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► **To cite this version:**

Siegfried M. Rump, Florian Bünger, Claude-Pierre Jeannerod. Improved error bounds for floating-point products and Horner's scheme. BIT Numerical Mathematics, Springer Verlag, 2016, 56 (1), pp.293 - 307. 10.1007/s10543-015-0555-z . hal-01137652

**HAL Id: hal-01137652**

**<https://hal.inria.fr/hal-01137652>**

Submitted on 31 Mar 2015

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## Improved Error Bounds for Floating-Point Products and Horner's Scheme

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Received: date / Accepted: date

**Abstract** Let  $\mathbf{u}$  denote the relative rounding error of some floating-point format. Recently it has been shown that for a number of standard Wilkinson-type bounds the typical factors  $\gamma_k := k\mathbf{u}/(1-k\mathbf{u})$  can be improved into  $k\mathbf{u}$ , and that the bounds are valid without restriction on  $k$ . Problems include summation, dot products and thus matrix multiplication, residual bounds for  $LU$ - and Cholesky-decomposition, and triangular system solving by substitution.

In this note we show a similar result for the product  $\prod_{i=0}^k x_i$  of real and/or floating-point numbers  $x_i$ , for computation in any order, and for any base  $\beta \geq 2$ . The derived error bounds are valid under a mandatory restriction of  $k$ . Moreover, we prove a similar bound for Horner's polynomial evaluation scheme.

**Keywords** floating-point product · IEEE 754 standard · Wilkinson type error estimates · Horner scheme

**CR Subject Classification** 65G50 · 65F05

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## 1 Introduction and notation

Denote by  $\mathbb{F}$  a set of floating-point numbers with  $p$  digits precision in base  $\beta$ , and with operations according to IEEE 754 standard [3] in rounding to nearest with any tie breaking rule. Then,  $\mathbf{u} := \frac{1}{2}\beta^{1-p}$  denotes the relative rounding error unit. Throughout the paper we assume that  $\beta \geq 2$  and  $p \geq 1$ , and that neither overflow nor underflow occurs.

As usual, for  $\circ \in \{+, -, \cdot, /\}$  and  $a, b \in \mathbb{F}$ , the floating-point result of an operation  $a \circ b$  is defined to be  $\text{fl}(a \circ b)$  for a rounding to nearest  $\text{fl}: \mathbb{R} \rightarrow \mathbb{F}$ . It follows [2, p. 38] that  $|\text{fl}(x) - x| \leq \mathbf{u}|x|$  for  $x \in \mathbb{R}$ , and in particular

$$|\text{fl}(a \circ b) - (a \circ b)| \leq \mathbf{u}|a \circ b|. \quad (1.1)$$

For matrices  $A \in \mathbb{F}^{m \times k}$  and  $B \in \mathbb{F}^{k \times n}$ , denote by  $\widehat{C}$  the floating-point result of the exact product  $C := AB$  computed using (blocked versions of) the classical algorithm, with any ordering for the inner products. A rounding error analysis *à la* Wilkinson then leads typically to  $|\widehat{C} - C| \leq \gamma_k |A||B|$  with  $\gamma_k := \frac{k\mathbf{u}}{1-k\mathbf{u}} = k\mathbf{u} + O(\mathbf{u}^2)$ ; see for example [2, p. 71]. This standard estimate has been improved in [4] into

$$|\widehat{C} - C| \leq k\mathbf{u}|A||B| \quad (1.2)$$

without restriction on the integer  $k$  and, in [9], similar improvements have been obtained for the residuals of the computed LU and Cholesky factors as well as for triangular system solutions.

A similar result was recently shown by Graillat, Lefèvre, and Muller [1] for binary arithmetic:

**Theorem 1.1** *Assume  $\beta = 2$  and let  $x \in \mathbb{F}$  and  $k \in \mathbb{N}$  be given. If the power  $x^{k+1}$  is computed by successive multiplications by  $x$ , then, in absence of underflow and overflow, the computed approximation  $\widehat{r}$  satisfies*

$$|\widehat{r} - x^{k+1}| \leq k\mathbf{u}|x^{k+1}| \quad \text{if} \quad k+1 \leq \sqrt{2^{1/3} - 1} \cdot \mathbf{u}^{-1/2}. \quad (1.3)$$

This improves the classical Wilkinson-type estimate  $|\widehat{r} - x^{k+1}| \leq \gamma_k |x^{k+1}|$ . They also note that for  $k \approx \mathbf{u}^{-1}$  the relative error on  $\widehat{r}$  can indeed be larger than  $k\mathbf{u}$ , thus suggesting that in the case of integer powers, the price to be paid for the refined constant  $k\mathbf{u}$  is a necessary restriction on the range of  $k$ . This is in contrast with bounds like (1.2) and the results in [4, 9], where restrictions on  $k$  can be avoided.

As Muller [8] mentioned, repeated multiplication may not be the method of choice to evaluate  $x^{k+1}$ . However, for better methods like binary exponentiation no improvement on the classical constant  $\gamma_k$  seems to be known.

In this note we generalize Theorem 1.1 to products of real and/or floating-point numbers, to any base, and to any evaluation scheme using  $k$  multiplications. Our restriction on  $k$  is weaker than the one in (1.3), though of the same order, and we show that it is essentially sharp.

**Theorem 1.2** Let  $x_0, x_1, \dots, x_k \in \mathbb{R}$  be given and suppose that  $\ell$  of them are in  $\mathbb{F}$ . Let also

$$K := 2k + 1 - \ell \quad \text{and} \quad \omega := \begin{cases} 1 & \text{if } \beta \text{ is odd,} \\ 2 & \text{if } \beta \text{ is even.} \end{cases} \quad (1.4)$$

Then, any order of evaluation of the product of  $\prod_{i=0}^k \text{fl}(x_i)$  produces an approximation  $\widehat{r}$  such that, in absence of underflow and overflow,

$$\left| \widehat{r} - \prod_{i=0}^k x_i \right| \leq K \mathbf{u} \left| \prod_{i=0}^k x_i \right| \quad \text{if} \quad K < \sqrt{\frac{\omega}{\beta}} \mathbf{u}^{-1/2}. \quad (1.5)$$

In particular, if  $\beta = 2$  and all the  $x_i$  are in  $\mathbb{F}$ , then  $(K, \omega) = (k, \beta)$  and (1.5) becomes

$$\left| \widehat{r} - \prod_{i=0}^k x_i \right| \leq k \mathbf{u} \left| \prod_{i=0}^k x_i \right| \quad \text{if} \quad k < \mathbf{u}^{-1/2}. \quad (1.6)$$

For  $\beta = 2$  and  $p \geq 4$ , the constraint in (1.6) cannot be replaced by  $k < 12\mathbf{u}^{-1/2}$ .

REMARK. Note that for  $\beta = 2$  and all the  $x_i$  in  $\mathbb{F}$  the restriction  $k < \mathbf{u}^{-1/2}$  improves on the restriction  $k + 1 \leq \sqrt{2^{1/3} - 1} \cdot \mathbf{u}^{-1/2} = 0.509\dots \cdot \mathbf{u}^{-1/2}$  in (1.3).

The techniques to prove Theorem 1.2 can be used to obtain similar results for other evaluation schemes. As an example we show how to improve the classical factor  $\gamma_{2n}$  for Horner's scheme [2, p. 95].

**Theorem 1.3** Let  $x, a_0, a_1, \dots, a_n \in \mathbb{F}$  be given and let  $\widehat{r}$  be the approximation to  $\sum_{i=0}^n a_i x^i$  produced by Horner's scheme. Then, in absence of underflow and overflow,

$$\left| \widehat{r} - \sum_{i=0}^n a_i x^i \right| \leq 2n \mathbf{u} \sum_{i=0}^n |a_i x^i| \quad \text{if} \quad n < \frac{1}{2} \left( \sqrt{\frac{\omega}{\beta}} \mathbf{u}^{-1/2} - 1 \right)$$

using  $\omega$  defined in (1.4).

## 2 Products

We need some preliminaries to prove Theorem 1.2. If some  $x_i$  is zero, then  $\widehat{r} = 0$  because no overflow occurs, and the results in Theorem 1.2 are trivial. If all the  $x_i$  are nonzero, then  $\widehat{r} \neq 0$  because, by assumption, no underflow occurs. Furthermore, using  $\mathbb{F} = -\mathbb{F}$  and  $\text{fl}(-x) = -\text{fl}(x)$ , we may henceforth assume without loss of generality that all the  $x_i$  are positive, so that all the  $\widehat{r}_i$  are positive as well.

The standard estimate (1.1) can be improved in two ways. First, it is known that

$$x \in \mathbb{R}: \quad |\text{fl}(x) - x| \leq \frac{\mathbf{u}}{1 + \mathbf{u}} |x| \quad (2.1)$$

and that this bound is sharp; see for example [6, p. 232] and [5]. Second, we use the *unit in the first place* (ufp): a real number  $x$  being given, we set  $\text{ufp}(0) = 0$  and, if

$x \neq 0$ ,  $\text{ufp}(x) := \beta^{\lfloor \log_\beta |x| \rfloor}$ . Thus,  $\text{ufp}(x)$  can be thought of as the weight of the first nonzero digit of  $x$  in its base- $\beta$  representation. Then,

$$x \in \mathbb{R}: \quad |\text{fl}(x) - x| \leq \mathbf{u} \text{ufp}(x). \quad (2.2)$$

This estimate is sharp as well; for more details, see [10]. Combining (2.1) and (2.2) yields the improved estimate

$$x \in \mathbb{R} \setminus \{0\}: \quad \text{fl}(x) = x(1 + \varepsilon) \quad \text{with} \quad |\varepsilon| \leq \min \left[ \frac{\mathbf{u}}{1 + \mathbf{u}}, \mathbf{u} \frac{\text{ufp}(x)}{|x|} \right]. \quad (2.3)$$

In the following we will use

$$x \in \mathbb{R} \setminus \{0\}: \quad \text{ufp}(x) \leq |x| < \beta \text{ufp}(x), \quad (2.4)$$

as well as

$$\begin{aligned} f \in \mathbb{F} \cap [1, \beta] &\Rightarrow f = 1 + 2n\mathbf{u} \quad \text{with } n \in \mathbb{N}_0, \\ f \in \mathbb{F} \cap [\beta^{-1}, 1] &\Rightarrow f = 1 - \frac{2n}{\beta}\mathbf{u} \quad \text{with } n \in \mathbb{N}_0. \end{aligned} \quad (2.5)$$

Some notation is necessary to formalize the computation of the floating-point approximation  $\widehat{r}$  in (1.5). The evaluation of  $\prod_{i=0}^k \text{fl}(x_i)$  in any given order by means of  $k$  floating-point multiplications is represented by a binary tree  $B$  whose  $k + 1$  leafs correspond to the  $\text{fl}(x_i)$  and whose  $k$  inner nodes correspond to the multiplications. Thus,  $B$  has  $2k + 1$  nodes  $N_i$  in total.

Since the order of evaluation is arbitrary, we may assume without loss of generality that  $x_0, \dots, x_L \in \mathbb{F}$  with  $L := \ell - 1$ . The numbering of the nodes shall be such that  $N_i$  corresponds to  $x_{i+L}$  for  $i = -L, \dots, k - L$ , and  $N_{k-L+1}, \dots, N_K$  are the inner nodes. Moreover,  $N_K$  shall be the root of  $B$ .

Each node  $N_i$  is the root of a tree  $B_i$  and is identified with the floating-point value  $\widehat{r}_i = \text{fl}(r_i)$  computed by  $B_i$ . It follows in particular that  $\widehat{r} = \widehat{r}_K$ . More precisely, define  $r_i := x_{i+L}$  for  $i = -L, \dots, k - L$  and, by means of a recursive definition, if an inner node  $N_i$ ,  $i \in \{k - L + 1, \dots, K\}$ , has children  $N_{i_1}, N_{i_2}$ ,  $1 \leq \nu \leq 2$ , for which  $\widehat{r}_{i_1}, \widehat{r}_{i_2}$  are already known, define  $r_i := \widehat{r}_{i_1} \cdot \widehat{r}_{i_2}$ . Since the  $x_i$  and  $\widehat{r}_i$  have been assumed to be positive, the same holds for the  $r_i$ .

By assumption,  $\widehat{r}_i = \text{fl}(r_i) = x_{i+L}$  for  $i = -L, \dots, 0$ . Moreover, for  $i = 1, \dots, K$  we have

$$\widehat{r}_i = \text{fl}(r_i) =: (1 + \varepsilon_i)r_i \quad \text{with} \quad |\varepsilon_i| \leq \min \left[ \frac{\mathbf{u}}{1 + \mathbf{u}}, \mathbf{u} \frac{\text{ufp}(r_i)}{r_i} \right] < \mathbf{u}. \quad (2.6)$$

For  $i \in \{1, \dots, k - L\}$ , the relative errors  $\varepsilon_i$  correspond to the rounding of  $x_{i+L}$  into  $\text{fl}(x_{i+L})$ , while for the remaining indices  $i \in \{k - L + 1, \dots, K\}$  they correspond to the  $k$  multiplications. This implies  $\prod_{i=0}^k \text{fl}(x_i) = \prod_{i=1}^{k-L} (1 + \varepsilon_i) \cdot \prod_{i=0}^k x_i$ , and therefore

$$\widehat{r}_K - \prod_{i=0}^k x_i = \left( \prod_{i=1}^{k-L} (1 + \varepsilon_i) - 1 \right) \cdot \prod_{i=0}^k x_i. \quad (2.7)$$

Since all factors  $x_i$  are positive, (1.5) is equivalent to  $|\prod_{i=1}^K (1 + \varepsilon_i) - 1| \leq K\mathbf{u}$ , and because  $\prod_{i=1}^K (1 + \varepsilon_i) \geq (1 - \mathbf{u})^K \geq 1 - K\mathbf{u}$  it suffices to prove

$$\prod_{i=1}^K (1 + \varepsilon_i) \leq 1 + K\mathbf{u}. \quad (2.8)$$

Hence, we need only upper bounds on the  $\varepsilon_i$  for the proof of Theorem 1.2.

Furthermore, the lemma below shows that, under weaker assumptions on the maximum  $K$ , the estimate (1.5) in Theorem 1.2 is true if a single  $\varepsilon_i$  is not positive, that is, if any of the  $k - L$  real  $x_i$  or any single intermediate product is not rounded upwards. A similar observation was already made in [1, Lemma 3].

**Lemma 2.1** *With the notation above, in particular (2.6), assume  $K \leq \sqrt{2} \mathbf{u}^{-1/2}$ .*

*If there exists an index  $i \in \{1, \dots, K\}$  with  $\varepsilon_i \leq 0$ , then (1.5) holds true.*

*Proof.* By (2.6) and (2.8), it suffices to show  $Z := (1 + \mathbf{u})^{K-1} \leq 1 + K\mathbf{u}$ . Using  $K^2\mathbf{u} \leq 2$  gives

$$\ln(Z) = (K - 1) \ln(1 + \mathbf{u}) \leq (K - 1)\mathbf{u} \leq K\mathbf{u} - \frac{1}{2}K^2\mathbf{u}^2 \leq \ln(1 + K\mathbf{u}). \quad \square$$

**Proof of Theorem 1.2.** With the notation above, in particular using (2.6), we have to prove (2.8). For  $K \in \{0, 1\}$  the assertion is trivial so that henceforth we assume  $K \geq 2$ . By Lemma 2.1 we can also assume that

$$\varepsilon_i > 0 \quad \text{for all } i \in \{1, \dots, K\}. \quad (2.9)$$

Let  $\varphi \in \mathbb{N}$  be the largest integer satisfying

$$\varphi < \sqrt{\frac{\omega}{\beta}} \mathbf{u}^{-1/2}. \quad (2.10)$$

Note that  $\varphi \geq 2$  because  $2 \leq K < \sqrt{\omega/\beta} \mathbf{u}^{-1/2} \leq \varphi + 1$ . Define  $I \subseteq \{1, \dots, K\}$  to be the index set with

$$i \in I \quad :\Leftrightarrow \quad \varepsilon_i > \frac{\mathbf{u}}{1 + \varphi\mathbf{u}}. \quad (2.11)$$

The following two properties will be proved for distinct  $i, j \in I$ :

$$\text{a) The nodes } N_i \text{ and } N_j \text{ are not adjacent in the tree } B. \quad (2.12)$$

$$\text{b) The nodes } N_i \text{ and } N_j \text{ do not have the same parent node in } B. \quad (2.13)$$

*Proof of (2.12).* In order to derive a contradiction suppose that  $N_i$  is a child of  $N_j$ . It follows that  $r_j = \widehat{r}_i \widehat{q}$ , where  $\widehat{q} \in \mathbb{F}$  is a (rounded)  $x_i$  or some intermediate result. If  $\text{ufp}(\widehat{r}_i) = \widehat{r}_i$ , then  $\widehat{r}_i$  is a power of  $\beta$  and  $\varepsilon_j = 0$  contradicting (2.9), so that (2.6) and  $i \in I$  imply

$$\text{ufp}(r_i) = \text{ufp}(\widehat{r}_i) < r_i < (1 + \varphi\mathbf{u})\text{ufp}(r_i) \quad \text{for } i \in I. \quad (2.14)$$

Since the second inequality is strict and  $1 + \varphi\mathbf{u} < 1 + \sqrt{\mathbf{u}} < \beta$ , it follows by (2.5), no matter whether  $\varphi$  is odd or even, that

$$\text{ufp}(r_i) = \text{ufp}(\widehat{r}_i) < \widehat{r}_i \leq (1 + \varphi\mathbf{u})\text{ufp}(r_i) \quad \text{for } i \in I. \quad (2.15)$$

By (2.15) and (2.5) we have

$$\widehat{r}_i = \text{ufp}(\widehat{r}_i)(1 + m\mathbf{u}) \quad \text{for even } m \in \mathbb{N} \text{ with } 2 \leq m \leq \varphi. \quad (2.16)$$

Hence,  $r_j = \widehat{r}_i \widehat{q}$ , (2.4), (2.16),  $j \in I$ , and (2.14) imply

$$\frac{R}{1 + m\mathbf{u}} \leq \frac{r_j}{\widehat{r}_i} = \widehat{q} \leq (1 + \varphi\mathbf{u})R \quad \text{abbreviating } R := \frac{\text{ufp}(r_j)}{\text{ufp}(\widehat{r}_i)}. \quad (2.17)$$

Since  $\widehat{q} \in \mathbb{F}$ ,  $R$  is a power of  $\beta$ , and  $R/(1 + m\mathbf{u}) > R(1 - m\mathbf{u}) \in \mathbb{F}$ , (2.5) implies that there exists  $\nu \in \mathbb{Q}$  such that

$$\widehat{q} = R(1 + \nu\mathbf{u}) \quad \text{and} \quad -m < \nu \leq \varphi. \quad (2.18)$$

Moreover, if  $\nu$  is non-negative, then  $\nu$  is a non-negative even integer by (2.5). From (2.18) and (2.16) we get  $|\nu| \leq \varphi$ . Now  $r_j = \widehat{r}_i \widehat{q}$ , (2.18), and (2.16) give

$$\text{ufp}(r_j) \leq r_j = \text{ufp}(r_j)(1 + (m + \nu)\mathbf{u} + m\nu\mathbf{u}^2), \quad (2.19)$$

and (2.14) together with  $j \in I$  yields

$$0 \leq (m + \nu)\mathbf{u} + m\nu\mathbf{u}^2 \leq \varphi\mathbf{u}. \quad (2.20)$$

First, assume that  $\nu$  is an even integer. Then,  $m + \nu > 0$  is also even by (2.16), so that  $1 + (m + \nu)\mathbf{u} \in \mathbb{F}$  and  $|m\nu\mathbf{u}^2| \leq \varphi^2\mathbf{u}^2 < \mathbf{u}$  imply  $\widehat{r}_j = \text{ufp}(r_j)(1 + (m + \nu)\mathbf{u})$  and

$$\varepsilon_j = \frac{\widehat{r}_j - r_j}{r_j} = -\frac{\text{ufp}(r_j)m\nu\mathbf{u}^2}{r_j} \leq \varphi|\nu|\mathbf{u}^2. \quad (2.21)$$

If  $\nu \geq 0$ , then  $\varepsilon_j \leq 0$ , a contradiction. Otherwise, (2.18) and  $-\nu \in \mathbb{N}$  give  $|\nu| = -\nu \leq m - 1 \leq \varphi - 1$ , so that  $\varphi < \mathbf{u}^{-1/2}$  implies

$$\varphi|\nu|\mathbf{u}^2(1 + \varphi\mathbf{u}) < \frac{1}{\sqrt{\mathbf{u}}} \left( \frac{1}{\sqrt{\mathbf{u}}} - 1 \right) \mathbf{u}^2(1 + \sqrt{\mathbf{u}}) = (1 - \sqrt{\mathbf{u}})\mathbf{u}(1 + \sqrt{\mathbf{u}}) \leq \mathbf{u}.$$

Hence,  $\varepsilon_j < \frac{\mathbf{u}}{1 + \varphi\mathbf{u}}$  by (2.21), again a contradiction to  $j \in I$  by (2.11).

Second, assume that  $\nu$  is not an even integer. Then, (2.18) and (2.5) give  $\nu < 0$ . Write  $\nu = 2n/\beta =: s + r/\beta$  with  $n, s, r \in \mathbb{Z}_{<0}$  with  $|r| := (2|n|) \bmod \beta$ . Since  $2n$  is even, necessarily

$$|r| \leq \begin{cases} \beta - 2 & \text{if } \beta \text{ is even,} \\ \beta - 1 & \text{if } \beta \text{ is odd,} \end{cases} \quad \Rightarrow \quad \left| \frac{r}{\beta} \right| \leq 1 - \frac{\omega}{\beta} \quad (2.22)$$

using  $\omega$  as in (1.4). In particular, for  $\beta = 2$  this means  $r = 0$ . Now, (2.19) becomes

$$r_j = \text{ufp}(r_j)(1 + (m + s + \delta)\mathbf{u}) \quad \text{with} \quad \delta := \frac{r}{\beta} + m\nu < 0 \quad (2.23)$$

because  $r \leq 0$  and  $-m \leq \nu < 0$ . Using (2.22) and (2.10) we obtain

$$|\delta|\mathbf{u} \leq \left( 1 - \frac{\omega}{\beta} + \varphi^2\mathbf{u} \right) \mathbf{u} < \mathbf{u}. \quad (2.24)$$

If  $s$  is odd, then  $\delta < 0$  and (2.24) yield  $\widehat{r}_j = \text{ufp}(r_j)(1 + (m + s - 1)\mathbf{u})$  and  $\varepsilon_j < 0$ , a contradiction. If  $s$  is even, then  $\widehat{r}_j = \text{ufp}(r_j)(1 + (m + s)\mathbf{u})$  and

$$\varepsilon_j = -\delta\mathbf{u} \frac{\text{ufp}(r_j)}{r_j} \leq |\delta|\mathbf{u}. \quad (2.25)$$

Note that  $s$  even implies  $r \neq 0$  as  $\nu$  is not an even integer.<sup>1</sup> By (2.18) we have  $-m < \nu = s + r/\beta$ . Since  $m, s$  are even integers and  $r/\beta < 0$ , it follows  $-m + 2 \leq s = \nu - r/\beta$ , so that (2.22) yields

$$|\nu| = -\nu \leq m - 2 - \frac{r}{\beta} \leq \varphi - 1 - \frac{\omega}{\beta}. \quad (2.26)$$

From (2.25), (2.23), (2.22), (2.16), (2.26), and (2.10) we deduce the final contradiction to  $j \in I$  and (2.11):

$$\frac{\varepsilon_j}{\mathbf{u}} \leq |\delta| \leq 1 - \frac{\omega}{\beta} + \varphi \left( \varphi - 1 - \frac{\omega}{\beta} \right) \mathbf{u} < 1 - \varphi\mathbf{u} - \frac{\omega}{\beta} + \varphi^2\mathbf{u} < 1 - \varphi\mathbf{u} < \frac{1}{1 + \varphi\mathbf{u}}.$$

This finishes the proof of (2.12).

*Proof of (2.13).* Again, in order to derive a contradiction, assume that  $N_i$  and  $N_j$  are the left and right children of an inner node  $N_a$ ,  $a \in \{k - L + 1, \dots, K\}$ , that is,  $r_a = \widehat{r}_i \widehat{r}_j$  and  $\widehat{r}_a = \text{fl}(r_a)$ . Then, like in the proof of (2.12),  $i, j \in I$  implies

$$\begin{aligned} \text{ufp}(r_i) &= \text{ufp}(\widehat{r}_i) < r_i < \widehat{r}_i = (1 + m\mathbf{u})\text{ufp}(r_i) \leq (1 + \varphi\mathbf{u})\text{ufp}(r_i), \\ \text{ufp}(r_j) &= \text{ufp}(\widehat{r}_j) < r_j < \widehat{r}_j = (1 + n\mathbf{u})\text{ufp}(r_j) \leq (1 + \varphi\mathbf{u})\text{ufp}(r_j) \end{aligned}$$

with even  $m, n \in \mathbb{N}_{\leq \varphi}$ . Thus,

$$r_a = (1 + (m + n)\mathbf{u} + mn\mathbf{u}^2)\text{ufp}(r_i)\text{ufp}(r_j), \quad (2.27)$$

and  $(m + n)\mathbf{u} \leq 2\varphi\mathbf{u} < 2\sqrt{\omega/\beta}\mathbf{u}^{1/2} \leq \frac{2\omega}{K\beta} \leq \frac{2}{K} \leq 1$  because  $K \geq 2$ . Moreover,  $m + n$  is even and  $mn\mathbf{u}^2 \leq \varphi^2\mathbf{u}^2 < \mathbf{u}$ . Thus (2.27) yields  $\text{ufp}(r_a) = \text{ufp}(\widehat{r}_a) = \text{ufp}(r_i)\text{ufp}(r_j)$ ,  $\widehat{r}_a = (1 + (m + n)\mathbf{u})\text{ufp}(\widehat{r}_a)$ , and  $\varepsilon_a = -mn\mathbf{u}^2\text{ufp}(\widehat{r}_a)/r_a < 0$  contradicting (2.9). This finishes the proof of (2.13).

For  $I$  consisting of  $k'$  indices, (2.6) and (2.11) give

$$\prod_{i=1}^{k'} (1 + \varepsilon_i) \leq \left(1 + \frac{\mathbf{u}}{1 + \mathbf{u}}\right)^{k'} \left(1 + \frac{\mathbf{u}}{1 + \varphi\mathbf{u}}\right)^{K - k'}. \quad (2.28)$$

Using (2.12) and (2.13) we will show by Lemma 2.2 in Subsection 2.1 that  $k' \leq \lfloor \frac{K+1}{2} \rfloor$ . This implies

$$\prod_{i=1}^{k'} (1 + \varepsilon_i) \leq \left(1 + \frac{\mathbf{u}}{1 + \mathbf{u}}\right)^{\lfloor \frac{K+1}{2} \rfloor} \left(1 + \frac{\mathbf{u}}{1 + \varphi\mathbf{u}}\right)^{\lceil \frac{K-1}{2} \rceil}. \quad (2.29)$$

<sup>1</sup> Thus, for the classical case  $\beta = 2$  a contradiction to  $\{i, j\} \subseteq I$  is already obtained.



Hence, according to (2.8) and using  $\frac{\mathbf{u}}{1+\mathbf{u}} \geq \frac{\mathbf{u}}{1+\varphi\mathbf{u}}$ , the proof is finished if we show

$$F(K) := \left(1 + \frac{\mathbf{u}}{1+\mathbf{u}}\right)^{\frac{K+1}{2}} \left(1 + \frac{\mathbf{u}}{1+\varphi\mathbf{u}}\right)^{\frac{K-1}{2}} \leq 1 + K\mathbf{u}. \quad (2.30)$$

For later use, we do this by proving for real  $\psi$  the following stronger statement

$$G(\psi) := \left(1 + \frac{\mathbf{u}}{1+\mathbf{u}}\right)^{\frac{\psi+1}{2}} \left(1 + \frac{\mathbf{u}}{1+\psi\mathbf{u}}\right)^{\frac{\psi-1}{2}} \leq 1 + (\psi-1)\mathbf{u} \quad (2.31)$$

provided that  $1 \leq \psi \leq \sqrt{\frac{\omega}{\beta}} \mathbf{u}^{-1/2}$ . If this is true, then for  $1 \leq K \leq \varphi$  we obtain

$$F(K) \leq G(K) \left(1 + \frac{\mathbf{u}}{1+\varphi\mathbf{u}}\right) \leq (1 + (K-1)\mathbf{u}) \left(1 + \frac{\mathbf{u}}{1+(K-1)\mathbf{u}}\right) = 1 + K\mathbf{u}$$

which is (2.30). A computation yields the Taylor expansion

$$G(\psi) = 1 + (\psi-1)\mathbf{u} + \frac{1}{2}G''(\xi)\xi^2 \quad \text{with} \quad G''(\xi) =: \alpha N(\xi)$$

for some  $0 < \xi < \mathbf{u}$  and

$$\alpha := -\frac{(\psi-1) \left(\frac{1+(\psi+1)\xi}{1+\psi\xi}\right)^{\frac{\psi-1}{2}} \left(1 + \frac{\xi}{1+\xi}\right)^{\frac{\psi-1}{2}}}{4(1+2\xi)(1+\xi)^3(1+(\psi+1)\xi)^3(1+\psi\xi)} < 0.$$

It suffices to show  $N(\xi) \geq 0$  for  $0 < \xi < \mathbf{u}$ . Now  $N(\xi) = \sum_{v=0}^5 c_v$  and  $\psi^2\mathbf{u} \leq 1$  with

$$\begin{aligned} c_0 &= 60\xi^4 + 160\xi^3 + 144\xi^2 + 48\xi + 4 > 4 + 48\xi \\ c_1 &= (48\xi^5 + 192\xi^4 + 248\xi^3 + 124\xi^2 + 20\xi)\psi > 20\psi\xi + 124\psi\xi^2 \\ c_2 &= (72\xi^5 + 187\xi^4 + 140\xi^3 + 24\xi^2 - 4\xi)\psi^2 > -4 \\ c_3 &= (32\xi^5 + 41\xi^4 - 8\xi^2)\psi^3 > -8\psi\xi \\ c_4 &= (8\xi^5 + \xi^4 - 4\xi^3)\psi^4 > -4\xi \\ c_5 &= -\xi^4\psi^5 > -\psi\xi^2. \end{aligned}$$

The series expansions were computed by the Symbolic Math Toolbox of MATLAB [7]. It follows  $N(\xi) > 0$  for  $0 < \xi < \mathbf{u}$ , and this proves (1.5) and (1.6).

The assertion on possible constraints of  $k$  is deferred to the appendix. This finishes the proof of Theorem 1.2.  $\square$

In the proof of Theorem 1.2 we defined  $\varphi$  to be the largest integer less than  $\sqrt{\omega/\beta} \mathbf{u}^{-1/2}$ , which reduces to  $\varphi < \mathbf{u}^{-1/2}$  for binary arithmetic. Switching from binary arithmetic to another basis requires indeed an adapted definition of  $\varphi$ . Consider  $p := 5$  decimal digits, that is,  $\mathbf{u} = 0.5 \cdot 10^{-4}$ . Then,  $\widehat{r}_i := \text{fl}(1.3033 \cdot 0.7697) = 1.0032$  and  $\widehat{q} := 0.99696$  yield  $\widehat{r}_j = 1.0002$ . Moreover,  $\varphi = 63$  whilst the largest integer less than  $\mathbf{u}^{-1/2}$  is  $\varphi' = 141$ . However, both  $\varepsilon_i$  and  $\varepsilon_j$  would satisfy (2.11) if  $\varphi$  was replaced by  $\varphi'$ , and indices of adjacent nodes would belong to  $I$ .

## 2.1 A result on colored trees

In (2.29) in the proof of Theorem 1.2 we used the upper bound  $\lfloor \frac{K+1}{2} \rfloor$  for the number  $k'$  of nodes in the index set  $I$ . This bound is a consequence of the following lemma.

**Lemma 2.2** *Let  $T$  be a tree with  $M$  nodes, each having at most two children. Assume that  $C$  nodes of  $T$  are colored according to the following rules:*

- (i) *colored nodes are not adjacent;*
- (ii) *each node has at most one colored child.*

Then,

$$C \leq \begin{cases} \lfloor \frac{M+1}{2} \rfloor & \text{if the root of } T \text{ is colored,} \\ \lfloor \frac{M}{2} \rfloor & \text{otherwise.} \end{cases}$$

Furthermore, these inequalities are sharp for all  $M$ .

*Proof* The result is trivial for  $M = 1$ , so assume  $M \geq 2$  and that the result is true up to  $M - 1$ . The root  $R$  of  $T$  is then connected to a tree  $T_1$  and, possibly, also to another tree  $T_2$  disjoint from  $T_1$ . Let  $T_1$  have  $M_1$  nodes,  $C_1$  of which being colored. Define  $M_2$  and  $C_2$  similarly if  $T_2$  exists, and let  $C_2 = M_2 = 0$  otherwise. Clearly,  $M = M_1 + M_2 + 1$  and  $0 \leq C_i \leq M_i \leq M - 1$  for  $i = 1, 2$ .

If  $R$  is colored, then  $C = C_1 + C_2 + 1$  and (i) implies the root of  $T_1$  is not colored. Hence, by induction,  $C_1 \leq \lfloor M_1/2 \rfloor \leq M_1/2$ . Similarly,  $C_2 \leq M_2/2$ , so that

$$C \leq \frac{M_1}{2} + \frac{M_2}{2} + 1 = \frac{M+1}{2}.$$

If  $R$  is not colored, then  $C = C_1 + C_2$  and (ii) implies that  $R$  has at most one colored child. Hence, for  $M_2$  either zero or nonzero,

$$C \leq \frac{M_1}{2} + \frac{M_2}{2} + \frac{1}{2} = \frac{M}{2}.$$

Since  $C$  is an integer, the claimed bounds follow for  $M \geq 2$ . Finally, trees with all internal nodes having exactly one child ("linked lists") and whose colored and uncolored nodes alternate show that the bound is attained for any  $M$ .  $\square$

Now, the upper bound for  $k'$  in the proof of Theorem 1.2 is obtained as follows. First, we construct a tree  $T$  by removing from the binary tree  $B$  the leaves  $N_{-L}, \dots, N_0$  associated with the  $\ell$  operands  $x_i$  already in  $\mathbb{F}$ . The nodes of  $T$  are the nodes  $N_1, \dots, N_K$  of  $B$ , and the nodes  $N_i$  with  $i \in I$  are considered as colored. Then, (2.12) and (2.13) imply that  $T$  follows the rules (i) and (ii) of Lemma 2.2, so that  $|I| = k' \leq \lfloor \frac{K+1}{2} \rfloor$ .

Optimality of the bounds in Lemma 2.2 is established by linked lists which represent recursive multiplication of floating-point numbers. We note that optimal bounds are attained for other evaluation schemes as well. Examples for all  $M$  for trees with colored root are sketched in Figure 2.1; examples with uncolored root follow similarly.

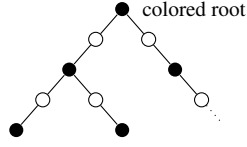


Fig. 2.1 Trees attaining the bound  $C = \lfloor \frac{M+1}{2} \rfloor$  for colored root.

### 3 Horner scheme

Using the techniques of the previous section we prove Theorem 1.3. For  $n = 0$  the assertion is trivial so that we may assume  $n \geq 1$ . The Horner scheme computes

$$\widehat{r}_0 := \text{fl}(a_n x); \quad \widehat{r}_i := \text{fl}(\text{fl}(\widehat{r}_{i-1} + a_{n-i})x), \quad i = 1, \dots, n-1; \quad \widehat{r} = \widehat{r}_n := \text{fl}(\widehat{r}_{n-1} + a_0).$$

For  $i = 1, \dots, n$ , let the relative error of the  $i$ -th addition and multiplication be denoted by  $\varepsilon_i$  and  $\varepsilon'_{i-1}$ , respectively. Then,

$$\begin{aligned} \widehat{r}_0 &= a_n x (1 + \varepsilon'_0), \\ \widehat{r}_i &= (\widehat{r}_{i-1} + a_{n-i})x(1 + \varepsilon_i)(1 + \varepsilon'_i), \quad i = 1, \dots, n-1, \\ \widehat{r} &= (\widehat{r}_{n-1} + a_0)(1 + \varepsilon_n). \end{aligned} \quad (3.1)$$

For each  $i \in \{1, \dots, n-1\}$  we apply Theorem 1.2 to the product  $x_0 x_1$  with  $x_0 := \widehat{r}_{i-1} + a_{n-i} \in \mathbb{R}$  and  $x_1 := x \in \mathbb{F}$ . Then,  $k = 1$ ,  $\ell = 1$  and therefore  $K = 2$ , so that (2.29) with the constant  $\varphi$  defined in (2.10) yields<sup>2</sup>

$$(1 + \varepsilon_i)(1 + \varepsilon'_i) \leq \left(1 + \frac{\mathbf{u}}{1 + \mathbf{u}}\right) \left(1 + \frac{\mathbf{u}}{1 + \varphi \mathbf{u}}\right), \quad i = 1, \dots, n-1. \quad (3.2)$$

Furthermore, (2.6) gives

$$(1 + \varepsilon'_0)(1 + \varepsilon_n) \leq \left(1 + \frac{\mathbf{u}}{1 + \mathbf{u}}\right)^2. \quad (3.3)$$

From the equalities in (3.1) we deduce that  $\widehat{r} = \sum_{i=0}^n a_i (1 + \alpha_i) x^i$ , where

$$\begin{aligned} 1 + \alpha_n &= (1 + \varepsilon'_0) \cdot \prod_{j=1}^{n-1} (1 + \varepsilon_j)(1 + \varepsilon'_j) \cdot (1 + \varepsilon_n), \\ 1 + \alpha_i &= \prod_{j=n-i}^{n-1} (1 + \varepsilon_j)(1 + \varepsilon'_j) \cdot (1 + \varepsilon_n), \quad i = 1, \dots, n-1, \\ 1 + \alpha_0 &= 1 + \varepsilon_n. \end{aligned}$$

Hence, (1.1), (3.2) and (3.3) imply

$$(1 - \mathbf{u})^{2n} \leq 1 + \alpha_n \leq \left(1 + \frac{\mathbf{u}}{1 + \mathbf{u}}\right)^{n+1} \left(1 + \frac{\mathbf{u}}{1 + \varphi \mathbf{u}}\right)^{n-1} =: H_n$$

<sup>2</sup> In fact, (2.29) is applied to  $|x_0|, |x_1|$  because the proof of Theorem 1.2 assumes positive factors.

and, for  $i = 0, \dots, n-1$ ,

$$(1 - \mathbf{u})^{2i+1} \leq 1 + \alpha_i \leq \left(1 + \frac{\mathbf{u}}{1 + \mathbf{u}}\right)^{i+1} \left(1 + \frac{\mathbf{u}}{1 + \varphi\mathbf{u}}\right)^i.$$

Then, using  $1 - 2n\mathbf{u} < (1 - \mathbf{u})^{2n}$ , we see that  $1 - 2n\mathbf{u} \leq 1 + \alpha_i \leq H_n$  for all  $i = 0, 1, \dots, n$ . The assumption  $n < \frac{1}{2} \left( \sqrt{\frac{\omega}{\beta}} \mathbf{u}^{-1/2} - 1 \right)$  implies  $2n + 1 \leq \varphi$ . Thus, (2.31) proves  $H_n \leq G(2n + 1) \leq 1 + 2n\mathbf{u}$ .  $\square$

We close this note with an application of Theorem 1.2.

### Corollary 3.1 (Evaluation of a polynomial given by its roots)

Given  $z, z_1, \dots, z_n, a_n \in \mathbb{F}$ , let  $\widehat{r} \in \mathbb{F}$  be a floating-point approximation to

$$r = a_n \prod_{i=1}^n (z - z_i)$$

obtained by first evaluating the  $n$  differences and then, in any order, a product of  $n+1$  terms. If  $n < \frac{1}{2} \sqrt{\frac{\omega}{\beta}} \mathbf{u}^{-1/2}$  then, in absence of underflow and overflow,

$$|\widehat{r} - r| \leq 2n\mathbf{u}|r|.$$

*Proof* Define  $x_0 := a_n \in \mathbb{F}$  and  $x_i := z - z_i \in \mathbb{R}$  for  $i = 1, \dots, n$ . Then, Theorem 1.2 with  $k = n$ ,  $\ell = 1$ ,  $K = 2k + 1 - \ell = 2n < \sqrt{\frac{\omega}{\beta}} \mathbf{u}^{-1/2}$  yields the assertion.  $\square$

## 4 Appendix

The goal of this appendix is to prove that for  $\beta = 2$  and  $p \geq 4$  the constraint  $k < \mathbf{u}^{-1/2}$  in Theorem 1.2 cannot be replaced by  $k < 12\mathbf{u}^{-1/2}$ . To do that<sup>3</sup> we construct  $x_0, x_1, x_2 \in \mathbb{F}$  for given precision  $p$  such that  $x_1 x_2 < 1$  and  $\text{fl}(\text{fl}(x_0 x_1) x_2) = x_0$ . Subsequent multiplications by  $x_1 x_2$  produce an exponential growth of the rounding error, eventually exceeding  $k\mathbf{u}$ .

Define  $s := \lfloor \mathbf{u}^{-1/2} \rfloor \in \mathbb{N}$ , so that  $s = \mathbf{u}^{-1/2} - \delta$  with  $0 \leq \delta < 1$ . We henceforth assume  $p \geq 15$  and treat the case  $p \leq 14$  later. Note that  $\beta = 2$  and  $p \geq 15$  imply  $s \geq 181$ . We distinguish two cases.

First, assume  $s$  is odd. Set

$$x_0 := 1 + (2s + 8)\mathbf{u}, \quad x_1 := 1 - (s - 4)\mathbf{u}, \quad \text{and} \quad x_2 := 1 + (s - 5)\mathbf{u},$$

so that  $x_i \in \mathbb{F}$ . Then,  $x_0 x_1 = 1 + (s + 10)\mathbf{u} + \mu_1 \mathbf{u}$  with  $\mu_1 := 4\delta \sqrt{\mathbf{u}} + (32 - 2\delta^2)\mathbf{u}$ , so that  $0 < \mu_1 < 1$  and  $s$  odd imply  $\text{fl}(x_0 x_1) = 1 + (s + 11)\mathbf{u}$ . Moreover,  $\text{fl}(x_0 x_1) x_2 = 1 + (2s + 7)\mathbf{u} + \mu_2 \mathbf{u}$  with

$$\mu_2 := \sqrt{\mathbf{u}}(6 - 55\sqrt{\mathbf{u}} + \Phi\delta) \quad \text{with} \quad \Phi := (\delta - 6)\sqrt{\mathbf{u}} - 2.$$

<sup>3</sup> In [1] long sequences  $x_i \in \mathbb{F}$  with  $\text{fl}(\dots(\text{fl}(x_0 x_1) x_2) \dots) x_k = x_0$  are constructed for some precisions.

Now  $\Phi < 0$  for any value of  $\delta$ , so that  $0 < 4\sqrt{\mathbf{u}} - 60\mathbf{u} \leq \mu_2 \leq 6\sqrt{\mathbf{u}} - 55\mathbf{u} < 1$ . Thus,

$$\text{fl}(\text{fl}(x_0 x_1) x_2) = x_0. \quad (4.1)$$

Define a vector  $X := [x_0 \ x \ x \dots x] \in \mathbb{F}^{2m+1}$  with  $m$  times repeating the row vector  $x = [x_1 \ x_2] \in \mathbb{F}^2$ . Denoting  $\widehat{r}_0 := x_0$  and  $\widehat{r}_i := \text{fl}(\widehat{r}_{i-1} X_i)$  for  $i \geq 1$  yields  $\widehat{r}_2 = v_0$ . Then, abbreviating  $\pi := x_1 x_2$  and using  $\widehat{r}_{2m} = \widehat{r}_2 = x_0$  gives

$$\widehat{r}_{2m} - \prod_{i=0}^{2m} X_i = x_0 - x_0 \pi^m = (\pi^{-m} - 1) \prod_{i=0}^{2m} X_i \quad \text{for } 1 \leq m \in \mathbb{N}. \quad (4.2)$$

Now,

$$\pi = 1 - (2 - (9 + 2\delta)\sqrt{\mathbf{u}})\mathbf{u} - (20 + 9\delta + \delta^2)\mathbf{u}^2 < 1 - (2 - 11\sqrt{\mathbf{u}})\mathbf{u} =: 1 - \gamma\mathbf{u},$$

and for  $m \in \mathbb{N}$ ,

$$\pi^{-m} > 1 + m\gamma\mathbf{u} + \frac{m(m-1)}{2}\gamma^2\mathbf{u}^2 = 1 + 2m\mathbf{u} + \frac{m\mathbf{u}\sqrt{\mathbf{u}}}{2}[(m-1)\gamma^2\sqrt{\mathbf{u}} - 22].$$

The assumption  $p \geq 15$  implies

$$(6 - 2\sqrt{\mathbf{u}})\gamma^2 - 22 = 2 - 272\sqrt{\mathbf{u}} + (814 - 242\sqrt{\mathbf{u}})\mathbf{u} > 2 - 272\sqrt{\mathbf{u}} > 0,$$

and therefore

$$m \geq 6\mathbf{u}^{-1/2} - 1 \quad \Rightarrow \quad \pi^{-m} > 1 + 2m\mathbf{u}. \quad (4.3)$$

Combining this with (4.2) shows that the error bound in (1.6) is not satisfied for  $k = 2 \lceil 6\mathbf{u}^{-1/2} - 1 \rceil < 12\mathbf{u}^{-1/2}$ , and that finishes the first part.

Second, assume  $s$  is even and define as before

$$y_0 := 1 + (2s + 6)\mathbf{u}, \quad y_1 := 1 - (s - 3)\mathbf{u}, \quad \text{and} \quad y_2 := 1 + (s - 4)\mathbf{u}. \quad (4.4)$$

Then,  $y_i \in \mathbb{F}$ . Furthermore,  $y_0 y_1 = 1 + (s + 7)\mathbf{u} + \mu_1 \mathbf{u}$  with  $\mu_1 := 4\delta\sqrt{\mathbf{u}} + (18 - 2\delta^2)\mathbf{u}$ , so that  $0 < \mu_1 < 1$  and  $s$  even imply  $\text{fl}(y_0 y_1) = 1 + (s + 8)\mathbf{u}$ . Moreover,  $\text{fl}(y_0 y_1) y_2 = 1 + (2s + 5)\mathbf{u} + \mu_2 \mathbf{u}$  with

$$\mu_2 := \sqrt{\mathbf{u}}(4 - 32\sqrt{\mathbf{u}} + \Phi\delta) \quad \text{with} \quad \Phi := (\delta - 4)\sqrt{\mathbf{u}} - 2.$$

As before,  $\Phi < 0$  for any value of  $\delta$ . Thus,  $0 < 2\sqrt{\mathbf{u}} - 35\mathbf{u} \leq \mu_2 \leq 4\sqrt{\mathbf{u}} - 32\mathbf{u} < 1$ . Hence, similar to (4.1),  $\text{fl}(\text{fl}(y_0 y_1) y_2) = y_0$  is again true. Now for the values  $y_1, y_2$  in (4.4) we obtain

$$y_1 y_2 = (1 - (s - 3)\mathbf{u})(1 + (s - 4)\mathbf{u}) < x_1 x_2,$$

and the result follows as before. Finally, for the cases  $4 \leq p \leq 14$ , consider

p	$m_0$	$m_1$	$m_2$	F
4	2	-4	4	9.6
5	20	-3	2	8.9
6	32	-14	16	5.8
7	28	-9	8	6.8
8	52	-39	44	5.8
9	48	-21	20	4.6
10	140	-117	130	5.2
11	94	-43	42	5.8
12	186	-154	158	4.0
13	184	-89	88	4.1
14	262	-125	124	7.2

For precision  $p$  define  $x_i := 1 + m_i \mathbf{u}$ . Then, (4.1) is satisfied, and the error bound in (1.6) is not true for  $k < F\mathbf{u}^{-1/2}$ . This finishes the proof.  $\square$

We finally mention that it is easy to see that, if  $1 \leq p \leq 2$ , then the error bound in (1.6) is satisfied for all  $k \in \mathbb{N}$ , and if  $p = 3$ , then the minimum value of  $k$  for which it is not satisfied is  $k = 72 \approx 25\mathbf{u}^{-1/2}$ .

## 5 Summary

In previous papers, the factor  $\gamma_k$  has been replaced by  $k\mathbf{u}$  in a number of classical error estimates in numerical analysis together with removing the restriction on  $k$ . We proved that  $k\mathbf{u}$  can be used for general products and for the Horner scheme, however, with a mandatory restriction on  $k$ . So, as by Theorem 1.2, a general principle to replace  $\gamma_k$  by  $k\mathbf{u}$  is necessarily restricted to  $k \lesssim \mathbf{u}^{-1/2}$ .

## 6 Acknowledgement

The authors wish to thank Marko Lange, Vincent Lefèvre, and Jean-Michel Muller for their fruitful and constructive comments on a preliminary version of this note.

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