in our opinion, the excessive reliance of this formalism on the evasive notion of quantum state. Indeed, in any standard textbook on quantum mechanics, the presentation begins with the postulate of its existence, and the rest of the exposition of the theory, including the important mechanism of measure, is based on this very notion. But, we insist, the existence of the quantum state is only postulated, so that the latter is an just abstract mathematical entity with no direct experimental counterpart, and hence doesn't have a clear status between being ontological or epistemological. This causes the aforementioned difficulties of interpretation which spread to other notions, such as that of measurement and the associated "measurement problem".

Yet, experimentally, the actual data that are obtained and dealt with are measurement outcomes and, in fact, any prediction regarding quantum mechanics is expressed in terms of measurement outcomes. This suggests, in order to gain a better understanding of quantum mechanics, to reverse the perspective by considering measurement outcomes as the primary component of the theory, instead of the quantum state. This, obviously, is not a new position regarding quantum mechanics. For instance, in [18], Rovelli states that "one can take the view that outcomes of measurements are the physical content of the theory, and the quantum state is a secondary theoretical construction" and refers back to Heisenberg and Bohr. In this article, we will consider such an approach using the tools of formal logic. More precisely, we will present an axiomatization of the behaviour of quantum measurements, and we will then sketch a study of the way some meaning can be assigned to this logical construction, in the form of a model [12, 15, 16].

Formally, we will consider a basic statement of the form $\mathrm{Mes}(s,p,t)$ which intended meaning is that the system labelled by s has been measured, with outcome p, and that t labels the resulting system. For fixed s and p, the statement

$$\exists\, t:\ \mathrm{Mes}(s,p,t)$$

corresponds to the possibility of obtaining outcome p when measuring s. This type of statement will prove extremely important in the following, as it will be the basic way for expressing properties of quantum measurement. Here, we intentionally leave probabilities aside and only focus one the possibility of having a outcome. An outcome p will be said to be impossible (resp. certain, possible) is its probability equals 0 (resp. equals 1, is nonzero). This type of approach can clearly be seen as a *possibilistic* one, using the term coined by Fritz [10].

In the following, we will study the logic based on the "Mes" relation. We shall, for instance, consider statements such as

$$\forall\, s,t,\ \mathrm{Mes}(s,p,t) \implies \exists\, u:\ \mathrm{Mes}(t,p,u)$$

which indicates that outcome p is always possible when measuring a system (labelled here by t) obtained by a measurement which yielded outcome p. Here, the part "$\mathrm{Mes}(s,p,t)$" can be interpreted as a preparation phase, with t labelling the prepared system, and "$\exists\, u:\ \mathrm{Mes}(t,p,u)$" is a prediction regarding this prepared system. If we drop the labels s, t and u, this corresponds to stating that the sequence of outcomes (p,p) is possible.

Another typical example is the following, where p and q are two mutually orthogonal outcomes:

$$\forall\, s,t,\ \mathrm{Mes}(s,p,t) \implies \neg\big(\exists\, u:\ \mathrm{Mes}(t,q,u)\big) \tag{1}$$

In term of sequences of outcomes, this means that (p,q) is impossible: one cannot obtain outcomes p and q in a row when measuring the same system. If this statement is true for any q orthogonal to p, this impossibility can also be expressed by saying if t has been prepared by $\mathrm{Mes}(s,p,t)$, then p is a certain outcome for t: measuring it with an observable having p as possible outcome with yield outcome p with probability 1, since equation (1) states that any other outcome is impossible.

Our study will proceed as follows. First, we shall identify a collection of properties involving "Mes", and we will use quantum states so as to ensure that these properties are compatible with the prediction of quantum mechanics. However, we insist on the fact that in the end, the obtained properties are expressed using the "Mes" construction only, so that there remains no reference whatsoever to quantum states. This way, we will obtain a logical theory, i.e. a collection of logical sentences which describe some aspects of the behavior of measurements and their outcomes. We will then study the way one can assign some meaning, some semantics to these sentences, which is called a *model* of the theory. Obviously, the orthodox formulation of quantum mechanics based on quantum states provides such a model but, more generally, any theory attempting to formalize quantum mechanics (and compatible with its predictions) and involving a notion of measurement would lead to a model of this logical theory. Thus, the study of these models will allow us to address some important questions such as whether the "standard" model (based on quantum states) has a particular status or, on the contrary, there exists other models leading to different possibilistic predictions.

A note on formalism In the following, in the construction $\mathrm{Mes}(s, p, t)$, we will consider that the labels s and t belong to an unspecified set S.

Regarding outcomes,the orthodox formulation of quantum mechanics stipulates that a quantum system is modelled by a Hilbert space and, following the pioneering work of von Neumann and Birkhoff [2], "the mathematical representative of any experimental proposition is a closed [] subspace of Hilbert space". Subsequent works have led to the reasonable opinion that, as summarized by Dalla Chiara and Giuntini in [6], "the set $\mathcal{E}$ of events should be a good abstraction from the structure of all closed subspaces in a Hilbert space. As a consequence $\mathcal{E}$ should be at least a σ-complete orthomodular lattice (generally non distributive)." In this article, we will be however a bit more general by only considering that outcomes (corresponding to p in $\mathrm{Mes}(s, p, t)$) form a orthomodular lattice L (and later focus on the case where L is the Hilbert lattice $L(\mathcal{H})$ made of the closed subspaces of a Hilbert space $\mathcal{H}$).

We recall that an orthomodular lattice is an ortholattice $(L, \vee, \wedge, \top, \bot, \cdot^{\perp})$ such that the orthomodular law holds:

$$\forall\, p, q \in L, \quad \big((p \wedge q) \vee q^{\perp}\big) \wedge q = p \wedge q$$

The lattice operations $\vee$ and $\wedge$ induce a partial order relation on L, defined by

$$p \leq q \iff p \vee q = q \iff p \wedge q = p$$

As it is well known, an important example of complete orthomodular lattice is provided by the set $L(\mathcal{H})$ of closed subspaces of a Hilbert lattice $\mathcal{H}$. We invite the reader to refer to [6] for a more general presentation of orthomodular lattices.

Finally, given an orthomodular lattice L, we define an *observable* of L as a *finite* subset $\mathcal{O} = \{p_1, p_2, \ldots, p_n\}$ of L such that

$$\forall\, i,\, p_i \neq \bot \qquad \forall\, i \neq j,\, p_i \leq p_j^{\perp} \qquad \text{and} \qquad \bigvee_{i=1}^{n} p_i = \top$$

In the case where L is the lattice $L(\mathcal{H})$ associated to a Hilbert space $\mathcal{H}$, this definition corresponds to the set of eigenspaces of a Hermitian operator with finitely many eigenvalues. $\mathcal{M}(L)$ will denote the set of observables of L.

1 A Logic for Measurement

Let us return to the statement that if a quantum system is measured twice in a row, one cannot obtain two mutually orthogonal outcomes, regardless which observables were measured, i.e. if $q \leq p^\perp$, then

$$\forall\, s, t \in S,\ \mathrm{Mes}(s, p, t) \implies \neg\big(\exists\, u \in S\colon\ \mathrm{Mes}(t, q, u)\big) \tag{2}$$

This is obviously true in orthodox quantum mechanics (and thus experimentally), as a consequence of the Born rule and the projection postulate. More precisely, if a system labelled by s is in a state $|\varphi\rangle$, and if $\mathrm{Mes}(s, p, t)$, then $\Pi_p|\varphi\rangle \neq \vec{0}$ (otherwise outcome p would not be possible) and the state $|\psi\rangle$ of t is colinear to $\Pi_p|\varphi\rangle$. Now, since $q \leq p^\perp$, this implies that $\Pi_q\Pi_p = 0$, so that $\Pi_q|\psi\rangle = \vec{0}$ and hence q is not a possible outcome. Here, for $p \in L(\mathcal{H})$, Π_p obviously denotes the orthogonal projection on p.

We insist again on the fact that even though the justification of equation (2) relies on the notion of quantum state, it is stated in a way that does not involve those states. In other words, it is a statement regarding measurement outcomes only, and the previous justification only tells us that it is consistent with the predictions of orthodox quantum mechanics.

This first property suggest the following definition:

Definition 1 (Verification Statement) *For all $p \in L$ and $s \in S$, we define $s \blacktriangleright p$ by*[1]

$$s \blacktriangleright p \overset{\Delta}{\iff} \neg\big(\exists\, t \in S\colon\ \mathrm{Mes}(s, p^\perp, t)\big)$$

In that case, we will say that s verifies *p.*

With this definition, equation (2) becomes

$$\forall\, p \leq q, \quad \forall\, s, t,\ \mathrm{Mes}(s, p, t) \implies t \blacktriangleright q \tag{3}$$

Let us explore some other properties regarding measurements.

Valid Outcomes The least element $\perp$ of L cannot be obtained as an outcome, that is

$$\forall\, s, t \in S,\ \neg\,\mathrm{Mes}(s, \perp, t)$$

which can be expressed as

$$\forall\, s \in S,\ s \blacktriangleright \top$$

[1] A "Δ" on top of an equality or an equivalence indicates a definition.

Moreover, $\perp$ is *the only* element of L which cannot be obtained as an outcome:

$$\forall\, p \in L,\ \big(\forall\, s, t \in S,\ \neg\,\mathrm{Mes}(s,p,t)\big) \implies p = \perp$$

Equivalently, one can state this as

$$\forall\, p \neq \perp, \quad \exists\, s, t \in S:\ \mathrm{Mes}(s,p,t)$$

Measurability Any system is likely to be measured (or has already been measured), so that for every system s and for every observable $\mathcal{O}$, at least one of the outcomes has to be possible, which we temporarily write as

$$\forall\, s \in S,\, \forall\, \mathcal{O} \in \mathcal{M}(L),\, \exists\, p \in \mathcal{O}:\, \exists\, t \in S:\ \mathrm{Mes}(s,p,t) \tag{4}$$

Weak Non-contextuality The next property can be expressed as follows: If an outcome p is certain (resp. impossible) when measuring s with a particular observable containing p, then it is certain (resp. impossible) when measuring s with *any* observable containing p. In orthodox quantum mechanics, this observation follows from the Born rule, which states that the probability of obtaining outcome p when measuring a quantum system in state $|\varphi\rangle$ is

$$\frac{\langle\varphi|\Pi_p|\varphi\rangle}{\langle\varphi|\varphi\rangle}.$$

In particular, this probability is independent of the measured observable, so that in our possibilistic approach, the impossibility or certainty of an outcome is independant of the measured observable. This should not be confused the stronger notion of non-contextuality usually associated to results such as the Kochen-Specker theorem. Here, we consider a weaker form which only deals with impossible or certain outcomes. Let us derive some properties from this.

First, suppose that $p, q \in L$ are such that $p \leq q$, and that $s \blacktriangleright p$. We will show that $s \blacktriangleright q$. Consider observable $\mathcal{O} = \{p, q^\perp, p^\perp \wedge q\}$. With the assumption that $s \blacktriangleright p$, measuring s with observable $\mathcal{O}$ would yield outcome p with certainty so that, from non-contextuality, it is not possible to obtained $q^\perp$ as outcome when measuring s (regardless of the measured observable). We thus have shown than

$$\forall\, p \leq q, \quad \forall\, s \in S,\ s \blacktriangleright p \implies s \blacktriangleright q. \tag{5}$$

Suppose now that $s \blacktriangleright p$ and $s \blacktriangleright q$ and that $p, q \in L$ are compatible (which we denote $p \mathrel{\mathrm{C}} q$), meaning that they both belong to a single boolean subalgebra of L or, equivalently, that they are possible outcomes of a single observable, and consider observable $\mathcal{O} = \{p \wedge q, p^\perp, p \wedge q^\perp\}$. Measuring s with $\mathcal{O}$ would yield $p \wedge q$ with certainty since $p^\perp$ is orthogonal to p and is thus impossible as follows from $s \blacktriangleright p$, and $p \wedge q^\perp$ is also impossible, being orthogonal to q.

But again, from non-contextuality, $p \wedge q$ is also certain if measuring s with observable $\mathcal{O}' = \{p \wedge q, (p \wedge q)^\perp\}$ so that $(p \wedge q)^\perp$ is impossible w.r.t $\mathcal{O}'$ and, using non-contextuality a last time, w.r.t. any observable. Thus, we have

$$\forall\, p \mathrel{\mathrm{C}} q, \quad \forall\, s,\ s \blacktriangleright p \text{ and } s \blacktriangleright q \implies s \blacktriangleright p \wedge q. \tag{6}$$

This property can be stated another way, using *Sasaki projections*[6]. We recall that this operation is defined on an orthomodular lattice as

$$\forall\, x, y \in L,\; x \,\&\, y \stackrel{\Delta}{=} y \wedge \left(x \vee y^{\perp}\right)$$

and it verifies the following basic properties:

$$\begin{aligned}
\forall\, p, q \in L, &\quad p \,\&\, q \leq q \\
\forall\, p_1, p_2, q \in L, &\quad p_1 \leq p_2 \implies p_1 \,\&\, q \leq p_2 \,\&\, q \\
\forall\, p, q \in L, &\quad p \,\&\, q = \perp \iff p \leq q^{\perp} \\
\forall\, p, q \in L, &\quad p \mathrel{\mathrm{C}} q \iff p \,\&\, q = p \wedge q
\end{aligned}$$

Using this operation, equation 6 can equivalently be states as

$$\forall\, p, q \in L, \quad \forall\, s,\; s \blacktriangleright p \text{ and } s \blacktriangleright q \implies s \blacktriangleright p \,\&\, q.$$

Weak non-contextuality allows us to reexpress some previous properties in a simpler way. For instance, using (5), equation (3) can be replaced by

$$\forall\, p \in L, \quad \forall\, s, t,\; \mathrm{Mes}(s, p, t) \implies t \blacktriangleright p.$$

Similarly, measurability was expressed as

$$\forall\, s \in S, \forall\, \mathcal{O} \in \mathcal{M}(L), \neg\left(\forall\, p \in \mathcal{O},\; s \blacktriangleright p^{\perp}\right).$$

But since an observable $\mathcal{O} \in \mathcal{M}(L)$ is a finite collection of mutually compatible elements of L, from equation (6), we have

$$\begin{aligned}
\left(\forall\, p \in \mathcal{O},\; s \blacktriangleright p^{\perp}\right) &\iff s \blacktriangleright \bigwedge \{p^{\perp} \mid p \in \mathcal{O}\} \\
&\iff s \blacktriangleright \left(\bigvee \{p \mid p \in \mathcal{O}\}\right)^{\perp} \\
&\iff s \blacktriangleright \top^{\perp} \\
&\iff s \blacktriangleright \perp
\end{aligned}$$

Thus, measurability simply becomes

$$\forall\, s,\; \neg(s \blacktriangleright \perp)$$

Compatible Preservation We now present a last property, relating the verification of properties before and after a measurement. In order to express it, let us first translate verifications statements in terms of quantum states. An outcome $p^{\perp}$ is impossible for a system in state $|\varphi\rangle$ iff its probability is 0, that is, using the Born rule, $\Pi_{p^{\perp}} |\varphi\rangle = \vec{0}$. In terms of verification statements, this means that s is in state $|\varphi\rangle$, so that one has[2]

$$s \blacktriangleright p \iff |\varphi\rangle \in p$$

[2]This "translation" will become clearer and more rigourous once we have introduced model-theoretic elements.

Suppose then that $s \blacktriangleright p$ and $\mathrm{Mes}(s,q,t)$ with p and q compatible. System t is then in a state $|\psi\rangle$ colinear to $\Pi_q|\varphi\rangle$. But having $s \blacktriangleright p$ means that $|\varphi\rangle \in p$ so that $\Pi_p|\varphi\rangle = |\varphi\rangle$, and the compatibility of p and q means that Π_p and Π_q commute. As a consequence,

$$\Pi_p \Pi_q |\varphi\rangle = \Pi_q \Pi_p |\varphi\rangle = \Pi_q |\varphi\rangle$$

and, similarly, $\Pi_p|\psi\rangle = |\psi\rangle$ so that $t \blacktriangleright p$. We have shown that the following property is compatible with the predictions of orthodox quantum mechanics:

$$\forall\, p \;\mathrm{C}\; q, \quad \forall\, s,t,\; s \blacktriangleright p \text{ and } \mathrm{Mes}(s,q,t) \implies t \blacktriangleright p$$

We call this *compatible preservation* since the verification of an element $p \in L$ is preserved during a measurement, provided that its outcome q is compatible with p.

Again, considering weak non-contextuality, this statement can be rewritten as

$$\forall\, p,q, \quad \forall\, s,t,\; s \blacktriangleright p \text{ and } \mathrm{Mes}(s,q,t) \implies t \blacktriangleright p \,\&\, q$$

We summarize all these properties by defining the following theory:

Definition 2 *Given an orthomodular lattice L, we define the theory $\mathcal{T}_L$ as the theory consisting of the following sentences:*

$$\forall\, s \in S,\; s \blacktriangleright \top \tag{7a}$$

$$\forall\, s \in S,\; \neg(s \blacktriangleright \bot) \tag{7b}$$

$$\forall\, p \neq \bot, \quad \exists\, s,t \in S\colon\; \mathrm{Mes}(s,p,t) \tag{7c}$$

$$\forall\, p, \quad \forall\, s,t \in S,\; \mathrm{Mes}(s,p,t) \implies t \blacktriangleright p \tag{7d}$$

$$\forall\, p \leq q, \quad \forall\, s \in S,\; s \blacktriangleright p \implies s \blacktriangleright q \tag{7e}$$

$$\forall\, p,q, \quad \forall\, s \in S,\; s \blacktriangleright p \textit{ and } s \blacktriangleright q \implies s \blacktriangleright p \,\&\, q \tag{7f}$$

$$\forall\, p,q, \quad \forall\, s,t \in S,\; s \blacktriangleright p \textit{ and } \mathrm{Mes}(s,q,t) \implies t \blacktriangleright p \,\&\, q \tag{7g}$$

It can be remarked that $\mathcal{T}_L$ can be seen as a one-sorted first-order theory since all the quantifications occur on the elements of S (the universal quantifications on elements of L only mean that there is a sentence for each possible assignement of p and q). Moreover, the definition of $\mathcal{T}_L$ is slightly redundant, since equation (7d) is a direct consequence of (7a) and (7g).

2 Models

All the sentences that constitute $\mathcal{T}_L$ are just sequences of characters, syntactical objects. They describe in mathematical terms some properties that "Mes" should verify. In order to give these sentences a meaning, we need to consider a structure made of a set A, called the *universe* and a ternary relation M on $A \times L \times A$ reflecting the syntactical construction "$\mathrm{Mes}(a,p,b)$". It can be remarked that such a structure (A, M) can be seen as a directed graph, labelled by elements of L, with A being its set of vertices, and a triple $(a,p,b) \in M$ denotes an arrow from a to b labelled by $p \in L$.

Now, intuitively, given a sentence φ, a graph (A,M) verifies this sentence (in which case we write $(A,M) \models \varphi$) if and only if the graph verifies the translation of φ in terms of A and M. For instance

$$(A,M) \models \forall\, p \in L,\, \forall\, s,t,\ \mathrm{Mes}(s,p,t) \implies \exists\, u\colon\ \mathrm{Mes}(t,p,u)$$

if and only if it verifies

$$\forall\, p \in L,\, \forall\, a,b \in A,\ M(a,p,b) \implies \exists\, c \in A\colon M(b,p,c).$$

Definition 3 *A graph $\mathfrak{G} = (A,M)$ is a* model *of $\mathcal{T}_L$ if and only $\mathfrak{G}$ verifies every sentence of $\mathcal{T}_L$. In that case, we write*

$$\mathfrak{G} \models \mathcal{T}_L.$$

We insist on the fact that $\mathcal{T}_L$ is only a set of syntactical elements, a set of sequences of characters, while a model of $\mathcal{T}_L$ is an actual set equipped with an actual relation, i.e. an actual L-labelled directed graph in which the properties expressed by $\mathcal{T}_L$ do hold. In such a model, elements of A can be seen as specifications of quantum states. To illustrate this, let us first introduce the model corresponding to the orthodox approach to quantum mechanics.

Definition 4 (Hilbert Graph) *Given a Hilbert space $\mathcal{H}$, we define the* Hilbert graph *$\mathfrak{H}_{\mathcal{H}} = (A_{\mathfrak{H}}, M_{\mathfrak{H}})$ by putting:*

$$A_{\mathfrak{H}} \triangleq \big\{ |\varphi\rangle \in \mathcal{H} \mid \big\| |\varphi\rangle \big\| = 1 \big\}$$
$$M_{\mathfrak{H}}\big(|\varphi\rangle, p, |\psi\rangle\big) \overset{\Delta}{\iff} \big\|\Pi_p|\varphi\rangle\big\| \neq 0 \textit{ and } |\psi\rangle = \frac{\Pi_p|\varphi\rangle}{\big\|\Pi_p|\varphi\rangle\big\|}$$

Obviously, we have :

Proposition 1 *For every Hilbert space $\mathcal{H}$, $\mathfrak{H}_{\mathcal{H}} \models \mathcal{T}_{L(\mathcal{H})}$.*

This result is the direct consequence of the fact that quantum states are actually the basic model of quantum mechanics, and that $\mathcal{T}_L$ has been defined considering quantum states. We recall that the verification relation $\blacktriangleright$ translates in this model as

$$|\varphi\rangle \blacktriangleright_{\mathfrak{H}} p \iff |\varphi\rangle \in p$$

However, elements of the universe of a model can also be uncomplete descriptions of a state. To illustrate this, we introduce another important model of $\mathcal{T}_L$.

Definition 5 (Lattice Graph) *Given an orthomodular lattice L, the* lattice graph $\mathfrak{L}_L = (A_{\mathfrak{L}}, M_{\mathfrak{L}})$ *is defined by*

$$A_{\mathfrak{L}} \triangleq L^\star \qquad \textit{where} \qquad L^\star \triangleq L \setminus \{\bot\}$$
$$M_{\mathfrak{L}}(a,p,b) \overset{\Delta}{\iff} b \leq a \,\&\, p$$

Proposition 2 *Given an orthomodular lattice L, $\mathfrak{L}_L \models \mathcal{T}_L$.*

Proof Let us first remark that

$$\begin{aligned} a \blacktriangleright_{\mathfrak{L}} p &\iff \neg\big(\exists\, b \in L^\star : b \leq a \,\&\, p^\perp\big) \\ &\iff a \,\&\, p^\perp = \bot \\ &\iff a \leq p \end{aligned}$$

With this in mind, it is easy to prove that the different sentences of $\mathcal{T}_L$ do hold. For instance, the first two sentences simply state that:

$$\forall\, a \in L^\star,\ \bot < a \leq \top.$$

The third sentence (7c) translates into the statement

$$\forall\, p \neq \bot,\ \exists\, a, b \in L^\star : b \leq a \,\&\, p$$

which is verified by putting, for instance, $a = \top$ and $b = p$. The other formulas can also be proven with no difficulties. $\square$

The way the logic $\mathcal{T}_L$ was designed – with a single relation "Mes" corresponding to the possible obtention of a given measurement outcomes – implies that any interpretation of quantum mechanics that includes a notion of measurement should, in order to comply with the predictions of orthodox quantum mechanics (corresponding in our approach to the model $\mathfrak{H}_{\mathcal{H}}$), provide a model of $\mathcal{T}_L$ for some orthomodular lattice L.

In particular, this is the case for many hidden variable-based approaches to quantum mechanics. Consider for instance the classical example given by Bell in [1]: a system, made of two spin one-half particles A and B, is assumed to have its state completely specified by a parameter λ belonging to some set Λ. Measuring the spin of particle A along direction $\vec{a}$ yields outcome $A(\vec{a}, \lambda) \in \{+1, -1\}$ and, similarly, measuring B along direction $\vec{b}$ yields outcome $B(\vec{b}, \lambda)$.

Such an approach would provide a model $\mathfrak{B}$ of $\mathcal{T}_{L(\mathbf{C}^2 \otimes \mathbf{C}^2)}$, with $A = \Lambda$. And even though the relation M is yet unspecified, we have some information about verification statements. In particular, we have, for all $\vec{a}$ and $\vec{b}$,

$$\lambda \blacktriangleright_{\mathfrak{B}} [A(\lambda, \vec{a})] \otimes [B(\lambda, \vec{b})]$$

where $[A(\lambda, \vec{a})]$ (resp. $[B(\lambda, \vec{b})]$) denotes the eigenspace corresponding to the indicated outcome.

Thus, the study of the models of $\mathcal{T}_L$ provides a general framework for understanding the meaning of the notion of quantum state and for determining the different ways of specifying (completely or partially) the state of a quantum system.

3 The Structure of Verified Properties

In order to study the models of $\mathcal{T}_L$ a bit further, let us go from elements of L to subsets of L. To that respect, we introduce a few additional definitions:

$$\forall\, E \subseteq L, \forall\, p \in L, \quad E \,\&^{\#}\, p \triangleq \{q \,\&\, p \mid q \in E\}$$

and, given a model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_L$,

$$\begin{aligned}
\forall\, E \subseteq L, \forall\, a \in A, \quad & a \blacktriangleright^{\#}_{\mathfrak{G}} E \overset{\Delta}{\iff} \forall\, p \in E,\, a \blacktriangleright_{\mathfrak{G}} p \\
\forall\, a \in A, \quad & [\![a]\!]_{\mathfrak{G}} \triangleq \{p \in \mathcal{L} \mid a \blacktriangleright_{\mathfrak{G}} p\} \\
\forall\, E \subseteq L, \quad & \mathrm{Cl}_{\mathfrak{G}}(E) \triangleq \bigcap \big\{ [\![a]\!]_{\mathfrak{G}} \bigm| a \in A \text{ and } E \subseteq [\![a]\!]_{\mathfrak{G}} \big\}
\end{aligned}$$

From those definitions, it is clear that

$$\begin{aligned}
\forall\, E \subseteq L, \forall\, a \in A, \quad & a \blacktriangleright^{\#}_{\mathfrak{G}} E \implies a \blacktriangleright^{\#}_{\mathfrak{G}} \mathrm{Cl}_{\mathfrak{G}}(E) \\
\forall\, E \subseteq L, \forall\, a, b \in A, \forall\, p \in L, \quad & a \blacktriangleright^{\#}_{\mathfrak{G}} E \text{ and } M(a, p, b) \implies b \blacktriangleright^{\#}_{\mathfrak{G}} E \,\&^{\#}\, p \\
\forall\, a \in A, \quad & \mathrm{Cl}_{\mathfrak{G}}([\![a]\!]_{\mathfrak{G}}) = [\![a]\!]_{\mathfrak{G}}
\end{aligned}$$

Let us recall that a *closure operator* [7, 9, 3] on a set E is a function $\mathrm{Cl}\colon \mathcal{P}(E) \to \mathcal{P}(E)$ which verifies the following three properties:

$$\begin{aligned}
\forall\, F \in \mathcal{P}(E), \quad & F \subseteq \mathrm{Cl}(F) \\
\forall\, F, G \in \mathcal{P}(E), \quad & F \subseteq G \implies \mathrm{Cl}(F) \subseteq \mathrm{Cl}(G) \\
\forall\, F \in \mathcal{P}(E), \quad & \mathrm{Cl}\big(\mathrm{Cl}(F)\big) = \mathrm{Cl}(F)
\end{aligned}$$

In that case, a subset $F \subseteq E$ is sais to be *closed w.r.t.* Cl iff $F = \mathrm{Cl}(F)$. It can also be remarked that a closure operator is uniquely determined by its closed subsets. It is straightforward to prove that:

Proposition 3 $\mathrm{Cl}_{\mathfrak{G}}$ *is a* closure operator *on* L.

Let us now present a few properties regarding $\mathrm{Cl}_{\mathfrak{G}}$ and based on the fact that $\mathfrak{G}$ is a model of $\mathcal{T}_L$:

1. From " $\forall\, a \in A,\, a \blacktriangleright_{\mathfrak{G}} \top$ " it follows that

$$\top \in \mathrm{Cl}_{\mathfrak{G}}(\emptyset)$$

Moreover, from " $\forall\, p \in L^\star,\ \exists\, a, b \in A\colon M(a,p,b)$ ", we deduce that if $p \neq \perp$, there exists a $a \in A$ such that $\neg\big(a \blacktriangleright p^\perp\big)$. This implies that $p^\perp \notin \mathrm{Cl}_{\mathfrak{G}}(\emptyset)$, so that we have

$$\mathrm{Cl}_{\mathfrak{G}}(\emptyset) = \{\top\}.$$

2. We also have " $\forall\, a \in A,\ \neg(a \blacktriangleright_{\mathfrak{G}} \perp)$ " and " $\forall\, p \in L^\star,\ \exists\, a, b \in A\colon M(a,p,b)$ ", so that

$$\forall\, p \in L,\ \perp \in \mathrm{Cl}_{\mathfrak{G}}(\{p\}) \implies p = \perp. \tag{8}$$

Indeed, if $\perp \in \mathrm{Cl}_{\mathfrak{G}}(\{p\})$ for some $p \in L^\star$, by considering $a, b \in A$ such that $M(a,p,b)$, we would have $b \blacktriangleright_{\mathfrak{G}} p$ so that $b \blacktriangleright_{\mathfrak{G}} \perp$, which is not possible.

3. From axioms (7e) and (7f) of $\mathcal{T}_L$, which, in terms of $\mathfrak{G}$, can be expressed as

$$\begin{aligned} \forall\, p \le q, &\quad \forall\, a \in A,\ a \blacktriangleright_{\mathfrak{G}} p \implies a \blacktriangleright_{\mathfrak{G}} q \\ \forall\, p, q, &\quad \forall\, a \in A,\ a \blacktriangleright_{\mathfrak{G}} p \text{ and } a \blacktriangleright_{\mathfrak{G}} q \implies a \blacktriangleright_{\mathfrak{G}} p \,\&\, q \end{aligned}$$

we directly deduce the corresponding properties of $\mathrm{Cl}_{\mathfrak{G}}$:

$$\begin{aligned} \forall\, p, q \in L, &\quad p \le q \implies q \in \mathrm{Cl}_{\mathfrak{G}}(\{p\}) \\ \forall\, p, q \in L, &\quad p \,\&\, q \in \mathrm{Cl}_{\mathfrak{G}}(\{p, q\}) \end{aligned}$$

4. Finally, suppose that $E \subseteq L$ and $p \in L$ are such that

$$\perp \in \mathrm{Cl}_{\mathfrak{G}}\big(\mathrm{Cl}_{\mathfrak{G}}(E) \,\&^{\#}\, p\big)$$

In that case, if $a \blacktriangleright^{\#}_{\mathfrak{G}} E$ and $M(a,p,b)$, then $a \blacktriangleright^{\#}_{\mathfrak{G}} \mathrm{Cl}_{\mathfrak{G}}(E)$ so that $b \blacktriangleright^{\#}{}_{\mathfrak{G}} \mathrm{Cl}_{\mathfrak{G}}(E) \,\&^{\#}\, p$ and finally

$$b \blacktriangleright^{\#}_{\mathfrak{G}} \mathrm{Cl}_{\mathfrak{G}}\big(\mathrm{Cl}_{\mathfrak{G}}(E) \,\&^{\#}\, p\big).$$

But from our hypothesis, we would then have $b \blacktriangleright_{\mathfrak{G}} \perp$, which is not possible. As a result, if $a \blacktriangleright^{\#}_{\mathfrak{G}} E$, then one cannot find a $b \in A$ such that $M(a,p,b)$, that is

$$a \blacktriangleright^{\#}_{\mathfrak{G}} E \implies a \blacktriangleright_{\mathfrak{G}} p^\perp$$

In terms of $\mathrm{Cl}_{\mathfrak{G}}$, we have shown

$$\perp \in \mathrm{Cl}_{\mathfrak{G}}\big(\mathrm{Cl}_{\mathfrak{G}}(E) \,\&^{\#}\, p\big) \implies p^\perp \in \mathrm{Cl}_{\mathfrak{G}}(E)$$

This discussion shows that $\mathrm{Cl}_{\mathfrak{G}}$ is an *admissible* closure operator on L, which we define as:

Definition 6 *A closure operator* Cl *on an orthomodular lattice L (or, more precisely, on its powerset $\mathcal{P}(L)$ ordered by inclusion) is said to be* admissible *if it verifies the previous properties, that is*

$$\mathrm{Cl}(\emptyset) = \{\top\} \tag{9a}$$

$$\forall\, p \in L,\ \perp \in \mathrm{Cl}(\{p\}) \implies p = \perp \tag{9b}$$

$$\forall\, p, q \in L,\ p \le q \implies q \in \mathrm{Cl}(\{p\}) \tag{9c}$$

$$\forall\, p, q \in L,\ p \,\&\, q \in \mathrm{Cl}(\{p, q\}) \tag{9d}$$

$$\forall\, E \subseteq L,\ \forall\, p \in L,\ \perp \in \mathrm{Cl}\big(\mathrm{Cl}(E) \,\&^{\#}\, p\big) \implies p^\perp \in \mathrm{Cl}(E) \tag{9e}$$

With this definition, we clearly have:

Proposition 4 *Given a model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_L$, the closure operator* $\mathrm{Cl}_{\mathfrak{G}}$ *defined by*

$$\forall\, E \subseteq L,\ \mathrm{Cl}_{\mathfrak{G}}(E) = \bigcap \big\{ [\![a]\!]_{\mathfrak{G}} \;\big|\; a \in A \textit{ and } E \subseteq [\![a]\!]_{\mathfrak{G}} \big\}$$

is an admissible closure operator on L.

Reciprocally, any admissible closure operator on L gives rise to a model of $\mathcal{T}_L$:

Proposition 5 *Given an admissible closure operator* Cl *on an orthomodular lattice L, the graph $\mathfrak{G}_{\mathrm{Cl}} = (A_{\mathrm{Cl}}, M_{\mathrm{Cl}})$ defined by*

$$A_{\mathrm{Cl}} = \big\{ \mathrm{Cl}(E) \;\big|\; E \subseteq L \textit{ and } \bot \notin \mathrm{Cl}(E) \big\}$$
$$M_{\mathrm{Cl}}(a, p, b) \overset{\Delta}{\iff} \mathrm{Cl}(a \,\&^{\#}\, p) \subseteq b$$

is a model of $\mathcal{T}_L$.

Proof It suffices to remark that $a \blacktriangleright_{\mathrm{Cl}} p \iff p \in a$, and that for all $p \neq \bot$, one has $\bot \notin \mathrm{Cl}(\{\top\} \,\&^{\#}\, p)$ so that $M_{\mathrm{Cl}}\big(\{\top\}, p, \mathrm{Cl}(\{\top\} \,\&^{\#}\, p)\big)$. □

We present next an admissible closure operator which can be defined on any orthomodular lattice L where, given an element $p \in L$, we have

$$p^{\uparrow} \overset{\Delta}{=} \{ q \in L \mid p \leq q \}$$

Proposition 6 *The operator* $\mathrm{Cl}_{\uparrow}$ *defined by*

$$\forall\, E \subseteq L,\ \mathrm{Cl}_{\uparrow}(E) = \big\{ p \in L \;\big|\; \textstyle\bigwedge E \leq p \big\} = (\textstyle\bigwedge E)^{\uparrow}$$

is an admissible closure operator on L.

Proof The only property worth proving is (9e). Suppose that $\mathrm{Cl}(E) = q^{\uparrow}$, and that $\bot \in \mathrm{Cl}(q^{\uparrow} \,\&^{\#}\, p)$. This implies that $q \,\&\, p = \bot$ (since $q^{\uparrow} \,\&^{\#}\, p = (q \,\&\, p)^{\uparrow}$) and hence that $q \leq p^{\perp}$ so that finally $p^{\perp} \in q^{\uparrow} = \mathrm{Cl}(E)$. □

Following from the previous discussion, a legitimate question is whether an orthomodular lattice has other admissible closure operators. In the next section, we will show that in a very important case, $\mathrm{Cl}_{\uparrow}$ is actually the only possible admissible closure operator.

4 The Hilbert Case

We now turn to the specific case where a quantum system is described by a Hilbert space $\mathcal{H}$ of dimension at least 3. This type of Hilbert space plays a very important role in the study of the foundations of quantum mechanics, since several important theorems are only valid if the dimension of the Hilbert space is at least 3. The most notable such theorems are, arguably, Gleason's theorem [11, 17] and the Kochen-Specker theorem [14].

Let us first present a result related to the Kochen-Specker theorem, which involves the notion of Sasaki filters.

Definition 7 *Given an orthomodular lattice L, a* Sasaki filter *of L is a non-empty subset F of L which verifies :*

$$\begin{aligned} \forall\, p, q \in L, &\quad p \in F \text{ and } p \leq q \implies q \in F \\ \forall\, p, q \in L, &\quad p \in F \text{ and } q \in F \implies p \,\&\, q \in F \end{aligned}$$

This notion is important in the present study of $\mathcal{T}_L$, since if Cl is an admissible closure operator on L, then every subset closed w.r.t. Cl (i.e. every subset E such that $E = \mathrm{Cl}(E)$) is a Sasaki filter as it follows directly from conditions (9a-9d). In particular, if a graph $\mathfrak{G} = (A, M)$ is a model of $\mathcal{T}_L$, then for all $a \in A$, the set $[\![a]\!]_{\mathfrak{G}}$ is a Sasaki filter of L. With this in mind, we recall the following theorem from [4, 5]:

Theorem 7 *If $\mathcal{H}$ is a Hilbert space of dimension at least 3, then every consistent Sasaki filter of $L(\mathcal{H})$ contains at most one atom.*

Here, by *atom*, we mean a one-dimensional subspace (that is, an element of the lattice which is "just" above the least element $\perp = \big\{|0\rangle\big\}$ in $L(\mathcal{H})$), and a Sasaki filter F is consistent if it does not contain $\perp$, that is if $F \neq L$.

Corollary 8 *If $\mathfrak{G} = (A, M)$ is a model of $\mathcal{T}_{L(\mathcal{H})}$ where $\dim \mathcal{H} \geq 3$, then for all $a \in A$, $[\![a]\!]_{\mathfrak{G}}$ contains at most one atom.*

This result has important consequences. Recall the possibility of having a hidden-variable model of $\mathcal{T}_L$ as the one sketched earlier after Bell's article. The previous corollary simply forbids the possibility of having such a model, since they would lead to the presence of more than one atom in a consistent Sasaki filter. For instance, with the previous notations, given any two non-colinear and non-orthogonal vectors $\vec{u}$ and $\vec{v}$, we had for all $\lambda \in \Lambda$,

$$\lambda \blacktriangleright_{\mathfrak{B}} [A(\vec{u}, \lambda)] \otimes [B(\vec{u}, \lambda)] \quad \text{and} \quad \lambda \blacktriangleright_{\mathfrak{B}} [A(\vec{v}, \lambda)] \otimes [B(\vec{v}, \lambda)]$$

which is precisely ruled out by corollary 8. More generally, this result rules out models where there exists at least one element in the universe of the model which verifies two distinct atomic outcomes. This includes (but is not restricted to) models involving counterfactual definiteness, and constructions such as the one presented by Bell. It is also worth mentionning that the interdiction of having such hidden variables models is independent of their possible locality, since this notion is not present in our approach. We can also derive a variant of the Kochen-Specker theorem from this result.

Corollary 9 *If $\mathcal{H}$ verifies $3 \leq \dim \mathcal{H}$, there is no model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_{L(\mathcal{H})}$ such that there exists an element $a \in A$ verifying*

$$\forall \mathcal{O} \in \mathcal{M}\big(L(\mathcal{H})\big),\, \exists p \in \mathcal{O} \colon a \blacktriangleright_{\mathfrak{G}} p$$

Another important consequence relates to the position and momentum of a particle. Heisenberg's uncertainty principle teaches us that the position and the momentum of a particle cannot be known simultaneously. Theorem 7 actually goes further, by stating that no model compatible with the predictions of orthodox quantum mechanics can simultaneously specify the position and the momentum of a particle, independently of whether these position and momentum are known (whatever the meaning of *being known*).

Theorem 7 only deals Sasaki filters, which corresponds to conditions (9a-9d) of admissible closure operators. Let us now present another result which follows moreover from conditions (9b) and (9e). However, this result only applies to the Hilbert lattice $L(\mathcal{H})$ associated to a Hilbert space $\mathcal{H}$ such that $3 \leq \dim \mathcal{H} < \infty$.

We start by some elementary bilinear algebra. For legibility reasons, elements of $L(\mathcal{H})$ will be denoted using capital letters and vectors with lower case letters, even though elements of an orthomodular lattice were previously denoted using lowercase letters. Moreover, if $P \in L(\mathcal{H})$, that is if P is a closed subspace of H, then Π_P denotes the orthogonal projection on P.

Proposition 10 *Given two closed subspaces P and Q of a Hilbert space, the restriction of P of $\Pi_P \circ \Pi_Q$ is self-adjoint.*

Proof It is a well-known fact that the orthogonal projection on a closed subspace of a Hilbert space is a self-adjoint operator. We then have, for $u, v \in P$:

$$\begin{aligned}
\langle u \mid \Pi_P \circ \Pi_Q(v)\rangle &= \langle \Pi_P(u) | \Pi_Q(v)\rangle \\
&= \langle u \mid \Pi_Q(v)\rangle \\
&= \langle \Pi_Q(u) | v\rangle \\
&= \langle \Pi_Q(u) | \Pi_P(v)\rangle \\
&= \langle \Pi_P \circ \Pi_Q(u) | v\rangle
\end{aligned}$$

□

As a consequence, if $\mathcal{H}$ is finite dimensional, which we will now assume, then P admits an orthonormal basis made of eigenvectors of $\Pi_P \circ \Pi_Q|_P$. Let $\{\alpha_i\}$ be such a basis, with corresponding eigenvalues $\{\lambda_i\}$, and let P_λ denote the eigenspace of $\Pi_P \circ \Pi_Q|_P$ associated to eigenvalue λ.

Proposition 11 *We have $P_1 = P \wedge Q$ and $P_0 = P \wedge Q^\perp$.*

Proof If $u \in P_1$, then $\big\|\Pi_Q(u)\big\| = \big\|u\big\|$ so that $\Pi_Q(u) = u$ and $P_1 \subseteq P \wedge Q$. Conversely, if $u \in P \wedge Q$, then $\Pi_P \circ \Pi_Q(u) = u$, so that

$$P_1 = P \wedge Q$$

Now, if $u \in P_0$, then $\Pi_P\big(\Pi_Q(u)\big) = |0\rangle$, which means that

$$\forall\, v \in P,\ \langle v|\Pi_Q(u)\rangle = 0$$

In particular, if $v = u$,

$$\langle \Pi_Q(u)|\Pi_Q(u)\rangle = \langle u|\Pi_Q^2(u)\rangle = \langle u|\Pi_Q(u)\rangle = 0$$

which implies that $u \in Q^\perp$. This shows that $P_0 \subseteq P \wedge Q^\perp$ and, obviously, if $u \in P \wedge Q^\perp$, then $\Pi_P \circ \Pi_Q(u) = \Pi_P(|0\rangle) = |0\rangle$ so that $P_0 = P \wedge Q^\perp$. $\square$

A consequence of these equalities is

Proposition 12 *Two subspaces P and Q are compatible if, and only if the spectrum of $\Pi_P \circ \Pi_Q|_P$ verifies*

$$\mathrm{sp}(\Pi_P \circ \Pi_Q|_P) \subseteq \{0, 1\}$$

Proof This follows directly from the previous proposition and the fact that $\Pi_P \circ \Pi_Q|_P$ is self-adjoint, so that $P = \bigvee\big\{P_\lambda \;\big|\; \lambda \in \mathbf{R}\big\}$:

$$\begin{aligned} P \text{ and } Q \text{ are compatible} &\iff P = (P \wedge Q) \vee (P \wedge Q^\perp) \\ &\iff P = P_1 \vee P_0 \\ &\iff \forall\, \lambda \notin \{0,1\},\ P_\lambda = \{\vec{0}\} \\ &\iff \mathrm{sp}\big(\Pi_P \circ \Pi_Q|_P\big) \subseteq \{0,1\} \end{aligned}$$

$\square$

Proposition 13 *If F is a closed w.r.t. an admissible closure operator* Cl*, and if P and Q are distinct incompatible elements of F, then neither P nor Q are minimal in F.*

Proof From the previous discussion, if P and Q are incompatible, then there exists an eigenvector u of $\Pi_P \circ \Pi_Q|_P$ associated with an eigenvalue $\lambda \notin \{0, 1\}$. As a consequence, by defining $v = \Pi_Q(u)$, then $u \notin Q$ and $v \notin P$. One can note moreover that $\Pi_P(v) = \lambda u$.

Let us now define $C = \mathrm{span}(u, v)$. Having $\lambda \neq 0$, one can write

$$C = \mathrm{span}(\lambda u, v - \lambda u) = \mathrm{span}\big(\Pi_P(v), v - \Pi_P(v)\big)$$

so that C is compatible with P, since $\Pi_P(v) \in P$ and $v - \Pi_P(v) \in P^\perp$. This implies that $P \,\&\, C = P \wedge C = \mathrm{span}\big(\Pi_P(v)\big) = \mathrm{span}(u)$. Similarly, one can write $C = \mathrm{span}(v, u - v) = \mathrm{span}\big(\Pi_Q(u), u - \Pi_Q(u)\big)$ so that C is compatible with Q and thus $Q \,\&\, C = \mathrm{span}(v)$.

As a consequence, $F \,\&^\# C$ contains two distinct vector rays — namely $\mathrm{span}(u)$ and $\mathrm{span}(v)$ — so that, following theorem 7, $\perp \in \mathrm{Cl}(F \,\&^\# C)$. But considering equation (9e), this applies that $C^\perp \in F$ and, as F is a Sasaki filter, it contains $C^\perp \,\&\, P$ so that P is not minimal in F since

$$C^\perp \,\&\, P = P \wedge (C^\perp \vee P^\perp) = P \wedge (C \wedge P)^\perp = P \wedge \big(\mathrm{span}(u)\big)^\perp < P$$

Similarly, F also contains $C^\perp \,\&\, Q$ which is strictly smaller than Q. □

Proposition 14 *If $\mathcal{H}$ is a Hilbert space such that $3 \leq \dim \mathcal{H} < \infty$, then any closed subset F of an admissible closure operator on $L(\mathcal{H})$ contains a most one minimal element.*

Proof Suppose that F contains two distinct minimal element P and Q. If P and Q are compatible, then $P \wedge Q = P \,\&\, Q$ belongs to F, so that neither P nor Q are minimal, which is absurd. If P and Q are incompatible, proposition 13 implies that neither P nor Q are minimal, which is absurd too. As a consequence, F contains at most one minimal element. □

Now, since $\mathcal{H}$ is finite dimensional, $L(\mathcal{H})$ has a finite height so that any non-empty subset of $L(\mathcal{H})$ has a minimal element. Combining this remark with the previous proposition, we obtain:

Theorem 15 *If $\mathcal{H}$ is a Hilbert space such that $3 \leq \dim \mathcal{H} < \infty$, the only admissible closure operator on $L(\mathcal{H})$ is $\mathrm{Cl}_\uparrow$.*

Proof Let Cl be an admissible closure operator on $L(\mathcal{H})$. We recall that a closure operator is entirely determined by its closed subsets. As a consequence, we only need to show that

$$\big\{\mathrm{Cl}(E) \mid E \subseteq L(\mathcal{H})\big\} = \big\{p^\uparrow \mid p \in L(\mathcal{H})\big\}$$

The previous discussion showed that any closed subset of $L(\mathcal{H})$ is of the form $p^\uparrow$, so that

$$\big\{\mathrm{Cl}(E) \mid E \subseteq L(\mathcal{H})\big\} \subseteq \big\{p^\uparrow \mid p \in L(\mathcal{H})\big\}$$

Conversely, for $p \in L(\mathcal{H})$, there exists a $q \in L(\mathcal{H})$ such that $\mathrm{Cl}(\{p\}) = q^\perp$. Obviously, $q \leq p$ since $p \in \mathrm{Cl}(\{p\})$ and, if $p = \perp$, then $q = \perp$. In the case where $\perp < p$, suppose that $q < p$. In that case, by orthomodularity, one has:

$$p = q \vee (p \wedge q^\perp)$$

so that $\perp < p \wedge q^\perp \leq p$. This implies that $\mathrm{Cl}(\{p\}) \subseteq \mathrm{Cl}(\{p \wedge q^\perp\})$ and hence $q \in \mathrm{Cl}(\{p \wedge q^\perp\})$. But then, from $p \wedge q^\perp \in \mathrm{Cl}(\{p \wedge q^\perp\})$ it follows that $(p \wedge q^\perp) \,\&\, q = \perp \in \mathrm{Cl}(\{p \wedge q^\perp\})$. But this is not possible, as shown by equation (8). As a consequence, we have $q = p$, so that

$$\forall\, p \in L(\mathcal{H}),\ \ \mathrm{Cl}(\{p\}) = p^\uparrow.$$

□

Proposition 16 *Given a model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_{L(\mathcal{H})}$ where $3 \leq \dim \mathcal{H} < \infty$, for all $a \in A$, there exists an element $e(a) \in L(\mathcal{H})$ such that $[\![a]\!]_{\mathfrak{G}} = e(a)^\uparrow$. Moreover, if $M(a, p, b)$, then $e(b) \leq e(a) \,\&\, p$.*

Proof The existence of the element $e(a)$ such that $[\![a]\!]_{\mathfrak{G}} = e(a)^\uparrow$ follows directly from the fact that $[\![a]\!]_{\mathfrak{G}}$ is closed w.r.t. $\mathrm{Cl}_{\mathfrak{G}}$ that is, following theorem 15, w.r.t. $\mathrm{Cl}_\uparrow$. Now, if $M(a, p, b)$, from $[\![a]\!]_{\mathfrak{G}} = e(a)^\uparrow$, it follows $a \blacktriangleright_{\mathfrak{G}} e(a)$. As a consequence, $b \blacktriangleright_{\mathfrak{G}} e(a)\&p$ and hence $e(a)\&p \in [\![b]\!]_{\mathfrak{G}}$ or, equivalently, $e(b) \leq e(a)\&p$ as $[\![b]\!]_{\mathfrak{G}} = e(b)^\uparrow$. □

It can be remarked that, in that case, for all $a \in A$, one has:

$$\begin{aligned}\forall\, p \in L, \quad \neg\big(\exists\, b \in A \colon M(a, p, b)\big) &\iff p^\perp \in [\![a]\!]_{\mathfrak{G}} \\ &\iff e(a) \leq p^\perp \\ &\iff p \leq e(a)^\perp\end{aligned}$$

so that

$$\forall\, p \in L, \quad \big(\exists\, b \in A \colon M(a, p, b)\big) \iff p \not\leq e(a)^\perp$$

As a result, in a state specified by a, an outcome p is *certain* if $e(a) \leq p$ and it is *possible* if $p \not\leq e(a)^\perp$.

Corollary 17 *If $\mathcal{H}$ is such that $3 \leq \dim \mathcal{H} < \infty$, then*

$$\forall\, p, q \in L(\mathcal{H}), \quad \mathcal{T}_{L(\mathcal{H})} \models \forall\, s \in S,\ s \blacktriangleright p \text{ and } s \blacktriangleright q \implies s \blacktriangleright p \wedge q$$

Proof Let $\mathfrak{G} = (A, M)$ be a model of $\mathcal{T}_{L(\mathcal{H})}$, and let a be such that $a \blacktriangleright_{\mathfrak{G}} p$ and $a \blacktriangleright_{\mathfrak{G}} q$. Since there exists an element $e(a) \in L(\mathcal{H})$ such that $[\![a]\!]_{\mathfrak{G}} = e(a)^\uparrow$, we have $e(a) \leq p$ and $e(a) \leq q$ so that $e(a) \leq p \wedge q$ and, finally, $a \blacktriangleright_{\mathfrak{G}} p \wedge q$. □

This can be compared to equation (6) where the deduction $s \blacktriangleright p \wedge q$ could only be made provided that p and q were compatible.

5 Sequences of Measurement Outcomes

We let us return to a question raised at the beginning of this article: given an orthomodular lattice L and elements $e_1, \ldots, e_n \in L$, is it possible to have these elements as successive outcomes when measuring a quantum system? In terms of $\mathcal{T}_L$, the question becomes whether

$$\mathcal{T}_L \models \exists\, s_0, s_1, \ldots, s_n \in S\colon\ \mathrm{Mes}(s_0, e_1, s_1) \text{ and } \mathrm{Mes}(s_1, e_2, s_2) \text{ and } \cdots \text{ and } \mathrm{Mes}(s_{n-1}, e_n, s_n)$$

Considering models of $\mathcal{T}_L$ as labelled graphs, this question can be easily translated in terms of their languages. We first introduce this notion:

Definition 8 *Given a complete orthomodular lattice L and a model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_L$, we define the language $\ell(\mathfrak{G})$ of $\mathfrak{G}$ as the language accepted by $\mathfrak{G}$ seen as a labelled automata where every state is both initial and final. Formally, a word on L (i.e. a finite sequence of elements of L) $\mathbf{p} = p_1 p_2 \cdots p_n$ is in $\ell(\mathfrak{G})$ if and only if there exists elements $a_0, a_1, \ldots, a_n \in A$ such that*

$$\forall\, k \in [\![1, n]\!],\ M(a_{k-1}, p_k, a_k)$$

In that case, given a model $\mathfrak{G}$ and a word $\mathbf{p} = p_1 \cdots p_n$, the previous question becomes to determine whether $\mathbf{p} \in \ell(\mathfrak{G})$? The answer to this question is *a priori* dependent of the graph. This fact is extremely interesting, since it might provide a method for discriminating models. Indeed, suppose that a given sequence of outcomes, i.e. that a word $\mathbf{p} = p_1 p_2 \cdots p_n$ is such that *i.* it can occur as a sequence of actual outcomes of a physical experiment, and that *ii.* it does not belong to the language $\ell(\mathfrak{G})$ of some model $\mathfrak{G}$. This provides a criteria for ruling out $\mathfrak{G}$ as a model of quantum mechanics.

However, in the case where we consider an orthomodular lattice of the form $L(\mathcal{H})$ with $3 \le \dim \mathcal{H} < \infty$, we shall see next that all the models of $\mathcal{T}_{L(\mathcal{H})}$ actually define the same language, so that it is not possible to discriminate them this way. We prove this by showing three successive inclusions:

$$\ell(\mathfrak{G}) \subseteq \ell(\mathfrak{L}_{L(\mathcal{H})}) \subseteq \ell(\mathfrak{H}_{\mathcal{H}}) \subseteq \ell(\mathfrak{G})$$

Proposition 18 *For any model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_{L(\mathcal{H})}$ with $3 \le \dim \mathcal{H} < \infty$, one has*

$$\ell(\mathfrak{G}) \subseteq \ell(\mathfrak{L}_{L(\mathcal{H})})$$

Proof With the previous notation, for any $p \in L(\mathcal{H})$, if $a, b \in A$ are such that $M(a, p, b)$, then $e(b) \le e(a) \,\&\, p$. As a consequence, if $\mathbf{p} = p_1 p_2 \cdots p_n \in \ell(\mathfrak{G})$, let $a_0, a_1, \ldots, a_n \in A$ such that $\forall\, k \in [\![1, n]\!]$, $M(a_{k-1}, p_k, a_k)$, i.e. $(a_0, \ldots, a_n)$ is a path labelled by $\mathbf{p}$. We then have

$$\forall\, k \in [\![1, n]\!],\ e(a_k) \le e(a_{k-1}) \,\&\, p_k$$

which we can write as $M_{\mathfrak{L}}\big(e(a_{k-1}), p_k, e(a_k)\big)$. As a consequence, $\mathbf{p}$ is the label in $\mathfrak{L}_{\mathcal{H}}$ of the path $\big(e(a_0), \ldots, e(a_n)\big)$, and hence $\mathbf{p} \in \ell\big(\mathfrak{L}_{\mathcal{H}}\big)$. $\square$

In order to show the second inclusion, we first recall that elements of $L(\mathcal{H})$ are subspaces of $\mathcal{H}$, i.e. they are sets of vectors. In particular, for all $a, p \in L(\mathcal{H})$, one has

$$a \,\&\, p = \big\{\Pi_p|\psi\rangle \;\big|\; |\psi\rangle \in a\big\}.$$

As a consequence, if $b \leq a \,\&\, p$, then for all $|\varphi\rangle \in b$ such that $\langle\varphi|\varphi\rangle = 1$, there exists a $|\psi\rangle \in a$ such that

$$|\varphi\rangle = \frac{\Pi_p|\psi\rangle}{\big\|\Pi_p|\psi\rangle\big\|}.$$

Proposition 19 *One has*

$$\ell\big(\mathfrak{L}_{L(\mathcal{H})}\big) \subseteq \ell(\mathfrak{H}_{\mathcal{H}})$$

Proof Let $\mathbf{p} = p_1 p_2 \cdots p_n \in \ell(\mathfrak{L}_{\mathcal{H}})$ and let $e_0, e_1, \ldots, e_n \in L(\mathcal{H})$ be such that the path $(e_0 e_1 \cdots e_n)$ is labelled by $\mathbf{p}$ or, equivalently,

$$\forall\, k \in [\![1, n]\!],\; e_k \leq e_{k-1} \,\&\, p_k$$

Let $|\varphi_n\rangle$ be any normalized element of e_n and define backwards $|\varphi_{n-1}\rangle \in e_{n-1}$, $\ldots$, $|\varphi_1\rangle \in e_1$ and $|\varphi_0\rangle \in e_0$ such that

$$\forall\, k,\; |\varphi_k\rangle = \frac{\Pi_{p_k}|\varphi_{k-1}\rangle}{\big\|\Pi_{p_k}|\varphi_{k-1}\rangle\big\|}$$

This means that for all k, $M_{\mathfrak{H}}\big(|\varphi_{k-1}\rangle, p_k, |\varphi_k\rangle\big)$, so that $\mathbf{p}$ is the label of the path $\big(|\varphi_0\rangle, |\varphi_1\rangle, \cdots, |\varphi_n\rangle\big)$ in $\mathfrak{H}_{\mathcal{H}}$. $\square$

Proposition 20 *For any model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_{L(\mathcal{H})}$ with $3 \leq \dim \mathcal{H} < \infty$, one has*

$$\ell(\mathfrak{H}_H) \subseteq \ell(\mathfrak{G})$$

Proof Let $\mathbf{p} = p_1 p_2 \cdots p_n$ be in $\ell(\mathfrak{H}_H)$, and let $\big(|\varphi_0\rangle, |\varphi_1\rangle, \ldots, |\varphi_n\rangle\big)$ be a path labelled by $\mathbf{p}$. Since $\mathfrak{G} \in \mathrm{MG}\big(L(\mathcal{H})\big)$ and $\mathrm{span}(|\varphi_0\rangle) \neq \bot$, there exists two elements $a_{-1}, a_0 \in A$ such that $M\big(a_{-1}, \mathrm{span}(|\varphi_0\rangle), a_0\big)$. With the previous notations,

$$\bot < e(a_0) \leq e(a_{-1}) \,\&\, \mathrm{span}(|\varphi_0\rangle)$$

But since $\mathrm{span}(|\varphi_0\rangle)$ is an atom of $L(\mathcal{H})$ and $e(a_{-1}) \,\&\, \mathrm{span}(|\varphi_0\rangle) \leq \mathrm{span}(|\varphi_0\rangle)$, it follows that

$$e(a_0) = \mathrm{span}(|\varphi_0\rangle)$$

Now, having $\Pi_{p_1}|\varphi_0\rangle \neq \vec{0}$ or, equivalently, $p_1 \not\leq e(a_0)^\perp$, there exists an element $a_1 \in A$ such that $M(a_0, p_1, a_1)$. It verifies:

$$e(a_1) \leq e(a_0) \,\&\, p_1$$

But again, $e(a_0)$ is an atom, so that $e(a_0) \,\&\, p_1$ is also an atom and

$$e(a_1) = e(a_0) \,\&\, p_1 = \mathrm{span}(|\varphi_0\rangle) \,\&\, p_1 = \mathrm{span}\big(\Pi_{p_1}|\varphi_0\rangle\big) = \mathrm{span}\big(|\varphi_1\rangle\big)$$

Iterating this process, it is possible to define elements $a_1, \ldots, a_n$ such that each time, $M(a_{k-1}, p_k, a_k)$ and $e(a_k) = \mathrm{span}\big(|\varphi_k\rangle\big)$.

As a consequence, we have exhibited a path $(a_0, \ldots, a_n)$ in $\mathfrak{G}$ labelled by $\mathbf{p}$, so that $\mathbf{p} \in \ell(\mathfrak{G})$. $\square$

To summarize this, we have shown:

Theorem 21 *If $3 \leq \dim \mathcal{H} < \infty$, then for all $\mathfrak{G} \in \mathrm{MG}\big(L(\mathcal{H})\big)$,*

$$\ell(\mathfrak{G}) = \ell\big(\mathfrak{L}_{L(\mathcal{H})}\big) = \ell(\mathfrak{H}_H)$$

This theorem states that the languages defined by the models of $\mathcal{T}_{L(\mathcal{H})}$ do only depend on $\mathcal{H}$, as soon as $3 \leq \dim \mathcal{H} < \infty$. If we denote this language $\ell(\mathcal{H})$, we have, considering $\mathfrak{L}_{L(\mathcal{H})}$ and $\mathfrak{H}_H$ respectively:

$$\begin{aligned}\forall\, p_1, \ldots, p_n \in L(\mathcal{H}), \quad p_1 p_2 \cdots p_n \in \ell(\mathcal{H}) &\iff p_1 \,\&\, p_2 \,\&\, \cdots \,\&\, p_n \neq \bot \\ &\iff \Pi_{p_1}\Pi_{p_2}\cdots\Pi_{p_n} \neq 0\end{aligned}$$

In particular, considering the Hilbert model $\mathfrak{H}_{\mathcal{H}}$ alone, one can say that regarding the possibility of sequences of outcomes, the description of a quantum system provided by a quantum state is complete, thus answering a decades-old question [8].

6 Conclusion and Perspectives

In this article, we have introduced a logical formulation of quantum mechanics based solely on the apparent behavior of the measurement process. The obtained logic, called $\mathcal{T}_L$, and the study of its models have led to the notion of *admissible closure operator* on an orthomodular lattice L which is closely related to the models of $\mathcal{T}_L$, as seen in propositions 4 and 5.

In the case where the orthomodular lattice is the one associated to a Hilbert space $\mathcal{H}$ of dimension at least 3, we have shown in theorem 7 that no model of $\mathcal{T}_{L(\mathcal{H})}$ can have any of its elements verify more than one atomic property, thus ruling out a large class of hidden-variable models of quantum mechanics.

Moreover, if $\mathcal{H}$ is finite dimensional, we have shown that there exists exactly one admissible closure operator, namely the one which closed subsets are the principal filters of $L(\mathcal{H})$. A consequence of this result is that, as studied in section 5, all the models of $\mathcal{T}_{L(\mathcal{H})}$ are equivalent regarding the prediction of obtaining a given sequence of outcomes, at least from a possibilistic point of view.

To that respect, quantum states appear to be a convenient tool for determining the possibility of a sequence of outcomes: such a sequence is possible if and only if one can assign a quantum state to the starting system so that the quantum state of the last system (obtained by successive orthogonal projections) is different from the null vector. Obviously, in probabilistic terms, this can be interpreted as saying that the probability of this sequence is non zero. But theorem 21 also teaches us that this method exactly captures *all* the possible sequences of outcomes: there exists no model of $\mathcal{T}_{L(\mathcal{H})}$ in which a given sequence is possible even though its probability using the Born rule is zero.

This article is only initiating the study of the logic $\mathcal{T}_L$ and the ensuing approach of quantum mechanics. A first direction for future developments is the study of the possible extension of theorem 15 to infinite dimensional Hilbert spaces. Another one is the study of the relation between the possibilistic approach developped in this article and the probabilistic one. Obviously, any probabilistic approach can lead to a possibilistic approach by considering events which probability is either zero or nonzero. But conversely, given a possibilistic model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_L$, how can one assign probabilities to the obtention of outcome p in a state $a \in A$?

A last important direction is the generalization of this approach to more complex settings. In this article, we have only considered sequences of outcomes, which correspond to a single quantum system being measured finitely many times in a row. An interesting generalization would be to consider directed acyclic graphs, which would correspond to composite quantum systems. This would, in particular, provide a way to understand the notion of quantum state in a relativistic framework, with the vertices of a graph being associated to spacetime events.

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