# Determining the distribution of calcium channels in the olfactory system

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# DETERMINING THE DISTRIBUTION OF CALCIUM CHANNELS IN THE OLFACTORY SYSTEM*

THIBAULT BOURGERON†, CARLOS CONCA‡, AND RODRIGO LECAROS§

**Abstract.** Cilia are long thin cylindrical structures that extend from an olfactory receptor into the nasal mucus. The transduction of an odor into an electrical signal occurs in the membranes of the cilia. The cyclic-nucleotide-gated (CNG) channels, activated by cyclic adenosine monophosphate (cAMP), allow a depolarizing influx of sodium ions, which initiate the electrical signal. A model for experiments involving the diffusion of cAMP into cilia and the resulting current activity has been developed in [8].

In this paper we study the inverse problem which consists in finding the spatial distribution of CNG ion channels from the experimental current data. This problem comes down to solving a Fredholm integral equation of the first kind. The Mellin transform allows us to write an explicit formula for its solution. Proving observability and continuity inequalities is then a question of estimating the Mellin transform of the kernel of this integral equation on vertical lines. We prove new estimates using arguments in the spirit of the stationary phase method. We complete the continuity and observability results obtained in [4] for an approximated kernel. We also introduce a better approximated kernel for which we prove continuity and observability results, and for which we perform numerical computations using experimental data. Lastly, we prove that no observability inequality, in certain weighted $L^2$ spaces, can be found for the original model. Throughout this paper we present numerical simulations illustrating the theoretical results.

**Key words.** Distribution of Calcium Channels, Olfactory System, Inverse Problem, Fredholm Integral Equation, Mellin Transform

**AMS subject classifications.** 35Q92, 45Q05, 65R20

**1. Introduction.** Identification of detailed features of neuronal systems is a major issue in the biosciences for the coming years. In this respect, inverse problem methods have already proved to be efficient. As a contribution to this field, making use of a model developed in [8], this article proposes an improved method to determine the spatial distribution of CNG ion channels along the length of a cilium. We also apply our method to experimental data published in [9].

Cilia are long thin cylindrical structures that extend from an olfactory receptor into the nasal mucus. When an odorant molecule binds to an olfactory receptor in a cilium membrane, it successively activates two enzymes, which results in an increase in the concentration of cyclic adenosine monophosphate (cAMP) concentration within the olfactory receptor neuron. Some of the cAMP binds to cyclic nucleotide-gated (CNG) ion channels, causing them to open. This allows a depolarizing influx of $Na^+$ ions to flow into the cell, which causes the neuron to depolarize. Although the single-channel properties have been well described, the distribution of these channels

*This work has begun during a visit by the first author to the CMM in Santiago, Chile. This was made possible thanks to ECOS no. C11E07. The authors received partial support from Ecos-Conicyt Grant C13E05.

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along the cilia remains still widely unknown and may well turn out to be crucial in determining the kinetics of the neuronal response.

Experimental procedures to isolate a single (grass frog) cilium have been developed, [10, 12, 11, 13, 6]. One olfactory cilium is drawn into a pipette which is then moved to a pseudointracellular bath which contains no cAMP. The pipette containing the cilium is then transferred to a bath containing cAMP. Contact with the bath initiates the diffusion of cAMP into the cilium. The current through the cilium is recorded.

A very natural issue is whether it is possible to determine the CNG channel distribution along the length of a cilium from current experimental data. In [8] the authors propose a mathematical model for the dynamics of cAMP concentration in this experiment, consisting of two nonlinear differential equations and a constrained Fredholm integral equation of the first kind. Numerical methods to compute the channel distribution were proposed in [8], [7]. Some computations indicated that this mathematical problem is highly ill-conditioned.

To tackle the problem mathematically, [9] introduced some simplifications in the model of the experiment. The simplified inverse problem, which interests us in this work, consists in determining the function $\rho$, representing the distribution of the CNG channels, from the measurement of the electrical current $\mathrm{I}_0[\rho]$, given by

$$\mathrm{I}_0[\rho](t) = \int_0^L \rho(x) H_0(c(t,x))\,\mathrm{d}x, \tag{1.1}$$

for $t \geqslant 0$, where the Hill function $H_0$ is defined by

$$H_0(x) = \frac{x^n}{x^n + K_{1/2}^n}. \tag{1.2}$$

In (1.1), $c(t,x)$ denotes the concentration of cAMP which diffuses along the cilium. The Hill function appearing here is widely used in biochemistry to model the binding of a ligand to a macromolecule, here the binding of cAMP molecules to the cilium.

After [9, 4] we will assume that $c(t,x)$ is known, analytically, and given by:

$$c(t,x) = c_0 \,\mathrm{erfc}\left(\frac{x}{2\sqrt{Dt}}\right). \tag{1.3}$$

Here, $\mathrm{erfc} = 1 - \mathrm{erf}$ and erf is the Gauss error function.

In (1.2), $K_{1/2} > 0$ is a constant corresponding to the half-bulk concentration, and $c_0$ the concentration of cAMP in the solution with which the pipette comes into contact.

All the parameters have been experimentally estimated, see [9, 2] and the references therein, to be: $n = 1.7$, $K_{1/2} = 0.17$ $\mu$M. For the numerical simulations throughout this paper we will also use $c_0 = 1$ $\mu$M, $D = 1$ m$^2$ s$^{-1}$. This value of $c_0$ has been chosen so as to show the interesting behaviours in the figures. The value of $D > 0$ does not play an important role in the analysis we propose, *cf.* (2.6).

In [9] an explicit formula for $\rho$ is given if $H$ is taken to be a single Heaviside function ($H(x) = a\mathbf{1}_{x>\alpha}$). Paper [4] presents identifiability, stability and reconstruction results when $H$ is a finite sum of Heaviside functions.

The aim of this article is to give a complete solution of this inverse problem. We make use of the Mellin transform to invert (1.1), which allows us to complete the results obtained in [4], in a simple way. We then introduce a better approximation of $H_0$ and we prove continuity and observability results for the corresponding operator I. For this operator we also perform numerical computations using experimental data. Lastly we prove that for the original model (1.1), (1.2), (1.3) presented above, no observability inequality can be found, in certain weighted $\mathrm{L}^2$ spaces.

As noted in [4], (1.1) is a Fredholm integral equation of the first kind. The associated inverse problem is in general ill-posed. For instance, if the kernel of an integral operator is sufficiently smooth, then this operator is compact from $\mathrm{L}^2$ to $\mathrm{L}^2$, thus, even if it is injective, its inverse cannot be continuous. The mathematical results contained in this article can be understood in this way. Depending on the regularity of the kernel of the operator (1.1) (*i.e.* depending on the choice for the approximation of the Hill function $H_0$), the integral operator (1.1) has a more or less strong regularizing effect. Therefore the inverse problem is more or less ill-posed, see [5]. Precisely in Propositions 3.4, 3.5, the kernel is at most continuous and the degree of ill-posedness of the inverse problem is 1. Whereas in Proposition 5.7 the kernel belongs to some space of smooth functions (denoted $\mathcal{S}[0,\infty)$, see Definition 5.4) and its degree of ill-posedness is infinite.

The paper is organized as follows. Section 2 introduces the main idea for using the Mellin transform to invert the integral operator I. Section 3 states continuity and observability inequalities for two different approximations of the Hill function and a non observability result for the genuine Hill function. Section 4 presents some numerical simulations performed with experimental data. Sections 5 and A (in the Appendix) give the proofs of these results. Finally the Mellin transform is briefly recalled Section B, in the Appendix.

**2. Notations and use of the Mellin transform.** The aim of this section is to present the strategy used to find the function $\rho > 0$, which is the distribution of the CNG channels, from the measurement of the current $\mathrm{I}_0[\rho]$. Before studying this problem we introduce two approximations of $\mathrm{I}_0[\rho]$, obtained by approximating the Hill function $H_0$.

The first approximation of the Hill function $H_0$ is linear. This case has already been studied in [4]. We focus on it here both to explain our method and because all the computations can be performed explicitly.

Definition 2.1. *For $(\alpha_k)$ a finite increasing sequence of elements of $(0, c_0)$ and $a_k$ positive numbers, let:*

$$H_1(x) = \sum_{k=1}^{m} a_k \mathbf{1}_{x \geqslant \alpha_k}. \tag{2.1}$$

As $t$ tends to $\infty$ the factor $x/\sqrt{Dt}$ appearing in (1.3) tends to 0. Therefore we consider the following approximation of $H_0$.

Definition 2.2. *For positive real numbers $a < c_0, K$, let:*

$$H_2^{(a)}(x) = \frac{x^n}{x^n + K_{1/2}^n} \mathbf{1}_{x \leqslant a} + K \mathbf{1}_{a < x \leqslant c_0}. \tag{2.2}$$

The function $H_2^{(a)}$ is represented in Figure 2.1. For the sake of simplicity we will often drop the superscript in the notations $H_2^{(a)}, I_2^{(a)}, J_2^{(a)}$.

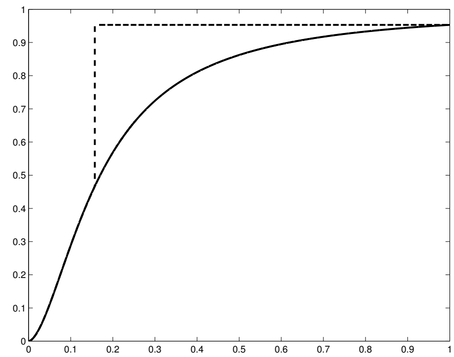


FIG. 2.1. *Functions* $H_0$ *and* $H_2^{(a)}$ *(dashed line) defined by* (1.2), (2.2), *with* $a = 0.1573$.

DEFINITION 2.3. *For* $l = 1, 2$ *we define* $\mathrm{I}_l[\rho]$ *as* $\mathrm{I}_0[\rho]$ *is defined:*

$$\mathrm{I}_l[\rho](t) = \int_0^L \rho(x) H_l(c(t,x))\,\mathrm{d}x, \tag{2.3}$$

*where* $c(t, x)$ *is still defined by* (1.3).

Appendix B contains some details regarding the Mellin transform and the definition of multiplicative convolution. As the Mellin transform of a multiplicative convolution is the pointwise product of Mellin transforms, it can be used to invert the operator $\mathrm{I}_l[\rho]$. Before going further, let us introduce some convenient notations.

**Notations.** For $l = 0, 1, 2$, let:

$$G_l(x) = H_l(c_0\,\mathrm{erfc}(2^{-1}D^{-1/2}x^{-1})) \qquad \text{and} \qquad J_l(x) = H_l(c_0\,\mathrm{erfc}(x)). \tag{2.4}$$

The functions $G_l$ and $J_l$ and their Mellin transforms are linked by the relations:

$$G_l(x) = J_l(2^{-1}D^{-1/2}x^{-1}) \qquad \text{and} \qquad \mathcal{M}G_l(s) = 2^s D^{s/2}\mathcal{M}J_l(-s). \tag{2.5}$$

The relation (2.3) defining $I_l$ can then be inverted:

$$\rho(x) = 2^{-q-1}D^{-q/2}\mathcal{M}_q^{-1}\left(\frac{\mathcal{M}\,\mathrm{I}_l[\rho]\left(\frac{s}{2}\right)}{\mathcal{M}J_l(-s)}\right). \tag{2.6}$$

The main idea of this article is that the quantities $\mathrm{I}_l[\rho]$ can be written as multiplicative convolutions, and the following formal identity holds. For $l = 0, 1, 2$:

$$\mathrm{I}_l[\rho](t^2) = \int_0^\infty x\rho(x)\mathbf{1}_{x\in[0,L]}G_l\left(\frac{t}{x}\right)\frac{\mathrm{d}x}{x} = \left(x\rho(x)\mathbf{1}_{x\in[0,L]}\right) * G_l, \tag{2.7}$$

where $*$ is the multiplicative convolution.

Therefore finding continuity or observability inequalities for the operator $\mathrm{I}_l[\rho]$ is reduced to bounding $\mathcal{M}J_l(s)$, from above or from below respectively, on the vertical line on which the inverse Mellin transform is taken. The next section presents the bounds obtained in each case $l = 0, 1, 2$, and the continuity/observability results they imply.

**3. Main results.** Before stating our main results, let us introduce some weighted $\mathrm{L}^p$ spaces for which the Mellin transform is well-suited.

**Notations.** For real numbers $p \geqslant 1, q$ and for a function $f : \mathbb{R} \to \mathbb{R}$ we denote $\|f\|_{\mathrm{L}^p_q} = \left(\int_{\mathbb{R}} |f(x)|^p \, |x|^q \, \mathrm{d}x\right)^{1/p}$. We denote $\mathrm{L}^p_q$ the Banach space $\mathrm{L}^p\left([0,\infty), x^q \, \mathrm{d}x\right)$ endowed with the norm $\|\ \|_{\mathrm{L}^p_q}$.

For a function $f : q + i\mathbb{R} \to \mathbb{C}$ we denote $\|f\|_{\mathrm{L}^p(q+i\mathbb{R})} = \left(\int_{\mathbb{R}} |f(q+it)|^p \, \mathrm{d}t\right)^{1/p}$. We also denote $\mathrm{L}^p(q+i\mathbb{R})$ the Banach space $\mathrm{L}^p\left(q+i\mathbb{R}, \mathrm{d}x\right)$ with the norm $\|\ \|_{\mathrm{L}^p(q+i\mathbb{R})}$.

As stated in the previous section, continuity/observability results for the operators $I_l$ $(l = 0,1,2)$, are consequences of the behaviours of $\mathcal{M}J_l$ on some vertical lines. The following lemma makes this link precise.

Lemma 3.1. *For $l = 0,1,2$, the operators $I_l$ are defined by* (1.2), (1.3), (1.1), *and* (2.1), (2.2), (2.3), *and the functions $J_l$ are defined by* (2.1), (2.2), (2.4). *Let $k \in \mathbb{N}, q \in \mathbb{R}, C > 0$ such that*

$$\left|\prod_{j=0}^{k-1} (s+2j)\ \mathcal{M}J_l(s)\right| \leqslant C \quad (\text{resp. } \geqslant C) \qquad \textit{on} \quad -\frac{q+1}{2} + i\mathbb{R}.$$

*Then:*

$$\left\|(\mathrm{I}_l[\rho])^{(k)}\right\|_{\mathrm{L}^2_{2k-2+\frac{q+1}{2}}} \leqslant C 2^{-k+q/2} D^{(q-1)/4} \, \|\rho\|_{\mathrm{L}^2_q} \qquad \left(\textit{resp.} \geqslant C 2^{-k+q/2} D^{(q-1)/4} \, \|\rho\|_{\mathrm{L}^2_q}\right).$$

As a consequence of Lemma 3.1 the strategy to prove continuity/observability results for the operators $I_l$, in a weighted $\mathrm{L}^2$ space, is to find an integer power $k \in \mathbb{N}$ such that $\left|s^k \mathcal{M}J_l(s)\right|$ is bounded from above/below on a vertical line, whose abscissa depends on the weight.

For the approximated operators $\mathrm{I}_1$ and $\mathrm{I}_2$ the integer $k$ appearing in the lemma can be taken to be 1, whereas for the non approximated operator $\mathrm{I}_0$, $\mathcal{M}J_0(s)$ decays faster than any $s^k$. The converse of Lemma 3.1, which is true under an extra assumption on $\mathcal{M}J_l$ (*cf.* Proposition 3.6 and its proof, Section 5.5), implies that no observability inequality of the previous type holds. These results are proved in Section 5. We now state the results for each case.

First, let us consider the linear approximation $\mathrm{I}_1[\rho]$ of the operator $\mathrm{I}_0[\rho]$. In this case the Mellin transform $\mathcal{M}J_1$ can be computed explicitly. Lemma 3.3 shows explicit bounds.

Definition 3.2. *Define:*

$$\beta_k = \operatorname{erfc}^{-1}\left(\frac{\alpha_k}{c_0}\right), \qquad q_1 = \frac{\ln\left(\frac{1}{a_1}\sum_{k=2}^m a_k\right)}{\ln\beta_1 - \ln\beta_2} > 0, \tag{3.1}$$

*where the numbers $a_k$, $\alpha_k$ are defined by* (2.1).

Lemma 3.3. *Let the $\beta_k$'s and $q_1$ be defined by* (2.1), (3.1) *and, let $J_1$ be defined by* (2.1), (2.4). *For $s$ in the right half-plane, the Mellin transform of $J_1$ is given by :*

$$\mathcal{M}J_1(s) = \frac{1}{s}\sum_{k=1}^m a_k \beta_k^s.$$

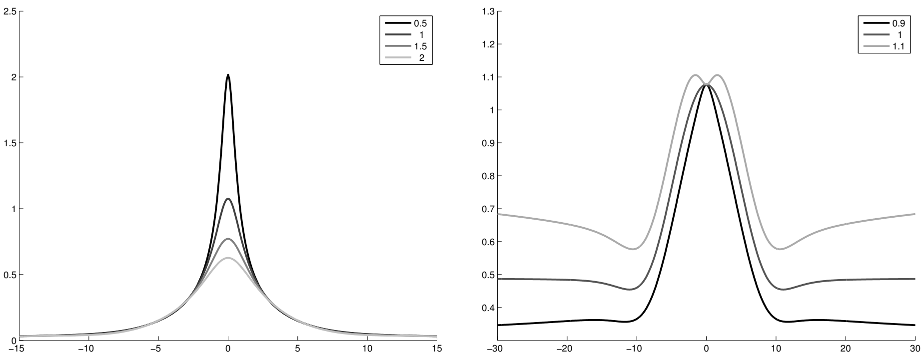


FIG. 3.1. $\left|\mathcal{M}J_2^{(a)}(s)\right|$ *(left) and* $\left|\mathcal{M}J_2^{(a)}(s)s^k\right|$*, for some values of* $k$ *(right), for* $s \in 1 + i\mathbb{R}$*,* $a = 0.1573$*,* $K = \frac{c_0^n}{K_{1/2}^n + c_0^n}$.

$$\forall q > q_1\ \exists c > 0\ \forall s \in q + i\mathbb{R} : |s\mathcal{M}J_1(s)| \geqslant c$$

This lemma can be rephrased. On vertical lines of the right half-plane the function $\left|s^k\mathcal{M}J_1(s)\right|$ is bounded from above for $k \in [0, 1]$ and is bounded from below for $k \geqslant 1$, if the vertical lines are far enough to the right in the complex plane. This allows us to establish Proposition 3.4.

PROPOSITION 3.4. *Let* $\mathrm{I}_1[\rho]$ *be defined by* (1.3), (2.1), (2.3).

- *Let* $q < -1$. *If* $\rho$ *is in* $\mathrm{L}^2_q$*, there exist constants* $C, C' > 0$ *such that:*

$$\|\mathrm{I}_1[\rho]\|_{\mathrm{L}^2_{\frac{q-3}{2}}} \leqslant C\,\|\rho\|_{\mathrm{L}^2_q} \quad \textit{and} \quad \|(\mathrm{I}_1[\rho])'\|_{\mathrm{L}^2_{\frac{q+1}{2}}} \leqslant C'\,\|\rho\|_{\mathrm{L}^2_q}.$$

- *Let* $q_1$ *be defined by* (2.1), (3.1) *and, let* $q < -2q_1 - 1$. *If* $(\mathrm{I}_1[\rho])'$ *is in* $\mathrm{L}^2_{\frac{q+1}{2}}$*, there exists a constant* $C > 0$ *such that:*

$$\|(\mathrm{I}_1[\rho])'\|_{\mathrm{L}^2_{\frac{q+1}{2}}} \geqslant C\,\|\rho\|_{\mathrm{L}^2_q}.$$

The proofs of these statements can be found in Section 5.2. This proposition is the same as the one obtained in [4], except that the weight $q$ appearing in that paper is positive.

Second, let us consider the non linear approximation $\mathrm{I}_2[\rho]$ of the operator $\mathrm{I}_0[\rho]$. In this case the computations are no longer explicit. Nevertheless an argument in the spirit of the stationary phase method proves that $\mathcal{M}J_2^{(a)}$ has the same behaviour as $\mathcal{M}J_1(s)$, if the real number $a$ has a small enough value. This is illustrated in Figure 3.1 and this allows to establish Proposition 3.5.

PROPOSITION 3.5. *Let* $a > 0$ *and* $\mathrm{I}_2^{(a)}[\rho]$ *be defined by* (2.2), (1.3), (2.3). *Let* $q < -1$.

- *If* $\rho$ *is in* $\mathrm{L}^2_q$*, there exists a constant* $C > 0$ *such that:*

$$\left\|\mathrm{I}_2^{(a)}[\rho]\right\|_{\mathrm{L}^2_{\frac{q-3}{2}}} \leqslant C\,\|\rho\|_{\mathrm{L}^2_q}.$$

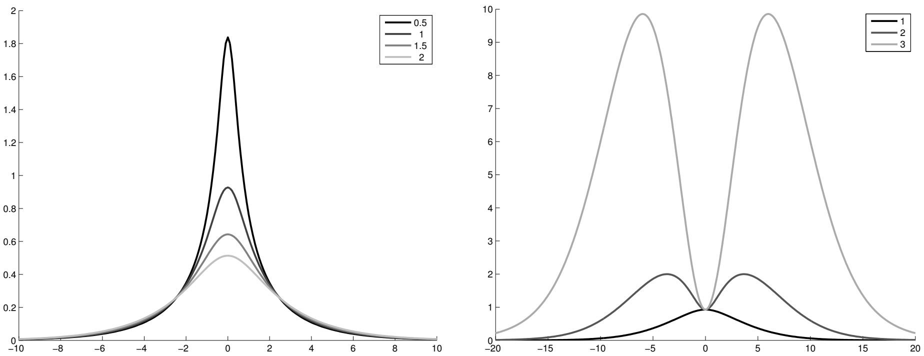


FIG. 3.2. $|\mathcal{M}J_0(s)|$ *(left) and* $\left|\mathcal{M}J_0(s)s^k\right|$, *for some values of* $k$ *(right), for* $s \in 1 + i\mathbb{R}$.

- *If* $\rho$ *is in* $\mathrm{L}^2_q$, *and* $a$ *has a small enough value, then there exists a constant* $C > 0$ *such that:*

$$\left\| \left( \mathrm{I}_2^{(a)}[\rho] \right)' \right\|_{\mathrm{L}^2_{\frac{q+1}{2}}} \leqslant C' \left\| \rho \right\|_{\mathrm{L}^2_q}.$$

- *If* $\left( \mathrm{I}_2^{(a)}[\rho] \right)'$ *is in* $\mathrm{L}^2_{\frac{q+1}{2}}$, *and* $a$ *has a small enough value, then there exists a constant* $C > 0$ *such that:*

$$\left\| \left( \mathrm{I}_2^{(a)}[\rho] \right)' \right\|_{\mathrm{L}^2_{\frac{q+1}{2}}} \geqslant C \left\| \rho \right\|_{\mathrm{L}^2_q}.$$

The proofs of these statements can be found in Section 5.4.

The smallness condition on $a$ is needed for the observability inequality to be true. Indeed for $J_2^{(c_0)} = J_0$ (because $H_2^{(c_0)} = H_0$) no inequality of the last type holds. This fact is a consequence of Proposition 3.6.

Finally let us consider the case of the Hill function defined by (1.2). Explicitly computing the derivatives of $J_0$ shows that the Mellin transform $\mathcal{M}J_0$ has a fast decay on vertical lines. Consequently on vertical lines of the right half-plane, and for any $k \in \mathbb{N}$, the function $\left|s^k \mathcal{M}J_0(s)\right|$ is bounded from above and is not bounded from below. This fact is illustrated in Figure 3.2. As a consequence, no observability inequality of the previous type holds.

PROPOSITION 3.6. *Let* $q < -1$.

- *If* $\rho$ *is in* $\mathrm{L}^2_q$, *there exists* $C > 0$ *such that:*

$$\left\| \mathrm{I}_0[\rho] \right\|_{\mathrm{L}^2_{\frac{q-3}{2}}} \leqslant C \left\| \rho \right\|_{\mathrm{L}^2_q}.$$

- *Furthermore, let us assume that* $\mu\left(\left\{ s \in \frac{q+1}{2} + i\mathbb{R} \mid \mathcal{M}\rho(s) = 0 \right\}\right) = 0$ *where* $\mu$ *is the Lebesgue measure on* $\frac{q+1}{2} + i\mathbb{R}$. *For* $n > 0$, *there is no* $k \in \mathbb{N}$, $C > 0$ *such that:*

$$\left\| \left( \mathrm{I}_0[\rho] \right)^{(k)} \right\|_{\mathrm{L}^2_{2k + \frac{q-3}{2}}} \geqslant C \left\| \rho \right\|_{\mathrm{L}^2_q}.$$

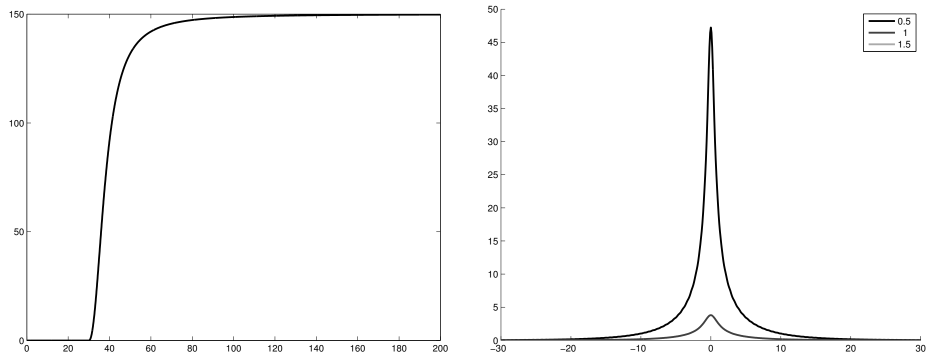


FIG. 4.1. *Functions* $\mathrm{I}_{exp}$ *defined by* (4.1) *and* $t \mapsto |\mathcal{M}\,\mathrm{I}_{exp}(-q+it)|$ *for some values of* $q$.

For instance, the condition $\mu\left(\left\{s \in \frac{q+1}{2} + i\mathbb{R} \mid \mathcal{M}\rho(s) = 0\right\}\right) = 0$ holds if $\mathcal{M}\rho$ is non-zero and holomorphic on a neighborhood of $\frac{q+1}{2} + i\mathbb{R}$ in $\mathbb{C}$ and this is satisfied if $\rho$ is in a non-zero $\mathrm{L}^1_w$ function for $w$ in a neighborhood of $\frac{q-1}{2}$, see Section B.

The proofs of these statements can be found in Section 5.5.

**4. Numerical simulations.** In this section we present some numerical simulations based on experimental data. The aim is to use formula (2.6) to compute the distribution $\rho$ of the CNG channels from an experimentally measured current $\mathrm{I}_{exp}$.

We consider that the experimentally observed current is given by the following formula (see [9] and the references therein).

$$\mathrm{I}_{exp}(t) = \begin{cases} 0 & \text{if } t \leqslant t_D \\ I_{max}\left[1 + \left(\frac{K_I}{t-t_D}\right)^{n_I}\right]^{-1} & \text{if } t > t_D \end{cases} \tag{4.1}$$

The parameters are given by: $I_{max} = 150$ pA, $n_I = 2.2$, $K_I = 100$ ms, $t_D = 30$ ms. The function $\mathrm{I}_{exp}$ is shown in Figure 4.1.

Note that the derivative of $\mathrm{I}_{exp}$ is in $\mathrm{L}^2_q$, for some $q < 0$ so Proposition 3.5 (or Proposition 3.4) can be applied directly. If the data were noisy, a standard regularization method for an inverse problem with a finite degree of ill-posedness (see [5] for instance) could be applied to the data before applying Proposition 3.5. In this extend our numerical method consisting of Proposition 3.5 can be applied to any noisy experimental data. For the approximated operator $\mathrm{I}_2^{(a)}[\rho]$ the distribution $\rho$ can be found from the current $\mathrm{I}_{exp}$, as shown in Figure 4.2.

For the operator $\mathrm{I}_0[\rho]$ the distribution $\rho$ cannot be found from the current $\mathrm{I}_{exp}$, even after a regularization, as shown in Figure 4.3. This is coherent with Proposition 3.6, which states that no observability inequality can be found for $\mathrm{I}_0$ in some weighted $\mathrm{L}^2_q$ spaces.

**Technical aspects**

As seen in Section 5 the functions $\mathcal{M}J_0, \mathcal{M}J_1, \mathcal{M}J_2$ are holomorphic on the right half-plane. As $t_D > 0, n_I > 0$, $\mathrm{I}_{exp}$ is in $\mathrm{L}^1_k$ for every $k < -1$, then, $\mathcal{M}\,\mathrm{I}_{exp}$ is holomorphic on the left half-plane. Therefore the quotient $\frac{\mathcal{M}\,\mathrm{I}_{exp}\left(\frac{s}{2}\right)}{\mathcal{M}J_l(-s)}$ is meromorphic on the left half-plane. The inverse Mellin transform of this function has been computed on different vertical lines, whose abscissas are $q < 0$.

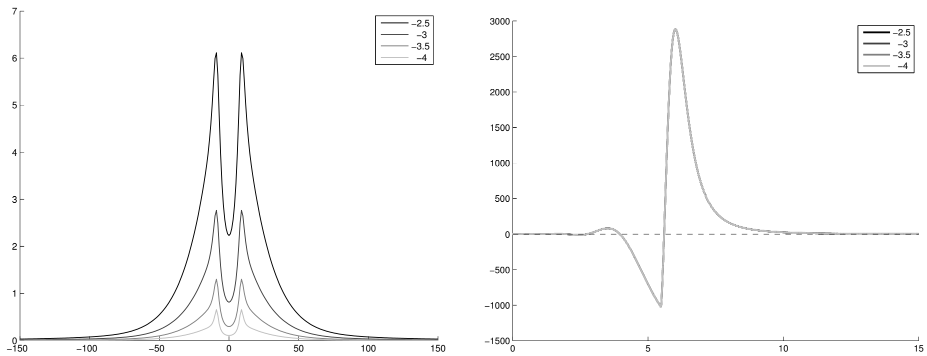


FIG. 4.2. *For $s \in q + i\mathbb{R}$, for some values of $q < 0$: - (left)* $\left|\frac{\mathcal{M}\,\mathrm{I}_{exp}\left(\frac{s}{2}\right)}{\mathcal{M}J_2^{(a)}(-s)}\right|$, *- (right)* $\rho_{exp} = \mathcal{M}_q^{-1}\left(\frac{\mathcal{M}\,\mathrm{I}_{exp}\left(\frac{s}{2}\right)}{\mathcal{M}J_2^{(a)}(-s)}\right)$.

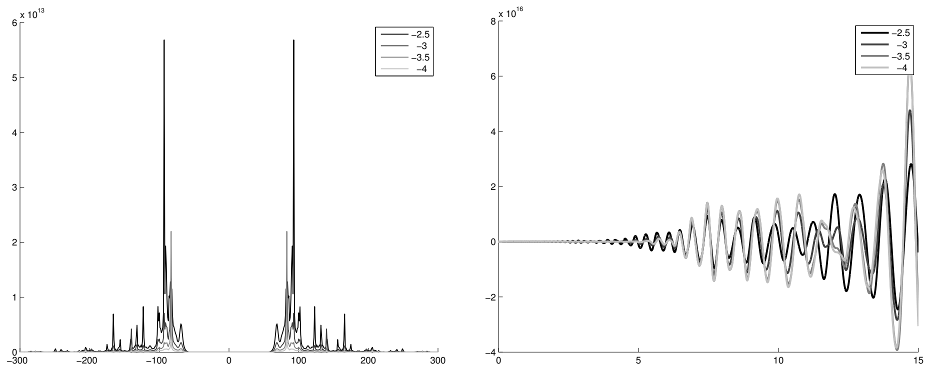


FIG. 4.3. *For $s \in q + i\mathbb{R}$, for some values of $q < 0$: - (left)* $\left|\frac{\mathcal{M}\,\mathrm{I}_{exp}\left(\frac{s}{2}\right)}{\mathcal{M}J_0(-s)}\right|$, *- (right)* $\mathcal{M}_q^{-1}\left(\frac{\mathcal{M}\,\mathrm{I}_{exp}\left(\frac{s}{2}\right)}{\mathcal{M}J_0(-s)}\right)$.

As the quotient $\frac{\mathcal{M}\,\mathrm{I}_{exp}\left(\frac{s}{2}\right)}{\mathcal{M}J_2^{(a)}(-s)}$ does not vanish in the left half-plane, the distribution $\rho$ does not depend on $q < 0$. More details regarding this fact can be found in [3]. Figure 4.2 illustrates Proposition 3.5. For the numerical simulations, the parameter $a$ is taken to be the biggest for which the quotient defined above does not vanish in the strip $\{x + it \mid x \in [-4, -2], t \in \mathbb{R}\}$ (that is on the vertical $-4 + i\mathbb{R}$ thanks to ), that is $a = 0.1573$. If the negative part of $\rho_{exp}$ is removed then its profile is similar to the one exhibited in [8], [7], [9].

**5. Proofs. Notation.** (Pochhammer symbol) For a real number $x$ and a non-negative integer $n$ we write:

$$(x)_n = x \cdots (x - n + 1) = \prod_{j=0}^{n-1} (x - j). \tag{5.1}$$

For instance $\forall q \in \mathbb{R}$ and $k \in \mathbb{N}$: $(x^q)^{(k)} = (q)_k \; x^{q-k}$.

**5.1. Proof of Lemma 3.1.** Let us apply the Mellin transform to (2.7) and use Proposition B.4 (of the Appendix):

$$1/2\,\mathcal{M}\,\mathrm{I}[\rho](s/2) = \mathcal{M}(x\rho(x))(s)\,\mathcal{M}G_l(s) = \mathcal{M}\rho(s+1)\,2^s D^{s/2}\,\mathcal{M}J_l(-s).$$

This leads to:

$$\begin{aligned}\mathcal{M}\,\mathrm{I}[\rho](s) &= 2\,2^{2s} D^s\,\mathcal{M}J_l(-2s)\,\mathcal{M}\rho(2s+1)\\ (s-k)_k\,\mathcal{M}\,\mathrm{I}[\rho](s-k) &= 2\,2^{2(s-k)} D^{s-k}(s-k)_k\,\mathcal{M}J_l(-2(s-k))\,\mathcal{M}\rho(2(s-k)+1).\end{aligned}$$

As the Mellin transform is an isometry (up to the factor $(2\pi)^{-1/2}$) of $\mathrm{L}^2_{2k-1+\frac{q-1}{2}}$ onto $\mathrm{L}^2\left(k+\frac{q-1}{4}+i\mathbb{R}\right)$, for $s$ in $k+\frac{q-1}{4}+i\mathbb{R}$ the previous relation implies:

$$\begin{aligned}\left\|(\mathrm{I}[\rho])^{(k)}\right\|_{\mathrm{L}^2_{2k-2+\frac{q+1}{2}}} &= (2\pi)^{-1/2}\left\|(-1)^k(s-k)_k\mathcal{M}\,\mathrm{I}[\rho](s-k)\right\|_{\mathrm{L}^2\left(k+\frac{q-1}{4}+i\mathbb{R}\right)}\\ &= (2\pi)^{-1/2}2\,2^{(q-1)/2}D^{(q-1)/4}\left\|(s-k)_k\mathcal{M}J_l(-2(s-k))\mathcal{M}\rho(2(s-k)+1)\right\|_{\mathrm{L}^2\left(k+\frac{q-1}{4}+i\mathbb{R}\right)}\\ &= (2\pi)^{-1/2}2\,2^{(q-1)/2}D^{(q-1)/4}\left\|(s)_k\mathcal{M}J_l(-2s)\mathcal{M}\rho(2s+1)\right\|_{\mathrm{L}^2\left(\frac{q-1}{4}+i\mathbb{R}\right)}.\end{aligned}$$

The inequality

$$|(s)_k\,\mathcal{M}J_l(-2s)| \leqslant C \quad (\text{resp. } \geqslant C) \qquad \text{on} \qquad \frac{q-1}{4}+i\mathbb{R}$$

is equivalent to:

$$\left|\left(\frac{-s}{2}\right)_k\mathcal{M}J_l(s)\right| \leqslant C \quad (\text{resp. } \geqslant C) \qquad \text{on} \qquad -\frac{q-1}{2}+i\mathbb{R}.$$

And:

$$\left(\frac{-s}{2}\right)_k = \prod_{j=0}^{k-1}\left(-\frac{s}{2}-j\right) = (-1)^k 2^{-k}\prod_{j=0}^{k-1}(s+2j).$$

Therefore, as $\mathcal{M}$ is an isometry of $\mathrm{L}^2_q$ onto $\mathrm{L}^2\left(\frac{q+1}{2}+i\mathbb{R}\right)$, the assumption implies:

$$\begin{aligned}\left\|(\mathrm{I}[\rho])^{(k)}\right\|_{\mathrm{L}^2_{2k-2+\frac{q+1}{2}}} &\leqslant (\text{resp. } \geqslant)\ C2^{-k}(2\pi)^{-1/2}2\,2^{(q-1)/2}D^{(q-1)/4}\left\|\mathcal{M}\rho(2s+1)\right\|_{\mathrm{L}^2\left(\frac{q-1}{4}+i\mathbb{R}\right)}\\ &= C2^{-k}(2\pi)^{-1/2}2\,2^{(q-1)/2}D^{(q-1)/4}2^{-1/2}\left\|\mathcal{M}\rho(s+1)\right\|_{\mathrm{L}^2\left(\frac{q-1}{2}+i\mathbb{R}\right)}\\ &= C2^{-k}(2\pi)^{-1/2}2^{q/2}D^{(q-1)/4}\left\|\mathcal{M}\rho(s)\right\|_{\mathrm{L}^2\left(\frac{q+1}{2}+i\mathbb{R}\right)}\\ &= C2^{-k+q/2}D^{(q-1)/4}\left\|\rho\right\|_{\mathrm{L}^2_q}.\end{aligned}$$

**5.2. Approximation of the Hill function by a step function.** *Proof.* [Proof of Lemma 3.3] The function $J_1$ is in $\mathrm{L}^1_k$ for every $k > -1$ thus $\mathcal{M}J_1$ is holomorphic on the right half-plane, see Proposition B.3 of the Appendix.

As erfc is a decreasing function, for $c, a > 0$ we have: $\mathbf{1}_{c\,\mathrm{erfc}(t)-a\geqslant 0} = \mathbf{1}_{\mathrm{erfc}^{-1}\left(\frac{a}{c}\right)-t\leqslant 0}$. The explicit formula for $\mathcal{M}J_1$ is then a consequence of: $\mathcal{M}(\mathbf{1}_{0\leqslant x\leqslant a})(s) = \frac{1}{s}a^s$ for $\operatorname{Re} s > 0$; and of the linearity of $\mathcal{M}$ .

As $a_k > 0$ and $0 < \alpha_1 < \cdots < \alpha_n$ the $\beta_k$ satisfy: $0 < \beta_n < \cdots < \beta_1 < +\infty$. As $q > 0$:

$$\left|\sum_{k=1}^{m} a_k \beta_k^{q+it}\right| \geqslant a_1\beta_1^q - \sum_{k=2}^{m} a_k\beta_k^q \geqslant a_1\beta_1^q - \beta_2^q \sum_{k=2}^{m} a_k.$$

This last term is bounded from below by a positive constant for $q > q_1$. ☐

*Proof.* [Proof of Proposition 3.4]

- Let $q > 0, s \in q + i\mathbb{R}, k \in [0,1]$. The explicit formula of Lemma 3.3 implies:

$$\left|s^k\ \mathcal{M}J_1(s)\right| \leqslant |s|^{k-1} \sum_{k=1}^{m} a_k\beta_k^q \leqslant C\,|s|^{k-1} \leqslant C.$$

  Now let $q < -1$, that is $-\frac{q+1}{2} > 0$. Applying Lemma 3.1 for $k = 0, 1$ on the vertical line $-\frac{q+1}{2} + i\mathbb{R}$ leads to the continuity inequalities.
- Let $q > q_1 > 0, s \in q + i\mathbb{R}, k \geqslant 1$. The explicit formula of Lemma 3.3 leads to the bounds:

$$\left|s^k \mathcal{M}J_1(s)\right| \geqslant |s|^{k-1}\left(a_1\beta_1^q - \beta_2^q \sum_{k=2}^{m} a_k\right) \geqslant C > 0.$$

  Now let $q < -1 - 2q_1$, that is $-\frac{q+1}{2} > q_1$. Lemma 3.1 applied for $k = 1$ on the vertical line $-\frac{q+1}{2} + i\mathbb{R}$ concludes the proof.

☐

**5.3. Lemmas.** LEMMA 5.1. *Let $A$ and $B$ be two elements of $[0,\infty]$, $k \in \mathbb{N}$ an integer and $f$ a function such that $f^{(j)}$ is in $\mathrm{L}^1_j(A,B)$ for every $j = 0,\ldots,k$. For every real number $t$, we have:*

$$\int_A^B f(x)x^{it}\,\mathrm{d}x = \sum_{j=0}^{k-1}(-1)^j Q_j(t)\left[x^{j+1}f^{(j)}(x)x^{it}\right]_A^B + (-1)^k Q_{k-1}(t)\int_A^B x^k f^{(k)}(x)x^{it}\,\mathrm{d}x,$$

*where $Q_j(t) = \left(\prod_{l=0}^{j} 1 + l + it\right)^{-1}$.*

*Proof.* The proof is by induction on $k \in \mathbb{N}$. For $k = 0$ there is nothing to prove. We assume that the formula is true for an integer $k \in \mathbb{N}$. As $(k + 1 + it)Q_k = Q_{k-1}$ it remains to be proved that:

$$(k+1+it)\int_A^B x^k f^{(k)}(x)x^{it}\,\mathrm{d}x = \left[x^{k+1}f^{(k)}x^{it}\right]_A^B - \int_A^B x^{k+1}f^{(k+1)}(x)x^{it}\,\mathrm{d}x.$$

As $\frac{\mathrm{d}}{\mathrm{d}x}x^{it} = \frac{it}{x}x^{it}$, the previous relation follows by integration by parts:

$$it\int_A^B x^k f^{(k)}(x)x^{it}\,\mathrm{d}x = \int_A^B x^{k+1}f^{(k)}(x)(x^{it})'\,\mathrm{d}x$$
$$= \left[x^{k+1}f^{(k)}(x)x^{it}\right]_A^B - (k+1)\int_A^B x^k f^{(k)}(x)x^{it}\,\mathrm{d}x - \int_A^B x^{k+1}f^{(k+1)}(x)x^{it}\,\mathrm{d}x.$$

☐

Corollary 5.2. *Let $f : [A,B] \to \mathbb{R}$ with $A, B \in [0,\infty]$ be a piecewise $C^1$ function. If $f$ is non-negative, $f'$ is non-positive, $f \in \mathrm{L}^1(A,B)$, $f' \in \mathrm{L}^1_1(A,B)$ and for all $t \in \mathbb{R}$: $[xf(x)x^{it}]_A^B = 0$, then:*

$$\langle t\rangle \left| \int_A^B f(x)x^{it}\,\mathrm{d}x \right| \leqslant \int_A^B f(x)\,\mathrm{d}x,$$

*where $\langle t\rangle = (1+t^2)^{1/2}$.*

*Proof.* From Lemma 5.1 with $k=1$ one obtains:

$$\forall t \in \mathbb{R} \quad (1+it)\int_A^B f(x)x^{it}\,\mathrm{d}x = -\int_A^B xf'(x)x^{it}\,\mathrm{d}x.$$

As $A, B \geqslant 0$ and $f' \leqslant 0$, using the formula for $t=0$:

$$\langle t\rangle \left| \int_A^B f(x)x^{it}\,\mathrm{d}x \right| \leqslant \int_A^B |xf'(x)|\,\mathrm{d}x = \int_A^B f(x)\,\mathrm{d}x.$$

□

Lemma 5.3. *Let $n, K > 0$ and $f = \frac{\mathrm{erfc}^n}{\mathrm{erfc}^n + K}$. Let $q \in \mathbb{R}$. There exists $x_q > 0$ such that the function $g_q : x \in [x_q,\infty) \mapsto f(x)x^{q-1}$ is decreasing. Let $\widetilde{q} = \inf E_q$ where $E_q = \{c \geqslant 0 \mid \forall x \geqslant c : g_q'(x) < 0\}$. The function $q \mapsto \widetilde{q}$ is increasing and $\widetilde{q} \sim (q/(2n))^{1/2}$ as $q \to +\infty$.*

*Proof.* Let $n, K > 0$. Let $q \in \mathbb{R}$. As $f > 0$, the inequality $g_q'(x) \leqslant 0$ is equivalent to

$$\frac{f'(x)}{f(x)} \leqslant -\frac{q}{x}. \tag{5.2}$$

Let us compute $\frac{f'}{f}$. To do so, let $u = \mathrm{erfc}^n$, so that: $f = \frac{u}{u+K}$. We have:

$$\frac{f'}{f} = \frac{u'}{u}\frac{K}{u+K} = n\frac{\mathrm{erfc}'}{\mathrm{erfc}}\frac{K}{u+K}. \tag{5.3}$$

The derivative of erfc is given by: $\mathrm{erfc}'(x) = -2\pi^{-1/2}e^{-x^2}$. And, as $x$ tends to $+\infty$: $\mathrm{erfc}(x) \sim \pi^{-1/2}e^{-x^2}\frac{1}{x}$. Thus, as $x$ tends to $+\infty$:

$$\frac{f'(x)}{f(x)} \sim n\frac{\mathrm{erfc}'(x)}{\mathrm{erfc}(x)} \sim -2nx. \tag{5.4}$$

This asymptotics proves that the inequality (5.2) is satisfied for large enough values of $x$. As a consequence for every $q$ in $\mathbb{R}$ the set $E_q$ is not empty, which justifies the definition of $\widetilde{q}$. Note that the definition of $\widetilde{q}$ implies: $g_q'(\widetilde{q}) = 0$, that is: $\frac{f'(\widetilde{q})}{f(\widetilde{q})} = -\frac{q}{\widetilde{q}}$, using (5.2).

Let $q_1 \geqslant q_2$ be two real numbers. In order to show that $\widetilde{q_2} \leqslant \widetilde{q_1}$, it is enough to prove that $g_{q_1}'(\widetilde{q_2}) \geqslant 0$. This holds because:

$$g_{q_1}'(\widetilde{q_2}) = \widetilde{q_2}^{q_1-1}(f'(\widetilde{q_2})\widetilde{q_2} + f(\widetilde{q_2})q_1) \geqslant \widetilde{q_2}^{q_1-1}(f'(\widetilde{q_2})\widetilde{q_2} + f(\widetilde{q_2})q_2) = \widetilde{q_2}^{q_1-q_2}g_{q_2}'(\widetilde{q_2}) = 0.$$

To find the asymptotics on $\widetilde{q}$, let us recall a lower bound on $\operatorname{erfc}(x)$ for $x \geqslant 0$:

$$\left(x + (x^2+2)^{1/2}\right)^{-1} \leqslant \frac{1}{2}\pi^{1/2}\exp(x^2)\operatorname{erfc}(x).$$

As the function $u = \operatorname{erfc}^n$ takes its values in $(0,1]$, we have $\frac{nK}{1+K} \leqslant \frac{nK}{u+K} \leqslant n$. Consequently, thanks to (5.3):

$$-n\left(x + (x^2+2)^{1/2}\right) \leqslant \frac{f'(x)}{f(x)}. \tag{5.5}$$

Let $q > 0$ and set $x_q = \frac{q}{(2n)^{1/2}(n+q)^{1/2}}$. The inequality $-\frac{q}{x} \leqslant -n\left(x + (x^2+2)^{1/2}\right)$ is equivalent to $x\left(x + (x^2+2)^{1/2}\right) \leqslant \frac{q}{n}$. A simple computation shows that this inequality is satisfied for $x = x_q$ (and becomes and equality). Thanks to (5.5), we conclude that $x_q$ satisfies: $\frac{f'(x_q)}{f(x_q)} \geqslant -\frac{q}{x_q}$, which leads to $\widetilde{q} \geqslant x_q$, by definition of $\widetilde{q}$ and by (5.2). This last inequality implies that $\widetilde{q}$ tends to $+\infty$ as $q$ tends to $+\infty$. Finally we get the asymptotics for $\widetilde{q}$, using (5.4):

$$-2n\widetilde{q} \sim \frac{f'(\widetilde{q})}{f(\widetilde{q})} = -\frac{q}{\widetilde{q}}.$$

□

**5.4. Non linear approximation of the Hill function.** *Proof.* [Proof of Proposition 3.5]

From the estimate for erfc at $\infty$, given in the proof of Lemma 5.3, the function $J_2^{(a)}$ is in $\mathrm{L}_k^1$ for every $k > -1$. Thus $\mathcal{M}J_2^{(a)}$ is holomorphic on the right half-plane, see Proposition B.3 of the Appendix. Using Lemma 3.1 on the vertical line $-\frac{q+1}{2} + i\mathbb{R}$ with $-\frac{q+1}{2} > 0$, as for the proof of Proposition 3.4, it amounts to bounding $\left|s\mathcal{M}J_2^{(a)}(s)\right|$, from above or from below, on the vertical lines $q + i\mathbb{R}$, for $q > 0$.

The function $J_2^{(a)}$ is written:

$$J_2^{(a)}(x) = H_2(c_0\operatorname{erfc}(x)) = \frac{c_0^n\operatorname{erfc}(x)^n}{c_0^n\operatorname{erfc}(x)^n + K_{1/2}^n}\mathbf{1}_{x\geqslant\alpha} + K\mathbf{1}_{0<x<\alpha},$$

with: $\alpha = \operatorname{erfc}^{-1}(a/c_0)$. Then the Mellin transform of $J_2^{(a)}$ at $s = q + it$ is given by:

$$\mathcal{M}J_2^{(a)}(s) = K\int_0^\alpha x^{s-1}\,\mathrm{d}x + \int_\alpha^\infty f(x)x^{s-1}\,\mathrm{d}x = K\frac{\alpha^s}{s} + \int_\alpha^\infty f(x)x^{q-1}x^{it}\,\mathrm{d}x,$$

where $f(x) = \frac{\operatorname{erfc}(x)^n}{\operatorname{erfc}(x)^n + c_0^{-n}K_{1/2}^n}$.

For any $a \geqslant 0, q > 0$ and $s \in q + i\mathbb{R}$ we have:

$$\left|\mathcal{M}J_2^{(a)}(s)\right| \leqslant K\frac{\alpha^q}{q} + \int_\alpha^\infty f(x)x^{q-1}\,\mathrm{d}x,$$

which is finite.

Let $q > 0$. According to Lemma 5.3 the function $x \mapsto f(x)x^{q-1}$ is decreasing for $x \geqslant x_0$. Let $a < c_0\operatorname{erfc}(x_0)$. Then $\alpha = \operatorname{erfc}^{-1}(a/c_0) \geqslant x_0$. Let $g(x) = f(x)x^{q-1}\mathbf{1}_{x\geqslant\alpha}$.

For every $t \in \mathbb{R}$ : $\left[f(x)x^{it}\right]_{x_0}^{\infty} = 0$ because $f$ vanishes for $x \leqslant \alpha$ and $x_0 \leqslant \alpha$, and $g(x) \sim_{+\infty} \pi^{-n/2}x^{-n+q-1}e^{-nx^2}$. Then Corollary 5.2 can be applied to the function $g$, with $A = \alpha, B = +\infty$, for $s \in q + i\mathbb{R}$, to give:

$$\begin{aligned}\left|s\mathcal{M}J_2^{(a)}(s)\right| &\leqslant K\left|\alpha^s\right| + \frac{|s|}{\langle t\rangle}\langle t\rangle\left|\int_\alpha^\infty f(x)\, x^{s-1}\,\mathrm{d}x\right| \\ &\leqslant K\alpha^q + \max(1,q)\int_\alpha^\infty f(x)x^{q-1}\,\mathrm{d}x < \infty,\end{aligned}$$

because: $\frac{|s|}{\langle t\rangle} \in [q,1] \cup [1,q]$, either $q \leqslant 1$ or $q \geqslant 1$.

For small values of $a$, the first part dominates the second one. The same calculation as above leads to:

$$\left|s\mathcal{M}J_2^{(a)}(s)\right| \geqslant K\alpha^q - \max(1,q)\int_\alpha^\infty f(x)x^{q-1}\,\mathrm{d}x < \infty,$$

The last expression is equivalent to $K\alpha^q$ as $\alpha$ tends to $\infty$, therefore, it is positive for large values of $\alpha$.

□

**5.5. The Hill function.** Definition 5.4. *Let $\mathcal{S}[0,\infty)$ be the space of functions $f$ in $C^\infty([0,\infty),\mathbb{C})$ which statisfy:*

$$\forall j \in \mathbb{N}, k \in \mathbb{N} \quad \lim_{x\to\infty} f^{(j)}(x)x^k = 0.$$

If $f$ is a function of the Schwartz space $\mathcal{S}(\mathbb{R})$ then $f\mathbf{1}_{x\geqslant 0}$ is in $\mathcal{S}[0,\infty)$.

*Remark.* The converse is also true. Let $f$ be in $\mathcal{S}[0,\infty)$. By Borel's lemma there exists a function $g \in C^\infty(\mathbb{R})$ such that for every $k \in \mathbb{N}$: $f^{(k)}(0) = g^{(k)}(0)$. Then the function $\psi(g\mathbf{1}_{x<0} + f\mathbf{1}_{x\geqslant 0})$, where $\psi$ is a smooth function vanishing for $x < -1$ and equal to 1 for $x \geqslant 0$, is in $\mathcal{S}(\mathbb{R})$ and its restriction to $[0,\infty)$ is $f$.

Lemma 5.5. *Let $f \in \mathcal{S}[0,\infty)$. Its Mellin transform $\mathcal{M}f$ is holomorphic on the right half-plane, and:*

$$\forall q > 0\ \forall k \in \mathbb{N}\ \exists C \geqslant 0\ \forall t \in \mathbb{R} : |\mathcal{M}J_0(q+it)| \leqslant C\langle t\rangle^{-k},$$

*where $\langle t\rangle = (1+t^2)^{1/2}$.*

*Proof.* [Proof of Lemma 5.5] Let $f \in \mathcal{S}[0,\infty), q > 0$. By the definition of $\mathcal{S}[0,\infty)$ for every $l$ in $\mathbb{N}$ and $k > -1$, the function $x \mapsto x^k f^{(l)}(x)$ is in $\mathrm{L}^1$. Proposition B.3 of the Appendix implies that $\mathcal{M}f$ is holomorphic on the right half-plane.

Lemma 5.1 with $g(x) = f(x)x^{q-1}$ is written:

$$\mathcal{M}f(q+it) = \int_0^\infty f(x)x^{q-1}x^{it}\,\mathrm{d}x = \sum_{j=0}^{k-1}(-1)^j Q_j(t)\left[x^{j+1}g^{(j)}x^{it}\right]_0^\infty + (-1)^k Q_{k-1}(t)\int_0^\infty x^k g^{(k)}(x)x^{it}\,\mathrm{d}x,$$

where $Q_j(t) = \left(\prod_{l=0}^j 1+l+it\right)^{-1}$.

To prove the result it is enough to show that the terms between brackets vanish and that the last integral is finite.

Let $l, k \in \mathbb{N}$. By the Leibniz rule:

$$x^l g^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} f^{(k-j)}(x)(x^{q-1})^{(j)} x^l = \sum_{j=0}^{k} \binom{k}{j} (q-1)_{(j)} f^{(k-j)}(x)\, x^{q+l-1-j}.$$

For $l = k+1$ and for $x = 0$ this expression vanishes because $f^{(k-j)}(0)$ is finite and: $q+k-j \geqslant q > 0$. As $x$ tends to $\infty$ the expression tends to 0 as: $f^{(k-j)}(x)\, x^{q+k-j} \to 0$.

For $l = k$ this expression shows that the integral $\int_0^\infty x^k \left|g^{(k)}(x)\right| \mathrm{d}x$ is finite because for every $j \in \{0, \dots, k\}$: $x \mapsto x^{q-1+j} f^{(j)}(x)$ is in $\mathrm{L}^1$ because $q - 1 + j \geqslant q - 1 > -1$. Then:

$$|\mathcal{M}f(q+it)| \leqslant C\, |Q_{k-1}(t)| \sim_{t\to\infty} C\langle t\rangle^{-k}.$$

□

Lemma 5.6. *Let $J_0$ be the function defined by* (1.2), (2.4)*. For $n > 0$, the function $J_0$ is in $\mathcal{S}[0, \infty)$.*

The proof of Lemma 5.6 is given in Appendix A. Proposition 3.6 can be strenghened in Proposition 5.7 below (in the sense that Lemma 5.6 and Proposition 5.7 imply Proposition 3.6).

Proposition 5.7. *Let $q < -1$, $l$ in $\{0, 1, 2\}$ such that $J_l$, defined by* (2.1), (2.2), (2.4)*, is in $\mathcal{S}[0, \infty)$.*

- *If $\rho$ is in $\mathrm{L}^2_q$, there exists $C > 0$ such that:*

$$\|\mathrm{I}_l[\rho]\|_{\mathrm{L}^2_{\frac{q-3}{2}}} \leqslant C\, \|\rho\|_{\mathrm{L}^2_q}.$$

- *Furthermore assume that $\mu\left(\left\{s \in \frac{q+1}{2} + i\mathbb{R} \mid \mathcal{M}\rho(s) = 0\right\}\right) = 0$ where $\mu$ is the Lebesgue measure on $\frac{q+1}{2} + i\mathbb{R}$. There is no $k \in \mathbb{N}$, $C > 0$ such that:*

$$\left\|(\mathrm{I}_l[\rho])^{(k)}\right\|_{\mathrm{L}^2_{2k+\frac{q-3}{2}}} \geqslant C\, \|\rho\|_{\mathrm{L}^2_q}.$$

*Proof.* [Proof of Proposition 5.7] It is enough to show that the two inequalities:

$$\left\|(\mathrm{I}[\rho])^{(k)}\right\|_{\mathrm{L}^2_{2k+\frac{q-3}{2}}} \leqslant C\, \|\rho\|_{\mathrm{L}^2_q} \qquad \left(\text{resp.} \ \geqslant C\, \|\rho\|_{\mathrm{L}^2_q}\right) \tag{5.6}$$

and

$$\left|\prod_{j=0}^{k-1}(s+2j)\ \mathcal{M}J_l(s)\right| \leqslant C \quad (\text{resp.} \ \geqslant C) \qquad \text{on} \qquad -\frac{q-1}{2} + i\mathbb{R} \tag{5.7}$$

are equivalent.

This equivalence having been proved, Lemma 5.5 implies that, for any $k \in \mathbb{N}$, the inequality $\leqslant$ is true and the inequality $\geqslant$ is not satisfied.

The implication ($\Leftarrow$) is Lemma 3.1. For the other implication ($\Rightarrow$), let us rewrite equations (5.6), (5.7) in equivalent forms (*cf.* proof of Lemma 3.1). Respectively, we have:

$$\|(s-k)_k\ \mathcal{M}J_l(-2(s-k))\ \mathcal{M}\rho(2(s-k)+1)\|_{\mathrm{L}^2\left(\frac{q-1}{4}+k+i\mathbb{R}\right)} \leqslant C\, \|\mathcal{M}\rho(2(s-k)+1)\|_{\mathrm{L}^2\left(\frac{q-1}{4}+k+i\mathbb{R}\right)} \tag{5.8}$$

$\left(\text{resp.} \geqslant C \left\| \mathcal{M}\rho(2(s-k)+1) \right\|_{\mathrm{L}^2\left(\frac{q-1}{4}+k+i\mathbb{R}\right)}\right)$ and

$$|(s-k)_k \mathcal{M}J_l(-2(s-k))| \leqslant C \quad (\text{resp. } \geqslant C) \qquad \text{on} \qquad \frac{q-1}{4}+k+i\mathbb{R}. \tag{5.9}$$

If (5.8) is satisfied then (5.9) is satisfied on the set:

$$\left\{ s \in \frac{q-1}{4}+k+i\mathbb{R} \mid |\mathcal{M}\rho(2(s-k)+1)| > 0 \right\},$$

that is (5.7) is satisfied on the set:

$$\left\{ s \in -\frac{q-1}{2}+i\mathbb{R} \mid |\mathcal{M}\rho(-s+1)| > 0 \right\}.$$

As $q<1$, and as $J_l$ is in $\mathcal{S}[0,\infty)$, the function $J_l$ is in $\mathrm{L}^1_{-\frac{q+1}{2}}$. By Theorem B.2 (in the Appendix), $\mathcal{M}J_l$ is continuous on $-\frac{q-1}{2}+i\mathbb{R}$. The left-hand side of (5.7) is then a continuous function of $s$. As $\mu\left(\left\{s \in \frac{q+1}{2}+i\mathbb{R} \mid \mathcal{M}\rho(s)=0\right\}\right)=0$, the inequality (5.7) holds for every $s$ in $-\frac{q-1}{2}+i\mathbb{R}$, which concludes the proof. ☐

**6. Conclusion.** We focussed on the problem of finding the spatial distribution of CNG ion channels from the experimental current data, following [4, 7, 9, 8, 2]. The self-similar structure of the integral inverse problem (1.1), (1.2), (1.3) allowed us to use the Mellin transform, and to obtain a thorough comprehension of it. It permited us to reduce its study to estimating some Mellin transform, on vertical lines of the complex plane. To do so, explicit computations were carried out using techniques inspired by the stationary phase method.

As a result the inverse problem studied has been shown not to be controlable in some weighted $\mathrm{L}^2$ spaces. The kernel of the integral operator is smooth and the associated inverse problem has an infinite degree of ill-posedness. This conclusion could probably be linked to the fact that for the original problem, introduced in [8], certain numerical computations indicate that it is ill-conditioned. We also introduced a better approximation than the one already studied, for which we performed numerical simulations and provided estimates. In this case the kernel of the integral operator is at most continuous and the associated inverse problem has a finite degree of ill-posedness (which is 1). The profiles obtained from the experimental current data consolidate the ones obtained in [8], [7], [9].

To go further, one could study the inverse problem (1.1), (1.2) where $c$ is defined as the solution of the linear heat equation

$$\begin{cases} \partial_t c - D\,\partial_{xx} c &= 0, \quad t>0,\, x\in(0,L), \\ c(t,0) &= c_0, \quad t>0, \\ \partial_x(t,L) &= 0, \quad t>0, \\ c(0,x) &= 0, \quad x\in(0,L). \end{cases}$$

If the Hill function (1.2) is changed into its Taylor polynomial extension around $c_0$, and if this polynomial has a degree $m \leqslant 8$, then [4] proves that the resulting integral operator I is identifiable in $\mathrm{L}^2$. To go further, a first step, could be to study this problem without approximating the Hill function. Studying the complete inverse problem for the original model could be a last step. It would seem that these two problems cannot be solved using the Mellin transform technique alone, as no self similar structure is involved.

**Appendix A. Proof of Lemma 5.6.**

We recall the Faà di Bruno formula, first proved by L. F. A. Arbogast see [1].

$$(f \circ g)^{(k)} = \sum_{\substack{(m_1,\dots,m_k)\in\mathbb{N}^k \\ \sum_{j=1}^k j m_j = k}} \frac{k!}{\prod_{j=1}^k m_j!\, j!^{m_j}}\; f^{\left(\sum_{j=1}^k m_j\right)} \circ g \prod_{j=1}^{k} \left(g^{(j)}\right)^{m_j}.$$

This formula allows us to make explicit computations, to prove that the functions appearing in Lemmas 5.6 are in $\mathcal{S}[0,\infty)$.

Lemma A.1. *The function* erfc *is of the class $C^\infty$ on $[0,\infty)$. For every $k \in \mathbb{N}^*$:*

$$\mathrm{erfc}^{(k)}(x) = P_k(x)\exp(-x^2),$$

*where $P_k$ is a polynomial which has a degree of $k-1$ whose leading coefficient is $\pi^{-1/2}(-2)^k$. In particular, as $x$ tends to $\infty$ :*

$$\mathrm{erfc}^{(k)}(x) \sim \pi^{-1/2}(-2)^k x^{k-1} e^{-x^2}.$$

*Proof.* The proof is by induction on $k \in \mathbb{N}^*$. By the definition of erfc: $\mathrm{erfc}'(x) = -2\pi^{-1/2}\exp(-x^2)$ and with simple calculations we have:

$$P_{k+1} = -2XP_k + P_k'.$$

□

Lemma A.2. *The function $H$ defined by* (1.2) *is of the class $C^\infty$ on $[0,\infty)$. For $n \geqslant 0$ it satisfies for every $k \in \mathbb{N}^*$:*

$$H^{(k)}(x) = \frac{P_k(x)}{(x^n + K_{1/2}^n)^{k+1}},$$

*where* $P_k(x) = (-1)^{k-1}(n-k-1)_k K_{1/2}^n\, x^{(n-1)k} + \cdots + (n)_k K_{1/2}^{nk}\, x^{n-k}$.

*In particular:*

$$H^{(k)}(x) \sim_0 K_{1/2}^{nk}(n)_k\, x^{n-k}.$$

*Proof.* The proof is by induction on $k \in \mathbb{N}^*$. Simple calculations lead to: $P_1(x) = nK_{1/2}^n x^{n-1}$ and:

$$P_{k+1}(x) = x^n P_k'(x) - (k+1)nx^{n-1}P_k(x) + K_{1/2}^n P_k'(x).$$

It follows that the leading term in $P_k(x)$ as $x$ tends to $+\infty$ is of the form $a_k x^{(n-1)k}$ where $a_k$ satisfies:

$$a_1 = nK_{1/2}^n, \qquad a_{k+1} = a_k((n-1)k - n(k+1)) = -a_k(n+k).$$

As $n \geqslant 0$, the leading term in $P_k(x)$ as $x$ tends to 0 is of the form $b_k x^{n-k}$ where $b_k$ satisfies:

$$b_1 = nK_{1/2}^n, \qquad b_{k+1} = b_k(n-k)K_{1/2}^n.$$

□

LEMMA A.3. *For $n > 0$, the function $J_0$ defined by (1.2), (1.3), (2.4) is in $\mathcal{S}[0,\infty)$.*

*Proof.* The function is of the class $C^\infty(0,+\infty)$ because erfc is of this class and $n > 0$. For every integer $k \in \mathbb{N}$, $H^{(k)}(c_0)$ and $\operatorname{erfc}^{(k)}(0)$ are finite, thus for every $k \in \mathbb{N}$, $J_0^{(k)}(0)$ is finite.

Applying the Faà di Bruno formula one gets, for $x \geqslant 0$:

$$J_0^{(k)}(x) = \sum_{\substack{(m_1,\dots,m_k)\in\mathbb{N}^k \\ \sum_{j=1}^k jm_j = k}} \frac{k!}{\prod_{j=1}^k m_j!\, j!^{m_j}}\, H^{\left(\sum_{j=1}^k m_j\right)}(c_0 \operatorname{erfc}(x)) \prod_{j=1}^k \left(c_0 \operatorname{erfc}^{(j)}(x)\right)^{m_j}.$$

As $x$ tends to $+\infty$, from the previous lemmas, we have, for $S = \sum_{j=1}^k m_j$:

$$H^{(S)}(c_0 \operatorname{erfc}(x)) \sim_{+\infty} K_{1/2}^{-n}(n)_S (c_0 \operatorname{erfc}(x))^{n-S} \sim_{+\infty} K_{1/2}^{-n}(n)_S (c_0 \pi^{-1/2} x^{-1} \exp(-x^2))^{n-S}.$$

Then, for $\sum_{j=1}^k jm_j = k$ and $\sum_{j=1}^k m_j = S$:

$$\begin{aligned} & H^{\left(\sum_{j=1}^k m_j\right)}(c_0 \operatorname{erfc}(x)) \prod_{j=1}^k \left(c_0 \operatorname{erfc}^{(j)}(x)\right)^{m_j} \\ \sim_{+\infty} & K_{1/2}^{nS}(n)_S (c_0 \pi^{-1/2} x^{-1} \exp(-x^2))^{n-S} \prod_{j=1}^k c_0^{m_j} (\pi^{-1/2}(-2)^j x^{j-1} \exp(-x^2))^{m_j} \\ = & (-2)^k K_{1/2}^{nS}(n)_S c_0^n \pi^{-n/2} x^{k-n} \exp(-nx^2) \end{aligned}$$

Let us denote:

$$C(n,k) = (-2)^k k! c_0^n \pi^{-n/2} \sum_{\substack{(m_1,\dots,m_k)\in\mathbb{N}^k \\ \sum_{j=1}^k jm_j = k}} (n)_{\sum_{j=1}^k m_j} K_{1/2}^{n\sum_{j=1}^k m_j} \left(\prod_{j=1}^k m_j!\, j!^{m_j}\right)^{-1}.$$

If $C(n,k) \neq 0$, the previous calculations lead to:

$$J_0^{(k)}(x) \sim C(n,k) x^{k-n} \exp(-nx^2).$$

If $C(n,k) = 0$: $J_0^{(k)}(x) = o\left(x^{k-n}\exp(-nx^2)\right)$. In both cases, for every $j \in \mathbb{N}$: $\left(J_2^{(c_0)}\right)^{(k)} = o(x^j)$, which concludes the proof.

□

**Appendix B. The Mellin transform.**

The construction of the Mellin transform on $i\mathbb{R}$ can be done in the general context of the Fourier transform on a locally compact abelian group, see [14]. Here we consider the *multiplicative* abelian group $G = (0, \infty)$ (with unit 1), equipped with the topology inherited from $\mathbb{R}$ and with the *Haar measure* $\frac{\mathrm{d}x}{x}$ (that is the unique measure on $G$, up to a positive multiplicative constant, which is translation-invariant). It is easy to show that the dual group $\Gamma$ of all the characters of $G$ with the Gelfand topology is isomorphic to $i\mathbb{R}$ with the topology inherited from $\mathbb{C}$ via $i\mathbb{R} \to \Gamma \quad it \mapsto (x \mapsto x^{-it})$. In this general context the $\mathrm{L}^1$ and $\mathrm{L}^2$ theories can be built, with the same results, as the Fourier transform on the topological group $(\mathbb{R}^n, +, \mathrm{d}x)$. The extension of the Mellin transform to a vertical strip $q + i\mathbb{R}$ of the complex plane $\mathbb{C}$ is simply obtained by defining $\mathcal{M}f(q + it) = \mathcal{M}g(it)$ with $g(x) = x^q f(x)$.

DEFINITION B.1. *Let $f$ be in $\mathrm{L}^1_q$. The Mellin transform of $f$ is a complex valued function defined on the vertical strip $q^* + i\mathbb{R}$, where $q^* = q + 1$; writing $s - 1 = q + it$ for a real number $t$ it is defined by:*

$$\mathcal{M}f(s) = \int_0^\infty x^s f(x) \frac{\mathrm{d}x}{x}.$$

THEOREM B.2 (Riemann-Lebesgue). *The Mellin transform is a linear continuous map of $\mathrm{L}^1_q$ into $C_0(q^* + i\mathbb{R}) \subset \mathrm{L}^\infty(q^* + i\mathbb{R})$, where $q^* = q + 1$, its norm is 1.*

PROPOSITION B.3. *If $f$ is in $L^1_q$ for every real number $q$ in $(a, b)$ then its Mellin transform $\mathcal{M}f(s) = \int_0^\infty f(x)x^{s-1}\,\mathrm{d}x$ is holomorphic in the strip $S = \{s \in \mathbb{C} \mid a + 1 < \operatorname{Re} s < b + 1\}$.*

PROPOSITION B.4.

| **function** | **Mellin transform** |
|---|---|
| $f(at),\ a > 0$ | $a^{-s}\mathcal{M}f(s)$ |
| $f(t^a),\ a \neq 0$ | $\lvert a\rvert^{-1}\mathcal{M}f(a^{-1}s)$ |
| $f^{(k)}(t)$ | $(-1)^k(s-k)_k\mathcal{M}f(s-k)$ |

***Example.***

- For $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{+\infty}\exp(-t^2)\,\mathrm{d}t$ one obtains for $\operatorname{Re} s > 0$: $\mathcal{M}\operatorname{erfc}(s) = \frac{1}{\sqrt{\pi}}\frac{1}{s}\Gamma\left(\frac{s+1}{2}\right)$.
- If $H_0$ is the Hill function $H_0(x) = \frac{x^n}{x^n + K^n_{1/2}}$ then for $s \notin n\mathbb{Z}$: $\mathcal{M}H_0(s) = -\frac{\pi}{n}\frac{K^s_{1/2}}{\sin\frac{\pi s}{n}}$.

THEOREM B.5 (inversion theorem). *If $f$ is in $\mathrm{L}^1_q$ and if $\|\mathcal{M}f\|_{\mathrm{L}^1(q^*+i\mathbb{R})}$ is finite, where $q^* = q + 1$, then almost everywhere $x > 0$ we have:*

$$f = \mathcal{M}^{-1}_{q^*}(\mathcal{M}f),$$

*where*

$$\mathcal{M}^{-1}_q\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\varphi(q + it)x^{-(q+it)}\,\mathrm{d}t.$$

Definition B.6. *For two functions $f, g$ we define the multiplicative convolution $f * g$ as:*

$$(f * g)(x) = \int_0^\infty f(y) g\left(\frac{x}{y}\right) \frac{\mathrm{d}y}{y}.$$

Proposition B.7.

$$\mathcal{M}(f * g)(s) = \mathcal{M}f(s)\mathcal{M}g(s),$$

*whenever this expression is well defined.*

Proposition B.8. *For a function $f$ in $\mathrm{L}^1_{q-1} \cap \mathrm{L}^2_{2q-1}$ we have:*

$$\|f\|_{\mathrm{L}^2_{2q-1}} = (2\pi)^{-1/2} \|\mathcal{M}f\|_{\mathrm{L}^2(q+i\mathbb{R})}.$$

As the subspace $\mathrm{L}^1_{q-1} \cap \mathrm{L}^2_{2q-1}$ is dense in $\mathrm{L}^2_{2q-1}$ this identity allows us to extend the Mellin transform to $\mathrm{L}^2_{2q-1}$.

Theorem B.9 (Plancherel transform). *According to the previous formula the Mellin transform can be extended, in a unique manner, to an isometry (up to the multiplicative constant $(2\pi)^{-1/2}$) of $\mathrm{L}^2_q$* onto $\mathrm{L}^2(q^* + i\mathbb{R})$, *where $q^* = \frac{q+1}{2}$.*

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