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L^1 CONVERGENCE OF A SL SCHEME FOR THE EIKONAL EQUATION WITH DISCONTINUOUS COEFFICIENTS

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Abstract. We consider the stationary eikonal equation where the coefficients are allowed to be discontinuous. The discontinuities must belong to a special class for which the notion of viscosity solutions in the sense of Ishii is suitable. We present a semi–Lagrangian scheme for the approximation of the viscosity solution also studying its properties. The main result is an *a-priori* error estimate in the L^1-norm. In the last section, we illustrate some tests and applications where the scheme is able to compute the right solution.

1. **Introduction.** In this paper we study the following boundary value problem. Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with a Lipschitz boundary $\partial\Omega$, we consider the Dirichlet problem

$$\begin{cases} c(x)|Du(x)| = f(x) & x \in \Omega, \\ u(x) = g(x) & x \in \partial\Omega, \end{cases} \tag{1}$$

where f, c and g are given real functions defined on Ω. We focus our attention on the case where f is positive, Borel measurable and possibly discontinuous.

In the most classical case, where $c(x)$ is constantly equal to one, $f(x) \equiv 1$ and $g(x) \equiv 0$, we get the eikonal equation giving the characterization of the distance from $\partial\Omega$. In other applications, e.g. in geometrical optics, computer vision, control theory and robotic navigation, c and f can vary but have typically a constant sign (e.g. positive). It is worth to note that in the study of many problems motivated by real world applications a discontinuous f and/or a degenerate c can appear in a natural way. In fact, one can easily imagine that the velocity of a front in a medium is affected by the physical properties of the medium and can be discontinuous if the medium is stratified by different materials. In the famous Shape-from-Shading problem the right-hand side is $f(x) = [(1-I^2(x))/I^2(x)]^{1/2}$ where I is the brightness of the image. Depending on the shape of the object represented in the image I can be discontinuous.

Another motivation to deal with discontinuous Hamiltonians comes directly from

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control theory. In the control framework discontinuous functions can be used to represent targets (using f as a characteristic function) and/or state constraints (using f as an indicator function). It is interesting to point out that, when the Hamiltonian is discontinuous, the knowledge of f at every point will not guarantee the well-posedness of the problem, even in the framework of viscosity solution. To deal with this problem we will adopt the notion of discontinuous viscosity solutions via semicontinuous envelopes of f introduced by Ishii in [13]. Other results of well-posedness of Hamilton-Jacobi equations in presence of discontinuous coefficients have been presented by various authors in several works (see [4, 11, 2]) and in the specific case of the eikonal equation [19, 16].

Our primary goal is to prove convergence for a semi–Lagrangian scheme. The typical convergence result, given in the L^∞ norm, is natural when dealing with classical viscosity solutions (see e.g. Crandall and Lions [6], Barles and Souganidis [3] and Falcone and Ferretti [10]). It is clear that, dealing with discontinuous coefficients and/or discontinuous solutions, the classical assumptions for convergence in the uniform norm are not satisfied. Then, it seems more natural to look for convergence in the L^1 norm as it happens in the analysis of approximation schemes for conservation laws. However, the list of contributions on this topic is rather short. At our knowledge, the only two convergence results in L^1 has been proved by Lin and Tadmor [18, 15] for a central finite difference scheme and by Bokanowski et al. [5] in dimension one. Deckelnick and Elliott [8] studied a problem where the solution is still Lipschitz continuous although the Hamiltonian is discontinuous. In particular, they proposed a finite difference scheme and an *a-priori* error estimate. Although our work has been also inspired by their results, we use different techniques and our analysis is devoted to a scheme of semi–Lagrangian type ($SL-scheme$). The benefits of a SL-scheme with respect to a finite difference scheme are a better ability to follow the informations driven by the characteristics and the fact that they do not require a structured grid. These peculiarities give us a faster and more accurate approximation in many cases as it has been reported in the literature (see e.g. [9, 7] or appendix A of [1]). It is also important to note that we prove an *a-priori* error estimate in the general case where also discontinuous viscosity solutions may appear.

2. **The model problem and previous results.** Let us start introducing the definition of discontinuous viscosity solution and summarize for readers convenience some well-posedness results.

Assumption A0. The boundary data

$$g : \partial\Omega \to [0, +\infty[\text{ is continuous,} \tag{2}$$

$c : \Omega \to \mathbb{R}$ is a non negative and continuous function such that $c(x) \leq M_c$ for all $x \in \Omega$. Additional hypotheses will be added on the set where c vanishes later in this paper. Moreover, the function $f : \mathbb{R}^N \to [\rho, +\infty[$, $\rho > 0$ is Borel measurable and possibly discontinuous.

Let us remind the definition of *discontinuous viscosity solution* for (1) introduced by Ishii in [13]. Let f be bounded in Ω, we define f_* and f^* which are respectively the lower semicontinuous and the upper semicontinuous envelope of f as

$$f_*(x) = \liminf_{r\to 0^+}\{f(y) : |y-x| \leq r\}, \quad f^*(x) = \limsup_{r\to 0^+}\{f(y) : |y-x| \leq r\}. \tag{3}$$

Definition 2.1. A lower (resp. upper) semicontinuous function $u : \Omega \to \mathbb{R}\cup\{+\infty\}$ (resp. $u : \Omega \to \mathbb{R}$) is a viscosity super- (resp. sub-) solution of the equation (1) if for every $\phi \in C^1(\Omega)$, $u(x) < +\infty$, and $x \in \arg\min_{x\in\Omega}(u-\phi)$, (resp. $x \in \arg\max_{x\in\Omega}(u-\phi)$), we have

$$c(x)|D\phi(x)| \geq f_*(x), \quad \text{(resp. } c(x)|D\phi(x)| \leq f^*(x)).$$

A function u is a discontinuous viscosity solution of (1) if u^* is a subsolution and u_* is a supersolution.

Note that also Dirichlet boundary conditions must be interpreted in the following weak sense.

Definition 2.2. An upper semicontinuous function $u : \overline{\Omega} \to \mathbb{R}$, subsolution of (1), satisfies the Dirichlet type boundary condition in the viscosity sense if for all $\phi \in C^1(\overline{\Omega})$ and $x \in \partial\Omega$, $x \in \arg\max_{x\in\overline{\Omega}}(u-\phi)$ such that $u(x) > g(x)$, we have

$$c(x)|D\phi(x)| \leq f^*(x).$$

Lower semicontinuous functions that satisfy a Dirichlet type boundary condition are defined accordingly.

In order to guarantee uniqueness we add the following assumption on f.

Assumption A1. Let us assume that there exist $\eta > 0$ and $K \geq 0$ such that for every $x \in \Omega$ there is a direction $n = n_x \in \mathcal{S}^N$ ($\mathcal{S}^N$ is the unit ball of dimension N centered at 0) such that

$$f(y + rd) - f(y) \leq Kr, \tag{4}$$

for every $y \in \Omega$, $d \in \mathcal{S}^N$, $r > 0$ with $|y - x| < \eta$, $|d - n| < \eta$ and $y + rd \in \Omega$.

Under Assumptions A0–A1, a comparison theorem between sub- and supersolution holds [8] (a more general result can be found in [17]). It is important to highlight that adopting the concept of discontinuous viscosity solution, that comparison result is not enough to prove uniqueness. Multiple discontinuous solutions may exist without any contradiction. In that case, an important role is played by two special elements of the class of solutions, the minimal supersolution and the maximal subsolution, defined respectively (see [17] for details) as

$$\begin{aligned} V_m &= \inf_{a\in\mathcal{A}} \int_0^{\tau_x(a)} f_*(y(t,a))dt + g(y(\tau_x(a)), a), \\ V_M &= \inf_{a\in\mathcal{A}} \int_0^{\tau_x(a)} f^*(y(t,a))dt + g(y(\tau_x(a)), a); \end{aligned} \tag{5}$$

where $\tau_x(a)$ is the first exit time from the domain of a trajectory starting from x and subject to the controlled dynamics $\dot{y}(t) = a(t)$, with control a in $\mathcal{S}^N$. It is also important to remark that in the case of existence of a continuous viscosity solution automatically $V_m \equiv V_M$ and the family of solutions reduces to just one solution. For an example of this case let us consider (1) in $\Omega = (-1, 1)$, $f(x) = 0$, for $x < 0$, and $f(x) = x$ for $x \geq 0$ and $c(x) = x$. Let us fix the boundary condition $u(-1) = u(1) = 0$. It is easy to verify that the piecewise continuous function,

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 - x & x \geq 0 \end{cases} \tag{6}$$

is a viscosity solution of the problem. Indeed, we can change at $x = 0$ the value for the solution in $[0, 1]$ obtaining a family of discontinuous viscosity solutions whose upper semicontinuous envelope is always V_M whereas the lower semicontinuous envelope coincides with V_m.

In order to state a more precise result, let us restrict ourselves to the special case $N = 2$ where c vanishes on an interface Σ_0 splitting the domain in two parts. This choice is made to simplify the presentation, more general situations can be treated in a similar way.

Let us denote by $\ell(C)$ the length of a curve C and assume the existence of a regular curve Σ_0 which splits the domain Ω into two subset Ω_j, $j = 1, 2$, where c does not vanish.

Assumption A2. Let $\Sigma_0 := \{x \in \Omega | c(x) = 0\}$, we assume that $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma_0$

$$\ell(\Sigma_0) < +\infty \text{ and } \Omega_j \cap \partial\Omega \neq \emptyset \text{ for } j = \{1, 2\}.$$

We conclude this section with a regularity result, which can be derived by adapting the classical proof by Ishii [14]:

Theorem 2.3. *Let Ω be an open domain with Lipschitz boundary. Assume A0, A1, A2. Let $u : \overline{\Omega} \to \mathbb{R}$ be a bounded viscosity solution of the problem (1), then u is Lipschitz continuous in every set Ω_1 and Ω_2.*

3. **The semi–Lagrangian approximation scheme and its properties.** We construct a semi–Lagrangian approximation scheme for the equation (1) following the approach illustrated in [9].

Using the Kruzkov's change of variable, $v(x) = 1 - e^{-u(x)}$, problem (1) becomes

$$\begin{cases} \max_{a\in\mathcal{S}^2}\{c(x)a \cdot Dv(x)\} = f(x)(1 - v(x)) & x \in \Omega, \\ v(x) = 1 - e^{-g(x)} & x \in \partial\Omega. \end{cases} \tag{7}$$

There exists a clear interpretation of this equation as the value function of an optimization problem with constant running cost and discount factor equal to one, and the dynamics given by $\frac{c(x)}{f(x)}a$ (see [1] for more details).

We discretize the left-hand side term of the first equation in(7) as a directional derivative getting the following discrete problem:

$$\begin{cases} v_h(x) = \frac{1}{1+h} \inf_{a\in B(0,1)} \left\{ v_h\left(x - \frac{h}{f(x)}c(x)a\right)\right\} + \frac{h}{1+h} & x \in \Omega, \\ v_h(x) = 1 - e^{-g(x)} & x \in \partial\Omega, \end{cases} \tag{8}$$

where h is a positive real number and we will assume (to simplify the presentation) that $x - \frac{h}{f(x)}c(x)a \in \overline{\Omega}$ for every $a \in \mathcal{S}^2$.

Let introduce a space discretization of (8) yielding a fully discrete scheme. We construct a regular triangulation of Ω made by a family of simplices S_j, such that $\overline{\Omega} = \cup_j S_j$, denoting x_m, $m = 1, ..., L$, the nodes of the triangulation, by $\Delta x := \max_j diam(S_j)$ the size of the mesh ($diam(B)$ denotes the diameter of the set B) and by G the set of the knots of the grid. We look for a solution of

$$\begin{cases} W(x_m) = \frac{1}{1+h} \min_{a\in B(0,1)} I[W](x_m - \frac{h}{f(x_m)}c(x_m)a) + \frac{h}{1+h} & x_m \in G, \\ W(x_m) = 1 - e^{-g(x_m)} & x_m \in G \cap \partial\Omega, \end{cases} \tag{9}$$

where $I[W](x)$ is a linear interpolation of W at the point x. Therefore, we look for the solution of equation (9) in the space of piecewise linear functions

$$\mathcal{W}^{\Delta x} := \left\{ w : \overline{\Omega} \to \mathbb{R} | w \in C(\Omega) \text{ and } Dw(x) = cost_j \text{ for any } x \in S_j \right\},$$

the existence and uniqueness of a solution in such space is an easy application of the Contraction Mapping Theorem.

$\Delta x = h$	$V_m error \ \|\| \cdot \|\|_1$	$Ord(L^1)$	$V_M error \ \|\| \cdot \|\|_1$	$Ord(L^1)$
0.2	0.1729		0.2204	
0.1	0.1166	0.5684	0.1054	1.0642
0.05	0.0765	0.6080	0.0517	1.0276
0.025	0.0495	0.6280	0.0257	1.0084
0.0125	0.0349	0.5042	0.013	0.9833

TABLE 1. Test 1: experimental errors in $L^1(\Omega)$ for the approximation of V_m and V_M.

Proposition 1. *Let* $x_m - \frac{h}{f(x_m)}c(x_m)a \in \overline{\Omega}$, *for every* $x_m \in G$ *and for any* $a \in B(0,1)$, *then there exists a unique solution* W *of* (9) *in* $\mathcal{W}^{\Delta x}$.

We can also prove that this scheme is monotone and consistent with the equation, and the following *a-priori* error estimate in $L^1(\Omega)$ holds true.

Theorem 3.1. *Let* $A0$, $A1$ *and* $A2$ *hold true. Let* $v(x)$ *be a viscosity solution of* (7) *and* $W(x) \in \mathcal{W}^{\Delta x}$ *be a piecewise linear function satisfying* (9). *Then, there exist two positive constants* C, C' *(independent from* h *and* Δx*) such that for* h *and* Δx *satisfying* $\frac{h}{\Delta x} \le \frac{\rho}{M_c}$, *(*$\rho$ *and* M_c *appear in Assumption* $A0$*) we have:*

$$||v(x) - W(x)||_{L^1(\Omega)} \le C\sqrt{h} + C'\Delta x. \tag{10}$$

Proof. We just sketch the main steps of the proof (which cen be found in [12]). We start introducing the set $\Sigma_{\Delta x}$ defined as

$$\Sigma_{\Delta x} := \left\{x \in \Omega | \mathcal{S}^2_{x,\Delta x} \cap \Sigma_0 \neq \emptyset \right\},$$

where $\mathcal{S}^2_{x,\Delta x}$ denotes the ball of radius Δx centred at x. We observe that

$$||v(x) - W(x)||_{L^1(\Omega)} \le \sum_{j=1,2} \int_{\Omega_j} |v(x) - W(x)| dx + \int_{\Sigma_{\Delta x}} |v(x) - W(x)| dx, \tag{11}$$

where $\Omega := \cup_j \Omega_j$ is the partition of Ω generated from Σ_0.

By the definition of Kruzkov's transform, we know that $|v(x) - W(x)| \le 2$ for every $x \in \Omega$ and adding the assumptions on the set Σ_0 we get, for a fixed $C' > 0$,

$$\int_{\Sigma_{\Delta x}} |v(x) - W(x)| dx \le 2 \int_{\Sigma_{\Delta x}} dx \le 2\ell(\Sigma_0)\Delta x \le C'\Delta x. \tag{12}$$

To prove the statement, we need to prove an estimate for the term $\int_{\Omega_j} |v(x) - W(x)| dx$ for every j. To this end, we remind that for Theorem 2.3, both $v(x)$ and $W(x)$ are Lipschitz continuous, so we can use a modification of the classical argument based on the variable doubling. □

4. **Numerical experiments and applications.** In this section we present some test problems pointing out the main features of our numerical scheme.

4.1. **Test 1: a 1D example.** We want to solve the following equation on the interval $[-1, 1]$

$$|xu'| = f(x), \tag{13}$$

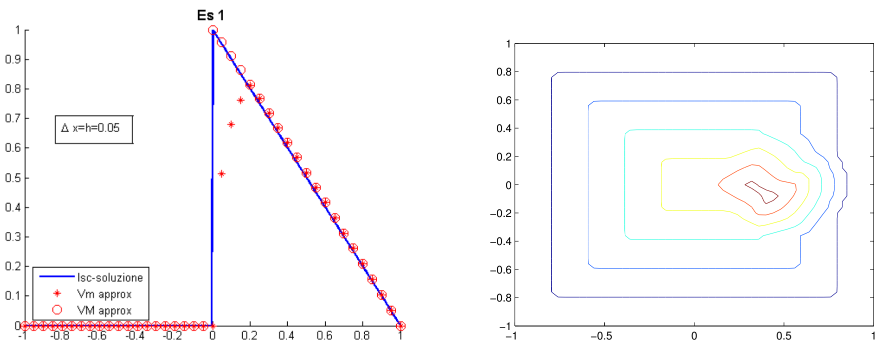

FIGURE 1. Test 1 (left): two approximations of the value function, Test 2 (right): level sets of the approximated value function.

with $u(\pm 1) = 0$ and

$$f(x) := \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases} \tag{14}$$

We denote by f^* the function defined above where $f(0) = 1$ and by f_* the same function where $f(0) = 0$. As we explained in the previous section, to have a bounded solution we use the Kruzkov transform to pass to the equation

$$|xv'| = f(x)(1 - v(x)). \tag{15}$$

In this case we can get easily a correct solution (which is not unique) and we can explicitly calculate V_m and V_M. In this particular case the solutions of the dynamics are very simple, they are $y(t) = xe^{-at}$. Then we have

$$V_m(x) = \begin{cases} \int_0^{\ln -x^{-1}} 0 \;\; dt = 0 & x \leq 0 \\ \int_0^{\ln x^{-1}} e^{-t} dt = [-e^{-t}]_0^{\ln x^{-1}} = 1 - x & x > 0 \end{cases} \tag{16}$$

It is simple to show with the same reckoning, that $V_m(x) = V_M(x)$ for $x \in \overline{\Omega} \setminus \{0\}$. The functions will diverge only at $x = 0$ where $V_m(0) = 0 \neq 1 = V_M(0)$ accordingly to the fact that V_m is l.s.c. and V_M is u.s.c.

Therefore, in this case we do not have a unique viscosity solutions, however

$$V_m(x) = (V_M)_*(x) = \begin{cases} 0 & x \leq 0, \\ 1 - x & x > 0, \end{cases} \tag{17}$$

is our unique lower semicontinuous solution. We want to verify if the numerical approximation introduced in the previous section, converges to all the elements of the solutions class. Let make a test obtaining the results contained in Figure 1 and in Table 1 where we perform the scheme using f_* (in some sense, we are computing V_m) and f^* (computing V_M). We can see in both cases a good convergence in L^1-norm although there is no convergence in the L^∞-norm.

4.2. **Test 2: a 2D example.** Let us consider, the equation $|Du(x)| = f(x)$ in $\Omega = (-1, 1)^2$, where

$$f(x_1, x_2) := \begin{cases} 2, & (x_1 - \frac{1}{2})^2 + x_2^2 \leq \frac{1}{8} \text{ and } x_2 \geq x_1 - \frac{1}{2}, \\ 3, & (x_1 - \frac{1}{2})^2 + x_2^2 \leq \frac{1}{8} \text{ and } x_2 < x_1 - \frac{1}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$\Delta x = h$	$\|\|\cdot\|\|_\infty$	$Ord(L^\infty)$	$\|\|\cdot\|\|_1$	$Ord(L^1)$
0.2	0.3281		0.6134	
0.1	0.1732	0.9217	0.3355	0.8705
0.05	0.0858	1.0134	0.1691	0.9884
0.025	0.0417	1.0409	0.0776	1.1237
0.0125	0.0192	1.1189	0.0296	1.3905

TABLE 2. Test 2: experimental error.

and we apply homogeneous boundary conditions $u(x) = 0$ on $\partial\Omega$. Note that in this case, discontinuities of f occur both along curved lines and straight lines. This shows the good capability of the semi–Lagrangian scheme to approximate the solution (Figure 1), this is due to the fact that the directions used in the scheme are not aligned with the geometry of the grid. Note that in this case we have existence of a continuous solution (therefore uniqueness of the viscosity solution) and that the error estimate in L^1–norm applies. We can see in Table 2 that in this case we have also convergence in the L^∞–norm. Due to the fact that in this case an analytic solution is not available we have used as exact solution a numerical approximation obtained on a very fine grid ($\Delta x = 0.005$).

4.3. **Test 3: finding the exit from a labyrinth.** We apply our scheme to find the exit path from a labyrinth Q. We can write this problem as a minimum time problem with state constraints (the walls can not be crossed). The geometry of the labyrinth is shown in Figure 2 where the gray square is the exit (the target $\mathcal{T}$ for the minimum time problem). We have computed the solution of

$$|Du(x)| = f(x) \quad x \in Q \setminus \mathcal{T}, \tag{18}$$

with Dirichlet boundary conditions $u(x) = 0$ on $\partial\mathcal{T}$ and discontinuous running cost

$$f(x) = \begin{cases} M & \text{if } x \text{ is on the walls,} \\ 1 & \text{otherwise.} \end{cases} \tag{19}$$

In the test we have chosen $\Delta x = h = 0.0078$, $M = 10^{10}$. In Figure 2 we can see the plot of the value function obtained.

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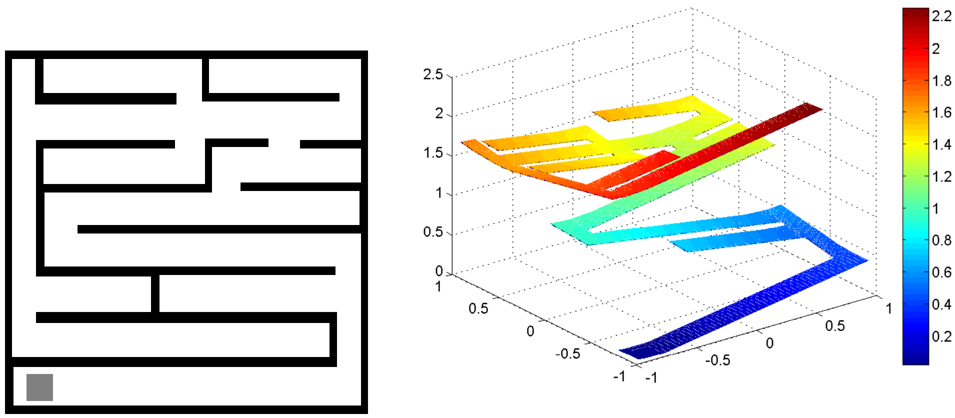

FIGURE 2. Test 3: The labyrinth (left) and the value function of the minimum time problem (right).

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