Mobile 4R and 5R loops

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Abstract

So-called 4R and 5R loops are mechanisms consisting of 4 or 5 links, pairwise joined by revolute joins. Here, they are supposed to be without offsets, that is, the common perpendiculars to the joins axes of two neighbour links are concurrent. A complete classification of 4R loops is given, showing, in particular, that all mobile 4R loops were already known. Also, in each case, the assembly conditions on the design parameters have been computed and are explicitly given.

For general 5R loops, the assembly condition is a homogeneous polynomial in 10 variable of degree probably higher than 100, too large to be computed, and even too large to be stored in available memory devices. Instead, the *configuration ideal* of the relations between the design parameters and the position variables has been computed. It is the prime ideal of a projective variety of dimension 8 and degree 1072 in the projective space of dimension 18. This allows us to classify the mobile 5R loops, which appear to be all already known, under a conjecture for which heuristic evidence is provided.

These result have been obtained by associating Gröbner basis computations with considerations of classical geometry and algebraic geometry.

Key words: 4R linkage; 5R linkage; Gröbner basis; mobility of overconstrained mechanisms.

1 Introduction

The discovery by Bennett [1] of a non-trivial mobile mechanism, consisting in a loop of four links linked by revolute joins has motivated several researches on this kind of mechanisms, commonly

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called nR loops; see, for example the bibliography of [7]. However we are not aware of any systematic classification of this kind of mechanisms.

In fact there are several natural questions about them, which are not yet completely answered, even for the 4R loops:

- Classify these mechanisms by their properties
- Which equations must satisfy the design parameters for allowing mounting the mechanism?
- Are all the mobile nR loops known?

The aim of this article is to solve these problems for 4R an 5R loops without offsets. By *without offsets*, we mean the following: for each link, we consider the common perpendicular of the two axes of its revolute joins; the linkage is without offsets if, for each pair of neighbour links, the common perpendiculars are concurrent. All known mobile 4R, 5R and 6R loops satisfy this condition.

Our results about the 4R loops are the following.

- A complete classification of 4R loops without offset (Proposition 6)
- For each class in this classification, we give explicitly the assembly conditions, that is the equations that the design parameters of the links must satisfy for allowing the assembly of the loop. This appears to be new in the skew case where the four join points are not coplanar. (Proposition 10).
- The proof that all mobile 4R loops without offset are known. If all join may move simultaneously, a mobile 4R loop is either planar (that is all join axes are parallel), or spherical (that is all join axes are concurrent), or is a Bennett linkage [1]. If all join can move, but not simultaneously, then all links are identical, and the trajectory splits into two components that each behave as two superposed revolute joins. Otherwise, a mobile 4R loop behaves as two superposed revolute joins (Theorem 11).

The characterization of mobile loops that are neither spherical nor coplanar results from a new general algorithm. This algorithm may apply to any mechanism such that the design parameters and the position variables are related by polynomial equations. The output of this algorithm are the conditions that the design parameters must satisfy for having a mobile mechanism (Section 5).

The methods that we have used for 4R loops may not be applied directly for 5R loops, because of a much higher computational complexity. While all computer computations, which are needed for 4R loops, may be done in a few CPU seconds, the similar computations for 5R loops usually fail after hours or days of computations, by reaching the limit of one hundred millions (10^8) for the number of columns of the matrices to be reduced. Nevertheless, we have obtained the following results.

For the study of the assembly conditions, we have restricted ourselves to the configurations, such that no four join points are coplanar. This implies that all links have a non-zero length and a non-zero angle. We call *configuration variety* the set of values of the design parameters and of the position variables corresponding to such configurations. This configuration variety is an irreducible projective algebraic variety. [1] We have computed a Gröbner basis of its definition ideal, and it results that this variety is embedded in the projective space of dimension 18, and has dimension 8 and degree 1072. The assembly condition is a single homogeneous irreducible polynomial in the 10 design parameters, which, theoretically, could be obtained by eliminating the position variables from the configuration ideal. However, this computation is far outside the state of the art: this polynomial is almost certainly of degree higher than 100, and has thus too many monomials for being stored in available storage devices.

As the algorithm for mobility produces, as side output, the assembly conditions, it can not be used for general 5R loops. Thus another approach is needed for studying the mobile 5R loops. We have used the following property of algebraic systems: if one accepts non-real complex positions, the movement of a mobile mechanism may be pursued until the alignment of any pair of links connected by a join. However, it may theoretically occur that, when one tends toward this alignment, some join tends to the infinity. We have conjectured that this cannot occur for a mobile 5R loop (Conjecture 16). Appendix B provides heuristic arguments supporting this conjecture.

Under this conjecture, we can provide a complete classification of mobile 5R loops, which shows that there were all previously known. More precisely:

Theorem 1. *A mobile 5R loop may be*

- *If some link remains fixed during the movement: a mobile 4R loop in which a link has been broken and the broken point is replaced by a revolute join*
- *If all join axes are parallel: a planar 5R loop*
- *If all joins axes are concurrent: a spherical 5R loop*
- *A Golberg linkage [6]; this includes as special instances, the Myard linkages [9], which have a zero-length link, and some degenerate Goldberg linkages such that all join may move, but not simultaneously; the movement of these degenerate linkages split into two trajectories consisting in a Bennett movement (one join fixed) and two superposed revolute joins (three joins fixed); this may degenerate further in a linkage whose movement splits into three trajectories, each behaving as two superposed revolute joins.*

If Conjecture 16 is true, there is no other mobile 5R loop.

[1] In this paper, we use freely the basic terminology of algebraic geometry. If the reader is not accustomed with it, we recommend him to consult [2] or Wikipedia

Methods of proof: These results have been obtained in the framework of effective algebraic geometry. Therefore we use freely the corresponding language, for which we refer to [2] (Wikipedia may also be useful to the reader). Most of our proofs are based on Gröbner basis computation, for which we have used Faugère's FGB package [4, 3], which, as far as we know, is far most efficient than most available Gröbner basis engines. Even with it, we need various considerations of algebraic geometry for selecting computations strategies that are tractable, and of classical elementary geometry for limiting the combinatorial complexity. In particular, contrarily to habits in this field, we have systematically avoided to use trigonometrical functions, except in the presentation of some results.

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2 Terminology and notation

A nR *loop* is a mechanism consisting of n links that are linked together by *revolute joins* to form a loop. The two links, to which is connected a given link, are said its *neighbour links*.

The relative position of the axes of the two joins of a link characterize, from a behavioural point of view, the geometry of the link. The *length* and the *angle* of a link is thus the distance and the angle of the join axes of the link. It is important to note that the angle of a link is defined modulo π, and that two opposite angles define the same zero-length link.

The behaviour of a link strongly depends if its length and its angle are zero or not. A link with both a zero length and a zero angle may rotate freely with respect to its neighbour links; removing it from a mechanism (by replacing it by a single revolute join) does not change the behaviour of the mechanism. Therefore we exclude such links from this study, and consider only three kinds of links, the *zero-angle links*, which have a non-zero length, the *zero-length links*, which have a non-zero angle, and the *non-degenerate links*, which have a non-zero length and a non-zero angle.

The join axes of a link have a common perpendicular, whose length is the length of the link, and which is unique if the link angle is not zero. The intersections between this common perpendicular and the join axes are called *join points*. As a join axis is shared by two neighbour links, there are, *a priori*, two join points on each join axis. Their distance is usually called *offset*. In this article, we consider only mechanisms such that all offsets are zero. Thus there is only one join point on each join axis, which justifies the terminology.

We usually identify a link with the common perpendicular to its join axes, which is the segment delimited by its join points. In the case of a zero-angle link, the non-uniqueness of the common perpendicular is not a problem, as the position of the join points is determined by their position on the neighbour links. In the case of a zero-length link, the two join points are identical, and the link is thus identified with this point and the direction of the common perpendicular.

This identification allows us to say that a set of links is *colinear* or *coplanar* if all the common perpendiculars have this property. Two neighbour links are said *aligned*, if they are colinear. There is two ways for aligning two non-zero-length links, by *extension*, if the common join point is between the others or by *covering*, in the other case. A nR loop is in a *fully aligned* position, if all its links are colinear.

The identification of a link with the common perpendicular to its join axes is supported by the following easy lemma.

Lemma 2. *If two non-zero-length links are not aligned, the position of their three join points determines the direction of their common join axis.*

If a nR loop has at most one zero-length link and is not in a fully aligned position, the position of the join points determines the direction of the join axes. If a nR loop has at most one zero-length link and is in a fully aligned position, the direction of the zero-length link is the line of the join points, and the position of the join points determines the direction of the join axes up to the rotation of the whole loop around the line of the join points.

Proof. If two non-zero-length links are not aligned, their common join axis is perpendicular to the plane of the join points.

If a zero-length link has two aligned non-zero-length neighbours, its join axes are both perpendicular to the line defined by the join points of the neighbour links.

If a nR loop is not in a fully aligned position, the preceding assertion allows to fix one join axis. If it is in a fully aligned position, one join axis of a non-zero length link may be fixed arbitrarily up to a rotation of this link. Then, using the link angles of the non-zero-length links, the directions of the other join axes may be successively fixed. □

For studying the nR loops, we need some notation. We name L_i, for $i = 1, \dots n$ the links, in such a way that L_i and L_{i+1} are neighbour links for $i = 1, \dots n-1$; thus L_n and L_1 are also neighbour links. The length and the angle of L_i are denoted respectively l_i and α_i. If $\alpha_i \neq 0$, we set $t_i = \frac{l_i}{\tan \alpha_i}$. Using t_i instead of α_i allows to avoid the use of trigonometrical functions, and to apply tools and results of algebraic geometry. The l_i and the α_i (or t_i) define the links of a loop and are thus called the *design parameters* or simply the *parameters* of the mechanism.

A *configuration* of a nR loop consists of the value of the design parameters together with a relative position of the links. *Assembly conditions* are the relations that the design parameters must satisfy for the existence of a configuration. Therefore, a configuration consists of a relative position of links that are defined by parameters satisfying the assembly conditions.

For computing with configurations, we represent the position of the links by the coordinates of the join points. More precisely, we denote $J_{i,j}$ the joint point between L_i and L_j, where $j = 1 + 1 \mod n$, and $(x_{i,j}, y_{i,j}, z_{i,j})$ its coordinates. As we never need any computation when all links have a zero length, we choose as L_1 a link of non-zero length. For getting rid of displacements and scaling (change of unit length) of the whole linkage, we set $l_1 = 1$, $(x_{n,1}, y_{n,1}, z_{n,1}) = (0, 0, 0)$, $(x_{1,2}, y_{1,2}, z_{1,2}) = (1, 0, 0)$, and fix the join axis at $J_{1,2}$ parallel to the z-axis. This implies $z_{2,3} = 0$. It is easy to see that, if the design parameters and the join points are fixed, this completely determines the position of the links, assuming that the assembly condition are satisfied.

3 Basic results

In this section, we introduce some basic useful results, often sufficiently easy for allowing to omit proofs.

Lemma 3. *Given a non-degenerate link (i.e. having non-zero length and non-zero angle), whose neighbour links have non-zero lengths, if the three links are coplanar (i.e. the four join points are coplanar), then the central link is aligned with one of its neighbours)*

Lemma 4. *If a zero-length link is between two aligned links, then the three links are aligned.*

Proof. Immediate, when recalling that "alignment" refers to the common perpendiculars to the join axes. □

These lemmas allow a complete classification of 3R loop and 4R degenerate loops (here "degenerate" means that the join points are coplanar).

Proposition 5. *The 3R loops are never mobile and divide in three classes characterized by their assembly constraints (in what follow we suppose that the links are numbered in order that $l_1 \geq l_2 \geq l_3$).*

- $\alpha_1 = \alpha_2 = \alpha_3 = 0$ *and* $0 < l_1 < l_2 + l_3$*: the join points form a non-degenerated triangle.*
- $l_1 = l_2 + l_3$ *and* $\alpha_1 = \alpha_2 + \alpha_3 \mod \pi$*: fully aligned configuration.*

- $l_1 = l_2 = l_3 = 0$ *and a triangular inequality for the angles: the three join axes are concurrent.*

Proposition 6. *The degenerate configurations of 4R loops split also in several classes (in what follows, the sentence "There is a circular permutation of the indexes such that" is kept implicit before some items):*

- $l_1 + \varepsilon_2 l_2 + \varepsilon_3 l_3 + \varepsilon_4 l_4 = 0$ *and* $\alpha_1 + \varepsilon_2\alpha_2 + \varepsilon_3\alpha_3 + \varepsilon_4\alpha_4 = 0 \bmod \pi$ *for some* $\varepsilon_i = \pm 1$*; these configurations are called* fully aligned*, as all join points are aligned, and their line is perpendicular to all join axes.*
- $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$*, and some inequalities on the lengths; these configurations are called* planar configurations*, and are mobile.*
- $l_1 = l_2 = l_3 = l_4 = 0$*, and some inequalities on the angles; These configurations are called* spherical configurations*. All join axes are concurrents and the loop is mobile.*
- $l_1 = l_2 > 0$*,* $l_3 = l_4 = 0$*,* $\alpha_1 \neq \alpha_2$*, and triangular inequalities for the angles* $\alpha_1 - \alpha_2, \alpha_3, \alpha_4$*.*
- $l_4 = 0$*,* $\alpha_2 = 0$*,* $P_6 = 0$ *and* $|l_1 - l_2| < l_3 < l_1 + l_2$*, where* P_6 *is a polynomial of degree 6 in the* l_i *and the* t_i*; the join points form a non-degenerate triangle.*
- $l_1 l_2 l_3 l_4 \neq 0$*,* $\alpha_3 = \alpha_4 = 0$*, and either* $\alpha_1 + \alpha_2 = 0 \bmod \pi$ *or* $\alpha_1 = \alpha_2, l_1 \neq l_2$*; the links* L_1 *and* L_2 *are aligned, and the link points other than* $J_{1,2}$ *form a non-degenerate triangle.*
- $l_1 = l_2$*,* $\alpha_1 = \alpha_2$*,* $l_3 = l_4$*,* $\alpha_3 = \alpha_4$*; when* L_1 *and* L_2 *are superposed,* L_3 *and* L_4 *are also superposed, and the configuration is mobile and behaves as two superposed 1R linkages. Thus we call it* $1\text{R}\times 2$*.*
- $l_1 = l_2 = l_3 = l_4 \neq 0$ *and* $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \neq 0$*; this is a* double $1\text{R}\times 2$*, which has two different moving trajectories, depending if* L_1 *is superposed with* L_2 *or* L_4*.*

Proof. The shortest proof seems consisting in considering the join points positions, in function of the number of zero-length links.

If at least three links have a zero length, the fourth has also a zero length, and we have a spherical configuration.

If two non-neighbour links have a zero length, Lemma 4 shows that the configuration is fully aligned. If two neighbour links have a zero length, the two other links must have the same length and be superposed. If they have the same angle, we have a $1\text{R}\times 2$ configuration, otherwise, some triangular inequalities on the angles must be verified

If there is exactly one zero-length link, either the join points form a non-degenerate triangle, or Lemma 4 shows that the configuration is fully aligned.

If all links have non-zero lengths, and the configuration is not planar, there is a non-zero-angle link, say L_1. By Lemma 3, it must be aligned with one of its neighbour, say L_2. If this alignment form a link that has a non zero length or a non zero angle, Proposition 5 shows that, either the configuration is fully aligned, or we have a non-degenerate triangle with three zero-angle links. If the alignment of L_1 and L_2 results into a link which has both zero length and zero angle, then we have an 1R$\times$2 configuration.

The polynomial P_6 may be explicitly computed by using equations (1) (below) modified as described at the end of this section: saturating them by the coefficients of the t_i in these equations, and eliminating the position variables produces easily this polynomial P_6. □

We have thus a complete classification of degenerate 4R configurations. For studying the non-degenerate configurations, we need the relations between the design parameters and the position variables, which are provided by the following proposition.

Proposition 7. *Let L_{i-1}, L_i and L_{i+1} be three consecutive links of a nR loop such that L_i has a non-zero angle. Then the design parameters of L_i are related to the position of the join points by the relations*

$$\begin{aligned}
&l_i^2 - \overrightarrow{J_{i-1,i}J_{i,i+1}} \cdot \overrightarrow{J_{i-1,i}J_{i,i+1}} = 0\\
&\overrightarrow{J_{i-2,i-1}J_{i-1,i}} \cdot (\overrightarrow{J_{i-1,i}J_{i,i+1}} \times \overrightarrow{J_{i,i+1}J_{i+1,i+2}})\, t_i\\
&\quad - (\overrightarrow{J_{i-2,i-1}J_{i-1,i}} \times \overrightarrow{J_{i-1,i}J_{i,i+1}}) \cdot (\overrightarrow{J_{i-1,i}J_{i,i+1}} \times \overrightarrow{J_{i,i+1}J_{i+1,i+2}}) = 0,
\end{aligned} \tag{1}$$

where $\times$ denotes the cross product, and $\cdot$ denotes the dot product.

If the three links have non-zero length and L_i is not aligned with L_{i-1} or L_{i+1}, then these relations allows to express the design parameters of L_i as a rational function of the position of the join points. Otherwise, the second relation reduces to $0 = 0$.

Proof. The triple product of the coefficient of t_i in the second equation is equal to the product $l_{i-1}\, l_i\, l_{i+1} \sin(\alpha_{i-1}) \sin(\alpha_i) \sin(\alpha_{i+1})$, while the second term is equal to $l_{i-1}\, l_i^2\, l_{i+1} \sin(\alpha_{i-1}) \sin(\alpha_{i+1}) \cos(\alpha_i)$. □

Remark 8. In our computations, we always fix the join axis at $J_{1,2}$ parallel to the z-axis. It follows that the cross product $\overrightarrow{J_{n,1}J_{1,2}} \times \overrightarrow{J_{1,2}J_{2,3}}$, which is equal to $[0, 0, y_{2,3}]$ is advantageously replaced by the unit vector in the same direction, $[0, 0, 1]$. This amounts to divide the equations for t_1 and t_2 by $y_{2,3}$. The advantage is that these equations do not reduce to $0 = 0$ when the links L_1 and L_2 are aligned.

Similarly, when the link angle α_1 is zero, the cross product $\overrightarrow{J_{n-1,n}J_{n,1}} \times \overrightarrow{J_{n,1}J_{1,2}}$ may also be replaced by $[0, 0, 1]$, allowing to divide the equation for t_n by $y_{n-1,n}$.

Remark 9. If exactly one of the links has a zero length, we may also express the design parameters of L_i in terms of the positions. For this, if L_i has a zero length, we define $t_i = \frac{1}{\tan \alpha_i}$, and introduce a point J_i such that the vector $\overrightarrow{J_{i-1,i}J_i}=\overrightarrow{J_{i,i+1}J_i}$ has length one and is orthogonal to the join axes of L_i. Then, if $l_i = 0$ for $i = n, 1$ or 2, the replacement of $\overrightarrow{J_{i-1,i}J_{i,i+1}}$ by $\overrightarrow{J_{i-1,i}J_i}$ (indexes evaluated modulo n) in the second equation of the preceding proposition allows to expresses rationally t_i in terms of the positions.

4 Assembly conditions for the non-degenerated 4R loops

Proposition 6 describes all degenerate configurations of 4R loops, and, gives the equations that the design parameters must satisfy (assembly conditions).

The basic tool of study for non-degenerate configuration is equations (1). Using the position variables described at the end of Section 2, the equations, or more exactly their left-hand side, become, after using Remark 8:

$$
\begin{aligned}
\mathit{eqs} :=\, & t_1 z_{3,4} + y_{3,4}, \\
& t_2 z_{3,4} + y_{2,3}x_{3,4} - y_{3,4}x_{2,3} - y_{2,3} + y_{3,4}, \\
& t_3 z_{3,4}y_{2,3} + z_{3,4}^2 y_{2,3}^2 + z_{3,4}^2 x_{2,3}^2 + y_{3,4}^2 x_{2,3}^2 + y_{2,3}^2 x_{3,4}^2 - z_{3,4}^2 x_{2,3} + y_{2,3}^2 x_{3,4} \\
& \qquad - y_{3,4}^2 x_{2,3} + y_{2,3}y_{3,4}x_{2,3} + y_{2,3}y_{3,4}x_{3,4} - 2y_{2,3}y_{3,4}x_{2,3}x_{3,4}, \\
& t_4 z_{3,4}y_{2,3} - z_{3,4}^2 x_{2,3} - y_{3,4}^2 x_{2,3} + y_{2,3}y_{3,4}x_{3,4}, \\
& l_2^2 - (x_{2,3}-1)^2 - y_{2,3}^2, \\
& l_3^2 - (x_{3,4}-x_{2,3})^2 - (y_{3,4}-y_{2,3})^2 - z_{3,4}^2, \\
& l_4^2 - x_{3,4}^2 - y_{3,4}^2 - z_{3,4}^2
\end{aligned}
$$

As these equations are rationally solvable in the design parameters (at least if one takes the squared lengths as design parameters), they define an irreducible variety of dimension five (the number of position variables) in a space of dimension twelve (the total number of variables). This variety is called the *configuration variety* of the non-degenerate 4R loops, as it is the Zariski closure of the set of non-degenerate 4R loops. Its points include also some degenerate configurations, which are limit of non-degenerate configurations. The definition ideal of the configuration variety (ideal of all the polynomials that are zero on this variety) is called the *configuration ideal* of the non degenerate 4R loops.

As the common zeros of the polynomials in *eqs* include many degenerate configurations, this set of the common zeros is much larger than the configuration variety. To obtain the configuration ideal, one has to saturate (see the appendix) these polynomials by the coefficients of t_1, t_2, t_3, t_4 in the four first equations. This is an easy computation (less than a CPU second) that provides a Gröbner basis of this definition ideal, consisting of 51 polynomials (this number may vary with the choice of the ordering of the variables).

Then the elimination of the position variables (see the appendix) from this configuration ideal provides two polynomials Q_2 and Q_4 of respective degrees 2 and 4 (again, less than one CPU second). By replacing Q_4 by $\frac{1}{2}Q_4 - \frac{1}{2}(1-l_2^2+l_3^2-l_4^2)\,Q_2$ and homogenizing (i.e. multiplying the terms of lower degree by the power of l_1 needed for having homogeneous polynomials; this allows formulas that are independent of the length unit), we get the two following polynomials:

$$P_2 = 2\,(t_3t_1 - t_4t_2) + l_1^2 - l_2^2 + l_3^2 - l_4^2,$$
$$P_4 = (t_3t_1 + t_4t_2)(l_1^2 - l_2^2 + l_3^2 - l_4^2) - 2(t_1^2l_3^2 + t_3^2l_1^2) + 2(t_2^2l_4^2 + t_4^2l_2^2) - 2\,(l_3^2l_1^2 - l_4^2l_2^2).$$

This proves the following.

Proposition 10. *The design parameters of any non-degenerate 4R configuration verify the relations $P_2 = P_4 = 0$. Conversely, every polynomial P in the design parameters, such that $P = 0$ for every non-degenerate configuration, belongs to the ideal generated by P_2 and P_4 (that is is a linear combination of P_2 and P_4, with polynomial coefficients).*

The characterization, among these non-degenerate configurations, of the mobile 4R loops requires an algorithm, which is the object of Section 5. Before, we have to recall that Bennett linkages are such mobile configuration.

4.1 Bennett Linkage

The Bennett linkages [1] are the only known mobile 4R loops built with links that have non-zero length and angles.

If l_1, l_2, l_3, l_4 are the length of the links, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the angles between their join axes, they are defined by

$$l_1 = l_3, \qquad l_2 = l_4$$
$$\alpha_1 = \alpha_3, \qquad \alpha_2 = \alpha_4$$
$$\frac{\sin^2\alpha_1}{l_1^2} = \frac{\sin^2\alpha_2}{l_2^2}$$

Using $t_i = \frac{l_i}{\tan\alpha_i}$, these equations may be rewritten

$$
\begin{aligned}
&l_1 = l_3, \qquad l_2 = l_4 \\
&t_1 = t_3, \qquad t_2 = t_4 \\
&t_1^2 + l_1^2 = t_2^2 + l_2^2
\end{aligned}
$$

A 4R loop satisfying these relations for the design parameters is mobile and all the joins move simultaneously, unless if $l_1 = l_2$ and $\alpha_1 = \alpha_2$ (or equivalently $t_1 = t_2$). In this case, the Bennett mechanism degenerates into a mechanism consisting of two revolute joins that are identical and superposed. We denote such a mechanism a 1R×2.

5 Mobility, a general algorithm and its application to 4R loops

Proposition 6 describes the mobile 4R loops that are degenerate in every position. These are the planar configurations, the spherical configurations and the 1R×2 configurations.

The other mobile 4R loops are non-degenerate for at least one position. It follows easily that they are non-degenerate in almost all positions, and that all their positions are zeros of the above configuration ideal.

The determination of the mobile 4R loops, which are not always degenerate, may be reduced to the following general problem.

Let us consider a mechanism modelled by polynomial equations involving a set, denoted param*, of design parameters, and a set, denoted* unknown*, of position variables. Find or characterize the set of values of the variables in* param *such the system of equations has infinitely many solutions.*

In terms of algebraic geometry, this may be restated as:

Given an algebraic variety defined on a product of affine spaces, compute the subvariety of its projection on the second space, which consists of the points such that the fiber has a positive dimension.

With such a specification, this problem may be interesting in many scientific areas, not only in mechanism theory. We do not know of any published algorithm for this problem. However, it is rather easy to deduce such an algorithm from [8].

In practice, we are interested in configurations such that all joins may move. Therefore, we may consider a slightly different problem, consisting in asking that a specific variable of *unknown* (let us call it x) is not constant on the fiber. This amounts to state that a specific join is moving. The solution of the general problem, where some variables are fixed, and some are not, may easily be deduced, as explained below.

For solving the problem, we use Gröbner basis computation for the monomial ordering called *lexdeg*([*unknown*], [*param*]) in MAPLE, and we choose x as the smaller (latest) variable in *unknown*. This ordering compares the monomials by considering first their *unknown* parts, and considers the *param* part only in case of equality of the *unknown* parts. For both comparisons, the *degree reverse lexicographical ordering* is used. This choice of ordering is fundamental; otherwise, the result of the algorithm may be wrong.

This ordering allows to consider a polynomial as a polynomial in the variables in *unknown*, which has polynomials in *param* as coefficients. Given such a polynomial P, we denote by LM(P) and LC(P) the leading monomial and the leading coefficients, as given by the MAPLE functions *LeadingMonomial* and *LeadingCoefficient*, called with the monomial ordering *tdeg*(*unknown*). Thus LM(P) depends only on the variables in *unknown* and LC(P) is a polynomial in *param*.

Given a Gröbner basis G for the above defined ordering, let us denote by $E_0(G)$ the list of the elements in G that do not depend on *unknown* (that is LM(P) $= 1$) and $E_x(G)$ the list of the LC(P) for the elements P in G such that LM(P) is a power of x. We have $E_0(G) \subset E_x(G)$, and $E_0(G)$ is a Gröbner basis.

The algorithm proceeds as follows, starting from a set E of generators of the configuration ideal of the problem.

Compute the Gröbner basis G of E, and deduce $E_0(G)$ *and* $E_x(G)$
While *the Gröbner basis of* $E_x(G)$ *differs from* $E_0(G)$ **do**
  Let $E = G \cup E_x(G)$
  Compute the Gröbner basis G of E, and deduce $E_0(G)$ *and* $E_x(G)$
  end do
end do
Output $E_0(G)$

Proof. The algorithm terminates eventually because the "while" loop is indexed by an increasing sequence of ideals. The remainder of the proof is an easy consequence of the results in [8] and is left to the reader. □

For getting also the mobile configurations leaving x fixed, one may change the ordering of the variables, but one may also not change the ordering, and run the algorithm with x replaced successively by each variable in *unknowns*.

When applied to the configuration ideal of non-degenerate 4R loops, this algorithm provides after one iteration of the loop an output of thirty polynomials (less that one CPU seconds). Some of these polynomials contain factors of the form $l_i + l_j$

or $t_2^2 + l_2^2$, which cannot be zero. Saturating this output by these factors, we get seven polynomials. The MAPLE function *PrimaryDecomposition* show that these polynomials generate an ideal that is the intersection of the ideals $\langle l_2-1, l_4-l_3, t_2-t_1, t_4-t_3\rangle$ and $\langle l_2-l_4, l_3-1, t_2-t_4, t_1-t_3, t_4^2+l_4^2-t_3^2-1\rangle$.

Recalling that we have set $l_1 = 1$, it is easy to recognize that the first ideal is generated by the equations of an 1R$\times$2 loop, and that the second one is generated by the equations of a Bennett linkage. This prove the following result.

Theorem 11. *Every mobile 4R loop that is built (without offsets) with links having non-zero length and angle is either a Bennett linkage or a 1R×2 linkage.*

It follows that every mobile 4R loop without offsets is either one of these linkages, or a planar linkage, or a spherical linkage.

Let us recall that "without offsets" means that the common perpendiculars to the axes of two neighbour links are concurrent. As far as with know, it is not known if there exist mobile 4R loops with non-zero offsets.

6 Non-degenerate 5R loops

With the tools that we have used, the complete study of the 4R loops is easy. It appears that a similar study for 5R loops is much more difficult.

For limiting the number cases to study, we restrict this study to non-degenerate configurations and to mobile configurations. Let us recall that a non-degenerate configuration is a loop such that three consecutive links are not coplanar. This implies that all links have non-zero length and non-zero angle.

With the choices described in Section 2, we have eight position variables and nine design parameters:

$$\begin{aligned} \textit{param} &:= t_1, t_2, t_3, t_4, t_5, l_2, l_3, l_4, l_5 \\ \textit{unknowns} &:= x_{2,3}, y_{2,3}, x_{3,4}, y_{3,4}, z_{3,4}, x_{4,5}, y_{4,5}, z_{4,5} \end{aligned}$$

Formulas (1) provides nine polynomials (the equation for l_1 disappear) that we call *eqs* (after having divided two of them by $y_{2,3}$, by Remark 8). Together, the coefficients, of the t_i in these polynomials have five irreducible factors, $f_1 := z_{3,4}$, $f_2 := z_{4,5}$, $f_3 := z_{4,5}y_{3,4} - z_{3,4}y_{4,5}$ and two polynomials of degree three, f_4 and f_5.

As for 4R loops, the configuration ideal is the ideal resulting of saturating *eqs* by f_1, f_2, f_3, f_4, f_5, and the assembly condition is the result of eliminating the position variables from the configuration ideal. As *eqs* allows to express the t_i and the l_i^2

rationally in terms of the position variables, the configuration ideal is a prime ideal of dimension 8. If its projection on the space of the design parameters (of dimension 9) would have a dimension less that 8, the fibers of this projection would have positive dimension, and all configurations would be mobile. It is easy to show that it is not the case, by choosing randomly position variables, deducing the design parameters, putting these parameters in *eqs*, and computing the dimension of the resulting ideal, which is, indeed, zero. It follows that the assembly conditions define an irreducible variety of dimension 8 in a space of dimension 9. It is thus a hypersurface, and *there is only one assembly condition*.

Unfortunately, the explicit computation of the configuration ideal is difficult and very close to the limit of the state of the art.

For computing the configuration ideal, we have to saturate *eqs* by the f_i. There are three standard methods for that, which give the same result when the computation succeeds. One may saturate *eqs* by the product of the f_i. One may saturate *eqs* by f_1, then saturate the result by f_2 and so on. One may also saturate simultaneously by all f_i, using five auxiliary variables. All these method fail, by reaching (sometimes after several days of computation) the limit of 10^8 for the number of columns of the matrices to be reduced.

Nevertheless, we succeeded to get the result by using the feature of FGB, which allows to stop the computation at some degree, and by mixing the two latter methods. More precisely, we have used the last method with the limit of 7 for the degree (option *"dlim"*=7), and we have verified the correctness of the result by verifying that the result does not change when it is saturated by each f_i. This consists in the following computation.

$uv := u_1, u_2, u_3, u_4, u_5;$

$up := 1 - u_1 f_1, 1 - u_2 f_2, 1 - u_3 f_3, 1 - u_4 f_4, 1 - u_5 f_5;$

$B := \mathit{fgb\_gbasis\_elim}([up, eqs], 0, [uv], [param, unknowns], \{\text{"index"} = 10^7, \text{"dlim"=7}\});$

for i *from* 1 *to* 5 *do*

  $\mathit{fgb\_gbasis\_elim}([[up][i], eqs], 0, [u_i], [param, unknowns], \{\text{"index"} = 10^7, \text{"dlim"=7}\});$

  $\mathit{print}(\mathit{evalb}(B = \%))$

od

The first Gröbner basis computation needs around 20 seconds. The five others need around 20 minutes together, and the equality test *evalb* returns always true. This proves that B is a Gröbner basis of the configuration ideal. It consists of 31 polynomials, six of degree 2, three of degree 3, eight of degree 4, twelve of degree 5, and two of degree 6.

Having a Gröbner basis of the configuration ideal, it is easy to computes its Hilbert series (MAPLE function *HilbertSeries*), which is

$$\frac{P(t)}{(1-t)^8} = \frac{6t^9 + 44t^8 + 138t^7 + 245t^6 + 275t^5 + 207t^4 + 108t^3 + 39t^2 + 9t + 1}{(1-t)^8}$$

Let us recall that the degree of a variety of dimension d is the number of (complex) intersection points with d hyperplanes in general position. If the variety is a hypersurface, its degree is equal to the degree of the polynomial that defines it. We have

Proposition 12. *The configuration variety has the dimension 8 and the degree 1072.*

Proof. The dimension is the degree of the denominator of the Hilbert series. This confirms the dimension, which has already been obtained, above in this section.

The degree is $P(1)$ where P is the numerator of the Hilbert series. □

The assembly condition should result in eliminating the position variables (*unknowns*) in the configuration ideal. Geometrically, this elimination corresponds to the projection of the configuration variety onto the space of the design parameters. A *fiber* of this projection is the inverse image of a point of its image. A *generic fiber* is the fiber over a generic point of the image, that is a point that does not satisfies any other equational constraint than the assembly condition. It is a basic result of algebraic geometry, that all the generic fibers have the same dimension and the same degree; moreover, the points where the degree or the dimension of the fiber change belong to a hypersurface. Over these points, either the dimension of the fiber is higher, or the degree is lower.

Proposition 13. *The generic fiber of the projection of the configuration variety onto the space of the design parameters has the dimension zero and the degree two.*

In other words, given a configuration, there is, in general *exactly one other configuration with the same design parameters. This configuration is obtained from the first one by rotating it by an angle of π around the x-axis.*

Proof. Let us choose random integer values for the positions variables. Using Equations (1), we may deduce values for the design parameters. Their substitution in the Gröbner basis B of the configuration ideal produces a system that is easy to solve and has exactly two solutions. Instead of solving, one may compute its Hilbert series, which is $t + 1$, implying the dimension zero and the degree two.

As the position variables have been chosen randomly, the probability is zero of a choice that belongs to the hypersurface where the degree or the dimension change. [2]

[2] In fact the probability is not exactly zero, because the integer values are chosen in some interval. However this probability is very small, and may be dramatically reduced by making the computation with several choices. It must be noted that, although this certifies the result, this is not a mathematical proof.

The last assertion is immediate, as the rotation of π of a configuration around the x-axis produces evidently another configuration with the same design parameters. $\square$

Given a projection, such as the one we are considering, the dimension of the projected variety is the sum of the dimensions of the generic fiber and the image of the projection (here $8 = 0 + 8$); if the generic fiber is zero-dimensional, the degree of the projected variety is the sum of two terms; the first one is the product of the degree of the generic fiber by the degree of the image of the projection; the second one measures the size of the set of the points at infinity of the variety, in the direction of the projection; if this set is zero-dimensional, this term is the sum of the multiplicities of its points.

If m is this measure, it follows from the previous results that the assembly constraint consists in a single irreducible polynomial in 9 variable of degree $\frac{1072-m}{2}$. It unlikely that m could be as large as 800. Thus the assembly constraint has probably more than 10^{13} coefficients, and is thus too large to be stored in the largest existing hard disk.

There is no hope to compute the assembly constraint for the 5R loops, even with dramatic improvements of hardware technology.

In particular, there is no hope of using the algorithm of Section 5 to determine the mobile non-degenerate configurations. Therefore, we have introduced another method, described below.

7 Basic results for mobility of 5R loops

7.1 Mobile configurations with some joins fixed

When a mobile configuration is moving, it is possible that some joins remain fixed. Such configurations are of low interest, because, by gluing together the links that are relatively fixed, one gets a simpler mobile configuration. Conversely, if, starting from a nR mobile configuration, one breaks one of the links and replaces the breaking point by a revolute join, one gets a mobile $(n+1)$R configuration, in which the added join is usually fixed.

Thus, we are interested only in mobile loops such that all joins may move, simultaneously or not. We call them *fully mobile*, and they are the only ones that are studied in the remainder of this article.

Except plane and spherical configurations, there is only one non-trivial class of fully mobile mobile 5R loops, the Goldberg linkages [6]. The Myard linkage [9] is a particular Goldberg linkage, in which a link has a zero length.

Goldberg linkages are constructed from two Bennett linkages in the following way.

Let us consider two Bennett linkages, and denote $l_1^{(i)}, l_2^{(i)}, l_3^{(i)}, l_4^{(i)}$ the lengths of the i-th link for $i = 1, 2$. Similarly, we denote $\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \alpha_4^{(i)}$ the corresponding angles of the links. We suppose that the links 1 and 3 of both linkages are all identical, that is, they have the same lengths and angle (as we are considering Bennett linkages, we know already that the links 1 and 3 of each linkage are identical). These two linkages are combined in order that the link 3 of the first linkage and the link 1 of second one become the same link. Then, the links 2 of the two Bennett linkages are aligned and fixed together for becoming a single link, with length $\pm l_2^{(1)} \pm l_2^{(2)}$. The signs $\pm$ depend on the nature of the alignment (by extension for an angle of 0 or by covering for an angle of π); in the case of alignment by covering, the two signs are different, and the $+$ appears with the longer link.

As the Bennett linkages are mobile, this linkage is mobile. The Goldberg linkage is obtained by removing the shared link.

Denoting by $l_1, \ldots, l_5$ and $\alpha_1, \ldots, \alpha_5$ the lengths and the angles of this linkage, we have thus, up to a circular permutation of the indexes, $l_1 = l_4^{(2)}$, $\alpha_1 = \alpha_4^{(2)}$, $l_2 = l_4^{(1)}$, $\alpha_2 = \alpha_4^{(1)}$, $l_3 = l_1^{(1)}$, $\alpha_3 = \alpha_1^{(1)}$, $l_4 = \varepsilon_1 l_2^{(1)} + \varepsilon_2 l_2^{(2)}$, $\alpha_4 = \varepsilon_1 \alpha_2^{(1)} + \varepsilon_2 \alpha_2^{(2)}$, $l_5 = l_3^{(2)}$, $\alpha_5 = \alpha_3^{(2)}$, , where ε_i is either 1 or -1, for $i = 1, 2$ (the circular permutation of the indexes has been chosen for the convenience of the recognition of Goldberg linkages in the results of our computations).

Putting these relations in the equations characterizing the Bennett linkages, we get that the Goldberg linkages are characterized, up to a circular permutation of the links, by the equations,

$$\begin{aligned}
&l_3 = l_5, \qquad \alpha_3 = \alpha_5, \\
&l_4 = \varepsilon_1 l_1 + \varepsilon_2 l_2, \qquad \alpha_4 = \varepsilon_1 \alpha_1 + \varepsilon_2 \alpha_2 \pmod{\pi}, \\
&\frac{\sin^2 \alpha_1}{l_1^2} = \frac{\sin^2 \alpha_2}{l_2^2} = \frac{\sin^2 \alpha_3}{l_3^2},
\end{aligned}$$

where ε_i denotes either 1 or -1.

The use of the parameters $t_i = \frac{l_i}{\tan \alpha_i}$ allows a formulation of these conditions, which is more convenient for the computation, as it avoids to consider several cases for the signs of the ε_i.

Proposition 14. *The parameters l_i and t_i are the parameters of a Goldberg linkage if and only if there are real numbers $\tilde{l}_i$ whose absolute values are l_i, such that, up to a circular permutation of the indexes, the following polynomials are all zero:*

$$\begin{array}{l}
\tilde{l}_5 + \tilde{l}_3 \\
t_5 - t_3 \\
\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_4 \\
(t_1^2 + \tilde{l}_1^2) - (t_3^2 + \tilde{l}_3^2) \\
(t_2^2 + \tilde{l}_2^2) - (t_3^2 + \tilde{l}_3^2) \\
t_1 t_2 \tilde{l}_4 + t_2 t_4 \tilde{l}_1 + t_4 t_1 \tilde{l}_2 - \tilde{l}_1 \tilde{l}_2 \tilde{l}_4
\end{array}$$

Proof. This an elementary application of the formulas for the tangent of a sum of angles. □

All the common zeros of these polynomials, define fully mobile 5R loops. However some of them are degenerated.

If either $l_1 = l_3, t_1 = t_3$, or $l_2 = l_3, t_2 = t_3$, one of the constitutive Bennett loops degenerates into a 1R$\times$2, and the movement splits into an 1R$\times$2 and a Bennett loop. If $l_1 = l_2 = l_3$ and $t_1 = t_2 = t_3$, then both Bennett loops are degenerated, and the movement splits into three 1R$\times$2 trajectories. In both cases, all joins may move, but not simultaneously.

If $l_1 = l_2$, then either $t_1 = t_2$ or $t_1 = -t_2$. If $l_1 = l_2, t_1 = t_2$ and $\varepsilon_1 = -\varepsilon_2$, the link L_4 has both a zero length and a zero angle, and the other links behave as a 1R$\times$2 linkage. Note that, in this case, the join between links 1 and 2 is fix.

If $l_1 = l_2$ and $t_1 = -t_2$, the link L_4 has either a zero length or a zero angle. The former case is the Myard mechanism. For the latter case (zero-angle for L_4), t_4 must be considered as infinite, and the last polynomial of preceding proposition must be replaced by $t_1 + t_2$.

8 Mobility of non-degenerate 5R loops

This section is devoted to one of the main results of this article:

Theorem 15. *Let us consider a mobile 5R loop such that every link has a non-zero length and a non-zero angle. If Conjecture 16 is true and all join may move, the loop is a Goldberg linkage or a degenerate specialization of it, which has a movement splitting either in three 1R×2 movements or in a Bennett and an 1R×2 movements.*

We have seen that we cannot use the algorithm of Section 5. We describe now another approach for mobility.

The mobility of a linkage means that the set of its configurations contains a curve in the space of the position variables. As we consider linkages modeled by polynomial equations, this curve is an algebraic curve. If we accept non-real solutions of the equations, this implies that for every value of a specific position variable, there is a position of the system where this variable has this value. In fact, for making true this assertion, we must accept both complex positions and positions at infinity. Because the lengths of the links are fixed, the positions at infinity are necessarily non-real.

Thus, a mobile linkage, such that all joins can move, has a position, not necessarily real nor finite, such that two given neighbour links are aligned in extension and another position where these joins are aligned by covering. We guess that, for every mobile 5R loop, the joint points are never at infinity when two neighbour links are aligned. More precisely:

Conjecture 16. *Given a mobile 5R loop, which is not planar nor spherical, when a join between two links moves toward the alignment of these links, the other join points do not tend toward infinity.*

Proof. For arguments supporting this conjecture, see Appendix B □

This conjecture implies that, for a 5R loop such that every join has a non-zero length and a non-zero angle, the alignment of two neighbour links produces a 4R loop with, at least, three links with non-zero length and non-zero angle. Proposition 6 shows that such a configuration is either non-degenerate, or fully aligned, or is a 1R$\times$2 configuration that can move to a fully aligned position.

This allows to classify the cases by the number of fully aligned configurations.

Here we are faced to a difficulty: there are two ways to align the two links of a join. There are therefore 32 ways of aligning all the links (obviously, if the lengths are fixed, most are unfeasible). Moreover, to write down the equations, we have to distinguish the different ways of aligning the joins. There is no way to algebraically (that is without inequalities) define the alignment in extension or by covering. However, we may distinguish the alignments by the following method that has already used in Proposition 14.

We assign a sign to every l_i by introducing a variable $\tilde{l}_i = \pm l_i$. These signs are chosen in order that, if there are fully aligned configurations, we have $\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 + \tilde{l}_4 + \tilde{l}_5 = 0$ for one of them.

These signed lengths allow to distinguish the two alignments of two neighbour links.

Proposition 17. *Given two links of parameters $\tilde{l}_1, t_1$ and $\tilde{l}_2, t_2$, the two ways of aligning them provide a link of parameters $\tilde{l} = \tilde{l}_1 + \tilde{l}_2, t = \frac{\tilde{l}\,(t_1 t_2 - \tilde{l}_1 \tilde{l}_2)}{t_1 \tilde{l}_2 + t_2 \tilde{l}_1}$ and a link of parameters $\tilde{l} = \tilde{l}_1 - \tilde{l}_2, t = \frac{\tilde{l}\,(t_1 t_2 + \tilde{l}_1 \tilde{l}_2)}{t_1 \tilde{l}_2 - t_2 \tilde{l}_1}$.*

Proof. Simple computation, but tedious because one has to consider all possible signs for the $\tilde{l}_i$. □

By putting these formulas in the assembly conditions for a non-degenerated 4R loop, we get necessary constraints of mobility for a 5R loop, assuming that the 4R loop obtained by aligning the two links is non-degenerated. These constraints consist of two polynomials of degrees 4 and 8. However, it is not a good idea to use these polynomial directly, because the ideal they generate has spurious components containing $\tilde{l}$, $\tilde{l}_1$, $\tilde{l}_2$ and $t_1^2 + \tilde{l}_1^2$. Saturating by these polynomial gives a Gröbner basis of 11 polynomials of degrees 4, 6, 7, 8, 8, 8, 9, 9, 9, 9, 10, which are better suited for the next computations.

8.1 At most one fully aligned position

If there is no position where all the links are aligned, there are 10 positions with two aligned links, such that each form a non-degenerate 4R configuration. We have therefore 20 independent constraints, represented by 110 polynomials. After writing these polynomials, and adding to their list the polynomials $1 - u_i \tilde{l}_i$ (for avoiding zero lengths), the elimination of the u_i (that is the saturation by the $\tilde{l}_i$) provides a Gröbner basis equal to $[1]$ in less than a CPU second. This shows that every mobile 5R loop has fully aligned positions.

If there is exactly one fully aligned position, a non-degenerate 4R loop is obtained by the other alignment of each pair of neighbour links. The choice of the signs of the $\tilde{l}_i$ implies that the assembly constraints of these 4R loops are obtained from the second formulas of Proposition 17. This gives 55 polynomials, to which one has to add $\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 + \tilde{l}_4 + \tilde{l}_5$ and the polynomial of degree five, which expresses that the signed sum of the angles of the links is zero.

Although overdetermined, this set of polynomials has common zeros. However, none may correspond to a mobile 5R loop, as we will show now.

Firstly the variety of the zeros of these 57 polynomials has components where some $\tilde{l}_i$ is zero (corresponding to degenerate 5R loops) or some $\tilde{l}_{i+1} - \tilde{l}_i$ is zero (corresponding to a degenerate 4R loop obtained by an alignment which is supposed to give a non-degenerate loop). There are also non real components where some $t_i^2 + \tilde{l}_i^2$

is zero. For removing these spurious components by saturation, we have proceeded as follows.

We have first saturated by the $\tilde{l}_i$, limiting the degree to 9 (option "*dlim*"=9 of FGB); this is done by adjoining to the 57 polynomials the polynomials $1-u_i\,\tilde{l}_i$, and calling *fgb_gbasis_elim* with the u_i in the first block of variables (3h 20mn of CPU time). Then, always limiting the degree to 9, we have saturated the result by the $\tilde{l}_{i+1} - \tilde{l}_i$ (less than 8mn of CPU time); the successive saturation by each $t_i^2 + \tilde{l}_i^2$ gives eventually a list of 119 polynomials, one of degree 5, one of degree 1, the other ones of degrees 3 and 4 (about 1mn of CPU time). Finally, we have verified the result by redoing these saturations, without limiting the degree, on the union of these 119 polynomials and the 57 initial ones (about 20s of CPU time).

These 119 polynomials define a curve (The MAPLE function *HilbertDimension* returns 1), which does not has any real point. To prove this, we have eliminated all the parameters except t_1 and t_2, which provides a single bivariate polynomial P of degree 28. The product of its discriminant with respect to t_1 and the coefficient of its highest power of t_1 has 33 different factors. Only two, namely t_2 and $16\,t_2^4 + 68\,t_2^2 - 9$, have real roots, which are 0 and $\pm 0.358\ldots$ It follows that in each interval delimited by these roots the number or real roots in t_1 is constant. Calling MAPLE function *RootFinding[Isolate]* on P with t_2 substituted by $-1, -1/4, 1/4, 1$ shows that this number is zero in all these intervals. Calling again this function on the list of 119 polynomials augmented by $t_2\,(16\,t_2^4 + 68\,t_2^2 - 9)$ shows that the 119 polynomials do not have any common real zero such that $t_2 = 0, \pm 0.358\ldots$. All together, this shows that the 119 polynomials do not have any common real zeros, and proves the following.

Proposition 18. *If Conjecture 16 is true, any mobile 5R loop, such that all links have non-zero length and non-zero angle, has at least two fully aligned position.*

8.2 Two fully aligned positions

In the case of two fully aligned positions, we have chosen the signs of the $\tilde{l}_i$ in order that their sum is zero for one of them. For the second fully aligned position, there are signs ε_i such $\varepsilon_1\tilde{l}_1 + \varepsilon_2\tilde{l}_2 + \varepsilon_3\tilde{l}_3 + \varepsilon_4\tilde{l}_4 + \varepsilon_5\tilde{l}_5 = 0$. The ε_i cannot are all equal, as, otherwise, the two fully aligned positions were the same. One cannot have four ε_i equal, as, otherwise, the fifth $\tilde{l}_i$ would be zero. As we may change simultaneously the signs of all $\tilde{l}_i$, we may thus suppose that three ε_i are $+1$, and the two other are -1.

By adding and subtracting the two relations between the $\tilde{l}_i$, we get that the sum of three $\tilde{l}_i$ is zero, and the sum of the two others is also zero.

There are therefore two cases, depending if the two $\tilde{l}_i$ that sum to zero correspond to neighbour links or not. Thus, up to a circular permutation of the links, we have either

$$\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 = \tilde{l}_4 + \tilde{l}_5 = 0$$

or

$$\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_4 = \tilde{l}_3 + \tilde{l}_5 = 0$$

When all the links are aligned, the signed sum of the angles of the links is zero modulo π, with the same signs as for the signed sum of the lengths. With two fully aligned positions, we have thus two linear relations between the angles α_i of the links. Adding and subtracting them, gives, in the first case, the relations $\frac{\tilde{l}_1}{l_1}\,\alpha_1 + \frac{\tilde{l}_2}{l_2}\,\alpha_2 + \frac{\tilde{l}_3}{l_3}\,\alpha_3 = 0 \pmod{\pi}$ and $\frac{\tilde{l}_4}{l_4}\,\alpha_4 + \frac{\tilde{l}_5}{l_5}\,\alpha_5 = 0 \pmod{\pi}$. Translating in terms of the t_i, this gives the relations

$$t_5 - t_4 = \tilde{l}_1 t_2 t_3 + \tilde{l}_2 t_3 t_1 + \tilde{l}_3 t_1 t_2 - \tilde{l}_1 \tilde{l}_2 \tilde{l}_3 = 0\,.$$

In the second cases, one obtains the same relations with the indexes 3 and 4 exchanged.

8.2.1 Case $\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 = \tilde{l}_4 + \tilde{l}_5 = 0$

We have just written four relations between the design parameters. We have also the relations expressing that the other alignments of the pair of links (L_1, L_2), (L_2, L_3) and (L_4, L_5) generate non-degenerate 4R loops. There are two relations for each pair, which give after a saturation, 11 polynomials (see the paragraph after Proposition 17). Thus, we have a total of 37 polynomials Substituting $\tilde{l}_1$ by 1 (choice of the unit length), and saturating the ideal generated by these 37 polynomials by the $\tilde{l}_i$ and the $\tilde{l}_i^2 + t_i^2$, we get in less than one CPU seconds a Gröbner basis of 13 polynomials. This Gröbner basis contains the polynomials $t_5^2 + \tilde{l}_5^2 - \tilde{l}_3$ and $\tilde{l}_3^2 + 1 - \tilde{l}_5^2 + 2\tilde{l}_3$, whose sum $t_5^2 + \tilde{l}_3^2 + \tilde{l}_3 + 1$ does not have any real zero.

It follows immediately that there is no mobile 5R loop in this case, such that all joins can move.

8.2.2 Case $\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_4 = \tilde{l}_3 + \tilde{l}_5 = 0$

The difference with the preceding case is that there is only one alignment giving a non-degenerate 4R loop: the other alignment of L_1 and L_2. We start thus with 15 polynomials. Substituting $\tilde{l}_1$ by 1 (choice of the unit length), and saturating by the $\tilde{l}_i$ and the $\tilde{l}_i^2 + t_i^2$, we get a Gröbner basis of 21 polynomials. Let us call it *cstr*. The dimension of the variety of the zeros of these polynomials, given by the function *HilbertDimension*, is three. As there are nine design parameters, and the ideal is obtained by saturating an ideal generated by six polynomials, the variety is equidimensional, that is, all its components have the same dimension three.

The dimension three implies that the projection of the variety on a linear space of dimension four is generally an hypersurface, that is, it is defined by a single polynomial. Such a projection is computed, for example, by *fgb_gbasis_elim* (*cstr*, 0, $[t_5, t_4, t_1, \tilde{l}_5, \tilde{l}_4], [t_2, t_3, \tilde{l}_2, \tilde{l}_3]$). This computation provides a Gröbner basis consisting in a single polynomial, which factors into

$$(\tilde{l}_2^2 t_2^2 - \tilde{l}_3^2 t_2^2 + \tilde{l}_2^4 - t_2^2 t_3^2)(\tilde{l}_2^2 + t_2^2 - \tilde{l}_3^2 - t_3^2)$$

This proves that the variety defined by *cstr* has two components on which exactly one of the factors is everywhere zero. Another component, where both factors are zero, is theoretically possible, but is excluded by the following argument. As such a component has the dimension three, if it would exist, the function *HilbertDimension* would returns three, when applied to the Gröbner basis of the union of *cstr* and the list of the two factors. An easy computation gives the dimension two, proving the nonexistence of this component.

The two components may be computed by saturating *cstr* by each factor.

Saturating by the first factor, one gets an ideal containing $\tilde{l}_2^2 + t_2^2 - \tilde{l}_3^2 - t_3^2$. One may recognize a relation that appears in Proposition 14, which characterizes the Goldberg linkages. As the variety of Golberg linkages has also the dimension three (when one of the lengths is fixed to one), one may guess that this component is the variety of Golberg linkages. In fact, when setting $\tilde{l}_1 = 1$ in the equations of Proposition 14, and saturating by $t_1^2 + 1$ and $\tilde{l}_4$, one gets exactly the Gröbner basis of this first component. We have thus:

Proposition 19. *The points of the first component are the design parameters of the Golberg linkages.*

It remains to study the second component. In fact, the points of this variety do not correspond to mobile 5R loops such that every join is mobile. For proving this, we express the mobility by the fact that if a 5R loop is mobile, each join may have any angle. This is done by applying recursively the following operations to a list of polynomials *cstr* (initially the Gröbner basis of the component):

$A :=$*fgb_gbasis_elim*([*config*, *cstr*, *angle*, *sat*], 0, [*uvars*], [*unknowns*, *param*]);
$B :=$ *fgb_gbasis_elim*(A, 0, [*unknowns*], [*param*]);
cstr := *op*(*fgb_gbasis_elim*([*op*(B), *sat2*], 0, [*uvars*], [*param*]));

where

- *unknowns* $:= x_3, y_3, x_4, y_4, z_4, x_5, y_5, z_5$, *param* $:= t_1, t_2, t_3, t_4, t_5, \tilde{l}_2, \tilde{l}_3, \tilde{l}_4, \tilde{l}_5$ and *uvars* $:= u, u_1, u_2, u_3, u_4, u_5$ are respectively the position variables, the design parameters and the auxiliary variables used in saturations,

- *cstr* is the current sequence of polynomials in the design parameters *param* (we have fixed $\tilde{l}_1 = 1$, and removed the brackets [ ] with MAPLE function *op*),
- *config* is the Gröbner basis of the configuration ideal, computed in section 6, also with the brackets removed; *config* consists in 31 polynomials in the variables *param* and *unknowns*,
- *sat* $:= 1 - u_2\tilde{l}_2, 1 - u_3\tilde{l}_3, 1 - u_4\tilde{l}_4, 1 - u_5\tilde{l}_5$ are saturation polynomials, aimed to remove possible components with some zero length,
- *sat2* $:= 1 - u(\tilde{l}_2 - 1), 1 - u_1(t_1^2 + 1), 1 - u_2(t_2^2 + \tilde{l}_2)^2, \ldots, 1 - u_5(t_5^2 + \tilde{l}_5)^2$ are saturation polynomials aimed to remove non real components and components where the other alignment of the join between links 1 and 2 leads to a degenerated 4R loop,
- *angle* is a polynomial or a pair of polynomials, which are zero for a given angle of some join; they are discussed below.

For fixing the angle between links L_5 and L_1, we choose *angle* $:= c\tilde{l}_5 - x_{5,1}$, where c is a positive rational number less than 1. Thus *angle* $= 0$ if the cosine of the angle between L_5 and L_1 is c; in particular, $c = 0$ corresponds to an angle of $\pi/2$. Running above program with $c = 0$, and then with the inverse of an integer for c, gives $[1]$ as result. This proves that the second component does not correspond to any mobile configuration.

However, the proof is incomplete, as we have not excluded that some join is rejected at infinity for the chosen angles. To show that it is not the case, it suffices to run the program with several independent sets of values of c. If one does not use the value 0 for c, the computation is longer (more than one hour of CPU time), and gives the result $[1]$ after the third iteration.

This is not a mathematical proof, but provides an evidence of the correctness of the result, as the probability of missing a mobile configuration is extremely small and may be as small as desired, by repeating the computation as many times as desired with independent values of c.

8.3 More than two fully aligned configurations

In Section 8.2, we have seen that we may chose the signs of the $\tilde{l}_i$ and renumber the links in order that $\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 + \tilde{l}_4 + \tilde{l}_5 = 0$ for the first alignment and $\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 - \tilde{l}_4 - \tilde{l}_5 = 0$ for the second one. For a third alignment, we have signs ε_i such that $\varepsilon_1\tilde{l}_1 + \varepsilon_2\tilde{l}_2 + \varepsilon_3\tilde{l}_3 + \varepsilon_4\tilde{l}_4 + \varepsilon_5\tilde{l}_5 = 0$. The same reasoning as in Section 8.2 shows that, after a possible change of all signs of the ε_i, three of them are equal to $+1$ and two are equal to -1. One of the two negative ε_i must have i in $\{1, 2, 3\}$ and the other in $\{4, 5\}$, as, otherwise, either the third alignment would be the same as one of the other, or one of the length would be zero. Renumbering again, we have therefore

$$\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 + \tilde{l}_4 + \tilde{l}_5 = 0$$
$$\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 - \tilde{l}_4 - \tilde{l}_5 = 0$$
$$\tilde{l}_1 + \tilde{l}_2 - \tilde{l}_3 + \tilde{l}_4 - \tilde{l}_5 = 0$$

This may be rewritten, after changing some signs of the $\tilde{l}_i$, as

$$\tilde{l}_1 + \tilde{l}_2 = -\tilde{l}_3 = \tilde{l}_4 = \tilde{l}_5$$

As in Section 8.2, we have the same relations (mod π) for the angles of the links. Translated in terms of t_i, they become:

$$\begin{aligned}&\tilde{l}_1 t_2 t_3 + \tilde{l}_2 t_3 t_1 + \tilde{l}_3 t_1 t_2 - \tilde{l}_1 \tilde{l}_2 \tilde{l}_3 = 0\\ &t_3 = t_4 = t_5\end{aligned}$$

Thus three links are identical, and there are two cases to consider, depending if the three identical links are neighbour or not. In the next sections, we come back to a numbering of the links that follow the neighbourhood relation.

8.3.1 Neighbour identical links

In this case, above numbering of the links follows the neighbourhood relation. It is easy to see that a 5R loop satisfying these relations behaves like a 1R$\times$2, with the links L_1 and L_2 fixed together to form a single link identical to the three others links.

Nevertheless, we have to look if the join between L_1 and L_2 may also move. If it may move, its other alignment gives either a non-degenerate 4R loop, of a fourth fully aligned position.

In the first case, we may add to above relations the assembly conditions of the 4R loop. Saturating these polynomials by the $\tilde{l}_i$ and the $t_i^2 + \tilde{l}_i^2$, we get the Gröbner basis [1] in a few seconds, showing the impossibility of this case.

In the second case, an argument similar to that of the beginning of Section 8.3 shows that there are four identical links, and the fifth one may be aligned with either of its neighbours to provide a link identical to the others. Thus, the length of this fifth link is the double of the other link lengths, and the alignments are covering alignments.

In other words, the alignment by covering of this fifth link with either of its neighbours transforms the loop into a 1R$\times$2, that has two mobility trajectories. It results in three mobility trajectories for the 5R loop, the one where the fifth link has a covering alignment with both its neighbours being common to both double 1R×2.

Proposition 20. *If a mobile 5R loop (without zero-length link nor zero-angle link) has three neighbour links that are identical, then either one of the joins is fixed during the movement, or four links are identical, and the movement has three trajectories, each behaving as a 1R×2. This is a special case of Golberg loops where the two constitutive Bennett loops are identical and degenerate into a double1R×2.*

8.3.2 Non-neighbour identical links

In this case, the we have, up a circular permutation of the indexes, the following relations between the design parameters:

$$\begin{aligned}
&\tilde{l}_3 + \tilde{l}_5 = -\tilde{l}_4 = \tilde{l}_2 = \tilde{l}_1 \\
&t_1 = t_2 = t_4 \\
&\tilde{l}_3 t_4 t_5 + \tilde{l}_4 t_5 t_3 + \tilde{l}_5 t_3 t_4 - \tilde{l}_3 \tilde{l}_4 \tilde{l}_5 = 0
\end{aligned}$$

A 5R loop satisfying these relations has clearly an 1R×2 movement, obtained by superposing links L_1 and L_2, and aligning accordingly the three other links. In this movement, the only moving joins are $J_{1,5}$ and $J_{2,3}$. We have thus to find conditions for the existence of other movement trajectories in which the other joins can move.

If there is a trajectory leaving fix another join, this is the trajectory of a mobile 4R loop, which may be either a Bennett loop or a 1R×2. If it is an 1R×2, then there are two neighbour links, other than L_1 and L_2, that are identical, and we are in the case studied in the preceding section.

Let us suppose that we obtain a Bennett loop by fixing some join. Let L be the link obtained by fixing this join. As the opposite links of a Bennett loop are identical, the neighbour links of L must be the L_1 and L_4, or L_2 and L_4. By symmetry, we may suppose that they are L_1 and L_4. It follows that L is obtained by aligning L_2 and L_3. The fact that this results in a Bennett loop implies that $\tilde{l}_5^2 + t_5^2 = \tilde{l}_1^2 + t_5^2$, and the formulas of Proposition 14 are verified. Therefore the loop is one of the degenerate Goldberg loop, where the movement splits into a 1R×2 and a Bennett loop. All the joins are moving, but not simultaneously.

This completes the proof of Theorem 15.

9 Mobility of 5R loops with links having zero length or angle

In this section, we consider the mobility of 5R loops having some links with a zero length or a zero angle.

As before, we are interested only in fully mobile loops where all joins may move, simultaneously or not.

We will prove first, by geometrical considerations, that such a fully mobile configurations, which is not planar nor spherical has, at most, one zero-length link and one zero-angle link. Moreover, if both exist, they are neighbour links. Then Theorem 15 will be extended to this case.

9.1 Several zero-length links

If a 5R loop has four zero-length links, then the five join points coincide, the fifth link has also a zero length, and the loop is a spherical linkage, which is fully mobile, with two degrees of freedom in its movement.

If a 5R loop has three zero-length links and two non-zero length links, there are only two distinct join points, A and B, and the relative position of the non-zero-length links is fix. If these links are neighbours, the join between them is fix and the loop is not fully mobile. If the non-zero-length links are not neighbour, they have, as a common neighbour, a zero-length link. Lemma 4 shows that the loop is not fully mobile.

If a 5R loop has two zero-length links and three non-zero length links, there are three distinct join points. If the loop would be mobile, the relative position of these join points would remain fixed during the movement. As at least one of these three join points does not belong to any zero-length link, it is a join point between two non-zero-length links, and the corresponding join will remain fix during the movement. Therefore the loop is not fully mobile.

We have thus proved the following.

Proposition 21. *If a fully mobile 5R loop has at least two zero-length links, then every link has a zero length and the loop is spherical.*

9.2 Several zero-angle links

Lemma 22. *If a mobile 5R loop has two neighbour zero-angle links, then all links have zero angles and the loop is planar.*

Proof. Let us choose as L_1 and L_2 the neighbour zero-angle links. Let P be the plane that contains the join point $J_{1,2}$, and is orthogonal to the three parallel join axes of L_1 and L_2. This plane must contain the direction of the two neighbour links of L_1 and L_2, and thus all five join points. As the movement may be done relatively to a fixed link, the plane P may be considered as fix and containing the five joins points in all positions of the mobile loop.

Let us consider first the case where all links have a non zero length. The join between L_3 and L_4 may move. For a position where the directions of L_3 and L_4 differ, their common join axis must be orthogonal to P. As, by definition of P, the other join axis of L_3 is orthogonal to P, L_3 has a zero angle. Repeating this reasoning after a circular permutation of the indexes, show that all the links have a zero angle, and thus that the loop is planar.

In the case of zero-length links, Proposition 21 shows that there is at most one such link. If this link is L_5, the reasoning of preceding paragraph show that L_3 has a zero angle. Similarly, if L_3 has a zero length, then L_5 has zero angle. In both cases, a circular permutation of the indexes allows to suppose that L_1 and L_2 have a zero angle and L_4 has a zero length.

Let us consider a movement fixing L_3 and P in the space. The axis $A_{3,4}$ of the join between L_3 and L_4 is fixed by this movement. Thus the axis $A_{4,5}$ of the join between L_4 and L_5 is on a cone with axis $A_{3,4}$. The other join axis $A_{5,1}$ of L_5 is orthogonal to P; as $J_{3,4} = J_{4,5}$, the axis $A_{4,5}$ is also on a cone whose axis is orthogonal to P. If the two cones were not identical, $A_{4,5}$, and thus L_4, would remain fix during the movement; this would contradict the hypothesis that the loop is fully mobile. Therefore, the two cones are identical. This implies that $A_{3,4}$ is orthogonal to P and thus that L_3 has a zero angle. The same reasoning, with indexes 3 and 5 exchanged, shows that L_5 is also a zero-angle link. This implies that all join axes are parallel and we have a planar loop with a zero-length link. □

Proposition 23. *A fully mobile 5R loop with at least two zero-angle links is planar*

Proof. By preceding lemma it suffices to prove the non existence of a fully mobile 5R loop with two zero-angle links which are not neighbour, and no other zero-angle link. In fact, with three zero angle links, at least two of them are neighbours.

Let us choose as L_1 the link between the two zero-angle links, which are thus L_2 and L_5. Using notation of Section 2, $J_{i,j}$ is the join point between L_i and L_j, and α_i be the angle of the L_i. As α_2 is zero, $J_{3,4}$ is in the plane P_2 containing $J_{1,2}$ and $J_{2,3}$, and orthogonal to the join axes of L_2; similarly $J_{3,4}$ is also in the plane P_1 orthogonal to the join axes of L_5 and containing $J_{1,2}$ and $J_{5,1}$.

Thus $J_{3,4}$ is on the common perpendicular of the join axes of L_1. As the join axis of $J_{2,3}$ is orthogonal to the plane P_2, the join axis of $J_{3,4}$ belongs to a cone of angle α_3

which has $J_{3,4}$ as a vertex, and a line orthogonal to P_2 as an axis. Similarly, the join axis of $J_{3,4}$ belongs also to the cone of angle α_4 with vertex $J_{3,4}$ and axis orthogonal to the plane P_1. As the planes P_1 and P_2 remain fixed during the movement, the axis of $J_{3,4}$, which is the intersection of two cones with fixed angle and different axis directions, keeps a fixed direction during the movement.

By Proposition 21, and using the symmetry of the problem, we may suppose that the length of L_3 is not zero, that is that $J_{2,3}$ and $J_{3,4}$ are distinct. The direction of the join axis of $J_{3,4}$ is well defined by the position of $J_{1,2}$, $J_{2,3}$ and $J_{3,4}$. As the angle of L_3 is not zero, this direction would change if the triangle $J_{1,2}J_{2,3}J_{3,4}$ could change of shape. As we have shown that this direction is fix, the shape of this triangle, and the join between L_1 and L_2 are fix. This proves that such a fully mobile loop cannot exist. □

9.3 One zero-length link

By Proposition 23, a fully mobile 5R loop with one zero-length link has at most one zero-angle link.

Lemma 24. *If a fully mobile 5R loop exist with a zero-angle link and a zero-length link, then these links are neighbours.*

Proof. Let L_1 be the zero-angle link. If the zero-length link is not a neighbour of L_1, it may be chosen as L_4, by symmetry of the problem. As the angle of L_1 is zero, and L_2 and L_5 have non-zero lengths, all the join points must be coplanar and belong to some plane P, orthogonal to the two join axes of L_1. If the loop is mobile, it has positions where $J_{1,2}$, $J_{2,3}$ and $J_{3,4}$ are not aligned, and thus the join axis of $J_{2,3}$ is orthogonal to P. As the join axis of $J_{1,2}$ is also orthogonal to P, this implies that L_1 has a zero angle, and such a mobile 5R loop cannot exist. □

It follows that we may renumber the links in order that L_5 becomes the zero-length link, and all other links have non-zero length and non-zero angle, except maybe L_1, which may have a zero-angle.

In the following, we consider such a 5R loop, that is fully mobile, and study the constraints on the design parameters that this mobility implies. As before, we will consider the alignments between neighbour links, and suppose Conjecture 16 true. We use the notation defined in Section 2.

If an alignment of L_1 and L_2 results in a non-zero-length link, the corresponding position is fully aligned. In fact, otherwise, the join points $J_{5,1} = J_{4,5}$, $J_{2,3}$ and $J_{3,4}$ would form a non-degenerate triangle, which would imply that $\alpha_3 = 0$.

In a fully aligned position we cannot have a link length which is the sum of the other link lengths, as, otherwise, the linkage could not move. Therefore, the fully aligned position obtained by aligning L_1 and L_2 in extension implies that $l_1 + l_2 = l_3 + l_4$. Similarly, the alignment by covering implies $l_1 - l_2 = \pm(l_3 - l_4)$, if $l_1 \neq l_2$. This relation is also true if $l_1 = l_2$, as this implies that $J_{2,3} = J_{4,5}$ for the alignment by covering, and thus $l_3 = l_4$.

These two relations between the lengths implies that either $l_1 = l_4$, $l_2 = l_3$ or $l_1 = l_3$, $l_2 = l_4$. We have therefore three cases to consider, $l_1 = l_2 = l_3 = l_4$, $l_1 = l_3 \neq l_2 = l_4$ and $l_1 = l_4 \neq l_2 = l_3$.

9.3.1 Case $l_1 = l_2 = l_3 = l_4$

The equality of the four lengths implies that the join points are aligned when L_2 is aligned in extension with either L_1 or L_3. These two fully aligned positions induces the following relations for the link angles:

$$\begin{aligned}
&\varepsilon_1\alpha_5 + \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0 \mod \pi \\
&\varepsilon_2\alpha_5 + \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 0 \mod \pi,
\end{aligned}$$

where ε_1 and ε_2 are either 1 or -1. By exchanging, if needed, the indexes 1 and 4 and the indexes 2 and 3 (which amounts to reverse the numbering of the links), we may suppose $\varepsilon_1 = \varepsilon_2$. As L_5 has a zero length, the sign (modulo π) of α_5 is arbitrary, and we may thus suppose $\varepsilon_1 = \varepsilon_2 = 1$. Therefore we have the constraints

$$\begin{aligned}
&l_1 = l_2 = l_3 = l_4 \\
&\alpha_2 = \alpha_4 \\
&\alpha_1 - \alpha_3 + \alpha_5 = 0 \mod \pi.
\end{aligned}$$

We may write Equations (1), modified as described after Proposition 7. In the case where $\alpha_1 = 0$, we replace the equation involving t_1 by the condition that the z-coordinates z_5 of J_5 is zero (recalling that we always suppose that the join axis at $J_{1,2}$ is parallel to the z-axis).

We have thus relations between design parameters and position variables. Computing their Gröbner basis, it appears that it contains $y_{2,3}y_5$ in the case $\alpha_1 = 0$, and $y_{2,3}(x_5^2 + y_5^2)$ in the other case (we recall that $(x_{2,3}, y_{2,3}, 0)$ and (x_5, y_5, z_5) are the coordinates of $J_{2,3}$ and J_5, respectively).

This shows that, if the 5R loop is fully mobile, its movement splits in two trajectories where L_1 remains aligned either with L_2 or L_5.

In the trajectory where L_1 and L_2 are aligned, this is an alignment by superposition, as, otherwise, no movement would be possible. Thus L_3 and L_4 are also superposed.

As the join points $J_{2,3}$ and $J_{5,1}$ are superposed, all the join points lies in some plane P during this trajectory of movement. We may suppose that P, $J_{5,1} = J_{2,3}$ and $J_{1,2}$ remain fixed during this movement. As the angle between L_2 and L_3 is not fixed, the join axis of $J_{2,3}$ remains perpendicular to P, and the links L_1 and L_2 remain fixed. It follows that L_3 and L_4 are superposed and rotate around the axis of $J_{2,3}$. This rotation involves the axis of $J_{4,5}$, which belongs thus to a cone whose angle is $\alpha_3 - \alpha_4$ and whose axis is perpendicular to P. On the other hand, as L_1 is fix, the same is true for the axis of $J_{5,1}$, and the axis of $J_{4,5}$ is also on a cone of angle α_5. When the join between L_1 and J_5 moves, the axis of $J_{4,5}$ moves also, and this is possible only if the two cones are identical. It follows that the axis of $J_{5,1}$ is perpendicular to P, and that $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4 \pm \alpha_5 \bmod \pi$. This trajectory is thus an 1R$\times$2 movement with L_3, L_4 and L_5 aligned. Combining with above relation, we get that $l_1 = l_2 = l_3 = l_4$ and $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_3 - \alpha_5$.

In the trajectory where L_1 and L_5 remain aligned, we may consider L_1 and L_5 as a single link, and the linkage becomes a mobile 4R loop. As we have shown that, if the loop is fully mobile then $\alpha_3 \neq 0$, all lengths and all angles of this 4R loop are different of zero. The loop is thus either a Bennett loop, or an 1R$\times$2 loop. As α_3 differs from α_2 and α_4, the loop cannot be an 1R$\times$2 loop, and must therefore be a Bennett loop, which implies that $\alpha_3 = \alpha_1 + \alpha_5 = \pm\alpha_2$. As $\alpha_1 = \alpha_2$ and $\alpha_5 \neq 0$, we get $\alpha_5 = -2\alpha_1$, and finally $\alpha_1 = \alpha_2 = \alpha_4 = -\alpha_3 = -\frac{1}{2}\alpha_5$. One may recognize a degenerate Myard linkage, whose movement trajectory splits into a Bennett and an 1R$\times$2 movement. This may be summarized as the following proposition.

Proposition 25. *If $l_1 = l_2 = l_3 = l_4$, a fully mobile 5R loop, with a zero length link, is a degenerate Myard configuration, whose mobility trajectory splits into the trajectories of a Bennett loop and an 1R$\times$2 loop.*

9.3.2 *Case $l_1 = l_3 \neq l_2 = l_4$*

The fully aligned positions imply, for the angles,

$$\begin{aligned} \varepsilon_1\alpha_5 + \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 &= 0 \mod \pi \\ \varepsilon_2\alpha_5 + \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 &= 0 \mod \pi, \end{aligned}$$

where $\varepsilon_i = \pm 1$ for $i = 1, 2$. As in the preceding case, we may suppose $\varepsilon_1 = \varepsilon_2 = 1$ and thus $\alpha_2 = \alpha_4$ and $\alpha_1 - \alpha_3 + \alpha_5 = 0 \bmod \pi$. As in the preceding section we may write equations for the configurations. Computing the Gröbner basis, saturated by $l_2^2 - 1$ (for the computation we set $l_1 = 1$, and we are in a case where $l_1^2 \neq l_2^2$), and then saturating by l_2 and the $l_i^2 + t_i^2$, we get in a few seconds a Gröbner basis containing z_5^3 if $\alpha_1 \neq 0$, or y_5^3 if $\alpha_1 = 0$, where (x_5, y_5, z_5) are the coordinates of the point A_5 introduced in the preceding section. Therefore the direction of L_5 is fix, and we have the following.

Proposition 26. *In this case there are no fully mobile 5R loop.*

9.3.3 Case $l_1 = l_4 \neq l_2 = l_3$

If the alignment in extension of L_2 and L_3 would produce a fully aligned position, we would have $l_2 + l_3 = l_1 + l_4$, and the four links would be equal. Therefore, in this position, the triangle $J_{1,2}J_{3,4}J_{4,5}$ is not degenerate, which implies $\alpha_2 + \alpha_3 = 0 \bmod \pi$.

Like in the preceding cases, the two alignments of L_1 and L_2 give the conditions

$$\begin{aligned} \varepsilon_1\alpha_5 + \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 &= 0 \quad \text{mod } \pi \\ \varepsilon_2\alpha_5 + \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 &= 0 \quad \text{mod } \pi. \end{aligned}$$

If we would have $\varepsilon_1 = \varepsilon_2$, we would deduce $\alpha_2 - \alpha_3 = 0$, and thus $\alpha_2 = \alpha_3 = 0$, which contradicts our assumptions. Thus we have $\varepsilon_1 = -\varepsilon_2$, which, after a possible change of the sign of α_5, implies $\alpha_1 = \alpha_4$ and $\alpha_2 - \alpha_3 - \alpha_5 = 0 \bmod \pi$.

Except for the lacking relation $\frac{sin^2\alpha_1}{l_1^2} = \frac{sin^2\alpha_2}{l_2^2}$, we have exactly the conditions of a Myard loop. The algorithm of section 5 will allow us to prove that, if the loop is mobile, this condition is satisfied.

To apply this algorithm, we have first to compute the configuration ideal. This requires first to show that, for a mobile loop that satisfies previous conditions, the directions of three consecutive links are not coplanar in almost all positions. In fact, this will show that we do not remove relevant configurations by saturating by the coefficients of the t_i in equations (1) modified as described after Proposition 7. To prove this, as all link angle are non-zero, it suffices to prove that all joins move simultaneously.

If the join between L_1 and L_2, or L_3 and L_4, would remain fixed during the movement, the other join points would form a non-degenerate triangle, making impossible the mobility. If the join between L_2 and L_3 remains fixed, the same argument works, unless if L_2 and L_3 are aligned by covering. In this case, the movement could only be an 1R×2 trajectory, but this is impossible, as $\alpha_2 \neq \alpha_3$. Thus the fixed join, if any, could only be one of the joins of L_5. Pushing the movement until alignment of L_1 and L_2, shows that L_5 would be aligned with either L_1 or L_4. The two components of the movement are therefore those of mobile 4R loops. As $l_1 \neq l_3$, none can be a Bennett loop. An 1R×2 movement, is also impossible, as is would keep fixed the join between L_2 and L_3, case already eliminated.

It follows that, in the case under consideration, all the joins move simultaneously, and in almost all positions, the configuration is not degenerated. This allows to make the following computation.

The first step consists of the computation of the configuration ideal. For this we put together equations (1) modified as described after Proposition 7, and the above constraints for the design parameters, expressed in terms of the l_i and $t_i = \frac{l_i}{\tan \alpha_i}$. Then we compute the Gröbner basis, saturating by l_2, $l_2^2 - 1$ and the variables that are factors of the coefficients of the t_1, t_2, t_3. This provides in less than half an hour a Gröbner basis of 212 polynomials. The saturation by the coefficients of t_4 and t_5 does not changes the basis, but needs almost three hours. As the configuration is not degenerated in almost all positions, these saturations are legitimate.

Then the Gröbner basis eliminating the position variables is computed (one CPU minute) for applying the algorithm of Section 5.

The polynomials independent of the position variables are those which have been explicitly introduced: $E_0 = [l_4 - 1, l_2 - l_3, t_2 + t_3, t_1 - t_4, 2t_5t_3l_3 + t_3^2 - l_3^2]$. The first E_x, contains nine polynomials, including $(t_3^2 l_3^2 - t_4^2 l_3^2 - l_3^2 + t_3^4)\,(t_3^2 + l_3^2 - t_4^2 - 1)$. Adding the second factor to the equations and proceeding to a step of the algorithm, the resulting E_0 and E_x are both equal to the Gröbner basis obtained by adding the second factor to above E_0. As these constraints are exactly those of a Myard linkage, this proves again that Myard linkages are mobile.

We have also to consider above first factor. For this we eliminate Myard linkages by saturating this first E_x by the second factor, and applying a step of the algorithm with the resulting polynomials added to the ideal. This results in a E_x containing $t_5(1+t_5^2)^2$. Saturating by $1+t_5^2$, (which cannot be equal to zero), we get $[l_4 - 1, l_2 - l_3, t_5, t_2 + t_3, -t_4 + t_1, 1 + t_4^2 - 2l_3^2, t_3^2 - l_3^2]$, which is a special instance of a Myard conditions. This can be verified by observing that the normal form (*Normal Form* function) of the Myard conditions by this Gröbner basis are all equal to zero. We have thus the following result.

Proposition 27. *In the case considered in this section, the mobile 5R loops are exactly the Myard mechanisms.*

9.4 One zero-angle link

We choose L_1 as the (unique) zero-angle link. Let P the plane containing the join points of L_1 and orthogonal to their (parallel) join axes. As the angle of L_1 is zero, the join points of L_2 and L_5 belong to P, and the join point $J_{3,4}$ is the only one that may not belong to P.

If an alignment of L_3 and L_4 results in a link of non-zero length, then $J_{3,4}$ belongs also to P. As L_2 and L_5 have non-zero angles, they must therefore be aligned with one of their neighbour links, by Lemma 3. This implies that the links split into two aligned set of links, and thus all links are aligned, unless if both sets, considered as single links, have a zero-length and a zero-angle. In this case, one of the sets contains two links and the other three. As we consider an alignment of L_3 and L_4, resulting into a non-zero-length link, the two links may not be L_3 and L_4. As we consider a position where L_3 and L_4 are aligned, the two links may not contain L_3 or L_4 and another link. Thus the two links consist in L_1 and one of its neighbours. As the angle of L_1 is zero and the angles of its neighbours is not zero, these alignment do not result into a zero-angle link, which shows that this case may not occur, and the position is fully aligned.

We have thus proved the following lemma.

Lemma 28. *If the lengths l_3 and l_4 of L_3 and L_4 are different, there are at least two fully aligned positions, corresponding to the two different alignments of L_3 and L_4. If $l_3 = l_4$, there is at least one fully aligned position, in which L_3 and L_4 are aligned in extension.*

There are therefore several cases, depending on the number of fully aligned position.

9.4.1 One fully aligned position

If there is exactly one fully aligned position, we have already seen that $l_3 = l_4$. The alignment by superposition of L_3 and L_4 provides a triangle $J_{5,1}J_{1,2}J_{2,3} = J_{5,1}J_{1,2}J_{4,5}$, which is not degenerate. In fact, otherwise, either the configuration would be fully aligned or the join axes at $J_{2,3}$ and $J_{4,5}$ would coincide, inducing an 1R$\times$2 configuration that can move to two different fully aligned positions. If the triangle is not degenerate, and the join axes at $J_{2,3}$ and $J_{4,5}$ would coincide, they would be orthogonal to the plane P of the triangle, which is impossible if the angles of L_2 and L_5 are non-zero. This implies that $\alpha_3 \neq \alpha_4$.

Let us consider the alignment of L_2 and L_3 which does not belong to the fully aligned position. This alignment implies that L_3 belongs to the above defined plane P, and all join points belong to P. Proposition 3 implies thus that either L_4 and L_5 are aligned, or L_2, L_3, L_4 as well as L_1, L_5 are aligned. In the second case, if the configuration is not fully aligned, above reasoning shows that L_1 and L_5 together have a zero angle, which is impossible. Thus, in the case of one fully aligned position, only the first case is possible.

Thus, this alignment of L_2 and L_3 implies that L_4 and L_5 are also aligned. The triangle $J_{5,1}J_{1,2}J_{3,4}$ is non-degenerate, as otherwise we would have a second fully aligned position. As the join axes at the vertexes are parallel, the link angles must be such that $\alpha_2 \pm \alpha_3 = 0 \bmod \pi$ and $\alpha_4 \pm \alpha_5 = 0 \bmod \pi$, the signs $\pm$ depending on the nature of the alignment of the links.

One alignment of L_1 and L_2 is provided by the fully aligned position. If the second alignment would superpose $J_{5,1}$ and $J_{2,3}$, the triangle $J_{2,3}J_{3,4}J_{4,5}$ would be degenerated, because of the non-zero angle of L_4. Thus, as the configuration is not fully aligned, the join axes at $J_{5,1}$ and $J_{2,3}$ would be superposed, and L_1 and L_2 would have the same angle, which is excluded by the hypotheses. It follows that this second alignment provides a non-degenerate 4R loop, for which one may write down the assembly conditions. Similarly, the same is true for the second alignment of L_1 and L_5.

Putting everything together, and using signed lengths as before, we get nine constraints for the eight design parameters: the angle constraint for the fully aligned position, the two assembly constraints for the second alignment of L_1 and L_2, the two assembly constraints for the second alignment of L_1 and L_5, and

$$\begin{aligned}
\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 + \tilde{l}_4 + \tilde{l}_5 &= 0 \\
\tilde{l}_3 - \tilde{l}_4 &= 0 \\
t_2\tilde{l}_3 - t_3\tilde{l}_2 &= 0 \\
t_4\tilde{l}_5 - t_5\tilde{l}_4 &= 0.
\end{aligned}$$

Saturating the ideal generated by these polynomials by the lengths and $t_3 - t_4$ (we have shown that $\alpha_3 \neq \alpha_4$), we get [1] as a Gröbner basis. This shows the following.

Proposition 29. *There is no mobile configuration with exactly one zero-angle link and at most one fully aligned position.*

9.4.2 Two fully aligned positions

In this section we consider the case of exactly two fully aligned positions.

In section 8.2 we have shown that, in the case of two fully aligned positions, the links split in a set of three links and a set of two links, such that the signed sum of the lengths and the signed sum of the angles are zero in each set. In particular, in each set, the links have the same relative orientation in the two fully aligned positions, and the links in the set of two elements have the same length and the same angle.

The latter property shows that the set of two links may not contain L_1, which has a zero angle, differing of the other link angles. Thus the set of two angles may be $\{L_2, L_3\}$, $\{L_2, L_4\}$, $\{L_2, L_5\}$, $\{L_3, L_4\}$, omitting $\{L_3, L_5\}$ and $\{L_4, L_5\}$, which may become one of the preceding pairs by applying the symmetry of the problem.

In the case $\{L_3, L_4\}$, these links are aligned by covering in both fully aligned positions. As, by Lemma 28 the alignment by extension of these links provides a fully aligned position, this case may not occur with only two fully aligned positions.

In the case $\{L_2, L_3\}$, these two links are aligned by covering in both fully aligned positions, and the alignment by extension of these links induces a non-fully-aligned position such that all join points are coplanar. By Lemma 3, L_4 and L_5 must each be aligned with one of its neighbours, and the configuration must by a non-degenerate triangle. It follows that $\alpha_2 + \alpha_3 = 0 \mod \pi$. This relation, together with $l_2 = l_3$ implies that, in all positions, the links L_2 and L_3 are symmetric with respect to their bisector plane Q (when L_2 and L_3 are aligned, we take, as bisector plane, the plane containing the join axis at $J_{2,3}$ and the join points of L_2 and L_3). In particular, the planes defined respectively by L_2, L_1 and L_3, l_4 are exchanged by this symmetry, and their intersection line lies in Q. In the positions where L_1 and L_5 are aligned, $J_{4,5}$ belongs to this intersection line, and therefore to Q. Thus, by symmetry, we get $l_4 = |l_5 \pm l_1|$, the sign $\pm$ depending on the alignment. As such a relation would imply that some of the lengths would be zero, this shows that this case may not occur.

In the case $\{L_2, L_4\}$, the links L_1 and L_5 have the same alignment in the two fully aligned positions. When L_1 and L_5 are aligned in the other way, the linkage behaves as a 4R loop which has, at least, three links with non-zero length and non-zero angle. If this 4R loop is degenerate (that is, all its join points are in the same plane), every link with a non-zero angle must be aligned with one of its neighbours (Proposition 3) . This implies that either the configurations is fully aligned or it is an 1R$\times$2 loop that may move to a fully aligned position. As there is only two fully aligned positions, this is impossible, and, thus, the second alignment of L_1 and L_5 results in a non degenerated 4R loop.

Putting together the assembly conditions of the 4R loop and the conditions on the lengths and the angles induced by the fully aligned positions, and saturating the resulting ideal by the lengths of the links (that are supposed to be non-zero), results in the Gröbner basis [1], showing that this case also cannot occur.

The only remaining case is when the set of two links is $\{L_2, L_5\}$. In this case, the alignment of L_3 and L_4 is the same in both fully aligned positions. It is an alignment by extension, as we have seen that the alignment by extension of L_3 and L_4 induces a fully aligned position. The fully aligned positions induce thus the relations $l_2 = l_5$, $\alpha_2 = \alpha_5$ and $\alpha_3 = -\alpha_4$. As the position, in which L_3 and L_4 are aligned by superposition, is not fully aligned, we have also $l_3 = l_4$. In this position, the triangle $J_{5,1}J_{1,2}J_{2,3} = J_{5,1}J_{1,2}J_{4,5}$ is non degenerated, as, otherwise we would have an 1R$\times$2 configuration that can move toward a third fully aligned position. This implies that $2l_5 = l_2 + l_5 > l_1$, and thus $l_1 \neq 2l_5$.

The fully aligned positions imply thus that the following polynomials are zero:

$l_2 - l_5, t_2 - t_5, l_3 + l_4 - 1, l_3 - l_4, t_3 + t_4$ (with as usual $l_1 = 1$). Now we may proceed to the following computation. Firstly we add to this list of polynomials the left-hand sides of equations (7); then we saturate by the lengths l_i; then we saturate the result by the $l_i^2 + t_i^2$, and we compute the Gröbner basis for the ordering eliminating the position variables. These computations, together, need less than a minute of CPU time. The first E_x provided by the algorithm of Section 5 is $[2l_4 - 1, 2l_3 - 1, l_2 - l_5, t_3 + t_4, t_2 - t_5, (1 - 4l_5^2)(4l_5^2 - 4t_4^2 - 1 + 4t_5^2)]$. AS $l_1 \neq 2l_5$, we can remove the factor $(1 - 4l_5^2)$, leading to the conditions:

$$
\begin{aligned}
&l_5 = l_4 = l_3 = l_2 = \frac{1}{2} l_1 \\
&\alpha_1 = \alpha_5 - \alpha_2 = \alpha_3 + \alpha_4 = 0 \qquad (\operatorname{mod} \pi) \\
&\frac{\sin^2 \alpha_2}{l_2^2} = \frac{\sin^2 \alpha_3}{l_3^2}
\end{aligned}
$$

This characterizes a Golberg linkage. Therefore we have:

Proposition 30. *A mobile 5R loop with one zero-length link and two fully aligned position is a Goldberg linkage.*

9.4.3 More than two fully aligned positions

We have shown in Section 8.3 that three links are identical and that the signed sums of the lengths and of the angles are zero for the two remaining links and one of the identical links. There are two cases, depending if the identical links are consecutive or not.

If the three identical links are consecutive in the loop, then, by symmetry of the indexing, we may suppose that they are L_3, L_4, L_5, that L_1 is the zero-angle link. We may also suppose that there are exactly three fully aligned positions, as, if there were four fully aligned positions, four links were identical, and the case may be considered with the case of three non-consecutive identical links. In the three fully aligned positions, L_1 and L_2 have the same relative orientation (because the signed sum of the lengths are zero in all fully aligned positions, and l_1 and l_2 have the same signs in all). For the other alignment of L_1 and L_2, these two links together form a link, which has a non-zero angle. It has also a non-zero length, as, otherwise the three other links would form an equilateral triangle, which is impossible by Lemma 3 applied to L_4. Therefore, with this alignment, the 5R loop becomes a 4R loop such all links have a non-zero length and a non-zero angle. As we have supposed that there is only three fully aligned positions, Proposition 6 implies that this alignment induces a non-degenerate 4R loop which must verify the constraints of Proposition 10.

We have thus eighth constraints on the design parameters, these two constraints and the constraints resulting from the fully aligned positions, namely $1 + \tilde{l}_2 + \tilde{l}_3 = 0$, $\tilde{l}_3 + \tilde{l}_5 = \tilde{l}_4 + \tilde{l}_5 = 0$, $t_3 = t_4 = t_5$, $\tilde{l}_3 t_2 + \tilde{l}_2 t_3 = 0$. Saturating these eighth polynomials by $\tilde{l}_2$ gives easily the Gröbner basis $[1]$, showing that there are no such configuration such that the join $J_{1,2}$ is mobile.

If there are three non-consecutive identical links they can be chosen, by symmetry, as L_2, L_4 and L_5, the link L_1 being the zero-length link. As said above, we do not exclude here the case where L_3 is also identical with L_2. This induces the relations $1 + \tilde{l}_2 + \tilde{l}_3 = 0$, $\tilde{l}_2 = \tilde{l}_4 = \tilde{l}_5$, $t_2 = t_4 = t_5$, $\tilde{l}_3 t_2 + \tilde{l}_2 t_3 = 0$ between the design parameters. Putting them with equations (1) modified as described in Remark 8, we may saturate by the lengths and apply the algorithm of Section 5, with $y_{2,3}$ as smallest position variable. The first E_x, computed in a few CPU seconds, contains a polynomial that has $t_5^2 + \tilde{l}_5^2$ as a factor. Saturating by this polynomial, we get the polynomials $[2\tilde{l}_5 - 1, 2\tilde{l}_4 + 1, 2\tilde{l}_3 + 1, 2\tilde{l}_2 + 1, t_4 - t_5, t_3 + t_5, t_2 - t_5]$. The design parameters satisfy thus the relations

$$
\begin{aligned}
& l_2 = l_3 = l_4 = l_5 = \frac{l_1}{2} \\
& \alpha_2 = \alpha_4 = \alpha_5 = -\alpha_3 \\
& \alpha_1 = 0.
\end{aligned}
$$

It easy to recognize a degenerate Goldberg linkage such that the movement trajectory splits into a Bennett and a 1R$\times$2 trajectories. If $\alpha_2 = \frac{\pi}{2}$, the four non-zero angles are equal modulo π, and the movement trajectory splits further into three 1R$\times$2 trajectories.

This completes the proof of Theorem 1.

A Computational tools

A.1 Software

The main computational tool for this work is Gröbner basis computation. For this purpose, we have used two MAPLE implementations of Faugère's F4 algorithm [5], both due to Faugère himself. The first one, is the built-in implementation (package GROEBNER, called with option `method = fgb`). It can only be used for easiest computations, because of a limitation of roughly 500,000 for the number of columns of the matrices to be reduced. The second implementation is Faugère's own version, named FGB, which can be downloaded as a MAPLE library [3].

FGB has several features that have been essential for the success of our computations. The first one is its efficiency. The second one is that the only limitation for the size of matrices generated by the program is the memory size of the computer. We have used, over a computer with 128 gigabytes of memory, a limit of 10^7 or 10^8 for the number of columns.

Finally, FGB allows to stop the computation when some degree is reached. This allows to know the polynomials of low degree in the Gröbner basis without completing the computation. Normally, the result of this truncated computation is not a Gröbner basis. However, in some cases, the truncated result may the desired Gröbner basis, and the verification of the correctness of the result may be much easier than the non-truncated computation. In this study, this occurred, for example, for the computation of the configuration ideal for the 5R loops.

For almost all Gröbner basis computation that were needed for this study, we have used the block ordering called *lexdeg* in MAPLE or the total degree ordering *tdeg*, which is the same as a *lexdeg* ordering such that one of the block of variables is empty. More precisely, most computations are called by *Basis(pol, lexdeg(var1, var2), method = fgb)*, where *pol* is a list of polynomials and *var1* and *var2* are disjoint lists of variables containing all variables appearing in *pol* (it does not matter if some other variables appear in *var1* and *var2*). With FGB package, the same result is obtained by *fgb_gbasis(pol, 0, var1, var2, options)*. Typically, *options* has the form $\{$*"verb"=3, "index"* $= 10^7\}$ for having information of the progress of the computation and allowing matrices of size 10^7.

For many computation, only the polynomials that do not depend on *var1* are needed. With FGB, this is obtained by using *fgb_gbasis_elim* instead of *fgb_gbasis*. With the built-in package *Groebner*, one has to use the function *select* for selecting the polynomials that have the degree zero in the variables *var1*.

A.2 Usage of Gröbner bases

Gröbner bases are rarely useful by themselves, as they generally consist of many large polynomials that are not human readable. A notable exception it when the Gröbner basis is $[1]$, which occurs if and only if the input polynomials do not have any common complex zero.

The Gröbner basis of a set of polynomials strongly depends on the choice of a *monomial ordering*. The most widely known monomial ordering is the lexicographical ordering. As this ordering usually induces longer computation and produces polynomials of larger degree and larger coefficients, we do not use this ordering, except for very small example.

The *graded reverse lexicographical ordering*, called *tdeg* in MAPLE produces generally many polynomials of relatively low degree. It is the best choice for computing the dimension and the degree of an algebraic variety (through the function called *HilbertSeries* in MAPLE). It is also used for solving the zero-dimensional systems, which are the systems that have a finite number of complex solutions (MAPLE functions *solve* and *RootFinding[Isolate].*

For the block ordering called *lexdeg* in MAPLE, the variables are divided in two blocks, and the monomials are compared by using first the ordering *tdeg* on the first block, and, in case of equality, by using *tdeg* on the second block. The computation of a Gröbner basis for this ordering implements the algebraic operation of elimination, which geometrically corresponds to a projection: given an ideal I, let G be the set of the polynomials that depend only on the second block of variables, in the Gröbner basis of I for this ordering; then G is a Gröbner basis of the ideal of the polynomials of I that depend only on the second block of variables. Geometrically, G is the Gröbner basis of the image by the projection, on the space of the second block of variables, of the variety defined by I.

An important example of elimination is the operation of *saturation*, which allows to remove, from the algebraic set of the zeros of an ideal, the components on which a given polynomial is zero. Commonly, the zeros of this given polynomial correspond to degenerate parasitic solutions. If f is the polynomial with respect to which one want to saturate, one adds $1 - uf$ to the considered equations, and one eliminates u; that is one keeps the polynomials independent of u in the Gröbner basis for the block ordering that has only u in the first block. For saturating with respect to several polynomials, there are several methods, see Section 6.

As the block ordering is reduced to *tdeg* when one of the blocks is empty, it follows that, generally, everything that may be obtained by a Gröbner basis computation is best obtained with a Gröbner basis computation with a block ordering. For this reason, this is the only ordering that has been used in this paper; also, this is the only available ordering in the standard distribution of FGB.

B Arguments supporting the conjecture

Conjecture 31. *Given a mobile 5R loop, which is not planar not spherical, when a join between two links moves toward the alignment of these links, the other join points do not tend toward infinity.*

To discuss this, we may suppose that the links to be aligned are L_1 and L_2. If a join point tends toward to infinity, we may suppose that it is $J_{3,4}$, by, if needed, reversing the indexing of the links.

A movement of the loop corresponds to a curve in the space of the positions. With the coordinates that we have used for representing the positions, the alignment of L_1 and L_2 corresponds to the intersection of this curve with the hyperplane $y_{2,3} = 0$. The positions with some join at infinity correspond to the intersection of the curve with the hyperplane at infinity. Thus, the conjecture is false if the curve passes through the intersection of two hyperplanes (in the projective space). It is unlikely that a given curve passes through the intersection of two independent hyperplanes, which are defined independently of the curve. This is therefore the first heuristic argument supporting the conjecture.

We give now another heuristic argument, which, at first glance, seems to disprove the conjecture, but does not in our particular case. Let us consider a curve C of mobile 5R loops, in the space of the design parameters. For each loop in C, the variety of its positions intersects the hyperplane at infinity in at least one point. For this position at infinity, the angle $\alpha_{2,3}$ between the links L_2 and L_3 has some value, which varies when the loop moves on C. Thus it reaches $0 \bmod \pi$ for some 5R loop L belonging to C.

If this situation can occur, the conjecture is false for this 5R loop. However, this particular loop must been the limit, under the deformation of the design parameters of loops satisfying the conjecture. The proof of Theorem 1 shows that this situation cannot occur, that is:

Proposition 32. *If the conjecture is false, the mobile 5R loops such that some join point tends to the infinity when $\alpha_{2,3}$ tends to zero modulo π form an algebraic set which is a connected component open and closed in the set of all mobile 5R loops.*

It follows that the existence of loops for which the conjecture is false is very constrained.

Here is another argument supporting the conjecture, that appears as much stronger than the preceding ones.

There are 10 design parameters (two for each link) that are rational functions (equations (1)) of 9 position variables (here we do not suppose $l_1 = 1$). If one removes any two of these 10 constraints, one gets, for fixed design parameters, less constraints than the number of position variables. Thus the resulting mechanism is always mobile. For examples, removing the equations involving t_i and t_{i+1} amounts to replace the revolute join at $J_{i,i+1}$ by a spherical join.

If one starts from a mobile 5R loop, and one removes two constraints, one gets thus a mechanism that has the same mobility trajectory. Applying the method of [8], one may determine the variety, in the space of the design parameters and of $y_{2,3}$, above which there is a position at infinity (W_∞ in the notation of this paper). Putting $y_{2,3} = 3$ in the equations of this variety, one gets at least one constraint that the design parameters of the mobile 5R loop must satisfy for having a join point at the infinity when L_1 and L_2 are aligned. Doing that for each pair of constraints to remove, give thus, at least, 45 constraints in the 10 design parameters.

It is highly unlikely, that these 45 constraints in 10 variables have a common solutions, and thus that the conjecture is false.

This gives, theoretically, an effective method for proving the conjecture, by computing these constraints an verifying that their Gröbner basis is [1]. Unfortunately we have been able to compute only two of these 45 constraints, that resulting of removing the equations involving t_3 and t_4 and that involving t_3 and t_5. The other computations fail by reaching, after days of computation, the limit of 10^8 for the number of columns of the matrices to be reduced.

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