Theoretical and Numerical Study of the Problem of Abort Landing in the Presence of Windshear

Ivan Cadena Guaqueta

THEORETICAL AND NUMERICAL STUDY OF THE PROBLEM OF ABORT LANDING IN THE PRESENCE OF WINDSHEAR

IVÁN CADENA GUÁQUETA

Master Project[1] *under the direction of J.F. Bonnans*[2 3]

Abstract. We analyze both theoretically and numerically the problem of abort landing in the presence of windshear. A pertinent model of optimal control is constructed in order to allow numerical simulations through an open source software. We verify that the numerical results correspond to the mathematical theory and in particular we analyze as much as possible the junction conditions in presence of state constraints. The code written to analyze the problem will be available as a contribution in `bocop.org`.

1. Introduction

Windshear is a difference in speed or direction of wind in two points sufficiently close to the atmosphere. It is a very dangerous problem when the aircraft is close to the ground at a low speed, that is at take-off and landing. This phenomenon can lead to a sudden lose of lift, resulting in a crash of the plane with the ground if the pilot does not have the time to react or if his efforts were not successfull. Between 1964 and 1985, windshear was directly or undirectly related to al least 26 major civil air disasters, leaving 820 casualties. Among these accidents, 15 occured during take-off, 3 during flight and 8 during landing. Having this context in mind, we would like to address mathematically this situation in order to understand which flight strategy the aircraft should adopt whenever windshear is present.

We rely on the models proposed in articles [4, 7] where optimal control techniques are used. In the following sections we present the mathematical model that describes a windshear situation while the aircraft is landing (Section 2), we state the necesarry optimality conditions for this specific problem (Section 3), we find with a numerical procedure a control that garantees a maximum and positive altitude of the plane with respect to the ground (Section 4) and we make an analysis betw.een the theory and the numerical results that have been obtained (Section 5).

2. Model

We consider a bidimensional model with coordinates (x, h). More precisely, we analyze the plane landing in the presence of windshear. We suppose that the decision taken by the aircraft has been to abort landing and regain altitude in order to retry landing.

2.1. **Wind structure.** Our model for the windshear phenomenon is inspired from [4], where nonsmooth are used. However, our numerical method only requires smooth functions. Thus, we consider the following representation of the wind, where the functions are smooth approximations of the ones used in reference [4]:

$$W_x(x) = -50\exp[-\beta_1(x)^2], \tag{1}$$

$$W_h(x) = 0.02\beta_2(x) - 50 \tag{2}$$

These functions are built through auxiliary fonctions β_1 and β_2 that stretch the gaussian W_x at its lowest point in order to form a valley, and to make W_h constant after 5000ft. These functions rely on the smooth

[1]Within the framework of the Master of Applied Mathematics and Information Sciences at École Centrale Paris, from 7/10/2014 to 7/02/2015

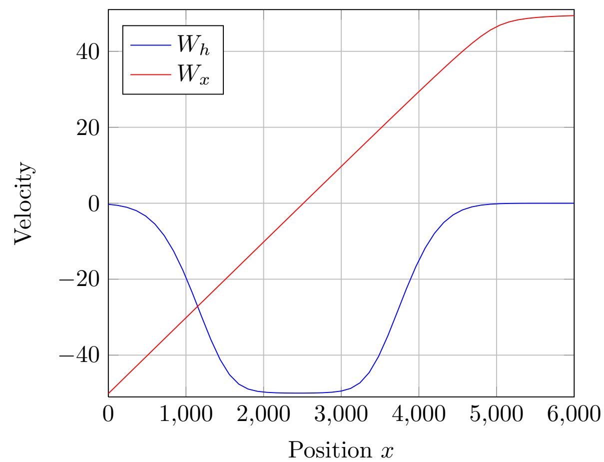
[2]CMAP, Centre de Mathématiques Appliquées - École Polytechnique

[3]INRIA Saclay - Île de France COMMANDS

approximation of the absolute value:

$$|x| \approx |x|_\epsilon := \sqrt{x^2 + \epsilon^2}.$$

This choice models a wind structure presenting windshear around 2000ft and 3000ft in the x coordinate.



2.2. **Aerodynamic Forces.** There are three forces that take part in the aerodynamic of the aircraft. They are the thrust T, the lift L and the drag D. Their equations, that depend on the speed of the plane v and the angle of attack α are the following:

$$T = A_0 + A_1 v + A_2 v^2, \tag{3}$$

$$L = (1/2)\rho S(C_0 + C_1\alpha)v^2, \tag{4}$$

$$D = (1/2)\rho S(B_0 + B_1\alpha + B_2\alpha^2)v^2 \tag{5}$$

The constants ρ are S correspond to the air density and the surface of the aircraft.

2.3. **Dynamics.** Once the wind structure and the areodynamic forces have been set, we can describe the dynamics of the aircraft where the position (x, h), the speed v, the path inclination γ and the angle of attack α are the state variables and the variation of the angel of attack u is the control.

$$\dot{x} = v\cos\gamma + W_x, \tag{6}$$

$$\dot{h} = v\sin\gamma + W_h, \tag{7}$$

$$\begin{aligned} \dot{v} &= T/m\cos(\alpha+\delta) - D/m - g\sin\gamma \\ &\quad -\dot{W}_x\cos\gamma - \dot{W}_h\sin\gamma, \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{\gamma} &= T/(mv)\sin(\alpha+\delta) + L/(mv) - g/v\cos\gamma \\ &\quad +(\dot{W}_x\sin\gamma - \dot{W}_h\cos\gamma)/v, \end{aligned} \tag{9}$$

$$\dot{\alpha} = u. \tag{10}$$

We will require, as in the reference [4] that the angle of attack and its variation are bounded in the following fashion:

$$\alpha \le \alpha_M, \tag{11}$$

$$|u| \le u_M. \tag{12}$$

The termes for the thrust inclination δ, the gravity constant g, α_M and u_M are fixed values.
Choosing $\dot{\alpha}$ as the control, we consider the case of a commercial airplane where it is forbidden to change baldly the angle of attack of the aircraft.

2.4. **Cost Function and Problem.** The danger of landing or taking off in the presence of windshear is to crash into the ground, and so in this situations the pilot has to maximize the altitude of the aircraft. In order to analyze this idea we introduce the following cost to be minimized:

$$J(u) := -\theta - \eta h(\tau), \quad \eta \geq 0, \tag{13}$$

where time varies between 0 and τ and the value θ verifies:

$$\theta \leq h(t) \text{ p.p } t \in [0, \tau]. \tag{14}$$

Here the idea of maximization of the altitude is captured by the value θ in the cost. However, if we do not add more terms the problem would have infinite solutions: after having reached the minimum altitude the aircraft does not have any criteria to set the rest of the trajectory. Adding the term $h(\tau)$ corresponding to the final altitude, we restrict ourselves to a much specific class of trajectories. On the other hand, the term η allows the values θ and $h(\tau)$ to have the same order of magnitude.

We fix the values of all the variables at $t = 0$ and the value of the path inclination and the final time, setting the initial conditions and a condition of stability at the end of the trajectory. This can be represented as follows:

$$x(0) = x_0, h(0) = h_0, v(0) = v_0 < 0, \tag{15}$$
$$\gamma(0) = \gamma_0, \alpha(0) = \alpha_0.$$
$$\gamma(\tau) = \gamma_\tau. \tag{16}$$

In summary, the optimal control problem to be studied is:

$$\min\ J(u) \text{ subject to (6)-(12), and (14)-(16).} \tag{17}$$

3. Necessary Optimality Conditions

This problem belongs to the class of optimal control with constraints on both the state and control. For an extensive treatment of optimal control theory reference [3] can be consulted. First of all, we verify that the problem that we are analysing is qualified. To check this, we write the problem in a fashion that allows us to assure the constraint qualification.

We note the C^∞ application that shows the solution of the ordinary differential equation problem that corresponds to the dynamics (6)-(12), and to the boundary conditions (14) - (16) like this:

$$u \longmapsto (\ldots, h_u, \alpha_u, \ldots).$$

With this notation the optimal control problem (17) is equivalent to:

$$\min\ \{\mathfrak{F}(u,\theta) \mid \quad (u,\theta) \in W, \quad \mathfrak{G}(u,\theta) \in K_1 \times K_2\}$$

Where the sets are:

$$\begin{aligned} W &= L^\infty[0,\tau] \times \mathbb{R}, \\ K_1 &= \{u \in L^\infty[0,\tau];\ |u| \leq u_M\}, \\ K_2 &= \{(w,v) \in C_-[0,\tau] \times C[0,\tau];\ v \leq \alpha_M\}, \end{aligned}$$

and the functions,

$$\begin{aligned} \mathfrak{F}(u,\theta) &= -\theta - \eta h_u(\tau), \\ \mathfrak{G}(u,\theta) &= (u, \theta - h_u, \alpha_u). \end{aligned}$$

with this structure we assure that the problem is qualified thanks to Corollary 2.101 in [1].

We look then for a trajectory u whose associated state $(x, h, v, \gamma, \alpha)$ and costate (μ, p) verify:

Hamiltonian inequality:

$$p_\alpha(t)u(t) = \min\ \{p_\alpha(t)v; |v| \leq u_M\} \text{ p.p } t \in [0, \tau] \tag{18}$$

Costate dynamics: Writing $p = (p_i)$ and $f = (f_i)$, where i makes reference to the states and f_i is the dynamic of the state i, the following relations must hold:

$$\begin{aligned}
(19)\qquad -\dot{p}_i &= \sum_j p_j \frac{\partial f_j}{\partial i}, \quad i \in \{x, v, \gamma\} \\
(20)\qquad -dp_h &= -d\mu_h \\
(21)\qquad -dp_\alpha &= p_v \frac{\partial f_v}{\partial \alpha} + p_\gamma \frac{\partial f_\gamma}{\partial \alpha} + d\mu_\alpha \\
(22)\qquad p_h(\tau) &= -\eta, \\
(23)\qquad p_i(\tau) &= 0, \quad i \in \{x, v, \alpha\}
\end{aligned}$$

Complementarity conditions: $\mu = (\mu_h, \mu_\alpha)$:

$$(24)\qquad d\mu_h \geq 0, \qquad \int_0^\tau [\theta - h(t)] d\mu_h(t) = 0,$$

$$(25)\qquad d\mu_\alpha \geq 0, \qquad \int_0^\tau [\alpha_M - \alpha(t)] d\mu_\alpha(t) = 0.$$

At last, we do not have to specify non-nullity conditions because we have the constraint qualification.

4. Numerical Solution

We take the problem (17) stated in the last section with the following numerical values:

Constants for the aerodynamic forces

$$\begin{aligned}
A_0 &= 4.456 \cdot 10^4 \text{ lb} \\
A_1 &= -2.4 \cdot 10^1 \text{ lb sec ft}^{-1} \\
A_2 &= 1.442 \cdot 10^{-2} \text{ lb sec}^2\text{ft}^{-2} \\
B_0 &= 1.552 \cdot 10^{-1} \\
B_1 &= 1.2369 \cdot 10^{-1} \text{ rad}^{-1} \\
B_2 &= 2.4203 \cdot 10^1 \text{ rad}^{-2} \\
C_0 &= 7.125 \cdot 10^{-1} \\
C_1 &= 6.0877 \cdot 10^1 \text{ rad}^{-1} \\
\rho &= 2.203 \cdot 10^{-3} \text{ lb sec}^2\text{ft}^{-4} \\
S &= 1.56 \cdot 10^3 \text{ ft}^2
\end{aligned}$$

Design Constants

$$\begin{aligned}
\eta &= 10^{11} \\
\epsilon &= 0.1 \\
\tau &= 40.
\end{aligned}$$

Constants for the dynamics and constraints

$$\begin{aligned}
m &= 4.66 \cdot 10^3 \text{ lb} \\
g &= 3.22 \cdot 10^1 \text{ ft sec}^{-2} \\
\delta &= 3.5 \cdot 10^{-2} \text{ rad} \\
\alpha_M &= 3 \cdot 10^{-1} \text{ rad sec}^{-1} \\
u_M &= 5.2 \cdot 10^{-2} \text{ rad}
\end{aligned}$$

Fixed values at the boundaries

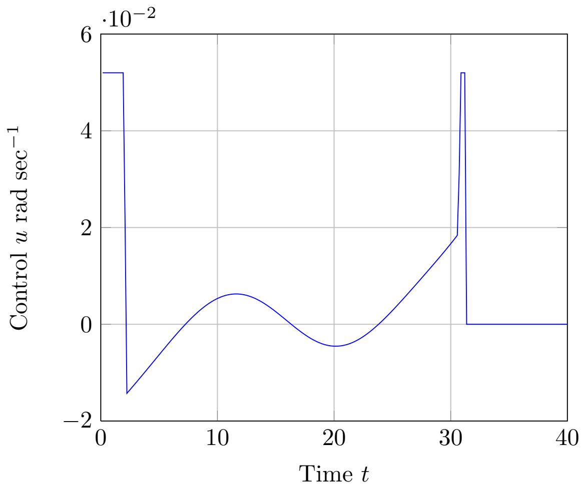
$$\begin{aligned}
x_0 &= 0 \cdot 10^1 \text{ ft} \\
h_0 &= 600 \text{ ft} \\
v_0 &= 240 \text{ ft sec}^{-1} \\
\gamma_0 &= -3.9 \cdot 10^{-2} \text{ rad} \\
\gamma_\tau &= 1.3 \cdot 10^{-1} \text{ rad} \\
\alpha_0 &= 1.28 \cdot 10^{-1} \text{ rad}
\end{aligned}$$

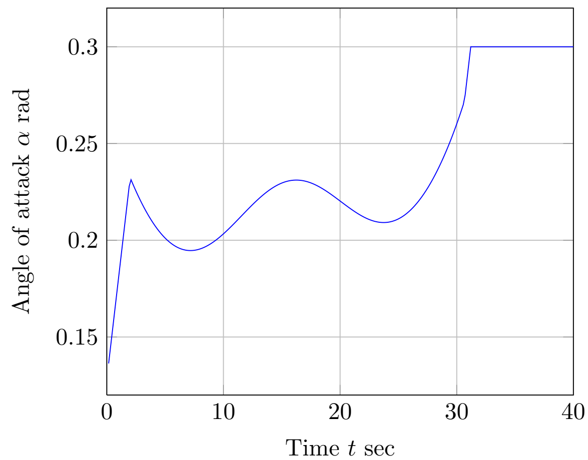
All these values were taken from reference [4].
We solve numerically the problem with the open-source software BOCOP [8]. As parameters for the numerical procedure we take 175 discretization points, with an implicit Euler method and an intial value for the control $u = u_M = 0.035$.

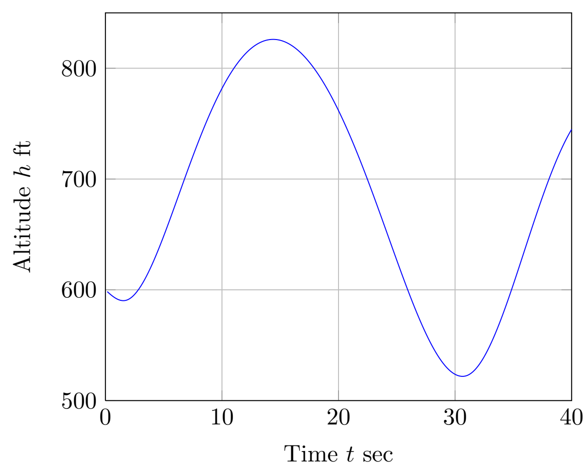
We find that the control follows the following structure:

- Within the first seconds of the trajectory the aircraft has to maximize the variation of the angle of attack, that is the control. This has to be done in order to compensate the upcoming force of the windshear.

- Once the aircraft has attained the maximum and constant vertical wind force, the control has to be near zero for the trajectory to be stable.
- In the region where the windshear loses magnitude, the aircraft should maximize once again the control in order to gain altitude taking advantage of the weak force of the wind. Then when the maximum allowed value of the angle of attack is reached, the control has to be zero.



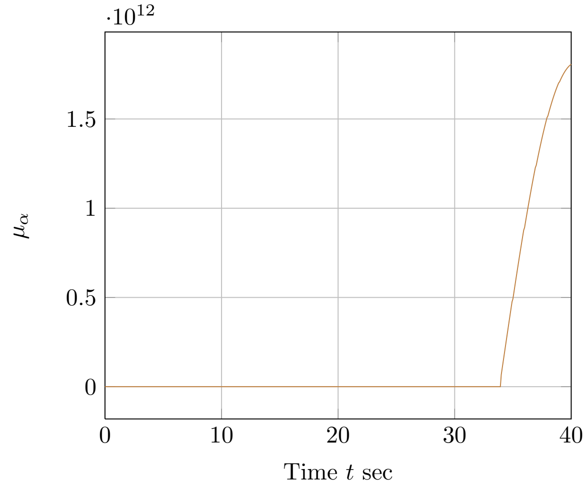




We find that the minimum altitude is 562ft, reached around second 30.65.

5. Analysis

We verify that the numerical results correspond with the underlying theory. For ease of reference we denote by $t_1 < t_2 < t_3$ the points where the control u is discontinuous. Concerning the measures of the constraints of the states, we find the following numerical results: 1) $\mu_h = c\delta_{t_2}, c > 0$, that is, it is a function with zero value except at time t_2 where the minimum altitude is reached, and 2) μ_α is zero valued only before t_3 where a possible discontinuity might appear. Its graphic is shown below:



Define the fonction of constraints ζ and ξ, where y is the state of the system as follows:

$$\zeta(y) = \theta - h, \quad \xi(y) = \alpha - \alpha_M.$$

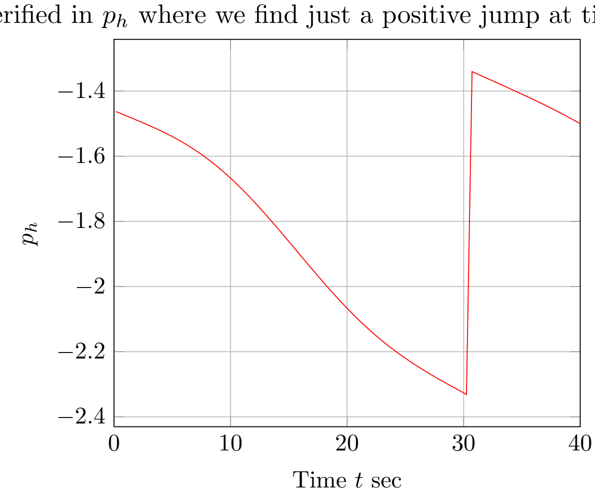
The costate dynamics show that the possible jumps of the costate satisfy

$$-[p(t)] = [\mu_h(t)]\nabla\zeta(y(t)) + [\mu_\alpha(t)]\nabla\xi(y(t)).$$

This expression yields both

$$[p_\alpha(t)] = -[\mu_\alpha(t)], \quad \text{and} \quad [p_h(t)] = [\mu_h(t)]. \tag{26}$$

Hence, the continuity of costates p_h and p_α will depend on the continuity of the corresponding measures. This behaviour is indeed verified in p_h where we find just a positive jump at time t_2:



This allows us to conclude that the minimal altitude is just attained in one point, otherwise we would have seen jumps in other points in both p_h and μ_h.

It is clear that μ_α is continuous in every point except possibly at t_3. The following analysis solves this issue.

Lemma 1. *In the current set-up, we have* $[\mu_\alpha(t_3)] = 0$.

Proof. By (18), and since our problem is affine in the control we have the following relation between the jumps of the switching function $\Xi = p_\alpha$ and u, for $t \in (0, \tau)$:

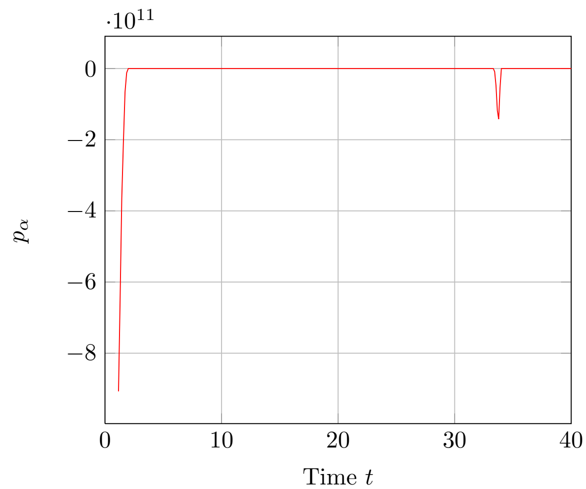
$$[p_\alpha(t)][u(t)] \leq 0.$$

However, using equation (26) we get:

$$[\mu_\alpha(t_3)][u(t_3)] \geq 0,$$

and since by inspection we have $[u(t_3)] < 0$, it must be the case that $[\mu_\alpha(t_3)] = 0$. $\square$

In the case of a scalar state constraint and of a Hamiltonian affine with respect to the an m-dimensional control similar results were obtained in [2], so the above lemma can be viewed as a partial extension of Lemme 8.4 in the former reference, and a special case of Lemma 4.1 in [6].
This analysis has shown that both p_α and μ_α are continuous in $(0, \tau)$. The following figure shows the continuous structure of p_α:



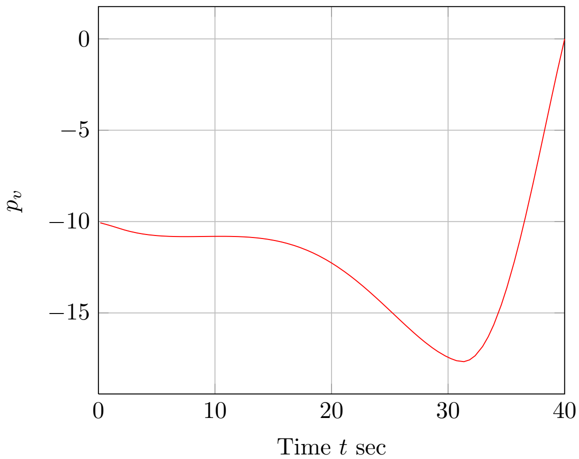
The observed structure of the previous function, negative-zero-negative-zero, corresponds to the obtained conditions of the control u in order for the Hamiltonian inequality to hold: whenever Ξ is negative, u is maximum-valued and whenever Ξ is zero, u has no restrictions.
We can examine, as well, the singular arc studying the switching function. We know that differentiating an even number of times Ξ we can recover an expression for u. Indeed in the first derivative, where $D_\alpha = \partial D / \partial \alpha$,

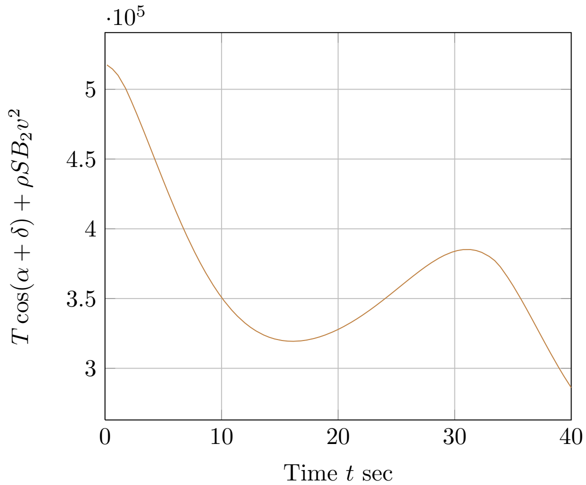
$$\begin{aligned} \dot{\Xi} &= -p_v \frac{\partial f_v}{\partial \alpha} - p_\gamma \frac{\partial f_\gamma}{\partial \alpha} \\ &= -p_v \left[-\frac{T}{m} \sin(\alpha + \delta) - \frac{D_\alpha}{m} \right] - p_\gamma \left[\frac{T}{mv} + \frac{L_\alpha}{mv} \right] \end{aligned}$$

there is no dependence on u. To analyze the second derivative, since $u = \dot{\alpha}$, we can differentiate once again to obtain:

$$\begin{aligned} \ddot{\Xi} &= \frac{\partial \dot{\Xi}}{\partial \alpha} \dot{\alpha} + \Phi \\ &= -\frac{p_v}{m} \left[T \cos(\alpha + \delta) + \rho S B_2 v^2 \right] u + \Phi, \end{aligned}$$

where the terme Φ is independant of u. We can check numerically that the coefficient of the control u has nonzero values, as shown by the following two figures:





References

[1] J.F. Bonnans, A. Shapiro. *Perturbation Analysis of Optimization Problems.* Springer, 2000 edition.

[2] J.F. Bonnans, *Commande Optimale.* Article pour l'encyclopédie des techniques de l'ingénieur. To appear.

[3] J.F. Bonnans, *Lecture Notes in Optimal Control.* `http://www.cmap.polytechnique.fr/~bonnans/notes.html`. Consulted in January 2015.

[4] R. Bulirsch, F. Montrone, H.J. Pesch, *Abort Landing in the Presence of Windshear as a Minimax Optimal Control Problem, Part 1: Necessary Conditions.* J. Optim. Theory Appl., 70(1): 1-23, 1991.

[5] R. Bulirsch, F. Montrone, H.J. Pesch, *Abort Landing in the Presence of Windshear as a Minimax Optimal Control Problem, Part 2: Multiple Shooting and Homotopy.* J. Optim. Theory Appl., 70(2): 223-254, 1991.

[6] H. Maurer, *On Optimal Control Problems with Bounded State Variables and Control Appearing Linearly.* SIAM J. Control and Optimization, 15(3): 345-362, 1977.

[7] A. Miele, T. Wang, C.Y. Tzeng, W.W. Melvin, *Optimal Abort Landing Trajectories in the presence of Windshear.* J. Optim. Theory Appl., 55(2): 165-202, 1987.

[8] BOCOP 2.0.0, Open-souce Software. `www.bocop.org`.