Large deviations for some fast stochastic volatility models by viscosity methods

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LARGE DEVIATIONS FOR SOME FAST STOCHASTIC VOLATILITY MODELS BY VISCOSITY METHODS

MARTINO BARDI , ANNALISA CESARONI , DARIA GHILLI

Abstract. We consider the short time behaviour of stochastic systems affected by a stochastic volatility evolving at a faster time scale. We study the asymptotics of a logarithmic functional of the process by methods of the theory of homogenisation and singular perturbations for fully nonlinear PDEs. We point out three regimes depending on how fast the volatility oscillates relative to the horizon length. We prove a large deviation principle for each regime and apply it to the asymptotics of option prices near maturity.

1. Introduction

In this paper we are interested in stochastic differential equations with two small parameters $\varepsilon > 0$ and $\delta > 0$ of the form

$$\begin{cases} dX_t = \varepsilon\phi(X_t, Y_t)dt + \sqrt{2\varepsilon}\sigma(X_t, Y_t)dW_t & X_0 = x \in \mathbb{R}^n, \\ dY_t = \frac{\varepsilon}{\delta}b(Y_t)dt + \sqrt{\frac{2\varepsilon}{\delta}}\tau(Y_t)dW_t & Y_0 = y \in \mathbb{R}^m, \end{cases} \tag{1.1}$$

where W_t is a standard r-dimensional Brownian motion, the functions $\phi(x, y)$, $\sigma(x, y), b(y), \tau(y)$ are $\mathbb{Z}^m$-periodic with respect to the variable y, and the matrix τ is non-degenerate. This is a model of systems where the variables Y_t evolve at a much faster time scale $s = \frac{t}{\delta}$ than the other variables X_t. The second parameter ε is added in order to study the small time behavior of the system, in particular the time has been rescaled in (1.1) as $t \mapsto \varepsilon t$. Passing to the limit as $\delta \to 0$, with ε fixed, is a classical singular perturbation problem, its solution leads to the elimination of the state variable Y_t and to the definition of an averaged system defined in $\mathbb{R}^n$ only. There is a large literature on the subject, see the monographs [32], [30], the memoir [3] and the references therein. Here we study the asymptotics as both parameters go to 0 and we expect different limit behaviors depending on the rate ε/δ. Therefore we put

$$\delta = \varepsilon^\alpha, \text{ with } \alpha > 1,$$

and consider a functional of the trajectories of (1.1) of the form

$$v^\varepsilon(t, x, y) := \varepsilon \log E\left[e^{h(X_t)/\varepsilon} | (X., Y.) \text{ satisfy (1.1)}\right], \tag{1.2}$$

where $h \in BC(\mathbb{R}^n)$. The logarithmic form of this payoff is motivated by the applications to large deviations that we want to give. It is known that v^ε solves the Cauchy problem with initial data $v^\varepsilon(0, x, y) = h(x)$ for a fully nonlinear parabolic equation. Letting $\varepsilon \to 0$ in this PDE is a regular perturbation of a singular perturbation problem, for which we can rely on the techniques of [4], stemming from

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Evans' perturbed test function method for homogenisation [19] and its extensions to singular perturbations [1, 2, 3]. We show that under suitable assumptions the functions $v^\varepsilon(t,x,y)$ converge to a function $v(t,x)$ characterised as the solution of the Cauchy problem for a first order Hamilton-Jacobi equation

$$v_t - \bar{H}(x, Dv) = 0 \ \text{ in }]0,T[\times\mathbb{R}^n, \quad v(0,x) = h(x). \tag{1.3}$$

A significant part of the paper is devoted to the analysis of the *effective Hamiltonian* $\bar{H}$, which is obtained by solving a suitable cell problem. As usual in the theory of homogenisation for fully nonlinear PDEs, this is an additive eigenvalue problem. It turns out to have different forms in the following three regimes depending on α:

$$\begin{cases} \alpha > 2 & \text{supercritical case,} \\ \alpha = 2 & \text{critical case,} \\ \alpha < 2 & \text{subcritical case.} \end{cases}$$

More precisely, in the supercritical case the cell problem involves a linear elliptic operator and $\bar{H}$ has the explicit formula

$$\bar{H}(x,p) = \int_{\mathbb{T}^m} |\sigma(x,y)^T p|^2 \, d\mu(y)$$

where μ is the invariant probability measure on the m-dimensional torus $\mathbb{T}^m$ of the stochastic process

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t.$$

In the critical case the cell problem is a fully nonlinear elliptic PDE and $\bar{H}$ can be represented in various ways based, e.g., on stochastic control. Finally, in the subcritical case the cell problem is of first order and nonlinear, and a representation formula for $\bar{H}$ can be given in terms of deterministic control. In particular, under the condition $\tau\sigma^T = 0$ of non-correlations among the components of the white noise acting on the slow and the fast variables in (1.1), we have

$$\bar{H}(x,p) = \max_{y\in\mathbb{R}^m} |\sigma^T(x,y)p|^2.$$

Let us mention that an important step of the method is the comparison principle for the limit Cauchy problem (1.3), ensuring that the weak convergence of the relaxed semilimits is indeed uniform, as well as the uniqueness of the limit. It is known that this property of the effective Hamiltonian may require additional conditions [3]. Here we show that no extra assumptions are needed in the super- and subcritical cases, whereas in the critical case the comparison principle holds if either the matrix σ is independent on x, or it is non-degenerate, or the non-correlation condition $\tau\sigma^T = 0$ holds.

The main application of the convergence results is a large deviations analysis of (1.1) in the three different regimes. We prove that the measures associated to the process X_t in (1.1) satisfy a Large Deviation Principle (briefly, LDP) with good rate function

$$I(x;x_0,t) := \inf\left[\int_0^t \bar{L}\left(\xi(s),\dot{\xi}(s)\right)\, ds \;\middle|\; \xi \in AC(0,t),\ \xi(0) = x_0, \xi(t) = x\right],$$

where $\bar{L}$ is the *effective Lagrangian* associated to $\bar{H}$ via convex duality. In particular we get that

$$P(X_t^\varepsilon \in B) = e^{-\inf_{x\in B} \frac{I(x;x_0,t)}{\varepsilon} + o(\frac{1}{\varepsilon})}, \quad \text{as } \varepsilon \to 0$$

for any open set $B \subseteq \mathbb{R}^n$. Following [22] we also apply this result to an estimate of option prices near maturity and an asymptotic formula for the implied volatility.

Our first motivation for the study of systems of the form (1.1) comes from financial models with stochastic volatility. In such models the vector X_t represents the log-prices of n assets (under a risk-neutral probability measure) whose volatility σ is affected by a process Y_t driven by another Brownian motion, which is often negatively correlated with the one driving the stock prices (this is the empirically observed leverage effect, i.e., asset prices tend to go down as volatility goes up). Fouque, Papanicolaou, and Sircar argued in [25] that the bursty behaviour of volatility observed in financial markets can be described by introducing a faster time scale for a mean-reverting process Y_t by means of the small parameter δ in (1.1). Several extensions, applications to a variety of financial problems, and rigorous justifications of the asymptotics can be found in [26, 27, 9, 10, 28], see also the references therein. On the other hand, Avellaneda et al. [5] used the theory of large deviations to give asymptotic estimates for the Black-Scholes implied volatility of option prices near maturity in models with constant volatility. In the recent paper [22], Feng, Fouque, and Kumar study the large deviations for system of the form (1.1) in the one-dimensional case $n = m = 1$, assuming that Y_t is an Ornstein-Uhlenbeck process and the coefficients in the equation for X_t do not depend on X_t. In their model ε represents a short maturity for the options, $1/\delta$ is the rate of mean reversion of Y_t, and the asymptotic analysis is performed for $\delta = \varepsilon^\alpha$ in the regimes $\alpha = 2$ and $\alpha = 4$. Their methods are based on the approach to large deviations developed in [23]. A related paper is [21] where the Heston model was studied in the regime $\delta = \varepsilon^2$ by methods different from [22].

Although sharing some motivations with [22] our results are quite different: we treat vector-valued processes under rather general conditions and discuss all the regimes depending on the parameter α; our methods are also different, mostly from the theory of viscosity solutions for fully nonlinear PDEs and from the theory of homogenisation and singular perturbations for such equations. Our assumption of periodicity with respect to the y variables may sound restrictive for the financial applications. It is made mostly for technical simplicity and can be relaxed to the ergodicity of the process Y_t as in [9, 10]: this will be treated in a paper in preparation.

Large deviation principles have a large literature for diffusions with vanishing noise; some of them were extended to two-scale systems with small noise in the slow variables, see [34], [37], and more recently [33], [18], and [35]. Our methods can be also applied to this different scaling. The paper by Spiliopoulos [35] also states some results for the scaling of (1.1) under the assumptions of periodicity and $n = m = 1$, but its methods based on weak convergence are completely different from ours. A related paper on homogenisation of a fully nonlinear PDE with vanishing viscosity is [13].

The paper is organized as follows. In Section 2 we give the precise assumptions and describe the parabolic PDEs satisfied by v^ε in the different regimes. In sections 3, 4, 5 we analyse the cell problem and the properties of the effective Hamiltonian in the critical ($\alpha = 2$), supercritical ($\alpha > 2$), and subcritical case ($\alpha < 2$), respectively. Section 6 is devoted to the convergence result for each regime of the functions (1.2) to the unique viscosity solution of the limit problem (1.3) with $\bar{H}$ identified in the previous sections, see Theorems 6.1 and 6.2. In section 7 we prove the Large

Deviation Principle for all the regimes, Theorem 7.1. Finally, in Section 8 we give some applications to option pricing.

2. The fast stochastic volatility problem

2.1. **The stochastic volatility model.** We consider fast-mean reverting stochastic volatility system that can be written in the form

$$\begin{cases} dX_t = \phi(X_t, Y_t)dt + \sqrt{2}\sigma(X_t, Y_t)dW_t, & X_0 = x \in \mathbb{R}^n \\ dY_t = \varepsilon^{-\alpha} b(Y_t)dt + \sqrt{2\varepsilon^{-\alpha}}\tau(Y_t)dW_t, & Y_0 = y \in \mathbb{R}^m. \end{cases} \tag{2.1}$$

where $\varepsilon > 0$, $\alpha > 1$ and W_t is an r-dimensional standard Brownian motion. We assume $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbf{M}^{n,r}$ are bounded continuous functions, Lipschitz continuous in (x, y) and periodic in y, where $\mathbf{M}^{n,r}$ denotes the set of $n \times r$ matrices. Moreover $b : \mathbb{R}^m \to \mathbb{R}^m, \tau : \mathbb{R}^m \to \mathbf{M}^{m,r}$ are locally Lipschitz continuous functions, periodic in y. These assumptions will hold throughout the paper. We will use the symbol $\mathbf{S}^k$ to denote the set of $k \times k$ symmetric matrices.
In the following we will assume the uniform nondegeneracy of the diffusion driving the fast variable Y_t, i.e for some $\theta > 0$

$$\xi^T \tau(y)\tau(y)^T \xi = |\tau^T(y)\xi|^2 > \theta|\xi|^2 \quad \text{for every } y \in \mathbb{R}, \xi \in \mathbb{R}^m. \tag{2.2}$$

In order to study small time behavior of the system (2.1), we rescale time $t \to \varepsilon t$ for $0 < \varepsilon \ll 1$, so that the typical maturity will be of order of ϵ. Denoting the rescaled processes by X_t^ε and Y_t^ε we get

$$\begin{cases} dX_t^\varepsilon = \varepsilon\phi(X_t^\varepsilon, Y_t^\varepsilon)dt + \sqrt{2\varepsilon}\sigma(X_t^\varepsilon, Y_t^\varepsilon)dW_t, & X_0^\varepsilon = x \in \mathbb{R}^n \\ dY_t^\varepsilon = \varepsilon^{1-\alpha} b(Y_t^\varepsilon)dt + \sqrt{2\varepsilon^{1-\alpha}}\tau(Y_t^\varepsilon)dW_t, & Y_0^\varepsilon = y \in \mathbb{R}^m. \end{cases} \tag{2.3}$$

Next we consider the functional

$$u^\varepsilon(t, x, y) := E\left[g(X_t) \,|\, (X^\varepsilon., Y^\varepsilon.) \text{ satisfy } (2.3)\right] \tag{2.4}$$

where $g \in BC(\mathbb{R}^n)$. We denote with $BC(\mathbb{R}^n)$ the space of bounded continuous functions in $\mathbb{R}^n$.

The partial differential equation associated to the functions u^ε is

$$\begin{aligned} u_t - \varepsilon \mathrm{tr}(\sigma\sigma^T D^2_{xx}u) - \varepsilon\phi \cdot D_x u - 2\varepsilon^{1-\frac{\alpha}{2}} \mathrm{tr}(\sigma\tau^T D^2_{xy}u) \\ - \varepsilon^{1-\alpha} b \cdot D_y u - \varepsilon^{1-\alpha}\mathrm{tr}(\tau\tau^T D^2_{yy}u) = 0 \end{aligned} \tag{2.5}$$

in $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$, where b and τ are computed in y, ϕ and σ are computed in (x, y). The equation is complemented with the initial condition:

$$u(0, x, y) = g(x).$$

Remark 1. Note that, since we assume the periodicity in y of the coefficients of the equation b, σ, τ, ϕ, we have that the solution u^ε of the equation (2.5) is periodic in y itself.

2.2. **The log-tranform and its HJB equation.** We introduce the logarithmic transformation method (see [24]). Assume that

$$g(x) = e^{h(x)/\varepsilon} \text{ with } h \in BC(\mathbb{R}^n)$$

and define

$$v^\varepsilon(t,x,y) := \varepsilon \log u^\varepsilon = \varepsilon \log E\left[e^{h(X_t^\varepsilon)/\varepsilon} \mid (X^\varepsilon., Y^\varepsilon.) \text{ satisfy } (2.3)\right], \tag{2.6}$$

where u^ε is defined in (2.4), $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $t \geq 0$. By (2.5) and some computations one sees that the equation associated to v^ε is

$$\begin{aligned} v_t = {} & |\sigma^T D_x v|^2 + \varepsilon \mathrm{tr}(\sigma\sigma^T D^2_{xx} v) + \varepsilon\phi \cdot D_x v + 2\varepsilon^{-\frac{\alpha}{2}}(\tau\sigma^T D_x v)\cdot D_y v + \\ & 2\varepsilon^{1-\frac{\alpha}{2}}\mathrm{tr}(\sigma\tau^T D^2_{xy} v) + \varepsilon^{1-\alpha} b \cdot D_y v + \varepsilon^{-\alpha}|\tau^T D_y v|^2 + \varepsilon^{1-\alpha}\mathrm{tr}(\tau\tau^T D^2_{yy} v), \end{aligned} \tag{2.7}$$

where b and τ are computed in y, ϕ and σ are computed in (x,y). In general, the functions u^ϵ are not smooth but one can check that v^ϵ is a viscosity solutions of (2.7) (see in particular Chapter VI and VII of [24]).

In the following proposition we characterize the value function v^ε as the unique continuous viscosity solution to a suitable parabolic problem with initial data for each of the three regimes. A general reference for these issue is [24]. The equation (2.7) satisfied by v^ϵ involves a quadratic nonlinearity in the gradient. This case was studied by Da Lio and Ley in [15], where the reader can find a proof of the next result.

Proposition 2.1. *i) Let $\alpha \geq 2$ and define*

$$\begin{aligned} H^\varepsilon(x,y,p,q,X,Y,Z) \quad := \quad & |\sigma^T p|^2 + b\cdot q + tr(\tau\tau^T Y) + \varepsilon\left(tr(\sigma\sigma^T X) + \phi\cdot p\right) \\ + \quad & 2\varepsilon^{\frac{\alpha}{2}-1}(\tau\sigma^T p)\cdot q + 2\varepsilon^{\frac{1}{2}} tr(\sigma\tau^T Z) + \varepsilon^{\alpha-2}|\tau^T q|^2. \end{aligned}$$

Then v^ε is the unique bounded continuous viscosity solution of the Cauchy problem

$$\begin{cases} \partial_t v^\varepsilon - H^\varepsilon\left(x,y,D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon^{\alpha-1}}, D^2_{xx}v^\varepsilon, \frac{D^2_{yy}v^\varepsilon}{\varepsilon^{\alpha-1}}, \frac{D^2_{xy}v^\varepsilon}{\varepsilon^{\frac{\alpha-1}{2}}}\right) = 0 & \text{in } [0,T]\times\mathbb{R}^n\times\mathbb{R}^m, \\ v^\varepsilon(0,x,y) = h(x) & \text{in } \mathbb{R}^n\times\mathbb{R}^m. \end{cases} \tag{2.8}$$

ii) Let $\alpha < 2$ and define

$$\begin{aligned} H_\varepsilon(x,y,p,q,X,Y,Z) \quad := \quad & |\sigma^T p|^2 + |\tau^T q|^2 + 2(\tau\sigma^T p)\cdot q + \varepsilon\left(tr(\sigma\sigma^T X) + \phi\cdot p\right) \\ + \quad & \varepsilon^{1-\frac{\alpha}{2}}(b\cdot q + tr(\tau\tau^T Y)) + 2\varepsilon^{1-\frac{\alpha}{4}} tr(\sigma\tau^T Z). \end{aligned}$$

Then v^ε is the unique bounded continous viscosity solution of the Cauchy problem

$$\begin{cases} \partial_t v^\varepsilon - H_\varepsilon\left(x,y,D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}, D^2_{xx}v^\varepsilon, \frac{D^2_{yy}v^\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}, \frac{D^2_{xy}v^\varepsilon}{\varepsilon^{\frac{\alpha}{4}}}\right) = 0 & \text{in } [0,T]\times\mathbb{R}^n\times\mathbb{R}^m, \\ v^\varepsilon(0,x,y) = h(x) & \text{in } \mathbb{R}^n\times\mathbb{R}^m. \end{cases} \tag{2.9}$$

Our goal is to study the limit as $\varepsilon \to 0$ of the functions v^ε described in Proposition 2.1. Following the viscosity solution apoproach to singular perturbation problems (see [3],[2]), we define a limit or effective Hamiltonian $\overline{H}$ and we characterize the limit of v^ε as the unique solution of an appropriate Cauchy problem with Hamiltonian $\overline{H}$. The first step in the procedure is the identification of the limit Hamiltonian. In order to define this operator, we make the ansatz that the function v^ε admits the formal asymptotic expansion

$$v^\varepsilon(t,x,y) = v^0(t,x) + \varepsilon^{\alpha-1}w(t,x,y) \tag{2.10}$$

and plug it into the equation. In the following sections we show that the limit Hamiltonian is different in the three different regimes: the critical case ($\alpha = 2$), the supercritical case (when $\alpha > 2$), and the subcritical case (when $\alpha < 2$).

Numerical experiments in [36] indicate that the first order approximation in the expansion (2.10) is sufficiently accurate to find option prices in a fast mean-reversion case of the volatility process.

3. The critical case: $\alpha = 2$

Equation (2.7) with $\alpha = 2$ becomes

$$\begin{aligned} v_t = |\sigma^T D_x v|^2 + \varepsilon \mathrm{tr}(\sigma\sigma^T D^2_{xx} v) + \varepsilon\phi \cdot D_x v + \frac{2}{\varepsilon}(\tau\sigma^T D_x v) \cdot D_y v \\ -2\mathrm{tr}(\sigma\tau^T D^2_{xy} v) + \frac{1}{\varepsilon} b \cdot D_y v + \frac{1}{\varepsilon^2}|\tau^T D_y v|^2 + \frac{1}{\varepsilon}\mathrm{tr}(\tau\tau^T D^2_{yy} v). \end{aligned} \tag{3.1}$$

3.1. The effective Hamiltonian.

We plug in the equation (3.1) the formal asymptotic expansion

$$v^\varepsilon(t, x, y) = v^0(t, x) + \varepsilon w(t, x, y)$$

and we obtain

$$v^0_t - |\sigma^T D_x v^0|^2 - 2(\tau\sigma^T D_x v^0) \cdot D_y w - b \cdot D_y w - |\tau^T D_y w|^2 - \mathrm{tr}(\tau\tau^T D^2_{yy} w) = O(\varepsilon).$$

We want to eliminate the *corrector* w and the dependence on y in this equation and remain with a left hand side of the form $v^0_t - \bar{H}(x, D_x v^0)$. Therefore we freeze $\bar{x}$ and $\bar{p} = D_x v^0(\bar{x})$ and define the *effective Hamiltonian* $\bar{H}(\bar{x}, \bar{p})$ as the unique constant such that the following stationary PDE in $\mathbb{R}^m$, called *cell problem*, has a viscosity solution w:

$$\bar{H}(\bar{x}, \bar{p}) - |\sigma^T \bar{p}|^2 - 2(\tau\sigma^T \bar{p}) \cdot D_y w(y) - b \cdot D_y w(y) - |\tau^T D_y w(y)|^2 - \mathrm{tr}(\tau\tau^T D^2_{yy} w(y)) = 0, \tag{3.2}$$

where σ is computed in $(\bar{x}, y)$ and τ, b in y. This is an additive eigenvalue problem that arises the theory of ergodic control and has a wide literature. Under our standing assumptions we have the following result.

Proposition 3.1. *For any fixed $(\bar{x}, \bar{p})$, there exists a unique $\bar{H}(\bar{x}, \bar{p})$ for which the equation (3.2) has a periodic viscosity solution w. Moreover $w \in C^{2,\alpha}$ for some $0 < \alpha < 1$ and satisfies for some $C > 0$ independent of $\bar{p}$ and $\forall \bar{x}, \bar{p} \in \mathbb{R}^n$*

$$\max_{y \in \mathbb{R}^m} |Dw(y; \bar{x}, \bar{p})| \le C(1 + |\bar{p}|). \tag{3.3}$$

To prove Proposition 3.1, we need the following lemma.

Lemma 3.2. *Let $\delta > 0$ and $w_\delta(\cdot; \bar{x}, \bar{p}) \in C^2(\mathbb{R}^m)$ be a periodic solution of*

$$\delta w_\delta + F(\bar{x}, y, \bar{p}, Dw_\delta, D^2 w_\delta) - |\sigma(\bar{x}, y)\bar{p}|^2 = 0, \tag{3.4}$$

where

$$F(\bar{x}, y, \bar{p}, q, Y) := -tr(\tau\tau^T(y)Y) - |\tau^T(y)q|^2 - b(y) \cdot q - 2(\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot q. \tag{3.5}$$

Then there exists $C > 0$ independent of $\bar{p}$ such that for all $\bar{x}, \bar{p} \in \mathbb{R}^n$ it holds

$$\max_{y \in \mathbb{R}^m} |D_y w_\delta(y; \bar{x}, \bar{p})| \le C(1 + |\bar{p}|). \tag{3.6}$$

Proof. The proof uses the Bernstein method, following the derivation of similar estimates in [20]. We carry out the computations in the case τ, σ, b are C^1. When τ, σ, b are Lipschitz the result can be proved by smooth approximation.
Denote by $w^\delta := w_\delta(y; \bar{x}, \bar{p})$ the solution of (3.4). By comparison with constant sub- and supersolutions we get the uniform bound

$$|\delta w^\delta| \le \max_{y\in\mathbb{R}^m} |\sigma^T(\bar{x}, y)\bar{p}|^2 \quad \forall y \in \mathbb{R}^m. \tag{3.7}$$

Define the function z as follows

$$z := |Dw^\delta|^2.$$

Should z attains its maximum at some point y_0, then at y_0

$$z_i = 2w_k^\delta w_{ki}^\delta = 0 \quad i = 1, \dots, m, \tag{3.8}$$

where we are adopting the summation convention, and

$$0 \le -(\tau\tau^T)_{ij} z_{ij} = -2(\tau\tau^T)_{ij} w_{ki}^\delta w_{kj}^\delta - 2w_k^\delta (\tau\tau^T)_{ij} w_{ijk}^\delta. \tag{3.9}$$

Then at y_0

$$\begin{aligned}\theta|D^2w^\delta|^2 &\le (\tau\tau^T)_{ij} w_{ki}^\delta w_{kj}^\delta \le \\ &- w_k^\delta(\tau\tau^T)_{ij} w_{ijk}^\delta = -w_k^\delta \left((\tau\tau^T)_{ij} w_{ij}^\delta\right)_k + w_k^\delta(\tau\tau^T)_{ij,k} w_{ij}^\delta,\end{aligned}$$

where we have used (3.9). Thus at y_0

$$\begin{aligned}&\theta|D^2w^\delta|^2 \le \\ &\quad w_k^\delta\left(-\delta w^\delta + (2\tau\sigma^T\bar{p} + b)\cdot Dw^\delta + |\tau^T Dw^\delta|^2 + |\sigma^T\bar{p}|^2\right)_k + w_k^\delta(\tau\tau^T)_{ij,k} w_{ij}^\delta,\end{aligned}$$

where we have used (3.4). Thanks to (3.8)

$$\begin{aligned}&w_k^\delta(|\tau^T Dw^\delta|^2)_k = w_k^\delta((\tau\tau^T)_{ij} w_i^\delta w_j^\delta)_k = \\ &\quad w_k^\delta(\tau\tau^T)_{ij,k} w_i^\delta w_j^\delta + w_k^\delta(\tau\tau^T)_{ij} w_{ik}^\delta w_j^\delta + w_k^\delta(\tau\tau^T)_{ij} w_i^\delta w_{jk}^\delta = w_k^\delta(\tau\tau^T)_{ij,k} w_i^\delta w_j^\delta.\end{aligned}$$

Moreover

$$w_k^\delta(\tau\tau^T)_{ij,k} w_{ij}^\delta \le \frac{\theta}{2}|D^2w^\delta|^2 + \frac{C}{2\theta}|Dw^\delta|^2.$$

Then

$$\theta|D^2w^\delta|^2 \le C(1+|\bar{p}|)|Dw^\delta|^2 + C|Dw^\delta|^3 + \frac{\theta}{2}|D^2w^\delta|^2 + C|\bar{p}|^2|Dw^\delta| \quad \text{at } y_0$$

and $C > 0$ depends only on the L^∞ norm of σ, b, τ and on the derivatives of σ, b and τ. Therefore

$$|D^2w^\delta|^2 \le C(1 + |Dw^\delta|^2 + |\bar{p}||Dw^\delta|^2 + |\bar{p}|^2|Dw^\delta|^2 + |Dw^\delta|^3) \quad \text{at } y_0. \tag{3.10}$$

Thanks to the uniform ellipticity of τ and using equation (3.4), we have

$$\theta|Dw^\delta|^2 \le |\tau^T Dw^\delta|^2 = \delta w^\delta - \mathrm{tr}(\tau\tau^T D^2w^\delta) - 2\tau\sigma^T\bar{p}\cdot Dw^\delta - b\cdot Dw^\delta \quad \text{at } y_0.$$

Using (3.7), we get at y_0

$$\begin{aligned}z^2 = |Dw^\delta|^4 \le C(&|\bar{p}|^4 + |D^2w^\delta|^2 + |\bar{p}|^2|Dw^\delta|^2 + |Dw^\delta|^2 + |\bar{p}||Dw^\delta|^2 + |\bar{p}|^2|D^2w^\delta| \\ &+ |\bar{p}|^2|Dw^\delta| + |\bar{p}|^3|Dw^\delta| + |D^2w^\delta||\bar{p}||Dw^\delta| + |D^2w^\delta||Dw^\delta|).\end{aligned} \tag{3.11}$$

Then (3.6) follows by dividing (3.11) by $|Dw^\delta|^3$ and noticing that the right member in (3.11) is polynomial of degree 4 in $|\bar{p}|$ and $|Dw^\delta|$.

□

Proof. We use the methods of [6] based on the small discount approximation

$$\delta w_\delta + F(\bar{x}, y, \bar{p}, D_y w_\delta, D^2_{yy} w_\delta) - |\sigma^T(\bar{x}, y)\bar{p}|^2 = 0 \quad \text{in } \mathbb{R}^m, \tag{3.12}$$

where F is defined in (3.5). Let $w_\delta := w_\delta(y, \bar{x}, \bar{p}) \in C^2(\mathbb{R}^m)$ be a solution of (3.12). We show that $\delta w_\delta(y)$ converges along a subsequence of $\delta \to 0$ to the constant $\bar{H}(\bar{x}, \bar{p})$ and $w_\delta(y) - w_\delta(0)$ converges to the corrector w. The hard part is proving equicontinuity estimates for δw_δ. Different from [6, 3], here the leading term in (3.2) is $|\tau^T Dw|^2$ rather than $\mathrm{tr}(\tau\tau^T D^2 w)$. Then the Krylov-Safonov estimates for elliptic PDEs must be replaced by the Lipschitz estimates proved in Lemma 3.2. In fact, thanks to (3.6), for some $C > 0$ independent of $\bar{p}$ and for all $y, z \in \mathbb{R}^m$ and $\delta > 0$

$$|\delta w_\delta(y) - \delta w_\delta(z)| \le C\delta(1 + |\bar{p}|)|y - z| \tag{3.13}$$

and the equicontinuity follows. The equiboundness follows from (3.7). Then by Ascoli-Arzela theorem, there is a sequence $\delta_n \to 0$ such that $\delta_n w_{\delta_n}$ converges locally uniformly to a constant thanks to (3.13). We call it $\bar{H}$. Similarly, we prove that $v_\delta := w_\delta(y) - w_\delta(0)$ is equibounded and equicontinuous and thus converges locally uniformly along a subsequence to a function w. Then, from (3.12) we get

$$\delta v_\delta + \delta w_\delta(0) + F(\bar{x}, y, \bar{p}, D_y v_\delta, D^2_{yy} v_\delta) - |\sigma^T(\bar{x}, y)\bar{p}|^2 = 0, \quad \text{in } \mathbb{R}^m.$$

Since v_δ is equibounded $\delta v_\delta \to 0$. Then from $\delta w_\delta \to \bar{H}$ we get that w is a solution of (3.2). Finally, by the comparison principle for (3.12), it is standard to see that $\bar{H}$ is unique.
Moreover the regularity theory for viscosity solutions of convex uniformly elliptic equations implies that $w \in C^{2,\alpha}$ for some $0 < \alpha < 1$.
Finally the corrector inherits (3.13) and satisfies for some $C > 0$ independent of $\bar{p}$ and for all $\bar{x}, \bar{p} \in \mathbb{R}^n$

$$\max_{y \in \mathbb{R}^m} |D_y w(y; \bar{x}, \bar{p})| \le C(1 + |\bar{p}|).$$

□

3.2. **Properties and formulas for $\bar{H}$.** The next result lists some elementary properties of the effective Hamiltonian $\bar{H}$.

Proposition 3.3. (a) *$\bar{H}$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$;*

(b) *the function $p \to \bar{H}(x, p)$ is convex;*

(c)

$$\min_{y \in \mathbb{R}^m} |\sigma^T(\bar{x}, y)\bar{p}|^2 \le \bar{H}(\bar{x}, \bar{p}) \le \max_{y \in \mathbb{R}^m} |\sigma^T(\bar{x}, y)\bar{p}|^2; \tag{3.14}$$

(d) *There exists $C > 0$ independent of p such that, for all $x, \bar{x}, p \in \mathbb{R}^n$,*

$$|\bar{H}(x, p) - \bar{H}(\bar{x}, p)| \le C(1 + |p|^2)|x - \bar{x}|; \tag{3.15}$$

(e) *if*

$$\tau(y)\sigma^T(x, y) = 0 \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m, \tag{3.16}$$

then, for all $x, \bar{x}, p, \bar{p} \in \mathbb{R}^n$,

$$
\begin{aligned}
(3.17)\qquad \min_{y\in\mathbb{R}^m}\left(|\sigma^T(x,y)p|^2 - |\sigma^T(\bar{x},y)\bar{p}|^2\right) &\le \bar{H}(x,p) - \bar{H}(\bar{x},\bar{p}) \\
&\le \max_{y\in\mathbb{R}^m}\left(|\sigma^T(x,y)p|^2 - |\sigma^T(\bar{x},y)\bar{p}|^2\right).
\end{aligned}
$$

Remark 2. The meaning of assumption (3.16) is that the components of the Brownian motion W_t influencing the slow variables X_t are not correlated with the components acting on the slow variables Y_t. In fact the condition is satisfied if the last m columns of σ and the first n columns of τ are indentically zero.

Proof. The results (a), (b), and (c) are obtained by standard methods in the theory of homogenisation, by means of comparison principles for the approximating equation (3.12), see, e.g., [19, 1]. Let us show one inequality in (3.15) (the other being symmetric). Let $w_\delta(y) := w_\delta(y;\bar{x},p)$ ans $v_\delta(y) := w_\delta(y;x,p)$. Then v_δ satisfies

$$
\begin{aligned}
(3.18)\qquad \delta v_\delta + F(\bar{x},y,p,D_y v_\delta, D^2_{yy}v_\delta) - |\sigma^T(\bar{x},y)\bar{p}|^2 &= |\sigma^T(x,y)p|^2 - |\sigma^T(\bar{x},y)\bar{p}|^2 + \\
&\quad (2\tau(y)\sigma^T(x,y)p - 2\tau(y)\sigma^T(\bar{x},y)p)\cdot Dv_\delta.
\end{aligned}
$$

Thanks to Lemma 3.2 we estimate Dv_δ, and then, using the Lipschitz continuity of σ, we get for some $C > 0$

$$
(3.19)\qquad \delta v_\delta + F(\bar{x},y,p,D_y v_\delta, D^2_{yy}v_\delta) - |\sigma^T(\bar{x},y)\bar{p}|^2 \le C(1+|p|^2)|x-\bar{x}|.
$$

Then the comparison principle gives

$$
\delta v_\delta(y) - \delta w_\delta(y) \le C(1+|p|^2)|x-\bar{x}| \quad \forall\, y\in\mathbb{R}^m.
$$

By letting $\delta \to 0$ we get the inequality for $\bar{H}(x,p) - \bar{H}(\bar{x},p)$ in (3.15), and by exchanging x and $\bar{x}$ we complete the proof.

If (3.16) holds, (3.18) simplifies to

$$
\delta v_\delta + F(\bar{x},y,p,D_y v_\delta, D^2_{yy}v_\delta) - |\sigma^T(\bar{x},y)\bar{p}|^2 \le \max_y\left\{|\sigma^T(x,y)p|^2 - |\sigma^T(\bar{x},y)\bar{p}|^2\right\}.
$$

Then, as before, we obtain by comparison the second inequality in (3.17), and the first is got in a symmetric way. □

Next we give some representation formulas for the effective Hamiltonian $\bar{H}$.

Proposition 3.4. *(i) $\bar{H}$ satisfies*

$$
(3.20)\qquad \bar{H}(\bar{x},\bar{p}) = \lim_{\delta\to 0}\sup_{\beta(\cdot)}\delta E\left[\int_0^\infty \left(|\sigma(\bar{x},Z_t)^T\bar{p}|^2 - |\beta(t)|^2\right)e^{-\delta t}dt \,|\, Z_0 = z\right]
$$

and

$$
(3.21)\qquad \bar{H}(\bar{x},\bar{p}) = \lim_{t\to\infty}\sup_{\beta(\cdot)}\frac{1}{t}E\left[\int_0^t (|\sigma^T(\bar{x},Z_s)\bar{p}|^2 - |\beta(s)|^2)ds \,|\, Z_0 = z\right],
$$

where $\beta(\cdot)$ is an admissible control process taking values in $\mathbb{R}^r$ for the stochastic control system

$$
(3.22)\qquad dZ_t = \left(b(Z_t) + 2\tau(Z_t)\sigma^T(\bar{x},Z_t)\bar{p} - 2\tau(Z_t)\beta(t)\right)dt + \sqrt{2}\tau(Z_t)dW_t;
$$

(ii) moreover

$$\bar{H}(\bar{x},\bar{p}) = \int_{\mathbb{T}^m} \left(|\sigma(\bar{x},z)^T\bar{p}|^2 - |\tau(z)^T Dw(z)|^2\right) d\mu(z), \tag{3.23}$$

where $w = w(\cdot;\bar{x},\bar{p})$ *is the corrector defined in Proposition 3.1 and* $\mu = \mu(\cdot;\bar{x},\bar{p})$ *is the invariant probability measure of the process* (3.29) *with the feedback* $\beta(z) = -\tau^T(z)Dw(z)$*;*
(iii) finally

$$\bar{H}(\bar{x},\bar{p}) = \lim_{t\to\infty}\frac{1}{t}\log E\left[e^{\int_0^t |\sigma^T(\bar{x},Y_s)\bar{p}|^2\,ds} \,|\, Y_0 = y\right], \tag{3.24}$$

where Y_t *is the stochastic process defined by*

$$dY_t = \left(b(Y_t) + 2\tau(Y_t)\sigma^T(\bar{x},Y_t)\bar{p}\right)dt + \sqrt{2}\tau(Y_t)dW_t. \tag{3.25}$$

Proof. (i) The first formula comes from a control interpretation of the approximating δ-cell problem (3.4). We write it as the Hamilton-Jacobi-Bellman equation

$$\delta w_\delta + \inf_{\beta\in\mathbb{R}^r}\left\{-\mathrm{tr}(\tau(y)\tau(y)^T D^2 w_\delta + \left(2\tau(y)\beta - 2\tau(y)\sigma(\bar{x},y)^T\bar{p} - b(y)\right)\cdot D_y w_\delta + |\beta|^2\right\} - |\sigma(\bar{x},y)^T\bar{p}|^2 = 0 \tag{3.26}$$

and we represent w_δ as the value function of the infinite horizon discounted stochastic control problem (see, e.g., [24])

$$w_\delta(z) = \sup_{\beta(\cdot)} E\left[\int_0^\infty (|\sigma^T(\bar{x},Z_t)\bar{p}|^2 - |\beta(t)|^2)e^{-\delta t}dt \,|\, Z_0 = z\right],$$

where Z_t is defined by (3.22). Then (3.20) follows from the proof of Proposition 3.1.

For the formula (3.21) we consider the *t-cell problem*

$$\begin{cases} \frac{\partial v}{\partial t} - \mathrm{tr}(\tau\tau^T D^2 v) - |\tau^T Dv|^2 - (b + 2\tau\sigma^T\bar{p})\cdot Dv - |\sigma^T\bar{p}|^2 = 0 & \text{in } (0,+\infty)\times\mathbb{R}^m, \\ v(0,z) = 0 & \text{on } \mathbb{R}^m. \end{cases} \tag{3.27}$$

This is also a HJB equation, whose solution is the value function

$$v(t,z;\bar{x},\bar{p}) = \sup_{\beta(\cdot)} E\left[\int_0^t (|\sigma^T(\bar{x},Z_s)\bar{p}|^2 - |\beta(s)|^2)ds \,|\, Z_0 = z\right],$$

where Z_t is defined by (3.22). Then a generalized Abelian-Tauberian theorem (see [2] for a general proof based only on the comparison principle for the Hamiltonian) states that

$$\bar{H}(\bar{x},\bar{p}) = \lim_{t\to+\infty}\frac{v(t,z;\bar{x},\bar{p})}{t} \quad \text{uniformly in } z. \tag{3.28}$$

(ii) The formula (3.23) is derived from a direct control interpretation of the cell problem (3.2). In fact, it is the HJB equation of the ergodic control problem of maximizing

$$\lim_{T\to\infty}\frac{1}{T}E\left[\int_0^T (|\sigma^T(\bar{x},Z_s)\bar{p}|^2 - |\beta(s)|^2)ds \,|\, Z_0 = z\right],$$

among admissible controls $\beta(\cdot)$ taking values in $\mathbb{R}^r$ for the system (3.22), as before. The process Z_t associated to each control is ergodic with a unique invariant measure

μ on $\mathbb{T}^m$ because it is a nondegenerate diffusion on $\mathbb{T}^m$, see, e.g., [3], so the limit in the payoff functional exists and it is the space average in $d\mu$ of the running payoff. Since the HJB PDE (3.2) has a smooth solution w, it is known from a classical verification theorem that the feedback control that achieves the minimum in the Hamiltonian, i.e., $\beta(z) = -\tau^T(z)Dw(z)$, is optimal. Then (3.23) holds with μ the invariant measure of the process

(3.29)
$$d\tilde{Z}_t = \Big(b(\tilde{Z}_t) + 2\tau(\tilde{Z}_t)\sigma^T(\bar{x},\tilde{Z}_t)\bar{p} + 2\tau(\tilde{Z}_t)\tau^T(\tilde{Z}_t)Dw(\tilde{Z}_t)\Big)\,dt + \sqrt{2}\tau(\tilde{Z}_t)dW_t.$$

(iii) To prove (3.24), take $v = v(t,x;\bar{x},\bar{p})$ a periodic solution of the t-cell problem and define the function $f(t,y) = e^{v(t,y)}$. Then f solves the following equation

$$\begin{cases} \frac{\partial f}{\partial t} - f|\sigma^T\bar{p}|^2 - (2\tau\sigma^T\bar{p} + b)\cdot Df - \mathrm{tr}(\tau\tau^T D^2 f) = 0 & \text{in } (0,\infty)\times\mathbb{R}^m \\ f(0,z) = 1 & \text{in } \mathbb{R}^m. \end{cases}$$

By the Feynman-Kac formula, we have

$$f(t,y) = E\left[e^{\int_0^t |\sigma^T(\bar{x},Y_s)\bar{p}|^2\,ds} \,|\, Y_0 = y\right],$$

where Y_t is defined by (3.25). Then

$$v(t,y) = \log E\left[e^{\int_0^t |\sigma^T(\bar{x},Y_s)\bar{p}|^2\,ds} \,|\, Y_0 = y\right]$$

and thanks to (3.28) we get (3.24). □

Remark 3. For $x, p \in \mathbb{R}^n$ define the following perturbed generator $L^{x,p}$

$$L^{x,p}g(y) := Lg(y) + 2(\tau\sigma(x,y)^T p)\cdot D_y g(y),$$

where

$$L = b\cdot D_y + \mathrm{tr}(\tau\tau^T D^2_{yy}).$$

Then the equation (3.2) becomes

$$\bar{H} - e^{-w}L^{\bar{x},\bar{p}}e^w - |\sigma^T\bar{p}|^2 = 0, \tag{3.30}$$

because $e^{-w}Le^w = Lw + |\tau^T D_y w|^2$ gives

$$e^{-w}L^{\bar{x},\bar{p}}e^w = e^{-w}Le^w + 2(\tau\sigma^T\bar{p})\cdot D_y w = Lw + |\tau^T D_y w|^2 + 2(\tau\sigma^T\bar{p})\cdot D_y w.$$

Multiplying (3.30) by e^w we get, for $g(y) = e^{w(y)}$,

$$\bar{H}g(y) - (L^{\bar{x},\bar{p}} + V^{\bar{x},\bar{p}})g(y) = 0, \tag{3.31}$$

where $V^{\bar{x},\bar{p}}(y) = |\sigma^T(\bar{x},y)\bar{p}|^2$ is a multiplicative potential operator.

We conclude that if w is a solution of (3.2), then $\bar{H}$ *is the first eigenvalue of the linear operator* $L^{\bar{x},\bar{p}} + V^{\bar{x},\bar{p}}$, *with eigenfunction* $g = e^w$.

Remark 4. Equations like (3.2) have been studied in an aperiodic setting by Khaise and Sheu in [31]. They prove the existence of a constant $\bar{H}$ such that there is a unique smooth solution w with prescribed growth of (3.2). Moreover they provide a representation formula for $\bar{H}$ as the convex conjugate of a suitable operator over a space of measures.

3.3. **Comparison principle for $\bar{H}$.** The comparison theorem among viscosity sub- and supersolutions of the limit PDE

$$v_t - \bar{H}(x, Dv) = 0 \quad \text{in } (0,T) \times \mathbb{R}^n \tag{3.32}$$

will be the crucial tool for proving that the convergence of v^ε is not only in the weak sense of semilimits but in fact uniform, and the limit is unique. It is known from [3] that in general the regularity of $\bar{H}$ with respect to x may be worse than that of H^ε and the comparison principle may fail. Next result gives three alternative additional conditions ensuring the comparison.

Theorem 3.5. *Assume either one of the following conditions:*
(i) σ is independent of x, i.e., $\sigma = \sigma(y)$, and $h \in BUC(\mathbb{R}^n)$; or
(ii) for some $\nu > 0$

$$|\sigma^T(x,y)p|^2 > \nu |p|^2 \quad \forall\, x, p \in \mathbb{R}^n,\, y \in \mathbb{R}^m; \tag{3.33}$$

or
(iii) $\tau(y)\sigma^T(x,y) = 0$ for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.
Let $u \in BUSC([0,T] \times \mathbb{R}^n)$ and $v \in BLSC([0,T] \times \mathbb{R}^n)$ be, respectively, a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution to (3.32) such that $u(0,x) \le h(x) \le v(0,x)$ for all $x \in \mathbb{R}^n$. Then $u(x,t) \le v(x,t)$ for all $x \in \mathbb{R}^n$ and $0 \le t \le T$.

Proof. In case (i) $\bar{H}$ is independent of x and it is continuous by Proposition 3.3. Then the result follows from standard theory, see e.g. [11].

In case (ii) the bounds (3.14) and (3.33) give

$$\bar{H}(x,p) \ge \nu |p|^2.$$

Then $\bar{H}$ is coercive and has the properties (a) and (b) and (c) of Proposition 3.3. The result follows from [15] once we prove that for some $C > 0$ and all $x, y, p \in \mathbb{R}^n$

$$|\bar{L}(x,p) - \bar{L}(y,p)| \le C(1 + |p|^2)|x - y|, \tag{3.34}$$

where $\bar{L}(x,p)$ is the effective Lagrangian, i.e. $\bar{L}(x,p) = \sup_{q \in \mathbb{R}^m}\{p \cdot q - \bar{H}(x,q)\}$. Take $\bar{q}$ such that

$$\bar{L}(x,p) = \bar{q} \cdot p - \bar{H}(x, \bar{q}).$$

Then

$$\bar{L}(x,p) - \bar{L}(y,p) \le \bar{H}(y,\bar{q}) - \bar{H}(x,\bar{q}) \le C(1 + |\bar{q}|^2)|x - y|,$$

where we have used the property (d) of Proposition 3.3. We want to estimate $|\bar{q}|$. If $|\bar{q}| > \frac{|p|}{\nu}$, then

$$0 \le \bar{L}(x,p) = \bar{q} \cdot p - \bar{H}(x,\bar{q}) \le \bar{q} \cdot p - \nu|\bar{q}|^2 < 0$$

and we reach a contradiction. Then

$$\bar{L}(x,p) - \bar{L}(y,p) \le C\left(1 + \frac{|p|^2}{\nu^2}\right)|x - y|.$$

By reversing the roles of x and y we get the full inequality (3.34).

In case (iii) we need the following semi-homogeneity of degree two of $\bar{H}$:

$$\mu \bar{H}(x, \frac{p}{\mu}) \ge \bar{H}(x, \frac{p}{\sqrt{\mu}}) \quad \forall 0 < \mu < 1, x, p \in \mathbb{R}^n. \tag{3.35}$$

This follows from the representation formula (3.24), because Jensen inequality gives

$$\mu \log E\left[e^{\int_0^t |\sigma^T(x,Y_s)\frac{p}{\mu}|^2\, ds} | Y_0 = y\right] \ge \log E\left[e^{\mu \int_0^t |\sigma^T(x,Y_s)\frac{p}{\mu}|^2\, ds} | Y_0 = y\right]$$

and the conclusion is reached after dividing by t and letting $t \to \infty$. The other ingredient of the proof is the first inequality in (3.17) that relates the regularity in x of $\bar{H}$ with that of the pseudo-coercive Hamiltonian $|\sigma^T(x,y)p|^2$. With these two inequalities one can repeat the proof of the comparison principle for the pseudo-coercive Hamiltonian by Barles and Perthame, see [12] for the stationary case and [7] for the evolutionary case. Let us give a sketch of the main points of the proof. We show that for $\mu < 1$, μ sufficiently near to 1, it holds

$$\sup_{\mathbb{R}^n \times [0,T]} (u - \mu v) \leq \sup_{\mathbb{R}^n} (u - \mu v)(\cdot, 0).$$

If this is true, then the inequality holds also for $\mu = 1$, proving the Theorem. By contradiction, we assume that for every $\mu < 1$, there exists $(\overline{x}, \overline{t})$ such that

$$u(\overline{x}, \overline{t}) - \mu v(\overline{x}, \overline{t}) > \sup_{\mathbb{R}^n} (u - \mu v)(\cdot, 0). \tag{3.36}$$

Let

$$\Phi(x,z,t,s) = u(x,t) - \mu v(z,s) - \frac{|x-z|^2}{\epsilon^2} - \frac{|t-s|^2}{\eta^2} - \delta \log(1 + |x|^2 + |z^2|) + \alpha\mu s.$$

For ε, η small enough, Φ has a maximum point, that we denote with (x', z', t', s'). By standard arguments, we get $\frac{|x'-z'|^2}{\epsilon^2}, \frac{|t'-s'|^2}{\eta^2} \longrightarrow 0$ as $\varepsilon, \eta \to 0$.

If either $s' = 0$ or $t' = 0$, it is easy to see that we get a contradiction with (3.36). So we consider the case $(x', z', t', s') \in \mathbb{R}^n \times \mathbb{R}^n \times (0,T) \times (0,T)$. Let

$$p = 2\frac{x'-z'}{\epsilon^2}, \quad q_x = \frac{2x'}{1 + |x'|^2 + |z'|^2}, \quad q_z = \frac{-2y'}{1 + |x'|^2 + |z'|^2}, \quad r = 2\frac{t'-s'}{\eta^2}.$$

Using the fact that u is a subsolution we get

$$r - \bar{H}(x', p + \delta q_x) \leq 0. \tag{3.37}$$

Since v is a supersolution and $\bar{H}$ satisfies (3.35), we get

$$\frac{r}{\mu} - \frac{1}{\mu}\bar{H}\left(z', \frac{p + \delta q_z}{\sqrt{\mu}}\right) \geq \alpha \tag{3.38}$$

So, we multiply (3.38) by $-\mu$ and sum up to (3.37) to obtain

$$\bar{H}\left(z', \frac{p + \delta q_z}{\sqrt{\mu}}\right) - \bar{H}(x', p + \delta q_x) \leq -\alpha\mu. \tag{3.39}$$

Using (3.17) we get

$$\bar{H}\left(z', \frac{p + \delta q_z}{\sqrt{\mu}}\right) - \bar{H}(x', p + \delta q_x) \geq \min_{y \in \mathbb{R}^n} \left(\frac{1}{\mu}|\sigma^T(z',y)(p + \delta q_z)|^2 - |\sigma^T(x',y)(p + \delta q_x)|^2\right). \tag{3.40}$$

Let

$$A(y) = |\sigma^T(z',y)(p + \delta q_z)|$$

$$\Delta(y) = ((\sigma(x',y) - \sigma(z',y))^T(p + \delta q_z), \quad J(y) = \delta\sigma^T(x',y)(q_x - q_z).$$

Note that $\Delta(y)$ goes to zero for $\epsilon, \eta \to 0$ and for all δ fixed uniformly in y, and $J(y)$ goes to zero for $\epsilon, \eta, \delta \to 0$ uniformly in y. Then we can rewrite the rhs of (3.40) as

$$\min_{y \in \mathbb{R}^m} \left(\frac{1-\mu}{\mu}A(y)^2 - \Delta(y)^2 - J(y)^2 - 2A(y)\Delta(y) - 2J(y)\Delta(y) - 2A(y)J(y)\right). \tag{3.41}$$

Moreover, for all $k_1, k_2 > 0$ and for all $y \in \mathbb{R}^m$ it holds

$$-2A(y)\Delta(y) \geq -k_1 A(y)^2 - \frac{1}{k_1}\Delta(y)^2, \quad \text{and} \quad -2A(y)J(y) \geq -k_2 A(y)^2 - \frac{1}{k_2} J(y)^2.$$

So, recalling (3.39), (3.40) and (3.41) we get

$$-\alpha\mu \geq \min_{y\in\mathbb{R}^m} \left(\left(\frac{1-\mu}{\mu} - k_1 - k_2 \right) A(y)^2 - \left(1 + \frac{1}{k_1}\right) \Delta(y)^2 - \left(1 + \frac{1}{k_2}\right) J(y)^2 - 2J(y)\Delta(y) \right).$$

If we choose $k_1, k_2 > 0$ such that $k_1 + k_2 < \frac{1-\mu}{\mu}$ then we obtain

$$0 > -\alpha\mu \geq \min_{y\in\mathbb{R}^m} \left(-\left(1 + \frac{1}{k_1}\right) \Delta(y)^2 - \left(1 + \frac{1}{k_2}\right) J(y)^2 - 2J(y)\Delta(y) \right) \to 0,$$

as $\varepsilon, \eta, \delta \to 0$, reaching a contradiction.

□

4. The supercritical case: $\alpha > 2$

As in Section 3, we prove the existence of an effective Hamiltonian giving the limit PDE and first we identify the cell problem that we wish to solve. Plugging the asymptotic expansion

$$v^\varepsilon(t,x,y) = v^0(t,x) + \varepsilon^{\alpha-1} w(t,x,y)$$

in the equation (2.7) we get

$$v_t^0 = |\sigma^T D_x v^0|^2 + b \cdot D_y w + \mathrm{tr}(\tau\tau^T D_{yy}^2 w) + O(\varepsilon).$$

We consider the δ-*cell problem* for fixed $(\bar{x}, \bar{p}, \bar{X})$

(4.1) $\delta w_\delta(y) - |\sigma(\bar{x}, y)^T \bar{p}|^2 - b(y) \cdot D_y w_\delta(y) - \mathrm{tr}(\tau(y)\tau(y)^T D_{yy}^2 w_\delta(y)) = 0$ in $\mathbb{R}^m$,

where w_δ is the *approximate corrector.*
The next result states that δw_δ converges to $\bar{H}$ and it is smooth.

Proposition 4.1. *For any fixed* $(\bar{x}, \bar{p})$ *there exists a constant* $\bar{H}(\bar{x}, \bar{p})$ *such that* $\bar{H}(\bar{x}, \bar{p}) = \lim_{\delta\to 0} \delta w_\delta(y)$ *uniformly, where* $w_\delta \in C^2(\mathbb{R}^m)$ *is the unique periodic solution of* (4.1). *Moreover*

$$\bar{H}(\bar{x}, \bar{p}) := \int_{\mathbb{T}^m} |\sigma(\bar{x}, y)^T \bar{p}|^2 \, d\mu(y) \quad \textit{uniformly in } \mathbb{T}^m, \tag{4.2}$$

where μ *is the invariant probability measure on* $\mathbb{T}^m$ *of the stochastic process*

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t,$$

that is, the periodic solution of

$$-\sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}((\tau\tau^T)_{ij}(y))\mu + \sum_i \frac{\partial}{\partial y_i}(b_i(y))\mu = 0 \qquad \textit{in } \mathbb{R}^m, \tag{4.3}$$

with $\int_{\mathbb{T}^n} \mu(y)\, dy = 1$.

Proof. The proof essentially follows the arguments presented in [6, 3] of ergodic control theory in periodic enviroments. □

Remark 5. Note that in dimension $n = 1$ the effective Hamiltonian assumes the form

$$H(\bar{x}, \bar{p}) = \int_{\mathbb{T}^m} \sigma(\bar{x}, y)^2 d\mu(y) \bar{p}^2 = (\bar{\sigma}\bar{p})^2,$$

where $\bar{\sigma} = \sqrt{\int_{\mathbb{T}^m} \sigma(\bar{x}, y)^2 d\mu(y)}$.

We list some elementary properties of the effective Hamiltonian $\bar{H}$.

Proposition 4.2. *$\bar{H}$ satisfies properties (a), (b), (c), (d) as in Proposition 3.3. Moreover*

(f) *for all $x, \bar{x}, p, \bar{p} \in \mathbb{R}^n$,*

$$\min_{y \in \mathbb{R}^m} \left(|\sigma^T(x, y)p|^2 - |\sigma^T(\bar{x}, y)\bar{p}|^2 \right) \leq \bar{H}(x, p) - \bar{H}(\bar{x}, \bar{p}) \leq \max_{y \in \mathbb{R}^m} \left(|\sigma^T(x, y)p|^2 - |\sigma^T(\bar{x}, y)\bar{p}|^2 \right); \tag{4.4}$$

(g) *for every $\lambda \in \mathbb{R}$, $x, p \in \mathbb{R}^n$,*

$$\bar{H}(x, \lambda p) = |\lambda|^2 \bar{H}(x, p). \tag{4.5}$$

Proof. For the proofs of (a), (b), (c), (d) we repeat the same arguments as in Proposition 3.3. Properties (f), (g) can be easily checked from the representation formula (4.2). □

We now state the comparison principle among viscosity sub- and supersolutions of the limit PDE

$$v_t - \int_{\mathbb{T}^m} |\sigma(x, y)^T Dv|^2 \, d\mu(y) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n. \tag{4.6}$$

In this case, differently from the critical case, we do not need additional assumptions for th e comparison principle to hold.

Theorem 4.3. *Let $u \in BUSC([0, T] \times \mathbb{R}^n)$ and $v \in BLSC([0, T] \times \mathbb{R}^n)$ be, respectively, a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution to* (4.6) *such that $u(0, x) \leq v(0, x)$ for all $x \in \mathbb{R}^n$. Then $u(x, t) \leq v(x, t)$ for all $x \in \mathbb{R}^n$ and $0 \leq t \leq T$.*

Proof. The homogeneity (4.5) of the Hamiltonian $\overline{H}$ implies (3.35), moreover (4.4) holds. Then the proof of Theorem 3.5, case (iii), applies here. □

5. The subcritical case: $\alpha < 2$

5.1. **The effective Hamiltonian.** In this case, the asymptotic expansion we plug in the equation is

$$v^\varepsilon(t, x, y) = v^0(t, x) + \varepsilon^{\frac{\alpha}{2}} w(t, x, y). \tag{5.1}$$

Plugging (5.1) into the equation (2.7) we get

$$v_t^0 = |\sigma^T D_x v^0|^2 + 2(\tau \sigma^T D_x v^0) \cdot D_y w + |\tau^T D_y w|^2 + O(\varepsilon). \tag{5.2}$$

Therefore the cell problem we want to solve is finding, for any fixed $(\bar{x}, \bar{p})$, a unique constant $\bar{H}$ such that there is a viscosity solution w of the following equation

$$\bar{H}(\bar{x}, \bar{p}) - 2(\tau(y)\sigma(\bar{x}, y)^T \bar{p}) \cdot D_y w(y) - |\tau(y)^T D_y w(y)|^2 - |\sigma(\bar{x}, y)^T \bar{p}|^2 = 0. \tag{5.3}$$

Since

$$2(\tau(y)\sigma^T(\bar{x}, y)\bar{p}) \cdot D_y w = 2(\sigma^T(\bar{x}, y)\bar{p}) \cdot (\tau^T(y) D_y w)$$

, we can restate the cell problem as

$$\bar{H}(\bar{x},\bar{p}) - |\tau^T(y)D_y w(y) + \sigma^T(\bar{x},y)\bar{p}|^2 = 0. \tag{5.4}$$

The following proposition deals with the existence and uniqueness of $\bar{H}$.

Proposition 5.1. *For any fixed $(\bar{x},\bar{p})$, there exists a unique constant $\bar{H}(\bar{x},\bar{p})$ such that the cell problem (5.3) admits a periodic viscosity solution w. Moreover w is Lipschitz continuous and there exists $C>0$ independent of $\bar{x},\bar{p}$ such that*

$$\max_y |Dw(y;\bar{x},\bar{p})| \le C(1+|\bar{p}|).$$

Proof. As for the other cases we introduce the following approximant problem, with $\delta>0$,

$$\delta w_\delta(y) - |\tau^T(y)D_y w_\delta(y) + \sigma^T(\bar{x},y)\bar{p}|^2 = 0 \text{ in } \mathbb{R}^m. \tag{5.5}$$

Let w_δ the unique periodic viscosity solution to (5.5). By standard comparison principle we get that

$$|\delta w_\delta| \le \max_{y\in\mathbb{R}^m} |\sigma^T(\bar{x},y)\bar{p}|^2 \le C(1+|\bar{p}|^2) \quad \forall y\in\mathbb{R}^m.$$

Moreover, using the coercivity of the Hamiltonian (see [8, Prop II.4.1]), we get that w_δ is Lipschitz continuous and there exists a constant C independent of δ and $\bar{p}$ such that

$$\max_{y\in\mathbb{R}^m} |Dw_\delta| \le C(1+|\bar{p}|).$$

So, we conclude as in the proof of Proposition 3.1. □

We give some representation formulas for the effective Hamiltonian $\bar{H}$.

Proposition 5.2. *(i) $\bar{H}$ satisfies*

$$\bar{H}(\bar{x},\bar{p}) = \lim_{\delta\to 0}\sup_{\beta(\cdot)} \delta\int_0^{+\infty} \left(|\sigma(\bar{x},y(t))^T\bar{p}|^2 - |\beta(t)|^2\right)e^{-\delta t}\,dt, \tag{5.6}$$

where $\beta(\cdot)$ varies over measurable functions taking values in $\mathbb{R}^r$, $y(\cdot)$ is the trajectory of the control system

$$\begin{cases} \dot{y}(t) = 2\tau(y(t))\sigma^T(\bar{x},y(t))\bar{p} - 2\tau(y(t))\beta, & t>0,\\ y(0)=y \end{cases}$$

and the limit is uniform with respect to the initial position y of the system.

(ii) If, in addition, $\tau(y)\sigma^T(x,y)=0$ for all x,y, then

$$\bar{H}(\bar{x},\bar{p}) = \max_{y\in\mathbb{R}^m} |\sigma^T(\bar{x},y)\bar{p}|^2. \tag{5.7}$$

(iii) If $n=m=r=1$, and $\sigma\ge 0$

$$\bar{H}(\bar{x},\bar{p}) = \left(\int_0^1 \frac{\sigma(\bar{x},y)}{\tau(y)}\,dy\right)^2 \left(\int_0^1 \frac{1}{\tau(y)}\,dy\right)^{-2} \bar{p}^2. \tag{5.8}$$

Proof. The formula (5.6) can be proved by writing (5.5) as a Bellman equation

$$\delta w_\delta(y) + \inf_{\beta\in\mathbb{R}^r}\left\{\left(2\tau(y)\beta - 2\tau(y)\sigma(\bar{x},y)^T\bar{p}\right)\cdot D_y w_\delta + |\beta|^2\right\} - |\sigma(\bar{x},y)^T\bar{p}|^2 = 0. \tag{5.9}$$

Then w_δ is the value function of the infinite horizon discounted deterministic control problem appearing in (5.6) (see, e.g., [8, 11]).

If $\tau(y)\sigma^T(x,y)=0$ for all x,y, then (5.4) reads

$$-|\tau^T(y)D_y w(y)|^2 = |\sigma^T(\bar{x},y)\bar{p}|^2 - \bar{H}(\bar{x},\bar{p}).$$

So, this gives immediately the inequality $\geq$ in (5.7). The other inequality is obtained by standard comparison principle arguments applied to the approximating problem (5.5).

Finally, in the case $n=m=r=1$, if $\bar{p}\geq 0$ we write explicitly the corrector as

$$w(y) = \int_0^y \frac{\bar{H}^{\frac{1}{2}} - \sigma(\bar{x},s)\bar{p}}{\tau(s)}\, ds.$$

Note that $w \in C^1$ is periodic and does the job. A similar construction works for $\bar{p}<0$.

□

For the comparison principle it is useful to define

$$H_0(\bar{x},\bar{p}) = \sqrt{\bar{H}(\bar{x},\bar{p})}$$

and observe that the cell problem (5.3) is equivalent to the following equation

$$H_0(\bar{x},\bar{p}) - |\tau^T(y)D_y w(y) + \sigma^T(\bar{x},y)\bar{p}| = 0. \tag{5.10}$$

Here are some properties of $\bar{H}$ and H_0.

Proposition 5.3. *$\bar{H}$ satisfies properties (a), (b), (c), (d) as in Proposition 3.3. Moreover $\bar{H}(x,p)=(H_0(x,p))^2$ with H_0 positively 1 homogeneous, i.e.*

$$H_0(x,\lambda p) = |\lambda| H_0(x,p) \qquad \forall \lambda \in \mathbb{R}, \tag{5.11}$$

there exists $C>0$ such that $|H_0(x,p)| \leq C|p|$, and

$$|H_0(x,p) - H_0(z,p)| \leq C(1+|p|)|x-z| \qquad \forall x,z \in \mathbb{R}^n,\ p \in \mathbb{R}^n. \tag{5.12}$$

Proof. For the proofs of (a), (b), (c) we repeat the same arguments as in Proposition 3.3. The properties of H_0 defined in (5.10) follow from standard theory, using comparison type argument in the approximating problem

$$\delta v_\delta(y) - |\tau^T(y)D_y v_\delta(y) + \sigma^T(x,y)p| = 0 \text{ in } \mathbb{R}^m.$$

□

5.2. **Comparison principle.** We consider the limit PDE

$$v_t - \bar{H}(x,Dv) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n. \tag{5.13}$$

We now state the comparison principle for the effective Hamiltonian $\bar{H}$.

Theorem 5.4. *Let $u \in BUSC([0,T]\times\mathbb{R}^n)$ and $v \in BLSC([0,T]\times\mathbb{R}^n)$ be, respectively, a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution to (3.32) such that $u(0,x)\leq h(x)\leq v(0,x)$ for all $x\in\mathbb{R}^n$. Then $u(x,t)\leq v(x,t)$ for all $x\in\mathbb{R}^n$ and $0\leq t\leq T$.*

Proof. Recall that $\bar{H}=H_0^2$ and H_0 is continuous and satisfies (5.11) and (5.12). So, we can apply Theorem 2.4 in [14]. □

6. The convergence result

In this Section we state the main result of the paper, namely, the convergence theorem for the singular perturbation problem. We will make use of the relaxed semi-limits which we define as follows. For the functions v_ε introduced in Section 2.2 the relaxed upper semi-limit $\bar v = \limsup^*_{\varepsilon\to 0}\sup_y v^\varepsilon$ is

$$\bar v(t,x) := \limsup_{\varepsilon\to 0,(t',x')\to(t,x)} \sup_y v^\varepsilon(t',x',y), \quad x\in\mathbb{R}^n,\ t\geq 0.$$

We define analogously the lower semi-limit $\underline{v} = \liminf_{*\varepsilon\to 0}\inf_y v^\varepsilon$ by replacing $\limsup$ with $\liminf$ and sup with inf. Since h is bounded the family v^ε is equi-bounded and we have $\bar v\in BUSC([0,T]\times\mathbb{R}^n)$ and $\underline{v}\in BLSC([0,T]\times\mathbb{R}^n)$.

The standing hypotheses of sections 2.1 and 2.2 are assumed in this section.

6.1. The convergence result: critical and supercritical case, $\alpha\geq 2$.

Recall that by Proposition 2.1 i) v^ε defined by (2.6) is the solution of

$$\begin{cases} \partial_t v^\varepsilon - H^\varepsilon\left(x,y,D_xv^\varepsilon,\frac{D_yv^\varepsilon}{\varepsilon^{\alpha-1}},D_{xx}v^\varepsilon,\frac{D^2_{yy}v^\varepsilon}{\varepsilon^{\alpha-1}},\frac{D_{xy}v^\varepsilon}{\varepsilon^{\frac{\alpha-1}{2}}}\right)=0 & (0,T)\times\mathbb{R}^n\times\mathbb{R}^m\\ v^\varepsilon(0,x,y)=h(x) & \mathbb{R}^n\times\mathbb{R}^m.\end{cases}$$

with

$$\begin{aligned} H^\varepsilon(x,y,p,q,X,Y,Z): &= |\sigma^Tp|^2+b\cdot q+\mathrm{tr}(\tau\tau^TY)+\varepsilon\left(\mathrm{tr}(\sigma\sigma^TX)+\phi\cdot p\right)\\ &+\ 2\varepsilon^{\frac{\alpha}{2}-1}(\tau\sigma^Tp)\cdot q+2\varepsilon^{\frac12}\mathrm{tr}(\sigma\tau^TZ)+\varepsilon^{\alpha-2}|\tau^Tq|^2.\end{aligned}$$

Theorem 6.1. *Assume $\alpha\geq 2$. Then*

i) The upper limit $\bar v$ (resp., the lower limit $\underline{v}$) of v^ε is a subsolution (resp., supersolution) of the effective equation

$$v_t-\overline{H}(x,Dv)=0 \text{ in } (0,T)\times\mathbb{R}^n \quad v(0,x)=h(x) \text{ on } \mathbb{R}^n \tag{6.1}$$

where $\bar H$ is given by (4.2) *for $\alpha>2$, and it is defined by Proposition 3.1 for $\alpha=2$ (with the formulas* (3.20), (3.21), (3.23), *and* (3.24)*);*

ii) if $\alpha>2$ then v^ε converges uniformly on the compact subsets of $[0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ to the unique viscosity solution of (6.1).

iii) if $\alpha=2$ and

$$\begin{cases}\text{either } \sigma=\sigma(y) \text{ is independent of } x \text{ and } h\in BUC(\mathbb{R}^n),\\ \text{or, for some } \nu>0,\ |\sigma^T(x,y)p|^2>\nu|p|^2 \quad \forall\, x,p\in\mathbb{R}^n,\ y\in\mathbb{R}^m,\\ \text{or, } \tau(y)\sigma^T(x,y)=0 \text{ for all } x,y,\end{cases} \tag{6.2}$$

then v^ε converges uniformly as in ii).

Proof. i) The inequalities $\underline{v}(0,x)\leq h(x)\leq\bar v(0,x)$ follow from the definitions. The problem of taking the limit in the PDE is a regular perturbation of a singular perturbation problem, in the terminology of [4]. The result can be proved by the methods developed in [4] for such problems, with minor modifications.

ii) By the definition of the semilimits $\underline{v}\leq\bar v$ in $[0,T)\times\mathbb{R}^n$. The comparison principle Proposition 4.3 for the effective equation (6.1) gives the inequality $\leq$ and therefore $\bar v=\underline{v}=v$ in $[0,T]\times\mathbb{R}^n$. Thanks to the properties of semilimits, we finally get that v^ε converges locally uniformly to the unique bounded solution of (6.1).

iii) The proof is the same as for *ii)*, but now we need the additional assumption (6.2) for the comparison principle Theorem 3.5. ☐

6.2. **The convergence result: subcritical case,** $\alpha < 2$**.** Recall that by Proposition 2.1 ii) v^ε defined by (2.6) is the solution of

$$\begin{cases} v_t^\varepsilon = H_\varepsilon\left(x, y, D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}, D_{xx} v^\varepsilon, \frac{D_{yy}^2 v^\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}, \frac{D_{xy} v^\varepsilon}{\varepsilon^{\frac{\alpha}{4}}}\right) & (0,T)\times\mathbb{R}^n\times\mathbb{R}^m \\ v^\varepsilon(0,x,y) = h(x) & \mathbb{R}^n\times\mathbb{R}^m. \end{cases}$$

with

$$\begin{aligned} H_\varepsilon(x,y,p,q,X,Y,Z): \;&=\; |\sigma^T p|^2 + 2(\tau\sigma^T p)\cdot q + |\tau^T q|^2 + \varepsilon\left(\operatorname{tr}(\sigma\sigma^T X) + \phi\cdot p\right) \\ &+\; 2\varepsilon^{1-\frac{\alpha}{4}}\operatorname{tr}(\sigma\tau^T Z) + \varepsilon^{1-\frac{\alpha}{2}} b\cdot q + \varepsilon^{1-\frac{\alpha}{2}}\operatorname{tr}(\tau\tau^T Y). \end{aligned}$$

Theorem 6.2. *Assume* $\alpha < 2$. *Then*

i) the upper limit $\bar v$ *(resp., the lower limit* $\underline{v}$*) of* v^ε *is a subsolution (resp., supersolution) of the effective equation* (6.1) *where* $\bar H$ *is defined by Proposition 5.1 (with the formula* (5.6)*);*

ii) v^ε *converges uniformly on the compact subsets of* $[0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ *to the unique viscosity solution of* (6.1).

Proof. The proof is the same as that of Theorem 6.1, by using the comparison principle Proposition 5.4. □

Remark 6. In the case $\alpha \le 2$ we can give a convergence result analogous to Theorem 6.1 and Theorem 6.2 for a terminal cost $h = h(x,y)$ depending also on the fast variable y, so that the payoffs is

$$v_\varepsilon(t,x,y) := \varepsilon \log E\left[e^{\frac{h(X_t,Y_t)}{\epsilon}} | (X_\cdot, Y_\cdot) \text{ satisfy (1.1)}\right], \tag{6.3}$$

In this case we must find a suitable *effective initial value* $\bar h$ depending only on the variable x; moreover the convergence cannot be up to time $t = 0$ but only on the compact subsets of $(0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ to the unique viscosity solution of

$$v_t - \overline{H}(x, Dv) = 0 \text{ in } (0,T)\times\mathbb{R}^n \quad v(0,x) = \bar h(x) \text{ on } \mathbb{R}^n.$$

The proof follows the methods of [2], where an asymptotic problem for finding $\bar h$ is given and the relaxed semi-limits are modified at $t = 0$ to deal with the expected initial layer. For further details and proofs we refer to [29].

7. The large deviation principle

In this section we derive a large deviation principle for the process X_t^ε defined in (2.3). Throughout the section we suppose that σ is uniformly non degenerate, that is, for some $\nu > 0$ and for all $x, p \in \mathbb{R}^n$

$$|\sigma^T(x,y)p|^2 > \nu |p|^2. \tag{7.1}$$

By (3.14), under (7.1), the effective Hamiltonian is coercive. Let $\bar L$ be the *effective Lagrangian*, i.e. for $x \in \mathbb{R}^n$

$$\bar L(x,q) = \max_{p\in\mathbb{R}^n}\{p\cdot q - \bar H(x,p)\}. \tag{7.2}$$

Note that $\bar L(x,\cdot)$ is a convex nonnegative function such that $\bar L(x,0) = 0$ for all $x \in \mathbb{R}^n$, since $\bar H(x,\cdot)$ is convex nonnegative and $\bar H(x,0) = 0$ for all $x \in \mathbb{R}$.
For each $x_0 \in \mathbb{R}^n$ and $t > 0$, define

$$I(x;x_0,t) := \inf\left[\int_0^t \bar L\left(\xi(s),\dot\xi(s)\right) ds \;\middle|\; \xi \in AC(0,t),\ \xi(0) = x_0, \xi(t) = x\right]. \tag{7.3}$$

Remark 7. (a) The function I defined in (7.3) is continuous in the variable x (see, e.g., [16]) and is a nonnegative function such that $I(x_0; x_0, t) = 0$.

(b) I satisfies the following growth condition for some $C > 0$ and all $x, x_0 \in \mathbb{R}^n$

$$\frac{1}{4C}\frac{|x - x_0|^2}{t} \le I(x; x_0, t) \le \frac{1}{4\nu}\frac{|x - x_0|^2}{t}, \tag{7.4}$$

where ν is defined in (7.1). In fact, thanks to the property (3.14) stated in Proposition 3.3, we get that

$$\frac{1}{4C}|p|^2 \le \bar{L}(x, p) \le \frac{1}{4\nu}|p|^2.$$

Then we have

$$\frac{1}{4C}\inf_{\xi(0)=x_0,\xi(t)=x}\int_0^t |\dot{\xi}(s)|^2 \le I(x; x_0, t) \le \frac{1}{4\nu}\inf_{\xi(0)=x_0,\xi(t)=x}\int_0^t |\dot{\xi}(s)|^2,$$

from which we get (7.4).

(c) If σ does not depend on x, i.e. $\bar{H} = \bar{H}(p)$, the rate function in (7.3) is

$$I(x; x_0, t) = t\bar{L}\left(\frac{x - x_0}{t}\right).$$

(d) If σ does not depend of x and $n = 1$, I is a monotone nondecreasing function of x when $x > x_0$. Analogously, I is a monotone nonincreasing function of x when $x < x_0$.

Theorem 7.1. *Let* $(X^\varepsilon, Y^\varepsilon)$ *be the process defined in* (2.3) *with initial position* $X_0^\varepsilon = x_0$ *and* $Y_0^\varepsilon = y_0$. *Then for every* $t > 0$, *a large deviation principle holds for* $\{X_t^\varepsilon : \varepsilon > 0\}$ *with speed* $\frac{1}{\varepsilon}$ *and good rate function* $I(x; x_0, t)$. *In particular, for any open set* $B \subseteq \mathbb{R}^n$

$$\lim_{\epsilon\to 0}\epsilon \log P(X_t^\varepsilon \in B) = -\inf_{x\in B} I(x; x_0, t). \tag{7.5}$$

Remark 8. Thanks to Remark 7, if σ does not depend on x and $n = 1$, we have $\inf_{y>x} I(y; x_0, t) = I(x; x_0, t)$ for $x \ge x_0$ and (7.5) can be written in the following way

$$\lim_{\varepsilon\to 0}\varepsilon \log P(X_t^\varepsilon > x) = -I(x; x_0, t) \quad \text{when } x > x_0$$

and analogously when $x < x_0$

$$\lim_{\varepsilon\to 0}\varepsilon \log P(X_t^\varepsilon < x) = -I(x; x_0, t).$$

Remark 9. We note that the rate function I defined in (7.3) does not depend on the drift ϕ of the log-price X_t^ε and it depends only on the volatility σ and on the fast process Y_t^ε. In fact, this holds for the effective Hamiltonian $\bar{H}$ by the representation formulas (3.20) for $\alpha = 2$, (4.2) for $\alpha > 2$ and (5.6) for $\alpha < 2$, and hence it holds for the Legendre transform $\bar{L}$.

Proof. We divide the proof in two steps, the first is the proof of the large deviation principle, while the second is the proof of the representation formula (7.3) for the good rate function.

Step. *1 (Large deviation principle)* The proof of this step is similar to that of Theorem 2.1 of [22] with some minor changes. The idea is to apply Bryc's inverse

Varadhan lemma (see Appendix A, Lemma A.1) with μ_ε given by the laws of $\{X_t^\varepsilon\}$ and Λ_h^ε given by v_ε. Recall that, for $h \in BC(\mathbb{R}^n)$, v_ε is defined as

$$v_\varepsilon(t,x,y) := \varepsilon \log E\left[e^{\frac{h(X_t^\varepsilon)}{\epsilon}} | (X_\cdot^\varepsilon, Y_\cdot^\varepsilon) \text{ satisfy (2.3)}\right].$$

We proved in Theorems 6.2, 6.2 that v_ε converge uniformly to a function v^h.
To apply Lemma A.1, we have to prove the exponential tightness of $\{X_t^\varepsilon\}$. Define the following function

$$f_\varepsilon(x,y) = \begin{cases} f(x) + \varepsilon^{\alpha-1}\zeta(y) & \text{if } \alpha \geq 2, \\ f(x) + \varepsilon^{\frac{\alpha}{2}}\zeta(y) & \text{if } \alpha < 2, \end{cases} \tag{7.6}$$

where

$$f(x) = \log(1+|x|^2)$$

and $\zeta(y)$ is a positive differentiable function with bounded first and second derivatives. Since $f(x)$ is an increasing function of $|x|$ and since $\zeta(y) \geq 0$, we have that for any $c > 0$ there exists a compact set $K_c \subset \mathbb{R}^n$ such that

$$f_\varepsilon(x,y) > c \quad \text{when } x \notin K_c. \tag{7.7}$$

We observe that $||\partial_{x_j} f||_\infty + ||\partial^2_{x_j x_i} f||_\infty < \infty$ for all $i = 1\cdots n, j = 1\cdots n$, and by our choice of ζ we therefore have that

$$\sup_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} H_\varepsilon(x,y,D_x f_\varepsilon, D_y f_\varepsilon, D^2_{xx} f_\varepsilon, D^2_{yy} f_\varepsilon, D^2_{xy} f_\varepsilon) = C < \infty, \tag{7.8}$$

where H_ε is defined as follows

$$\begin{aligned} H_\varepsilon(x,y,p,q,X,Y,Z) \;&=\; |\sigma^T p|^2 + \varepsilon \mathrm{tr}(\sigma\sigma^T X) + \varepsilon\phi\cdot p + 2\varepsilon^{-\frac{\alpha}{2}}\mathrm{tr}(\tau\sigma^T p)\cdot q \\ &+\; 2\varepsilon^{1-\frac{\alpha}{2}}\mathrm{tr}(\sigma\tau^T Z) + \varepsilon^{1-\alpha} b\cdot q + \varepsilon^{-\alpha}|\tau^T q|^2 + \varepsilon^{1-\alpha}\mathrm{tr}(\tau\tau^T Y). \end{aligned}$$

We will write $H_\epsilon f_\epsilon(x,y)$ to denote $H_\varepsilon(x,y,D_x f_\varepsilon, D_y f_\varepsilon, D^2_{xx} f_\varepsilon, D^2_{yy} f_\varepsilon, D^2_{xy} f_\varepsilon)$. The P and E in the following proof denote probability and expectation conditioned on (X,Y) starting at (x,y). Define the process

$$M_t^\varepsilon = \exp\left\{\frac{f_\varepsilon(X_t^\varepsilon, Y_t^\varepsilon)}{\varepsilon} - \frac{f_\varepsilon(x,y)}{\varepsilon} - \frac{1}{\epsilon}\int_0^t H_\varepsilon f_\varepsilon(X_s^\varepsilon, Y_s^\varepsilon)\, ds\right\}. \tag{7.9}$$

Then $M_{\varepsilon,t}$ is a supermartingale and hence we can apply the optional sampling theorem (see Appendix A, Theorem A.2), that is

$$1 \geq E\left[M_t^\varepsilon\right]. \tag{7.10}$$

Then

$$\begin{aligned} 1 \geq E\left[M_t^\varepsilon \,|\, X_t^\varepsilon \notin K_c\right] \;&\geq\; E\left[e^{\frac{(c - f_\varepsilon(x,y) - tC)}{\varepsilon}} \,|\, X_t^\varepsilon \notin K_c\right] \\ &=\; P(X_t^\varepsilon \notin K_c) e^{\frac{(c - f_\varepsilon(x,y) - tC)}{\varepsilon}}, \end{aligned} \tag{7.11}$$

where we have used (7.7) and (7.8) to estimate the first and third term in M_t^ε. Then we get

$$\varepsilon \log P(X_t^\varepsilon \notin K_c) \leq tC + f_\varepsilon(x,y) - c \leq \text{ const } - c$$

and this finally gives us the exponential tightness of X_t^ε.
So, by Bryc's inverse Varadhan lemma (see Appendix A, Lemma A.1), the measures associated to the process X_t^ε satisfy the LDP with the good rate function

$$I(x; x_0, t) = \sup_{h\in BC(\mathbb{R}^n)} \{h(x) - v^h(t, x_0)\} \tag{7.12}$$

and

$$v^h(t,x_0)=\sup_{x\in\mathbb{R}^n}\{h(x)-I(x;x_0,t)\}.$$

Step. *2 (Representation formula for the good rate function)* The solution v^h to the effective equation

$$\begin{cases} v_t-\bar{H}(x,Dv)=0 & \text{in } (0,T)\times\mathbb{R}^n \\ v(0,x)=h(x) & \text{in } \mathbb{R}^n \end{cases} \tag{7.13}$$

can be represented through the following formula

$$v^h(t,x)= \tag{7.14}$$
$$\sup\left\{h(y)-\int_0^t \bar{L}\left(\xi(s),\dot{\xi}(s)\right)\,ds \mid y\in\mathbb{R}^n, \xi\in AC(0,t), \xi(0)=x, \xi(t)=y\right\},$$

where $\bar{L}$ is the effective Lagrangian defined in (7.2). We refer to [16] where it is shown that v^h is continuous and is the solution of (7.13). We define

$$r(x;x_0,t)=\inf_{\xi(0)=x_0,\xi(t)=x}\int_0^t \bar{L}\left(\xi(s),\dot{\xi}(s)\right)\,ds \tag{7.15}$$

Thanks to (7.12) and (7.14), we can write

$$I(x;x_0,t)= \tag{7.16}$$
$$r(x;x_0,t)+\sup_{h\in BC(\mathbb{R})}\inf\left\{h(x)-h(y)+\int_0^t \bar{L}\left(\xi(s),\dot{\xi}(s)\right)\,ds-r(x;x_0,t)\right\},$$

where the infimum is over $y\in\mathbb{R}^n$ and absolutely continuous functions ξ such that $\xi(0)=x_0, \xi(t)=y$. Then

$$I(x;x_0,t)=r(x;x_0,t)+J(x;x_0,t),$$

where $J(x;x_0,t):=\sup_{h\in BC(\mathbb{R})} J_h(x;x_0,t)$ and

$$J_h(x;x_0,t)=\inf\left\{h(x)-h(y)+\int_0^t \bar{L}\left(\xi(s),\dot{\xi}(s)\right)\,ds-r(x;x_0,t)\right\}.$$

Taking $y=x$, we obtain $J_h(x;x_0,t)\le 0$ and therefore $J(x;x_0,t)\le 0$. Now we define a function $h_*\in BC(\mathbb{R})$ as follows:

$$h_*(y)=r(y;x_0,t)\wedge r(x;x_0,t).$$

We claim that h_* is continuous. Then $J_{h_*}(x;x_0,t)=0$ and therefore $J(x;x_0,t)=0$. In conclusion

$$I(x;x_0,t)=\inf_{\xi(0)=x_0,\xi(t)=x}\int_0^t \bar{L}\left(\xi(s),\dot{\xi}(s)\right)\,ds.$$

Finally, the claim follows from the continuity of the function $r(y;x_0,t)$ in the variable y, that can be found, e.g., in [16], Section 4, Proposition 3.1 and Corollary 3.4.

□

8. Out-of-the-money option pricing and asymptotic implied volatility

8.1. **Option price.** In this section, we give some applications of Theorem 7.1 in dimension 1 to out-of-the-money option pricing. In particular, in Corollary 8.1, we state an asymptotic estimate for the behaviour of the price of out-of-the-money European call option with strike price K and short maturity time $T = \varepsilon t$.

Let S_t^ε be the asset price, evolving according to the following stochastic differential system

$$\begin{cases} dS_t^\varepsilon = \varepsilon\xi(S_t^\varepsilon, Y_t^\varepsilon)S_t^\varepsilon dt + \sqrt{2\varepsilon}\zeta(S_t^\varepsilon, Y_t^\varepsilon)S_t^\varepsilon dW_t & S_0^\varepsilon = S_0 \in \mathbb{R}_+ \\ dY_t^\varepsilon = \varepsilon^{1-\alpha}b(Y_t^\varepsilon)dt + \sqrt{2\varepsilon^{1-\alpha}}\tau(Y_t^\varepsilon)dW_t & Y_0^\varepsilon = y_0 \in \mathbb{R}^m, \end{cases} \tag{8.1}$$

where $\alpha > 1$, τ, b are as in (2.3) and $\xi : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$, $\zeta : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbf{M}^{1,r}$ are Lipschitz continuous bounded functions, periodic in y. Observe that $S_t^\varepsilon > 0$ almost surely if $S_0 > 0$. We define $X_t^\varepsilon = \log S_t^\varepsilon$. Then $(X_t^\varepsilon, Y_t^\varepsilon)$ satisfies (2.3) with

$$\phi(x,y) = \xi(e^x, y) - \zeta(e^x, y)\zeta^T(e^x, y) \qquad \sigma(x,y) = \zeta(e^x, y).$$

We consider out-of-the-money call option by taking

$$S_0 < K \quad \text{or} \quad x_0 < \log K. \tag{8.2}$$

Following the argument used in [22], we can derive an option price estimates stated in Corollary 8.1. Similarly, by considering out-of-the-money put options, one can obtain the same formula for $S_0 > K$.

Corollary 8.1. *Suppose that $S_0 < K$. Then, for fixed $t > 0$*

$$\lim_{\varepsilon\to 0^+} \varepsilon \log E\left[(S_t^\varepsilon - K)^+\right] = -\inf_{y > \log K} I(y; x_0, t). \tag{8.3}$$

8.2. **Implied volatility.** We give an asymptotic estimate of the Black-Scholes implied volatility for out-of-the-money European call option, with strike price K, which we denote by $\sigma_\varepsilon(t, \log K, x_0)$.
We recall that given an observed European call option price for a contract with strike price K and expiration date T, the *implied volatility* σ is defined to be the value of the volatility parameter that must go into the Black-Scholes formula to match the observed price.
By arguments similar to those of the ones used in [22], we get the following asymptotic formula.

Corollary 8.2.

$$\lim_{\varepsilon\to 0^+} \sigma_\varepsilon^2(t, \log K, x_0) = \frac{(\log K - x_0)^2}{2\inf_{y>\log K} I(y; x_0, t)t}. \tag{8.4}$$

Note that the infimum in the right-hand side of (8.4), is always positive by assumption (8.2) and by (7.4).

Remark 10. When $\zeta(s,y) = \zeta(s)$, then thanks to Remark 8, (8.3) simplifies to

$$\lim_{\varepsilon\to 0^+} \varepsilon \log E\left[(S_t^\varepsilon - K)^+\right] = -I(\log K; x_0, t)$$

and (8.4) reads

$$\lim_{\varepsilon\to 0^+} \sigma_\varepsilon^2(t, \log K, x_0) = \frac{(\log K - x_0)^2}{2I(\log K; x_0, t)t}.$$

Proof. By the definition of implied volatility

$$
\begin{aligned}
E\left[(S_t^\varepsilon - K)^+\right] &= e^{r\varepsilon t} S_0 \Phi\left(\frac{x_0 - \log K + r\varepsilon t + \sigma_\varepsilon^2 \frac{\varepsilon t}{2}}{\sigma_\varepsilon \sqrt{\varepsilon t}}\right) \\
&- K\Phi\left(\frac{x_0 - \log K + r\varepsilon t - \sigma_\varepsilon^2 \frac{\varepsilon t}{2}}{\sigma_\varepsilon \sqrt{\varepsilon t}}\right),
\end{aligned} \tag{8.5}
$$

where Φ is the Gaussian cumulative distribution function. Then the proof follows as in [22], using (8.5) and Corollary 8.1. □

Appendix A

We recall some standard notions from large deviation theory that we need in section 7. Throughout the section, μ_ϵ will denote a family of probability measures defined on $\mathbb{R}^n$ with its Borel σ-field $\mathcal{B}$. For the definitions and theorems in a more general setting and for further details we refer to [17].
Given a family of probability measures $\{\mu_\epsilon\}$, a large deviation principle characterizes the limiting behavior, as $\epsilon \to 0$, of $\{\mu_\epsilon\}$ in terms of a rate function through asymptotic upper and lower exponential bounds on the values that μ_ϵ assigns to measurable subsets of $\mathbb{R}^n$.

Definition A.1. *A rate function I is a lower semicontinuous map $I : \mathbb{R}^n \to [0, \infty]$, and it is a good rate function if for all $\alpha \in [0, \infty)$, the level set $\Psi_I(\alpha) := \{x : I(x) \leq \alpha\}$ is compact.*

For any set $B \subseteq \mathbb{R}^n$, we denote by B° the interior of B.

Definition A.2. *A family of probability measures $\{\mu_\epsilon\}$ satisfies the large deviation principle with a rate function I if, for all $B \in \mathcal{B}$,*

$$
-\inf_{x \in B^\circ} I(x) \leq \liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(B) \leq \limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(B) \leq -\inf_{x \in \bar{B}} I(x). \tag{A.1}
$$

The right-and left-hand sides of (A.1) are referred to as the upper and lower bounds, respectively.

Definition A.3. *A family of probability measures $\{\mu_\epsilon\}$ on $\mathbb{R}^n$ is exponentially tight if for every $\alpha < \infty$, there exists a compact set $K_\alpha \subset \mathbb{R}^n$ such that*

$$
\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(K_\alpha^c) < -\alpha.
$$

Moreover, for each Borel measurable function $h : \mathbb{R}^n \to \mathbb{R}$, define

$$
\Lambda_h^\epsilon := \epsilon \log \int_{\mathbb{R}^n} e^{\frac{h(x)}{\epsilon}} \mu_\epsilon(dx).
$$

and

$$
\lim_{\epsilon \to 0} \epsilon \log \int_{\mathbb{R}^n} e^{\frac{h(x)}{\epsilon}} \mu_\epsilon(dx) = \Lambda_h \tag{A.2}
$$

provided the limit exists. Then, the so-called Bryc's inverse Varadhan Lemma permits to derive the large deviation principle as a consequence of exponential tightness of the measures μ_ϵ and the existence of the limits (A.2) for every $h \in BC(\mathbb{R}^n)$. The statement is the following.

Lemma A.1. *Suppose that the family $\{\mu_\epsilon\}$ is exponentially tight and that the limit in* (A.2) *exists for every $h \in BC(\mathbb{R}^n)$. Then $\{\mu_\epsilon\}$ satisfies the LDP with the good rate function*

$$I(x) = \sup_{h \in BC(\mathbb{R}^n)} \{h(x) - \Lambda_h\}.$$

Furthermore, for every $h \in BC(\mathbb{R}^n)$,

$$\Lambda_h = \sup_{x \in \mathbb{R}^n} \{h(x) - I(x)\}.$$

Finally we recall the optional sampling theorem. For further details see [38].

Theorem A.2. *Let $M = \{M_t\}_{t \geq 0}$ be a submartingale right-continuos and let τ be a stopping time, such that one of the following conditions is satisfied*

- *τ is a.s. bounded, i.e. there exists $T \in (0, \infty)$ such that $\tau \leq T$ a.s.;*
- *τ is a.s. finite and $M_{\tau \wedge t} \leq Y$ for all $t \geq 0$, where Y is an integrable variable (in particular $|M_{\tau \wedge n}| \leq K$ for a constant $K \in [0, \infty)$)*

Then the variable M_τ is integrable and

$$E(M_\tau) \geq E(M_0). \tag{A.3}$$

If, instead, M is a supermartingale, then

$$E(M_\tau) \leq E(M_0).$$

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