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Abstract

We prove the asymptotic normality of nonparametric estimator of pairwise interaction function for a stationary pairwise interaction point process characterized by the Papangelou conditional intensity and observed in a bounded window of a sequence of cubes growing up to $\mathbb{R}^d$. Formula for the variance of the resulting estimator can be obtained using Papangelou conditional intensity of the point process. This is a random function satisfying the counterpart of the Georgii-Nguyen-Zessin formula. The proof of the asymptotic normality of the resulting estimator is based on the m_n-approximation method in the setting of dependent random fields indexed by $\mathbb{Z}^d$ where d is a positive integer.

Keywords: Georgii-Nguyen-Zessin formula, nonparametric estimation, pairwise interaction point process, Papangelou conditional intensity, asymptotic normality.

1 Introduction

A special type of Gibbs processes that are noteworthy both for their abundant use in statistical physics are the pairwise interaction point processes. Several methods of estimation are available for parametric families of pairwise interaction point processes: approximate maximum likelihood ([29], [30]), maximum pseudo-likelihood ([4]), Monte Carlo likelihood ([14]).

In this paper we are concerned with nonparametric statistics for stationary pairwise interaction point processes. They have been introduced by [34], [7] and [13]. These provide a large variety of complex patterns starting from simple potential functions (or pairwise interaction function) which are easily interpretable as attractive or repulsive forces acting among points and are of practical importance because of their ability to model a wide variety of spatial point patterns, especially those displaying some degree of spatial regularity. A great deal is understood about pairwise interaction models because they are very natural with respect to the derivation of Papangelou conditional intensity ([25]).

The goal of the present paper is to establish the asymptotic normality of the nonparametric estimator of the pairwise interaction function for a stationary pairwise interaction point process characterized by the Papangelou conditional intensity. The asymptotic normality in statistics is based on central limit theorems for the sequences of random variables. These classical limit theorems have been extended to the setting of spatial processes. Some results on the central limit theorem and its functional versions are [2], [3], [9], [10] and [22]. Our proof is based on the m-approximation method ($m_n \to \infty$ as $n \to \infty$) in the setting of dependent random fields indexed by $\mathbb{Z}^d$ where d is a positive integer. We apply a central limit theorem for triangular arrays of stationary m-dependent random fields with unbounded m established by [39]. This result improves a central limit theorem established by [15]. The notion of m-dependence was first introduced by [17]. Central limit theorems for m-dependent random fields has already been considered by many researchers. For example [6], [38], [35] [24].

The paper is organized as follows. Section 2 sets up the basic tools of the Gibbs point processes in $\mathbb{R}^d$. Section 3 contains the main asymptotic results on proposed estimators of stationary pairwise interaction point process. This concerns firstly the asymptotic behaviour of variance and covariance of the kernel-type estimator under certain smoothness conditions. The latter is based on the knowledge of Papangelou conditional intensity and the iterated Georgii-Nguyen-Zessin formula. Secondly, we are able to study asymptotic Gaussianity of proposed estimators and prove it in Section 4.

2 Basic tools

Let $\mathscr{B}^d$ be the Borel σ-algebra (generated by open sets) in $\mathbb{R}^d$ (the d-dimensional space) and $\mathscr{B}^d_O \subseteq \mathscr{B}^d$ be the system of all bounded Borel sets. A point process $\mathbf{X}$ in $\mathbb{R}^d$ is a locally finite random subset of $\mathbb{R}^d$, i.e. the number of points $N(\Lambda) = n(\mathbf{X}_\Lambda)$ of the restriction of $\mathbf{X}$ to Λ is a finite random variable whenever Λ is a bounded Borel set of $\mathbb{R}^d$ (see [7]). We define the space of locally finite point configurations in $\mathbb{R}^d$ as $N_{lf} = \{\mathbf{x} \subseteq \mathbb{R}^d; n(\mathbf{x}_\Lambda) < \infty, \forall \Lambda \in \mathscr{B}^d_0\}$, where $\mathbf{x}_\Lambda = \mathbf{x} \cap \Lambda$. We equip N_{lf} with σ-algebra $\mathscr{N}_{lf} = \sigma\{\{\mathbf{x} \in N_{lf} : n(\mathbf{x}_\Lambda) = m\}, m \in \mathbb{N}_0, \Lambda \in \mathscr{B}^d_0\}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$. The volume of a bounded Borel set Λ of $\mathbb{R}^d$ is denoted by $|\Lambda|$. For any finite subset I of $\mathbb{Z}^d$, we denote $|I|$ the number of elements in I, $[\![a, b]\!] \equiv \{a, a+1, \ldots, b\}$ for $a, b \in \mathbb{Z}$ and the sum $\sum^{\neq}$ signifies summation over distinct pairs. For any Gibbs point processes in a bounded window, the conditional intensity at a location u given the configuration $\mathbf{x}$ is related to the probability density f by

$$\lambda(u,\mathbf{x}) = \frac{f(\mathbf{x}\cup\{u\})}{f(\mathbf{x})}$$

(for $u \notin \mathbf{x}$), the ratio of the probability densities for the configuration $\mathbf{x}$ with and without the point u added. The Papangelou conditional intensity can be interpreted as follows: for any $u \in \mathbb{R}^d$ and $\mathbf{x} \in N_{lf}$, $\lambda(u,\mathbf{x})\mathrm{d}u$ corresponds to the conditional probability of observing a point in a ball of volume $\mathrm{d}u$ around u given the rest of the point process is $\mathbf{x}$.

Pairwise interaction point processes in $\mathbb{R}^d$ can be defined and characterized through the Papangelou conditional intensity (see [25]) which is a function λ : $\mathbb{R}^d \times N_{lf} \to \mathbb{R}_+$. The Georgii-Nguyen-Zessin (GNZ) formula (see [31], [40], [12], [28]) states that for any nonnegative measurable function h on $\mathbb{R}^d \times N_{lf}$

$$\mathrm{E}\sum_{u\in\mathbf{X}} h(u,\mathbf{X}\setminus u) = \mathrm{E}\int_{\mathbb{R}^d} h(u,\mathbf{X})\lambda(u,\mathbf{X})\mathrm{d}u. \tag{2.1}$$

Using induction we obtain the iterated GNZ-formula: for nonnegative functions $h : (\mathbb{R}^d)^n \times N_{lf} \longrightarrow \mathbb{R}$

$$\mathrm{E}\sum_{u_1,\dots,u_n\in\mathbf{X}}^{\neq} h(u_1,\dots,u_n,\mathbf{X}\setminus\{u_1,\dots,u_n\})$$
$$= \int\dots\int \mathrm{E}\,h(u_1,\dots,u_n,\mathbf{X})\lambda(u_1,\dots,u_n,\mathbf{X})\mathrm{d}u_1\dots\mathrm{d}u_n \tag{2.2}$$

where $\lambda(u_1,\dots,u_n,\mathbf{x})$ is Papangelou conditional intensity and is defined (not uniquely) by $\lambda(u_1,\dots,u_n,\mathbf{x}) = \lambda(u_1,\mathbf{x})\lambda(u_2,\mathbf{x}\cup\{u_1\})\dots\lambda(u_n,\mathbf{x}\cup\{u_1,\dots,u_{n-1}\})$. Examples of Gibbs point process models and their conditional intensities are presented in [1], [25] and [26]. The formula (2.2) will also be used extensively later in this document.

3 Assumptions and the main results

For the general pairwise interaction process the conditional intensity is

$$\lambda(u,\mathbf{x}) = g_0(u)\exp\left(-\sum_{v\in\mathbf{x}\setminus u} g_0(u,v)\right)$$

and note that $g_0(u,v) = g_0(v,u)$ (i.e. symmetric pairwise interaction).

Alternatively, if $g_0(u)$ is a constant and $g_0(u,v) = g(v-u)$ is translation invariant, then a pairwise interaction point process is called stationary.

Throughout this paper, we define a stationary pairwise interaction point process via the Papangelou conditional intensity at a location u given by

$$\lambda_{\beta^\star}(u,\mathbf{x}) = \beta^\star \exp\left(-\sum_{v\in\mathbf{x}\setminus u} g(v-u)\right) \tag{3.1}$$

where $\beta^\star$ is the true value of the Poisson intensity parameter, g represents a pairwise interaction potential. Specifically, g is nonnegative measurable function; the practical significance of this assumption is that pairwise interaction processes are a wide and useful class of models for spatial point processes with inhibition between points; for discussion see [11] and [36]. We denote $G = \exp(-g)$ the pairwise interaction function. The basic assumption throughout this paper is the Papangelou conditional intensity (3.1) has a finite range R, i.e.

$$\lambda_{\beta^\star}(u,\mathbf{x}) = \lambda_{\beta^\star}(u,\mathbf{x}\cap B(u,R)), \tag{3.2}$$

for any $u\in\mathbb{R}^d$, $\mathbf{x}\in N_{lf}$, where $B(u,R)$ is the closed ball in $\mathbb{R}^d$ with center u and radius R.

Suppose that a single realization $\mathbf{x}$ of a point process $\mathbf{X}$ is observed in a bounded window $\Lambda_n \in \mathscr{B}_0^d$ where $(\Lambda_n)_{n\geq 1}$ is a sequence of cubes growing up to $\mathbb{R}^d$. We face a missing data problem, which in the spatial point process literature is referred to as a problem of edge effects, we can avoid this problem by reducing the window by introducing the $2R$-interior of the cubes Λ_n, i.e. $\Lambda_{n,R} = \{u\in\Lambda_n : B(u,2R)\subset\Lambda_n\}$ and assume this has non-zero area. The choice of $\Lambda_{n,R}$ is induced by the fact that when u is on the edge of $\Lambda_{n,R}$ and $v\in B(u,R)$, then we can observe the point v. Various edge corrections have been suggested by [33], [5], [20], [25] and [37]). We assume that the support of the interaction function $G=\exp(-g)$ is $T=\{t\in\mathbb{R}^d; g(t)>0, \text{ for } ||t||<R\}$. Throughout in this paper, h is a nonnegative measurable function defined for all $w\in\mathbb{R}^d$, $\mathbf{x}\in N_{lf}$, by

$$h(w,\mathbf{x}) = \mathbb{1}(\mathbf{x}\cap B(w,R)=\emptyset), \tag{3.3}$$

and we also introduce the following function

$$\bar{F}(o,w) = \mathrm{E}[h(o,\mathbf{X})h(w,\mathbf{X})] = \mathbb{P}(\mathbf{X}\cap B(o,R)=\emptyset, \mathbf{X}\cap B(w,R)=\emptyset). \tag{3.4}$$

To estimate the function $\beta^\star\bar{F}(o,t)$, we propose empiric estimator $\widehat{\bar{F}}_n(t)$ defined for $t\in T$ by

$$\widehat{\bar{F}}_n(t) = \frac{1}{|\Lambda_{n,R}|}\sum_{u\in\mathbf{X}_{\Lambda_{n,R}}} h(u,\mathbf{X}\setminus\{u\})h(t+u,\mathbf{X}\setminus\{u\}). \tag{3.5}$$

To estimate the function $\beta^{\star 2}G(t)\bar{F}(o,t)$, we propose kernel-type estimator $\widehat{H}_n(t)$ defined for $t \in T$ by

$$\widehat{H}_n(t) = \frac{1}{b_n^d|\Lambda_{n,R}|} \sum_{\substack{u,v\in\mathbf{X} \\ v-u\in B(o,R)}}^{\neq} 1\!\!1_{\Lambda_{n,R}}(u)h(u,\mathbf{X}\setminus\{u,v\})h(v,\mathbf{X}\setminus\{u,v\})K\left(\frac{v-u-t}{b_n}\right), \tag{3.6}$$

where $\neq$ over the summation sign means that the sum runs over all pairwise different points $u;v$ in $\mathbf{X}$ and $K:\mathbb{R}^d \to \mathbb{R}$ denotes a smoothing kernel function associated with a sequence $(b_n)_{n\geq 1}$ of bandwidths. Plugging in the above estimators (3.5) and (3.6) and with the convention $c/0 = 0$ for all real c, we suggest a new edge-corrected nonparametric estimator $\widehat{G}_n(t)$ for $\beta^\star G(t)$ for $t \in T$ by

$$\widehat{G}_n(t) = \frac{\widehat{H}_n(t)}{\widehat{\bar{F}}_n(t)}. \tag{3.7}$$

Moreover, we have to impose certain natural restrictions on the kernel function K and the sequence $(b_n)_{n\geq 1}$.

Condition $K(d)$**.**

The sequence of bandwidths $(b_n)_{n\geq 1}$, is a sequence of positive real numbers satisfying:

$$\lim_{n\to\infty} b_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n^d|\Lambda_{n,R}| = \infty.$$

The kernel function $K:\mathbb{R}^d \longrightarrow \mathbb{R}$ is nonnegative and bounded with bounded support and satisfies:

$$\int_{\mathbb{R}^d} K(z)\mathrm{d}z = 1, \int_{\mathbb{R}^d} K^2(z)\mathrm{d}z < \infty.$$

Condition $K(d,m)$. Let $z = (z_1,\ldots,z_d)'$, $z_i \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} z_1^{i_1}\ldots z_d^{i_d} K(z_1,\ldots,z_d)\mathrm{d}z_1\ldots\mathrm{d}z_d = 0, \quad \text{for} \quad 0 < \sum_{j=1}^d i_j < m,$$

and

$$\int_{\mathbb{R}^d} |z|^m K(z)\mathrm{d}z < \infty.$$

3.1 Asymptotic behaviour of the variance and the covariance of the kernel-type estimator

In this section, we investigate the asymptotic behaviour of the variance and the covariance of the kernel-type estimator (3.6) estimating $\beta^{\star 2}G(t)\bar{F}(o,t)$. The latter is based on the knowledge of the iterated GNZ-formula (2.2).

Theorem 3.1. *Consider a stationary pairwise interaction point process* $\mathbf{X}$ *in* $\mathbb{R}^d$ *with Papangelou conditional intensity* (3.1) *satisfying condition* (3.2)*. Further let the kernel function* K *satisfy Condition* $K(d)$*. We have*

$$\lim_{n\to\infty} b_n^d |\Lambda_{n,R}| \operatorname{Var}\left(\widehat{H}_n(t)\right) = \beta^{\star 2} G(t)\bar{F}(o,t)\int_{\mathbb{R}^d} K^2(z)\mathrm{d}z$$

at any continuity point $t \in T\backslash\{o\}$ *of* $G\bar{F}$*.*

The following theorem presents an asymptotic representation of the covariance of the kernel-type estimator (3.6) in two points t_1, t_2 in T.

Theorem 3.2. *Consider a stationary pairwise interaction point process* $\mathbf{X}$ *in* $\mathbb{R}^d$ *with Papangelou conditional intensity* (3.1) *satisfying condition* (3.2)*. Further let the kernel function* K *satisfy Condition* $K(d)$ *and let* $t_1 \neq t_2 \in T$*. We have*

$$\lim_{n\to\infty} b_n^d |\Lambda_{n,R}| \operatorname{Cov}\left(\widehat{H}_n(t_1), \widehat{H}_n(t_2)\right) = 0.$$

The proof of Theorem 3.2 is similar to that of Theorem 3.1, and is omitted.

3.2 Asymptotic normality of a nonparametric estimator

Asymptotic normality of a nonparametric estimate of the pairwise interaction function G for a stationary pairwise interaction point process is essentially based on a multivariate central limit theorem for the estimator $\widehat{H}_n(t)$ of $\beta^{\star 2}G(t)\bar{F}(o,t)$. The proving idea of the below Theorem 3.3 consists in approximating the sequence $\widehat{H}_n(t)$ by a triangular array of m_n-dependent random fields. The corresponding Lindeberg-type CLT for m_n-dependent random fields has been proved in [39].

Now, we assume that the domain Λ_n is divided into a fixed number of subdomains as follows $\Lambda_n = \cup_{i\in I_n}\Lambda_i$, following [32], [21], [18] we will describe a point process in $\mathbb{R}^d$ as lattice process by means of this decomposition $\Lambda_i = \{\xi \in \mathbb{R}^d; \widetilde{q}(i_j - \frac{1}{2}) \leq \xi_j \leq \widetilde{q}(i_j + \frac{1}{2}), j = 1,\ldots,d\}$ for a fixed number $\widetilde{q} > 0$, $i = (i_1,\ldots,i_d)$, and setting $\mathbf{X}_i = \mathbf{X}_{\Lambda_i}$, $i \in \mathbb{Z}^d$, this becomes a Gibbs lattice field. We will consider estimation of $\beta^{\star 2}G(t)\bar{F}(o,t)$ from $\widehat{H}_n(t)$, where the process is observed in $\Lambda_{n,R} = \cup_{i\in\widetilde{I}_n}\Lambda_i$, where $\widetilde{I}_n = \{i \in I_n; |i-j| \leq 1, \text{ for all } j \in I_n\}$, and the norm is $|j| = \max\{|j_1|,\ldots,|j_d|\}$ and assume that I_n increases towards $\mathbb{Z}^d$ and write

$$\widehat{H}_n(t) = \frac{1}{b_n^d|\Lambda_{n,R}|}\sum_{i\in\widetilde{I}_n}\sum_{\substack{u\in\mathbf{X}_i, v\in\mathbf{X}\\ v-u\in B(o,R)}}^{\neq} h(u,\mathbf{X}\setminus\{u,v\})h(v,\mathbf{X}\setminus\{u,v\})K\left(\frac{v-u-t}{b_n}\right).$$

We consider in this work the field $\{\mathbf{X}_i\}_{i\in\mathbb{Z}^d}$ $(d\in\mathbb{N})$ in form of

$$\mathbf{X}_i = \widetilde{F}(\varepsilon_{i-k}, k\in\mathbb{Z}^d), i\in\mathbb{Z}^d. \tag{3.8}$$

For each $m\in\mathbb{N}$, $i\in\mathbb{Z}^d$, we write $\mathbf{X}_{i,m} = \widetilde{F}(\varepsilon_{i-k}, k\in[\![0,m-1]\!]^d)$, where $\{\varepsilon_i\}_{i\in\mathbb{Z}^d}$ are i.i.d. random variables and $\widetilde{F}$ is measurable function. Let $\{\varepsilon_i'\}_{i\in\mathbb{Z}^d}$ be an i.i.d. copy of $\{\varepsilon_i\}_{i\in\mathbb{Z}^d}$ and consider for all positive integer n the coupled version $\mathbf{X}_i^\star$ defined by $\mathbf{X}_i^\star = \widetilde{F}(\varepsilon_{i-k}^\star, k\in\mathbb{Z}^d)$, where

$$\varepsilon_j^\star = \begin{cases} \varepsilon_j & \text{if} \quad j\neq 0 \\ \varepsilon_0' & \text{if} \quad j=0. \end{cases}$$

Let $i\in\mathbb{Z}^d$ and $p>0$ be fixed. We define $\delta_{i,p} = \|\mathbf{X}_i - \mathbf{X}_i^\star\|_p$, where $\|\cdot\|_p$ is the usual $\mathbb{L}^p$-norm. Thus, [23] obtained this sufficient condition:

$$\sum_{i\in\mathbb{Z}^d} |i|^{5d/2}\delta_{i,p} < \infty \tag{3.9}$$

for a random field of the form (3.8) for the kernel density estimator to be asymptotically normal.

The desired sequence $(m_n)_{n\geq 1}$ can be chosen as

$$m_n = \max\left\{ v_n, \left[\left(\frac{1}{b_n^3}\sum_{|i|>v_n} |i|^{\frac{5d}{2}}\delta_{i,2}\right)^{\frac{1}{3d}}\right] + 1 \right\}$$

where $v_n = [b_n^{-\frac{1}{2d}}]$ and $[\cdot]$ denotes the integer part function.

In order to establish the asymptotic normality of $\widehat{H}_n$, we need additional assumptions:

Condition $\mathscr{W}$. There exists a sequence of integers $(m_n)_{n\geq 1}$ such that $m_n\to\infty$ as $n\to\infty$, and the following limits hold

$$m_n^d b_n^d \to 0 \quad \text{and} \quad \frac{1}{(m_n b_n)^{3d/2}} \sum_{|i|>m_n} |i|^{\frac{5d}{2}}\delta_{i,2} \to 0 \quad \text{if (3.9) holds.} \tag{3.10}$$

$$\frac{m_n^{2d}}{b_n^d n^d} \to 0. \tag{3.11}$$

Theorem 3.3. *Consider a stationary pairwise interaction point process* $\mathbf{X}$ *in* $\mathbb{R}^d$ *with Papangelou conditional intensity (3.1) satisfying condition (3.2). Further let the kernel function* K *satisfy Condition* $K(d)$ *and assume that Condition* $\mathscr{W}$ *holds. Then, for any positive integer* s *and any distinct points* $t_1,\ldots,t_s$ *in* $T\backslash\{o\}$*, we have*

$$\sqrt{b_n^d|\Lambda_{n,R}|}\left(\widehat{H}_n(t_i) - \mathrm{E}\widehat{H}_n(t_i)\right)_{i=1}^s \xrightarrow{d} \mathcal{N}(0,\sigma_H)$$

where the covariance matrix $\sigma_H = [r_{i,j}]_{i,j=1,\dots,s}$ *is given by*

$$r_{i,i} = \beta^{\star 2} G(t_i) \bar{F}(o,t_i) \int_{\mathbb{R}^d} K^2(z)\mathrm{d}z, \quad \text{for} \quad i = 1,\dots,s \quad \text{and} \;\; r_{i,j} = 0 \quad \text{if} \;\; i \neq j.$$

Corollary 1. *Let, in addition to the assumptions of Theorem 3.3 and the kernel function* K *satisfy Condition* $K(d,m)$ *and let* $b_n^{d+2m}|\Lambda_{n,R}| \to 0$ *as* $n \to +\infty$*. Furthermore if* $G(t)\bar{F}(o,t)$ *has bounded and continuous partial derivatives of order* m *in some neighborhood of the points* $t_1,\dots,t_s$*. Then*

$$\sqrt{b_n^d |\Lambda_{n,R}|}\left(\widehat{H}_n(t_i) - \beta^{\star 2} G(t_i)\bar{F}(o,t_i)\right)_{i=1}^{s} \xrightarrow{d} \mathscr{N}(0,\sigma_H)$$

where the covariance matrix $\sigma_H = [r_{i,j}]_{i,j=1,\dots,s}$ *is given by*

$$r_{i,i} = \beta^{\star 2} G(t_i) \bar{F}(o,t_i) \int_{\mathbb{R}^d} K^2(z)\mathrm{d}z, \quad \text{for} \quad i = 1,\dots,s \quad \text{and} \;\; r_{i,j} = 0 \quad \text{if} \;\; i \neq j.$$

The estimator (3.5) turns out to be unbiased estimator of $\beta^\star \bar{F}(o,t)$ and strongly consistent (the uniform strong consistency) as n tends infinity, since a classical ergodic theorem for spatial point processes obtained in [27]. This implies the following:

Theorem 3.4. *Consider a stationary pairwise interaction point process* $\mathbf{X}$ *in* $\mathbb{R}^d$ *with Papangelou conditional intensity* (3.1) *satisfying condition* (3.2)*. Further let the kernel function* K *satisfy Condition* $K(d)$*, Condition* $K(d,m)$ *and assume that Condition* $\mathscr{W}$ *and* $b_n^{d+2m}|\Lambda_{n,R}| \to 0$ *as* $n \to \infty$ *hold. Then for any positive integer* s *and any distinct points* $t_1,\dots,t_s$ *in* $T\backslash\{o\}$*,*

$$\sqrt{b_n^d |\Lambda_{n,R}|}\left(\widehat{G}_n(t_i) - \beta^\star G(t_i)\right)_{i=1}^{s} \xrightarrow{d} \mathscr{N}(0,\sigma_G)$$

where the covariance matrix $\sigma_G = [\sigma_{i,j}]_{i,j=1,\dots,s}$ *is given by*

$$\sigma_{i,i} = \frac{G(t_i)}{\bar{F}(o,t_i)} \int_{\mathbb{R}^d} K^2(z)\mathrm{d}z \quad \text{for} \quad i = 1,\dots,s \quad \text{and} \;\; \sigma_{i,j} = 0 \quad \text{if} \;\; i \neq j.$$

4 Proofs

4.1 Proof of Theorem 3.1

Proof. We determine the asymptotic behaviour of the variance of the estimator $\widehat{H}_n(t)$. For this we use the following result.

Lemma 1. *Consider any Gibbs point process* $\mathbf{X}$ *in* $\mathbb{R}^d$ *with Papangelou conditional intensity* λ. *Let* $f : \mathbb{R}^d \times \mathbb{R}^d \times N_{lf} \longrightarrow \mathbb{R}$ *be a nonnegative and measurable function. Then*

$$
\begin{aligned}
&\operatorname{Var}\left(\sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\setminus\{u,v\})\right)\\
&= \mathrm{E}\int_{\mathbb{R}^{2d}} f(u,v,\mathbf{X})\left[f(u,v,\mathbf{X})+f(v,u,\mathbf{X})\right]\lambda(u,v,\mathbf{X})\mathrm{d}u\mathrm{d}v\\
&+\mathrm{E}\int_{\mathbb{R}^{3d}} f(u,v,\mathbf{X})\left[f(v,w,\mathbf{X})+f(w,v,\mathbf{X})\right]\lambda(u,v,w,\mathbf{X})\mathrm{d}u\mathrm{d}v\mathrm{d}w\\
&+\mathrm{E}\int_{\mathbb{R}^{3d}} f(u,v,\mathbf{X})\left[f(u,w,\mathbf{X})+f(w,u,\mathbf{X})\right]\lambda(u,v,w,\mathbf{X})\mathrm{d}u\mathrm{d}v\mathrm{d}w\\
&+\mathrm{E}\int_{\mathbb{R}^{4d}} f(u,v,\mathbf{X})f(w,y,\mathbf{X})\lambda(u,v,w,y,\mathbf{X})\mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y\\
&-\int_{\mathbb{R}^{4d}} \mathrm{E}\left[f(u,v,\mathbf{X})\lambda(u,v,\mathbf{X})\right]\mathrm{E}\left[f(w,y,\mathbf{X})\lambda(w,y,\mathbf{X})\right]\mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y.
\end{aligned}
$$

Proof. After a simple calculation, but prolix (see [19] and [16]), we obtain the following decomposition:

$$
\begin{aligned}
\left(\sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\setminus\{u,v\})\right)^2 &= \sum_{u,v\in\mathbf{X}}^{\neq} f^2(u,v,\mathbf{X}\setminus\{u,v\})\\
&+\sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\setminus\{u,v\})f(v,u,\mathbf{X}\setminus\{u,v\})\\
&+\sum_{u,v,w\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\setminus\{u,v,w\})f(v,w,\mathbf{X}\setminus\{u,v,w\})\\
&+\sum_{u,v,w\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\setminus\{u,v,w\})f(w,v,\mathbf{X}\setminus\{u,v,w\})\\
&+\sum_{u,v,w\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\setminus\{u,v,w\})f(u,w,\mathbf{X}\setminus\{u,v,w\})\\
&+\sum_{u,v,w\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\setminus\{u,v,w\})f(w,u,\mathbf{X}\setminus\{u,v,w\})\\
&+\sum_{u,v,w,y\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\setminus\{u,v,w,y\})f(w,y,\mathbf{X}\setminus\{u,v,w,y\}).
\end{aligned}
\tag{4.1}
$$

On the other hand using standard definition of variance, we have

$$\operatorname{Var} \sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\backslash\{u,v\}) = \mathrm{E}\left(\sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\backslash\{u,v\})\right)^2 - \left(\mathrm{E}\sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\backslash\{u,v\})\right)^2. \tag{4.2}$$

After rearrangement with the formula of GNZ (2.2) and formulas (4.1) and (4.2), we obtain the desired result. □

In the sequel, we denote

$$H(u_1,u_2,\ldots,u_s,\mathbf{X}) = h(u_1,\mathbf{X})h(u_2,\mathbf{X})\ldots h(u_s,\mathbf{X}), \tag{4.3}$$

where h is given by (3.3),

$$I_R(u_1,u_2,\ldots,u_s) = 1\!\!1\left(u_1 \in B(o,R), u_2 \in B(o,R), \ldots, u_s \in B(o,R)\right) \tag{4.4}$$

and we keep in mind the following function

$$\bar{F}(u_1,u_2,\ldots,u_s) = \mathrm{E}[H(u_1,u_2,\ldots,u_s,\mathbf{X})] = \mathrm{E}[h(u_1,\mathbf{X})h(u_2,\mathbf{X})\ldots h(u_s,\mathbf{X})]. \tag{4.5}$$

Substitute

$$f(u,v,\mathbf{x}) = 1\!\!1_{\Lambda_{n,R}}(u) I_R(v-u) H(u,v,\mathbf{x}) K\left(\frac{v-u-t}{b_n}\right)$$

the variance term of $\widehat{H}_n(t)$ is now expanded using Lemma 1, therefore we find that

$$\operatorname{Var}\widehat{H}_n(t) = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8,$$

with

$$T_1 = \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathrm{E} \int_{\mathbb{R}^{2d}} 1\!\!1_{\Lambda_{n,R}}(u) I_R(v-u) H(u,v,\mathbf{X}) K^2\left(\frac{v-u-t}{b_n}\right) \lambda_{\beta^\star}(u,v,\mathbf{X}) \mathrm{d}u\mathrm{d}v,$$

$$T_2 = \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathrm{E} \int_{\Lambda_{n,R}^2} I_R(v-u,u-v) H(u,v,\mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{u-v-t}{b_n}\right) \lambda_{\beta^\star}(u,v,\mathbf{X}) \mathrm{d}u\mathrm{d}v,$$

$$T_3 = \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathrm{E} \int_{\mathbb{R}^d} \int_{\Lambda_{n,R}^2} I_R(v-u,w-v) H(u,v,w,\mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{w-v-t}{b_n}\right) \times \lambda_{\beta^\star}(u,v,w,\mathbf{X}) \mathrm{d}u\mathrm{d}v\mathrm{d}w,$$

$$T_4 = \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathrm{E} \int_{\mathbb{R}^d} \int_{\Lambda_{n,R}^2} I_R(v-u,v-w) H(u,v,w,\mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{v-w-t}{b_n}\right) \times \lambda_{\beta^\star}(u,v,w,\mathbf{X}) \mathrm{d}u\mathrm{d}v\mathrm{d}w,$$

$$T_5 = \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathrm{E} \int_{\mathbb{R}^{2d}} \int_{\Lambda_{n,R}} I_R(v-u,w-u) H(u,v,w,\mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{w-u-t}{b_n}\right) \times \lambda_{\beta^\star}(u,v,w,\mathbf{X}) \mathrm{d}u\mathrm{d}v\mathrm{d}w,$$

$$T_6 = \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathrm{E} \int_{\mathbb{R}^d} \int_{\Lambda_{n,R}^2} I_R(v-u,u-w) H(u,v,w,\mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{u-w-t}{b_n}\right) \times \lambda_{\beta^\star}(u,v,w,\mathbf{X}) \mathrm{d}u\mathrm{d}v\mathrm{d}w,$$

$$T_7 = \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathrm{E} \int_{\mathbb{R}^{4d}} 1\!\!1_{\Lambda_{n,R}}(u) 1\!\!1_{\Lambda_{n,R}}(w) I_R(v-u,y-w) H(u,v,w,y,\mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) \times K\left(\frac{y-w-t}{b_n}\right) \lambda_{\beta^\star}(u,v,w,y,\mathbf{X}) \mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y$$

and

$$T_8 = -\frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \int_{\mathbb{R}^{4d}} 1\!\!1_{\Lambda_{n,R}}(u) 1\!\!1_{\Lambda_{n,R}}(w) I_R(v-u,y-w) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) \times \mathrm{E}\left[H(u,v,\mathbf{X})\lambda_{\beta^\star}(u,v,\mathbf{X})\right] \mathrm{E}\left[H(w,y,\mathbf{X})\lambda_{\beta^\star}(w,y,\mathbf{X})\right] \mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y.$$

The asymptotic behaviour of the leading term T_1 is obtained by applying the second order Papangelou conditional intensity given by:

$$\lambda_{\beta^\star}(u,v,\mathbf{x}) = \lambda_{\beta^\star}(u,\mathbf{x})\lambda_{\beta^\star}(v,\mathbf{x}\cup\{u\}) \quad \text{for any } u,v\in\mathbb{R}^d \quad \text{and } \mathbf{x}\in N_{lf}.$$

And using the finite range property (3.2) for each function $\lambda_{\beta^\star}(u,\mathbf{x})$ and $\lambda_{\beta^\star}(v,\mathbf{x}\cup\{u\})$. We recall when $\mathbf{x}=\emptyset$, this implies that

$$\lambda_{\beta^\star}(u,\emptyset) = \beta^\star \quad \text{and} \quad \lambda_{\beta^\star}(v,\emptyset\cup\{u\}) = \beta^\star G(v-u) \quad \text{for all} \quad u,v\in\mathbb{R}^d.$$

By stationarity of $\mathbf{X}$, it results

$$\begin{aligned}
T_1 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2}\,\mathrm{E}\int_{\mathbb{R}^{2d}} 1\!\!1_{\Lambda_{n,R}}(u) I_R(v-u) H(u,v,\mathbf{X}) K^2\left(\frac{v-u-t}{b_n}\right) \lambda_{\beta^\star}(u,v,\mathbf{X})\mathrm{d}u\mathrm{d}v \\
&= \frac{\beta^{\star 2}}{b_n^{2d}|\Lambda_{n,R}|^2}\int_{\mathbb{R}^{2d}} 1\!\!1_{\Lambda_{n,R}}(u) I_R(v-u)\bar{F}(u,v) K^2\left(\frac{v-u-t}{b_n}\right) G(v-u)\mathrm{d}u\mathrm{d}v \\
&= \frac{\beta^{\star 2}}{b_n^{2d}|\Lambda_{n,R}|}\int_{\mathbb{R}^{d}} I_R(s)\bar{F}(o,s) K^2\left(\frac{s-t}{b_n}\right) G(s)\mathrm{d}s \\
&= \frac{\beta^{\star 2}}{b_n^{d}|\Lambda_{n,R}|}\int_{\mathbb{R}^{d}} I_R(b_n z+t)\bar{F}(o,b_n z+t) K^2(z) G(b_n z+t)\mathrm{d}z.
\end{aligned}$$

The continuity of $G\bar{F}$ in t and the boundedness conditions on the kernel function yield the desired result by dominated convergence theorem, i.e. $\lim_{n\to\infty} b_n^d|\Lambda_{n,R}|T_1 = \beta^{\star 2}G(t)\bar{F}(o,t)\int_{\mathbb{R}^d} K^2(z)\mathrm{d}z$.

The following lemma will justify the applicability of dominated convergence theorem in the proofs of the next asymptotic results.

Lemma 2. *([8], Lemma A.2) Let $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of convex sets in $\mathbb{R}^d$ growing up to $\mathbb{R}^d$ and let $y,z\in\mathbb{R}^d$. Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers with $b_n \underset{n\to\infty}{\longrightarrow} 0$. Then we have*

$$\lim_{n\to\infty}\frac{\left|\Lambda_n\cap(\Lambda_n - b_n y - z)\right|}{|\Lambda_n|} = 1.$$

We will now show that all the other terms of the variance of the estimator $\widehat{H}_n(t)$ vanish. With similar arguments to those used to obtain the results above, we treat the term T_2.

$$T_2 = \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathrm{E} \int_{\mathbb{R}^{2d}} 1\!\!1_{\Lambda_{n,R}}(u) 1\!\!1_{\Lambda_{n,R}}(v) I_R(v-u, u-v) H(u,v,\mathbf{X}) K\left(\frac{v-u-t}{b_n}\right)$$
$$\times K\left(\frac{u-v-t}{b_n}\right) \lambda_{\beta^\star}(u,v,\mathbf{X}) \mathrm{d}u \mathrm{d}v$$
$$= \frac{\beta^{\star 2}}{b_n^{2d}|\Lambda_{n,R}|^2} \int_{\mathbb{R}^{2d}} 1\!\!1_{\Lambda_{n,R}}(u) 1\!\!1_{\Lambda_{n,R}}(v) I_R(v-u, u-v) \bar{F}(o, v-u) K\left(\frac{v-u-t}{b_n}\right)$$
$$\times K\left(\frac{u-v-t}{b_n}\right) G(v-u) \mathrm{d}u \mathrm{d}v$$
$$= \frac{\beta^{\star 2}}{b_n^{2d}|\Lambda_{n,R}|}$$
$$\int_{\mathbb{R}^d} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - s)|}{|\Lambda_{n,R}|} I_R(s,-s) \bar{F}(o,s) K\left(\frac{s-t}{b_n}\right) K\left(\frac{-s-t}{b_n}\right) G(s) \mathrm{d}s$$
$$= \frac{\beta^{\star 2}}{b_n^{d}|\Lambda_{n,R}|} \int_{\mathbb{R}^d} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - (b_n z + t))|}{|\Lambda_{n,R}|} I_R(b_n z + t, -b_n z - t) \bar{F}(o, b_n z + t) K(z)$$
$$\times K(-z - \frac{2t}{b_n}) G(b_n z + t) \mathrm{d}z.$$

The boundedness conditions on the kernel and under condition $t \neq o$ imply that $b_n^d |\Lambda_{n,R}| T_2 \longrightarrow 0$ as $n \to \infty$, by dominated convergence theorem and the geometric properties of $\Lambda_{n,R}$ (Lemma 2).

For the asymptotic behaviour of the third term T_3, we remember the third order Papangelou conditional intensity by

$$\lambda_{\beta^\star}(u,v,w,\mathbf{x}) = \lambda_{\beta^\star}(u,\mathbf{x}) \lambda_{\beta^\star}(v, \mathbf{x} \cup \{u\}) \lambda_{\beta^\star}(w, \mathbf{x} \cup \{u,v\})$$

for any $u,v,w \in \mathbb{R}^d$ and $\mathbf{x} \in N_{lf}$. Using the finite range property (3.2) for each function $\lambda_{\beta^\star}(u,\mathbf{x})$, $\lambda_{\beta^\star}(v,\mathbf{x} \cup \{u\})$ and $\lambda_{\beta^\star}(w,\mathbf{x} \cup \{u,v\})$, i.e.

$$\begin{aligned} \lambda_{\beta^\star}(u,\mathbf{X}) &= \lambda_{\beta^\star}(u, \mathbf{X} \cap B(u,R)) \\ &= \beta^\star \quad \text{when} \quad \mathbf{X} \cap B(u,R) = \emptyset, \end{aligned}$$

$$\begin{aligned} \lambda_{\beta^\star}(v, \mathbf{X} \cup \{u\}) &= \lambda_{\beta^\star}(v, (\mathbf{X} \cup \{u\}) \cap B(v,R)) \\ &= \beta^\star G(v-u) \quad \text{when} \quad \mathbf{X} \cap B(v,R) = \emptyset, v-u \in B(o,R), \end{aligned}$$

and when $\mathbf{X} \cap B(w,R) = \emptyset$ and $w - v \in B(o,R)$, we have

$$\begin{aligned} \lambda_{\beta^\star}(w, \mathbf{X} \cup \{u,v\}) &= \lambda_{\beta^\star}(w, (\mathbf{X} \cup \{u,v\}) \cap B(w,R)) \\ &= \lambda_{\beta^\star}(w, v, \{u\} \cap B(w,R)). \end{aligned}$$

Since $\mathbf{X}$ is a point process to interact in pairs, the interaction terms due to triplets or higher order are equal to one, i.e. $G(\mathbf{y}) = 1$ when $n(\mathbf{y}) > 2$, for $\emptyset \neq \mathbf{y} \subseteq \mathbf{x}$. Then,

$$\lambda_{\beta^\star}(w, v, \{u\} \cap B(w,R)) = \begin{cases} \beta^\star G(w-v) & \text{if} \quad u \notin B(w,R), \\ \beta^\star & \text{otherwise} \end{cases}$$

and

$$\lambda_{\beta^\star}(u, v, w, \emptyset) = \begin{cases} \beta^{\star 3} G(v-u) G(w-v) & \text{if} \quad u \notin B(w,R) \\ \beta^{\star 3} G(v-u) & \text{otherwise.} \end{cases} \tag{4.6}$$

Which ensures that $\lambda_{\beta^\star}(u,v,w,\emptyset)$ is a function that depends only variables $v-u, w-v$, denoted by $G_3(v-u, w-v)$. In this way from the definition of H (resp. I_R) given by (4.3) (resp.(4.4)) and by stationarity of $\mathbf{X}$, it follows that

$$\begin{aligned} T_3 &= \frac{1}{b_n^{2d} |\Lambda_{n,R}|} \int_{\mathbb{R}^{2d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - v)|}{|\Lambda_{n,R}|} \bar{G}_3(v, w-v) K\left(\frac{v-t}{b_n}\right) K\left(\frac{w-v-t}{b_n}\right) \mathrm{d}v \mathrm{d}w \\ &= \frac{1}{|\Lambda_{n,R}|} \int_{\mathbb{R}^{2d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - (b_n z + t))|}{|\Lambda_{n,R}|} \bar{G}_3(b_n z + t, b_n z' + t) K(z) K(z') \mathrm{d}z \mathrm{d}z', \end{aligned}$$

where $\bar{G}_3(v, w-v) = I_R(v, w-v) \bar{F}(-v, w-v) G_3(v, w-v)$, where $\bar{F}$ is given by (4.5). By dominated convergence theorem and using the Lemma 2, we get $b_n^d |\Lambda_{n,R}| T_3 \longrightarrow 0$ as $n \to \infty$. Analogously, one may show the other terms T_4, T_5 and T_6.

For the asymptotic behaviour of the leading term T_7, let u, v, w, y any points in $\mathbb{R}^d$ and $\mathbf{x} \in N_{lf}$, we define Papangelou conditional intensity of the fourth order by

$$\lambda_{\beta^\star}(u, v, w, y, \mathbf{x}) = \lambda_{\beta^\star}(u, \mathbf{x}) \lambda_{\beta^\star}(v, \mathbf{x} \cup \{u\}) \lambda_{\beta^\star}(w, \mathbf{x} \cup \{u, v\}) \lambda_{\beta^\star}(y, \mathbf{x} \cup \{u, v, w\}).$$

Next we introduce the finite range property (3.2) and reasoning analogous with the foregoing on $\lambda_{\beta^\star}(u,v,w,\emptyset)$ is defined by (4.6), which ensures that $\lambda_{\beta^\star}(u,v,w,y,\emptyset)$ is a function that depends only variables $v-u, y-w, w-u, w-v$, denoted by

$G_4(v-u, y-w, w-u, w-v)$. Further by the stationarity of $\mathbf{X}$, this implies that

$$
\begin{aligned}
T_7 &= \frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \mathrm{E} \int_{\mathbb{R}^{4d}} 1\!\!1_{\Lambda_{n,R}}(u) 1\!\!1_{\Lambda_{n,R}}(w) I_R(v-u, y-w) H(u,v,w,y,\mathbf{X}) K\left(\frac{v-u-t}{b_n}\right) \\
&\times K\left(\frac{y-w-t}{b_n}\right) \lambda_{\beta^\star}(u,v,w,y,\mathbf{X}) \mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y \\
&= \frac{1}{b_n^{2d}|\Lambda_{n,R}|} \\
&\int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_4(v, y-w, w, w-v) K\left(\frac{v-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) \mathrm{d}v\mathrm{d}w\mathrm{d}y \\
&= \frac{1}{|\Lambda_{n,R}|} \\
&\int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_4(b_n z + t, b_n z' + t, w, w - b_n z - t) K(z) K(z') \mathrm{d}z\mathrm{d}z'\mathrm{d}w,
\end{aligned}
$$

where $\bar{G}_4(v, y-w, w, w-v) = I_R(v, y-w)\bar{F}(v,w,y)G_4(v, y-w, w, w-v)$. Similar arguments show that $b_n^d|\Lambda_{n,R}|T_7 \longrightarrow 0$ as $n \to \infty$.

We now finish the proof of the asymptotic behaviour of the term T_8. Under the conditions imposed in Theorem 3.1, we have

$$
\begin{aligned}
T_8 &= -\frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \int_{\mathbb{R}^{4d}} 1\!\!1_{\Lambda_{n,R}}(u) 1\!\!1_{\Lambda_{n,R}}(w) I_R(v-u, y-w) K\left(\frac{v-u-t}{b_n}\right) \\
&\times K\left(\frac{y-w-t}{b_n}\right) \mathrm{E}\left[H(u,v,\mathbf{X})\lambda_{\beta^\star}(u,v,\mathbf{X})\right] \mathrm{E}\left[H(w,y,\mathbf{X})\lambda_{\beta^\star}(w,y,\mathbf{X})\right] \mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y \\
&= -\frac{1}{b_n^{2d}|\Lambda_{n,R}|^2} \int_{\mathbb{R}^{4d}} 1\!\!1_{\Lambda_{n,R}}(u) 1\!\!1_{\Lambda_{n,R}}(w) K\left(\frac{v-u-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) \\
&\times \bar{G}_5(u,v,w,y,v-u,y-w) \mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y \\
&= -\frac{1}{b_n^{2d}|\Lambda_{n,R}|} \\
&\times \int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_5(o,v,w,y,v,y-w) K\left(\frac{v-t}{b_n}\right) K\left(\frac{y-w-t}{b_n}\right) \mathrm{d}v\mathrm{d}w\mathrm{d}y \\
&= -\frac{1}{|\Lambda_{n,R}|} \int_{\mathbb{R}^{3d}} \frac{|\Lambda_{n,R} \cap (\Lambda_{n,R} - w)|}{|\Lambda_{n,R}|} \bar{G}_5(o, b_n z + t, w, b_n z' + t + w, b_n z + t, b_n z' + t) \\
&\times K(z) K(z') \mathrm{d}w\mathrm{d}z\mathrm{d}z',
\end{aligned}
$$

where

$\bar{G}_5(u,v,w,y,v-u,y-w) = I_R(v-u,y-w)\bar{F}(u,v)\bar{F}(w,y)G(v-u)G(y-w)$, where $\bar{F}$ is given by (4.5). By dominated convergence theorem, we get the desired result. □

4.2 Proof of Theorem 3.3

Proof. To prove the Theorem 3.3, we need (using the well-known Cramér-Wold (1936)). Without loss of generality, we consider only the case $s = 2$. Let λ_1 and λ_2 be two constants with $\lambda_1 + \lambda_2 \neq 0$ and let the points $t_1, t_2 \in T \backslash \{o\}$ be fixed, $t_1 \neq t_2$ and define the triangular array of random variables

$$\frac{1}{|\Lambda_{n,R}|^{1/2}} \sum_{i \in \widetilde{I}_n} \bar{Y}_{n,i} =$$
$$\lambda_1 (b_n^d |\Lambda_{n,R}|)^{1/2} \Big(\widehat{H}_n(t_1) - \mathrm{E}[\widehat{H}_n(t_1)] \Big) + \lambda_2 (b_n^d |\Lambda_{n,R}|)^{1/2} \Big(\widehat{H}_n(t_2) - \mathrm{E}[\widehat{H}_n(t_2)] \Big)$$

where

$$\bar{Y}_{n,i} = \lambda_1 \Big(Y_i(t_1) - \mathrm{E}[Y_i(t_1)] \Big) + \lambda_2 \Big(Y_i(t_2) - \mathrm{E}[Y_i(t_2)] \Big)$$

and for all $j = 1, 2$, we consider

$$Y_i(t_j) = (b_n^d)^{-1/2} \sum_{\substack{u \in \mathbf{X}_i, v \in \mathbf{X} \\ v - u \in B(o,R)}}^{\neq} h(u, \mathbf{X} \setminus \{u, v\}) h(v, \mathbf{X} \setminus \{u, v\}) K\left(\frac{v - u - t_j}{b_n} \right).$$

The proving idea the below Theorem 3.3 consists in approximating the sequence $\bar{Y}_{n,i}$ by a triangular array of m-dependent random fields. The corresponding Lindeberg-type CLT m-dependent random fields has been proved by [39]. Therefore, it proves convenient to switch notation from the text and to define

$$\widetilde{Y}_{n,i} = \lambda_1 \Big(\widetilde{Y}_i(t_1) - \mathrm{E}[\widetilde{Y}_i(t_1)] \Big) + \lambda_2 \Big(\widetilde{Y}_i(t_2) - \mathrm{E}[\widetilde{Y}_i(t_2)] \Big)$$

and for all $j = 1, 2$, we consider

$$\widetilde{Y}_i(t_j) = (b_n^d)^{-1/2} \sum_{\substack{u \in \mathbf{X}_{i,m_n}, v \in \mathbf{X} \\ v - u \in B(o,R)}}^{\neq} h(u, \mathbf{X} \setminus \{u, v\}) h(v, \mathbf{X} \setminus \{u, v\}) K\left(\frac{v - u - t_j}{b_n} \right).$$

By construction, $\{\widetilde{Y}_{n,i}\}_{i \in \mathbb{Z}^d}$ are m_n-dependent, in the sense that $\widetilde{Y}_{n,i}$ and $\widetilde{Y}_{n,j}$ are independent if $|i - j|_\infty \geq m_n$, where $|j|_\infty = \max\{|j_1|, \ldots, |j_d|\}$. We will use $\{\widetilde{Y}_{n,i}; i \in \mathbb{Z}^d\}_{n \in \mathbb{N}}$ to approximate $\{\bar{Y}_{n,i}; i \in \mathbb{Z}^d\}_{n \in \mathbb{N}}$. We decompose

$$\frac{1}{|\Lambda_{n,R}|^{1/2}} \sum_{i \in \widetilde{I}_n} \bar{Y}_{n,i} = \frac{1}{|\Lambda_{n,R}|^{1/2}} \sum_{i \in \widetilde{I}_n} \widetilde{Y}_{n,i} + \frac{1}{|\Lambda_{n,R}|^{1/2}} \sum_{i \in \widetilde{I}_n} \big(\bar{Y}_{n,i} - \widetilde{Y}_{n,i} \big).$$

Since K and h are bounded and applying Proposition 1 obtained [24] and under (3.10) of Condition $\mathscr{W}$ we know that

$$\frac{1}{|\Lambda_n|^{1/2}}\left\|\sum_{i\in\widetilde{I}_n}\left(\bar{Y}_{n,i}-\widetilde{Y}_{n,i}\right)\right\|_2 \le c\frac{|\lambda_1|+|\lambda_2|}{(m_nb_n)^{3d/2}}\sum_{|i|>m_n}|i|^{\frac{5d}{2}}\delta_{i,2}=o(1).$$

Then, it suffices to establish the following result.

Proposition 1. *Under (3.11) of Condition $\mathscr{W}$. As $n\to\infty$, we have*

$$\frac{1}{|\Lambda_{n,R}|^{1/2}}\sum_{i\in\widetilde{I}_n}\widetilde{Y}_{n,i}\xrightarrow{d}\mathscr{N}\left(0,\sigma^2\right),$$

where

$$\sigma^2=\left(\beta^{\star 2}\lambda_1^2\bar{F}(o,t_1)G(t_1)+\beta^{\star 2}\lambda_2^2\bar{F}(o,t_2)G(t_2)\right)\int_{\mathbb{R}^d}K^2(z)\mathrm{d}z. \tag{4.7}$$

Proof of Proposition 1

Proof. Taking into account of $\{\widetilde{Y}_{n,i}\}_{i\in\mathbb{Z}^d}$ are m_n-dependent random fields, we apply limit theorem for stationary triangular arrays of m-dependent random fields (Theorem 2 established by [39]). We need to verify that the two fundamental conditions (4.9) and (4.10), i.e. it suffices to show, for some sequence of increasing integers $\{l_n\}_{n\in\mathbb{N}}$ such that

$$m_n/l_n\to 0 \quad\text{and}\quad l_n/n\to 0 \quad\text{as}\quad n\to\infty, \tag{4.8}$$

we have

$$\lim_{n\to\infty}\frac{1}{l_n^d}\mathrm{E}\left[\left(\sum_{i\in[\![1,l_n]\!]^d}\widetilde{Y}_{n,i}\right)^2\right]=\sigma^2 \tag{4.9}$$

and

$$\lim_{n\to\infty}\frac{1}{l_n^d}\mathrm{E}\left[\left(\sum_{i\in[\![1,l_n]\!]^d}\widetilde{Y}_{n,i}\right)^2 1\!\!1\left\{\Big|\sum_{i\in[\![1,l_n]\!]^d}\widetilde{Y}_{n,i}\Big|>\varepsilon n^{d/2}\right\}\right]=0 \quad\text{for all}\quad \varepsilon>0 \tag{4.10}$$

where $[\![a,b]\!]\equiv\{a,a+1,\ldots,b\}$ for $a,b\in\mathbb{Z}$.

Taking into account the definition of $\sum_{i\in[\![1,l_n]\!]^d}\widetilde{Y}_{n,i}$ we get

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{l_n^d}\operatorname{Var}\left(\sum_{i\in[\![1,l_n]\!]^d}\widetilde{Y}_{n,i}\right)&\\ =\lim_{n\to\infty}b_n^dl_n^d\Big(\lambda_1^2\operatorname{Var}(\widehat{H}_{l_n}(t_1))+\lambda_2^2\operatorname{Var}(\widehat{H}_{l_n}(t_2))&+2\lambda_1\lambda_2\operatorname{Cov}(\widehat{H}_{l_n}(t_1),\widehat{H}_{l_n}(t_2))\Big).\end{aligned}$$

Using Theorem 3.1 entails replacing $\widehat{H}_n(t_j)$ by $\widehat{H}_{l_n}(t_j)$, we have

$$\lim_{n\to\infty} b_n^d l_n^d \operatorname{Var}(\widehat{H}_{l_n}(t_j)) = \beta^{\star 2} G(t_j)\bar{F}(o,t_j)\int_{\mathbb{R}^d} K^2(u)\mathrm{d}u.$$

Using Theorem 3.2, we have

$$\lim_{n\to\infty} b_n^d l_n^d \operatorname{Cov}\left(\widehat{H}_{l_n}(t_1), \widehat{H}_{l_n}(t_2)\right) = 0$$

then, it follows that

$$\lim_{n\to\infty} \frac{1}{l_n^d} \operatorname{Var}\left(\sum_{i\in[\![1,l_n]\!]^d} \widetilde{Y}_{n,i}\right) = \sigma^2 \tag{4.11}$$

where σ^2 is defined by (4.7).

To prove (4.10), we write $\xi_n = \sum_{i\in[\![1,l_n]\!]^d} \widetilde{Y}_{n,i}$, therefore we get

$$\begin{aligned}
\mathrm{E}\left[\xi_n^2 1\!\!1\{|\xi_n| > \varepsilon n^{d/2}\}\right] &\le ||\xi_n||_p^2 \left(\mathbb{P}(|\xi_n| > n^{d/2}\varepsilon)\right)^{(p-2)/p} \\
&\le ||\xi_n||_p^2 \left(\frac{||\xi_n||_2^2}{n^d \varepsilon^2}\right)^{(p-2)/p}.
\end{aligned}$$

Note that (4.11) yields $\|\xi_n\|_2 \le C l_n^{d/2}$ for all $n\in\mathbb{N}$. For $p>2$, observe that, since K and h are bounded,

$$\|\xi_n\|_p^p = \mathrm{E}|\xi_n|^p \le \mathrm{E}|\xi_n|^2 \left(\frac{C}{b_n^{d/2}} l_n^d\right)^{p-2}.$$

By the above inequalities, we have obtained

$$\frac{1}{l_n^d}\mathrm{E}\left[\xi_n^2 1\!\!1\{|\xi_n| > \varepsilon n^{d/2}\}\right] \le C\left(\frac{l_n^{2d}}{b_n^d n^d}\right)^{(p-2)/p}.$$

Now, (3.11) of Condition $\mathscr{W}$ entails that l_n can be chosen so that in addition to (4.8), $l_n^{2d}/(n^d b_n^d) \to 0$ as $n\to\infty$; hence (4.10) follows. □

This completes the proof of Theorem 3.3. □

4.3 Proof of Corollary 1

Proof. Let $t \in \{t_1,\ldots,t_s\}$ be fixed. Using the second-order Georgii-Nguyen-Zessin formula (2.2), and using the finite range property (3.2) and by stationarity of $\mathbf{X}$, we obtain

$$\begin{aligned}
\mathrm{E}\widehat{H}_n(t) &= \frac{\beta^{\star 2}}{b_n^d|\Lambda_{n,R}|}\,\mathrm{E}\int_{\mathbb{R}^{2d}} 1\!\!1_{\Lambda_{n,R}}(u) I_R(v-u) H(u,v) K\left(\frac{v-u-t}{b_n}\right) G(v-u)\mathrm{d}u\mathrm{d}v \\
&= \frac{\beta^{\star 2}}{b_n^d|\Lambda_{n,R}|}\int_{\mathbb{R}^{2d}} 1\!\!1_{\Lambda_{n,R}}(u) I_R(v-u) \bar{F}(o,v-u) K\left(\frac{v-u-t}{b_n}\right) G(v-u)\mathrm{d}u\mathrm{d}v \\
&= \beta^{\star 2}\int_{\mathbb{R}^d} I_R(b_n z+t)\bar{F}(o,b_n z+t)K(z)G(b_n z+t)\mathrm{d}z.
\end{aligned}$$

By Taylor expansion of the integrand in neighborhood of t and making use of Condition $K(d,m)$, Condition $K(d)$ and the function $G(t)\bar{F}(o,t)$ has bounded and continuous partial derivatives of order m of in some neighborhood of the point t, we get the following rate of convergence

$$\mathrm{E}\widehat{H}_n(t) = \beta^{\star 2}G(t)\bar{F}(o,t) + \mathcal{O}(b_n^m), \quad \text{as} \quad n\to\infty.$$

Under the condition $b_n^{d+2m}|\Lambda_{n,R}| \longrightarrow 0$, we get

$$\sqrt{b_n^d|\Lambda_{n,R}|}\left(\mathrm{E}\widehat{H}_n(t) - \beta^{\star 2}G(t)\bar{F}(o,t)\right) = \mathcal{O}\left(\sqrt{b_n^{d+2m}|\Lambda_{n,R}|}\right). \tag{4.12}$$

By the application of Theorem 3.3 we conclude the result announced. □

4.4 Proof of Theorem 3.4

Proof. Let $t\in\{t_1,\dots,t_s\}$ be fixed. We may split the difference $\widehat{G}_n(t)-\beta^\star G(t)$ as follows:

$$\begin{aligned}
\widehat{G}_n(t)-\beta^\star G(t) &= \frac{1}{\widehat{\bar{F}}_n(t)}\left(\left(\widehat{H}_n(t)-\mathrm{E}\widehat{H}_n(t)\right)+\left(\mathrm{E}\widehat{H}_n(t)-\beta^\star G(t)\widehat{\bar{F}}_n(t)\right)\right) \\
&= \frac{1}{\widehat{\bar{F}}_n(t)}\left(A_n^{(1)}+A_n^{(2)}\right).
\end{aligned}$$

Hence, from [27] ergodic theorem, and additionally assume that $\mathbb{P}$ is ergodic, $\widehat{\bar{F}}_n(t)$ converges almost surely to $\beta^\star\bar{F}(o,t)$. Note that there exists at least one stationary Gibbs measure. If this measure is unique, it is ergodic. Otherwise, it can be represented as a mixture of ergodic measures (see [13], Theorem 14.10). Therefore, we can assume for this proof, that $\mathbb{P}$ is ergodic. By Theorem 3.3, we have that as $n\to\infty$, $\sqrt{b_n^d|\Lambda_{n,R}|}A_n^{(1)}$ tends to a Gaussian distribution with zero mean. From strong consistency of the estimator $\widehat{\bar{F}}_n(t)$, and by inserting (4.12) and since $b_n^{d+2m}|\Lambda_{n,R}|\to 0$ as $n\to\infty$, we have that as $n\to\infty$,

$$\sqrt{b_n^d|\Lambda_{n,R}|}A_n^{(2)} \xrightarrow{\mathbb{P}} 0.$$

which completes the proof. □

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