CRITERION FOR RAYS LANDING TOGETHER

Jinsong Zeng

CRITERION FOR RAYS LANDING TOGETHER

JINSONG ZENG

Abstract. Let f be a polynomial with degree ≥ 2 and the Julia set J_f locally connected. We give a partition of complex plane $\mathbb{C}$ and show that, if z, z' in J_f have the same itinerary respect to the partition, then either $z = z'$ or both of them lie in the boundary of a Fatou component U, which is eventually iterated to a siegel disk. As an application, we prove the monotonicity of core entropy for the quadratic polynomial family $\{f_c = z^2 + c : f_c$ has no Siegel disks and J_{f_c} is locally connected $\}$.

1. Introduction

Let f be a polynomial with degree $d \geq 2$ in the complex plane $\mathbb{C}$. The *filled Julia set* is

$$K_f := \{z \in \mathbb{C} : \text{ The orbit } \{f^n(z)\}_{n\geq 0} \text{ is bounded }\}$$

and the *Julia set* is the topological boundary of the filled Julia set

$$J_f = \partial K_f.$$

Both of them are nonempty and compact, and the filled Julia set is *full*, i.e., the complement $\overline{\mathbb{C}} \setminus K_f$ is connected. We call $\Omega_f := \overline{\mathbb{C}} \setminus K_f$ the *basin of infinity* which consists of points with the orbit attracted by ∞. If J_f is connected. Then Ω_f is a simply connected and there exists an unique holomorphic parameterization $\Psi_f : \Omega_f \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ such that $\Psi_f(\infty) = \infty$, $\Psi'_f(\infty) = 1$ and

$$\Psi_f \circ f(z) = (\Psi_f(z))^d. \tag{1.1}$$

By the *external ray* $R(\theta)$ we mean the preimage of the radial line $\Psi_f^{-1}\{re^{2\pi i\theta} : r > 1\}$, where $\theta \in \mathbb{R}/\mathbb{Z}$ is the argument of the ray. We say that $R(\theta)$ *lands* at $z \in J_f$ if $\lim_{r\to 1}\Psi_f(re^{2\pi i\theta}) = z$. By the theorem of Carathéodory Ψ_f^{-1} extends continuous to $\partial\mathbb{D}$ with $\Psi_f^{-1}(\partial\mathbb{D}) = J_f$ if and only if J_f is locally connected.

Throughout this paper we consider the case, J_f is locally connected. Define $\alpha : \mathbb{R}/\mathbb{Z} \to J_f$, $\theta \mapsto \alpha(\theta)$ where $\alpha(\theta)$ is the landing point of ray $R(\theta)$. By (1.1), we have the following semi-conjugation,

$$f(\alpha(\theta)) = \alpha(\sigma_d(\theta)), \tag{1.2}$$

where $\sigma_d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ with $\theta \mapsto d\,\theta \bmod \mathbb{Z}$. Thus, to study the topology of the Julia set and the dynamics of f on J_f is necessarily to figure out the semi-conjugation α.

There are two questions arising naturally,

(1) For any z in J_f, is the fiber $\alpha^{-1}(z)$ finite ? In other words, are there only finite rays landing at z ?

Date: March 23, 2015.

2010 *Mathematics Subject Classification.* Primary: 37F45; Secondary: 37F10.

Key words and phrases. Polynomials; Julia sets; locally connected; External rays;

(2) Give a condition under which θ, θ' are in the same fiber. That is, when two external rays $R(\theta)$, $R(\theta')$ land at a same point ?

For the first question, if the orbit of z is finite, then the fiber $\alpha^{-1}(z)$ is finite [DH84]. If z is *wandering*, i.e., the orbit is infinite, J.Kiwi gave an upper bound $\#\alpha^{-1}(z) \leq 2^d$ [Ki02]. A.Blokh and G.Levin consider the more general problem: counting the number of external rays landing at distinct wandering points with disjoint forward orbits. Blokh and Levin worked the abstract modeling *invariant laminations* and introduced a new tool called *growing tree* [BL02]. In this paper, inspired by [Ki02], we reprove the inequality in a totally different way.

Theorem 1.1. *Let $z_1, \cdots, z_m$ be wandering branched points such that their forward orbits avoid the critical points and are pairwise disjoint. Then*

$$\sum_{1 \leq i \leq m} (v(z_i) - 2) \leq d - 2.$$

In the above theorem, a point z is called to be a *branched* point if the fiber $\alpha^{-1}(z)$ contains at least three angles and the *valence* $v(z)$ is cardinal number of $\alpha^{-1}(z)$.

For the existence, W. Thurston proved that for quadratic polynomials there is no wandering branched points. He asked a deep question concerning their existence for higher degree in the preprint [Th85]. A.Blokh and L.Oversteegen answered the question by constructing an uncountable family of cubic polynomials, the Julia set of each one is a *dendrite* and containing wandering branched points [BO08].

For the second question, following [BFH92], [Po93] and [Ki05] etc, we need a concept: *critical portrait* associated to a polynomial f.

• For critical point c in J_f, $\Theta(c)$ is the set of arguments of external rays which land at c and are inverse images of one ray landing at critical value $f(c)$. Obviously, $\#\Theta(c)$ is $\deg_f(c)$, the local degree of f at c.

• For strictly pre-periodic critical Fatou component U, $\Theta(U)$ is a collection of $\deg(f|_U)$ arguments whose rays *support* U and are inverse images of one ray *supporting* $f(U)$.

• For Fatou component cycle $U_0, \cdots, U_{p-1}$ with $f^i(U_0) = U_i, U_p := U_0$, let $U_{k_0}, \cdots, U_{k_l}$ with $0 \leq k_0 < \cdots < k_l \leq p-1$ be critical with degree $n_0, \cdots, n_l$. For $0 \leq i \leq p$, choose (z_i, θ_i), $z_i \in \partial U_i$ and $R(\theta_i)$ *supporting* U_i at z_i, such that $f^i(z_0) = z_i$, $f^p(z_p) = z_p$ and $f^i(R(\theta_0)) = R(\theta_i)$. Then $\Theta(U_{k_j})$ is the set of arguments whose external rays land on ∂U_{k_j} and are preimages of $R(\theta_{k_j+1})$, for $0 \leq j \leq l$.

Let $\mathcal{A} := \{\Theta(c_1), \cdots, \Theta(c_m), \Theta(U_1), \cdots, \Theta(U_n)\}$. For any $\Theta \in \mathcal{A}$, set

$$\widehat{\Theta} := \bigcup \{\Theta' : \exists \text{ a chain } \Theta_0 = \Theta, \cdots, \Theta_k = \Theta' \text{ in } \mathcal{A} \text{ such that } \Theta_i \bigcap \Theta_{i+1} \neq \emptyset\}.$$

The collection $\widehat{\mathcal{A}} := \{\widehat{\Theta}_1, \cdots, \widehat{\Theta}_N\}$ is called *critical portrait* associated to f. In the unit circle, there is a partition $\mathcal{P} := \{I_1, \cdots, I_d\}$ of $\mathbb{R}/\mathbb{Z} \setminus \bigcup_{1 \leq i \leq N} \widehat{\Theta}_i$. Each I_i is a finite union of open intervals with total length $1/d$.

Given a partition, we say x, x' *have the same itinerary* respect to the partition under a map g if and only if both $g^n(x)$ and $g^n(x')$ lie in the same piece of the partition, for any $n \geq 0$.

For polynomials with all critical points strictly preperiodic, B.Biefield, Y.Fisher and J.H.Hubbard showed that, if θ, θ' have the same sequence respect to the partition $\mathcal{P}$ then $\alpha(\theta) = \alpha(\theta')$ [BFH92]. A.Porier extends this result to crtitical finite polynomials, admitting periodic Fatou component [Po93]. Both of their proofs rely on the *orbifold metric* in

Julia set, on which f is expanding. In [Ki05], Kiwi considered the polynomials with all cycle repelling and Julia set connected. Based on the properties of *maximal lamination*, he proved that if θ, θ' have the same sequence respect to $\mathcal{P}$, then the *impressions* of $R(\theta)$ and $R(\theta')$ intersect.

We prove the following theorem, which is the main result of this paper.

Theorem 1.2 (Main Theorem)**.** *Let f be a polynomial with J_f locally connected. Let $\mathcal{P}$ be the partition induced by critical portrait $\widehat{\mathcal{A}}$. If θ, θ' have the same itinerary respect to $\mathcal{P}$, then either $R(\theta)$,$R(\theta')$ land at the same point or $R(\theta)$, $R(\theta')$ land at the boundary of a Fatou component U, which is eventually iterated to a siegel disk.*

Note that S.Zakeri in [Za00] proved that for *Siegel quadratic polynomial* f, i.e., $f : z \to z^2 + c$ has a fixed Siegel disk, no points has more that two rays landing at and if two rays landing at z then z must eventually hit the critical point 0.

The following Corollary holds immediately.

Corollary 1.3 (No wandering continua in J_f)**.** *Let f be a polynomial with J_f locally connected. Then there is no wandering continua in J_f.*

We have to point out that A.Blokh and G.Levin also proved the above corollary [BL02]. And J.Kiwi proved that, for polynomials without irrational neutral periodic orbits f, J_f is locally connected if and only if f has no wandering continua in J_f. Kiwi's proof relies on constructing a puzzle piece around each pre-periodic or periodic point of a polynomial f with all cycles repelling [Ki04].

1.1. **Motivation**

One of our motivation is to study the *core-entropy* of polynomials. Suppose X is a compact metric space and $g : X \to X$ is continuous. The topological entropy of g is measuring the complexity of iteration from the growth rate of the number of distinguishable orbits. The *core-entropy* of polynomial f is the topological entropy of f on its f-invaritant set *Hubbard tree*, i.e., the convex hull of the critical orbits within the (filled) Julia set. Let $Acc(f)$ be the set of all *biaccessible* angles θ, i.e., there exist at least two rays landing at $\alpha(\theta)$. Then the core-entropy $h(f)$ is related to the Hausdorff dimension of $Acc(f)$ in the following way,

$$h(f) = \log d \cdot \text{H.dim}\, Acc(f). \tag{1.3}$$

These quantities are according to W.Thurston who firstly introduced and explored the core-entropy of polynomials.

For quadratic polynomials, G.Tiozzo showed the continuity of core-entropy along *principal veins* of the Mandelbrot set $\mathcal{M}$ in [Ti13]. This result is generalized by W. Jung to all veins [Ju13]. Recently, G.Tiozzo proves that the function $\theta \mapsto h(f_\theta)$ with $f_\theta(z) = z^2 + c_\theta$ is continuous.

A.Douady proved the monotonicity of core-entropy along real vein $\mathcal{M} \bigcap \mathbb{R}$[Do95]. The monotonicity for all postcritically finite quadratic polynomials is proved in Tao Li's thesis [Li07]. As an application of theorem 1.2, we extend Tao Li's result to a quadratic family $\mathcal{F} := \{f_c = z^2 + c : f_c$ has no Siegel disks and J_{f_c} is locally connected $\}$.

Theorem 1.4 (Monotonicity of core-entropy)**.** *For any $f_c, f_{c'} \in \mathcal{F}$, if $f_c \prec f_{c'}$, then $Acc(f_c) \subseteq Acc(f_{c'})$ and so $h(f_c) \leq h(f_{c'})$.*

For any f_c, $f_{c'}$ in $\mathcal{F}$, we say $f_c \prec f_{c'}$ if and only if $I_c \supseteq I_{c'}$, where I_c is the *characteristic arc* of f_c. See section 7 for details.

1.2. **Sketch of the proof and outline of the paper**

The proof of main theorem 1.2 is based on the analysis in the dynamical plane. There is a partition $\{\Pi_i\}_{1\leq i\leq d}$ of $\mathbb{C}$, induced by critical portrait. It has nice properties: for any points $x, y \in \Pi_i \bigcap J_f$, the *regulated arc* $[x, y] \subseteq \overline{\Pi}_i$ and $F|_{[x,y]}$ is one-to-one, where F is a topological polynomial which takes the same value as f in $\overline{\Omega}_f$. Thus if $x \neq y$ have the same itinerary respect to $\{\Pi_i\}$, we obtain a sequence $\{F^n[x, y]\}$ of regulated arc. The sequence will eventually meet $\bigcup_{1\leq i\leq d} \partial\Pi_i \bigcap J_f$. However it is difficult to prove that the partition $\{\Pi_i\}_{1\leq i\leq d}$ separates $\overline{f^n(x), f^n(y)}$ for some n. To overcome this difficult, we use this sequence to construct a wandering arc in J_f, which is a contradiction.

In section 2, we prove theorem 1.1. This key result is useful to show the fact of no wandering regulated arcs in Lemma 6.1.

In section 3, we give the construction of regulated arcs and describe its properties.

In section 4, we explain how to get a desired topological polynomial F by modifying f in Fatou set.

Section 5 analysis the properties of partition induced by critical portrait in the dynamic plane.

The main Theorem 1.2 is proved in section 6.

In the last section, we discuss characteristic arcs in details and give an application of the main theorem to the monotonicity of core entropy for a quadratic polynomial family.

Acknowledgment. The author would like very much to thank Professors Weiyuan Qiu and Lei Tan for their introductions, support and suggestion over these years. The author also wants to thank China Scholarship Council for supports.

2. Wandering Orbit Portrait

If not otherwise stated, we assume f to be a polynomial with degree $d \geq 2$ and J_f locally connected. Our objective is to prove the Proposition 2.5.

2.1. **Portraits**

Now we give some definitions by following [Mi00][GM93][BFH92][Ki02] etc.

For a point z in J_f, the *valence* of z, written $v(z)$, is the number of external rays landing at z. Then $1 \leq v(z) \leq \infty$. If $v(z) \geq 3$, z is called to be a *branched* point. z is called to be *wandering* if $f^m(z) \neq f^n(z)$ for $m \neq n \geq 0$.

Let $T := \{\theta_1, \ldots, \theta_n\}$, $\theta_i \in \mathbb{R}/\mathbb{Z}$, $3 \leq n < \infty$. T is called to be a *portrait* of z if all $R(\theta_i)$ land at z. Denote by $\alpha(T) := z$ the base point and $v(T) := n$ the *valence* of T. Obviously, we have $3 \leq v(T) \leq v(z)$.

Let T be a portrait of z. Each connected components of $\mathbb{C} \setminus \bigcup_{\theta\in T} \overline{R}(\theta)$ is called a *sector* of T based at z. Evidently, any sector S of T is bounded by two rays $R(\theta_a), R(\theta_b)$ with $\theta_a, \theta_b \in T$. Let $I(S)$ be the segment of $\mathbb{R}/\mathbb{Z} \setminus \{\theta_a, \theta_b\}$ disjoint with T. Then there is a one-to-one correspondence betweens sectors based at z and the segments of $\mathbb{R}/\mathbb{Z} \setminus T$, characterized by the property that $R(t)$ is contained in S if and only if t is contained in $I(S)$. Denote the correspondence by $I : S \mapsto I(S)$.

We define the *annular size* of a sector S, written $l(S)$, by the length of the corresponding arc $I(S)$ in $\mathbb{R}/\mathbb{Z}$. Number the n sectors of T by $S_1(T), \cdots, S_n(T)$ according to their length:

$$l(S_1(T)) \leq l(S_2(T) \cdots \leq l(S_n(T)).$$

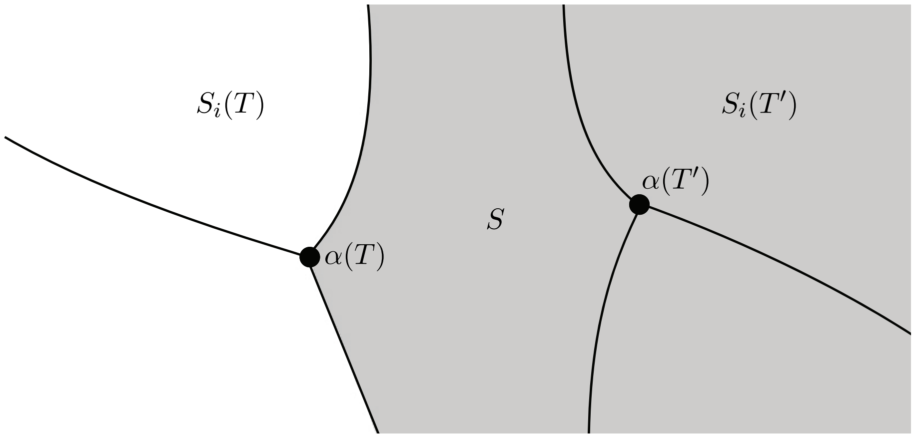

FIGURE 1. Portraits T, T' with distinct base points

By means of *critical sector* or *critical value sector* if a sector S contains critical points or critical values.

Lemma 2.1 (For portraits with distinct base points)**.** *Let T, T' be two portraits with $\alpha(T) \neq \alpha(T')$. Let S resp. S' be the sector of T resp. T' such that $\alpha(T')$ resp. $\alpha(T)$ is contained in S resp. S'. Then all but S' resp. S of the sectors of T' resp. S' are contained in S resp. S' and so we have*

$$l(S_i(T')) < l(S) \text{ for } S_i(T') \neq S' \quad \text{and} \quad l(S_i(T))) < l(S') \text{ for } S_i(T) \neq S.$$

Proof. Set $G := \bigcup_{1\leq i\leq v(T)} R(\theta_i) \cup \{\alpha(T)\}$ and $G' := \bigcup_{1\leq i\leq v(T')} R(\theta'_i) \cup \{\alpha(T')\}$. They are disjoint close connected subset of $\mathbb{C}$. So G' is contained in exactly one connected component of $\mathbb{C} \setminus G$, that is, some sector of T. Since $\alpha(T') \in S$ and $\alpha(T) \in S'$, we have $G' \subseteq S$ and $G \subseteq S'$. See figure 1. Thus all sectors of T' resp. T except S' resp. S are contained in S resp. S'. The lemma follows. $\square$

2.2. **Sector maps**

Lemma 2.2 (Properties of sector maps)**.** *Let $T = \{\theta_1, \cdots, \theta_{v(T)}\}$ be a portrait such that the base point $\alpha(T)$ is not a critical point of f, here θ_i are enumerated in cyclic order around the circle. Then*

(1) *The map $\sigma_d : t \mapsto dt \bmod \mathbb{Z}$ carries T bijectively onto the portrait $T' := \{\sigma_d(\theta_1), \cdots, \sigma_d(\theta_{v(T)})\}$ of $f(\alpha(T))$ preserving cyclic order. Define the **portrait map** to be*

$$\sigma_d : T \mapsto T'.$$

(2) *Let S be a sector of T bounded by $R(\theta_a)$ and $R(\theta_b)$. Then the **sector map***

$$\sigma_d : S \mapsto S',$$

where S' is the sector of T' bounded by $R(\sigma_d(\theta_a))$ and $R(\sigma_d(\theta_b))$, is well defined and one-to-one.

(3) *$l(\sigma_d(S)) = d\,l(S) \bmod \mathbb{Z}$. Moreover, the integer $n_0 := d\,l(S) - l(\sigma_d(S))$ is the number of critical points, counting multiplicity, of f contained in S.*

(4) *If $n_0 \geq 1$, then $\sigma_d(S)$ contains at least one critical values.*

(5) *If $l(S) < 1/d$, then $l(\sigma_d(S)) = d\,l(S)$ and the restriction of f on S is homeomorphic.*

Note that we distinguish the definitions of σ_d by acting on different categories.

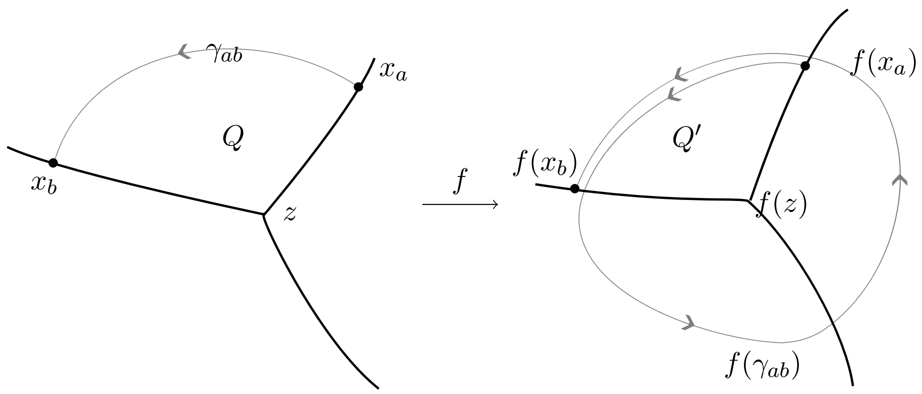

FIGURE 2. Sector maps

Proof. Let $z := \alpha(T)$. Since z is not critical. f is a locally orientation-preserving homeomorphism at z. Note that the $v(T)$ angles θ_i in $\mathbb{R}/\mathbb{Z}$ and rays $R(\theta_i)$ around z are identical in order. Moreover, the order of $R(\theta_i)$ can be measured within an arbitrarily small neighborhood of z. It follows that all rays with angels in T' land together at $f(\alpha(T)$ and σ_d sends angles in T onto T' bijectively and keeping the order. Thus (1) and (2) follows.

For (3), suppose S is bounded by $R(\theta_a)$, $R(\theta_b)$. Let $\gamma_{ab}(t)$ be a segment of equipotential curve $\{z \in \mathbb{C} : G_f(z) = 1\}$ which lies in $\overline{S}$ with $\gamma(0) = x_a$ and $\gamma(1) = x_b$, where $\{x_a\} := \gamma_{ab} \bigcap R(\theta_a)$ and $\{x_b\} := \gamma_{ab} \bigcap R(\theta_b)$. Let Q be the close domain bounded by $R(\theta_a), \gamma_{ab}$ and $R(\theta_b)$. See figure 2.

Consider the image $f(\partial Q)$. It starts at $f(z)$ and goes along the rays $R(\sigma(\theta_a))$ until it arrives at $f(x_a)$, then it rotates $d\ l(S)$ angles, parameterized by angles of external rays, along the equipotential curve $\{z \in \mathbb{C} : G_f(z) = d\}$ to $f(x_b)$, finally it turns to $f(z)$ along $R(\sigma(\theta_b))$ and stops.

Let $G_d := \{z \in \mathbb{C} : G_f(z) < d\}$. Let Q' be the domain $\sigma_d(S) \bigcap G_d$. By the arguments above, it is easy to see that $f(\gamma_{ab})$ surround ∂G_d in n_0 times and overlap $\partial G_d \bigcap \partial Q'$ one time more. Thus,

$$l(\sigma_d(S)) + n_0 = d\, l(S).$$

Moreover, $z \in \partial G_d \setminus \partial Q'$ has n_0 preimages in γ_{ab} and $z \in \partial G_d \bigcap \partial Q'$ has $n_0 + 1$ preimages in γ_{ab}. The winding number of points in $G_d \setminus f(\partial Q')$ are

$$w(z) = \begin{cases} n_0 + 1 & z \in Q' \\ n_0 & z \in G_d \setminus \overline{Q'}. \end{cases} \tag{2.1}$$

By the Arguments Principle, every point $z \in G_d \setminus f(\partial Q)$ has $w(z)$ preimages, counting multiplicity, in Q.

Now claim that every points z in $\partial Q' \setminus \partial G_d$, consisting of two segments of external rays, has $n_0 + 1$ preimages, counting multiplicity, in Q. Since such z can not be a critical value, choose sufficiently small enough neighborhood U_z such that the restriction of f on every component $f^{-1}U_z$ is homeomorphic. Since $U_z \bigcap Q'$ has $n_0 + 1$ components in Q and Q is closed, z must have $n_0 + 1$ preimages in Q as well.

Let $v_1, \cdots, v_n \in f(Q)$ be the critical value of $f|_Q$. Let μ_i be the total multiplicity of critical points in Q mapped to v_i. Choose a cell subdivision Δ of $f(Q)$ such that the set of its 0-cells contains $\{f(z), v_1, \cdots, v_n\}$ and the set of 1-cells contains $\partial Q'$. Let $\Delta_1 :=$ {complexes of Δ contained in $\overline{Q'}$} and $\Delta_2 := \Delta \setminus \Delta_1$. It follows that Δ_1 is a cell subdivision of Q'. Set x_i, y_i, z_i to be the number of 0-cell, 1-cell and 2-cell of Δ_i.

Computing the Euler characteristic, we have

$$\mathcal{X}(\overline{Q'}) = x_1 - y_1 + z_1 = +1 \tag{2.2}$$

and

$$\mathcal{X}(f(Q)) = (x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2) = +1. \tag{2.3}$$

After lifting every complexes in Δ by $f|_Q$, we obtain a cell subdivision Δ_0 of Q. Then

$$\begin{aligned}\mathcal{X}(Q) = &[(n_0+1)x_1 + n_0x_2 - \sum_{1\le i\le n} \mu_i] - [(n_0+1)y_1 + n_0y_2] \\ &+ [(n_0+1)z_1 + n_0z_2] = +1.\end{aligned} \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we have

$$\sum_{1\le i\le n} \mu_i = n_0.$$

Thus (3) is completed.

For (4), we use the notations as above. If not, assume Q' contains no critical values. Then every component of $f|_Q^{-1}(Q')$ is simply connected and f on the closure of which is homeomorphic. Consider the component C with $\partial Q \setminus \gamma_{ab} \subseteq \overline{C}$. $f|_{\overline{C}}$ cannot be one-to-one, a contradiction.

For (5), it follows directly by (3). □

2.3. **Dynamics of wandering portraits**

Portrait T is called to be *wandering* if and only if the point $\alpha(T)$ is wandering and not iterated to critical points of f. We denote by $T_n := \sigma_d^{\circ n}(T)$.

Recall that $S_1(T), \cdots, S_{v(T)}(T)$ are the $v(T)$ sectors of T enumerated by the order of their annular size. We have the following lemma. See also in [Ki02].

Lemma 2.3. *Let T be a wandering portrait. Then*

$$\lim_{n\to\infty} l(S_{v(T)-2}(T_n)) = 0.$$

Proof. If not, there exist a number $a > 0$ and an infinite sequence T_{n_k} such that

$$5a/6 < l(S_{v(T)-2}(T_{n_k}) < 7a/6.$$

The sectors $S_{v(T)-2}(T_{n_k})$ can not be pairwise disjoint. Because otherwise the total length of the infinite many intervals $I(S_{v(T)-2}(T_{n_k})$ would be greater than 1.

Then there exist $n_i \ne n_j$ such that $S_{v(T)-2}(T_{n_i}) \bigcap S_{v(T)-2}(T_{n_j}) \ne \emptyset$. By Lemma 2.1, we can assume $\alpha(T_{n_i}) \in S_{v(T)-2}(T_{n_j})$ and both sectors $S_{v(T)-2}(T_{n_i})$ and $S_{v(T)-1}(T_{n_i})$ are contained in $S_{v(T)-2}(T_{n_j})$. Thus,

$$l(S_{v(T)-2}(T_{n_j})) > l(S_{v(T)-2}(T_{n_i})) + l(S_{v(T)-1}(T_{n_j})) > 5a/3,$$

a contradiction. □

By lemma 2.3, for any wandering portrait T, the annular size of sectors T_n, except the two large ones, will converges to zero. Furthermore, a similar argument can show that $\liminf l(S_{v(T)-1}(T_n)) = 0$. We will not use this fact. We are more interested in the moment when a "wide" critical sector is mapped to a "narrow" critical value sector.

For any sufficiently small $\epsilon > 0$ and $1 \le k \le v(T) - 2$, Set

$$n_{\epsilon,k}(T) := \min\{n : l(S_k(T_n)) < \epsilon\}.$$

By lemma 2.3, $l(S_k(T_n))$ will eventually be smaller than ϵ as $n \to \infty$. Thus $n_{\epsilon,k}(T)$ is well defined. We have the following,

Lemma 2.4. *Let T be a wandering portrait. Then There exists $\delta > 0$ such that for any $\epsilon < \delta$, denote by $n_{\epsilon,k} := n_{\epsilon,k}(T)$, $1 \leq k \leq v(T) - 2$, we have $l(S_{k+1}(T_{n_{\epsilon,k}})) > \epsilon$ and there exists at least one critical value sector $S_{k_0}(T_{n_{\epsilon,k}})$ with $1 \leq k_0 \leq k$.*

Proof. By lemma 2.3, there exists an integer $N \geq 1$ such that, for any $n \geq N$,

$$l(S_{v(T)-2}(T_n)) < \frac{1}{2v(T)d}.$$

Set

$$\delta := \min_{1\leq i\leq N}\{\, l(S_1(T_i))\,\}.$$

For any $\epsilon < \delta$, since $n_{\epsilon,k}$ is the first time that the k^{th} sector has length strictly less than ϵ. We have

$$\epsilon \leq l(S_k(T_{n_{\epsilon,k}-1})) \leq l(S_{v(T)-2}(T_{n_{\epsilon,k}-1})) < \frac{1}{2v(T)d}.$$

By Lemma 2.2 (5), f maps the $v(T) - 2$ sectors $S_1(T_{n_{\epsilon,k}-1}), \cdots, S_{v(T)-2}(T_{n_{\epsilon,k}-1})$ onto sectors of $T_{n_{\epsilon,k}}$ homeomorphic with their length multiplied by d. Then

$$l(\sigma_d(S_k(T_{n_{\epsilon,k}-1}))) \geq d\epsilon > \epsilon \quad \text{and} \quad l(S_k(T_{n_{\epsilon,k}})) < \epsilon.$$

This means σ_d must map at least one of the two sectors $S_{v(T)-1}(T_{n_{\epsilon,k}-1})$ and $S_{v(T)}(T_{n_{\epsilon,k}-1})$ onto a "narrow" sector $S_{k_0}(T_{n_{\epsilon,k}})$ with $l(S_{k_0}(T_{n_{\epsilon,k}})) < \epsilon$. By lemma 2.2 (4), $S_{k_0}(T_{n_{\epsilon,k}})$ is a critical value sector. Actually, there are only one of the above two sectors mapped to such "narrow" sector. Because the total length of the $v(T) - 1$ images,

$$l(S_{k_0}(T_{n_{\epsilon,k}})) + \sum_{1\leq i\leq v(T)-2} l(\sigma_d(S_i(T_{n_{\epsilon,k}-1}))) < \frac{1}{2}.$$

It follows that the other sector is mapped to the widest sector $S_{v(T)}(T_{\epsilon,k})$ with length $> \frac{1}{2}$. Thus, we have

$$S_{k+1}(T_{n_{\epsilon,k}}) = \sigma_d(S_k(T_{n_{\epsilon,k}-1})) \geq d\epsilon > \epsilon \quad \text{and} \quad 1 \leq k_0 \leq k.$$

The proof is completed. □

2.4. Proof of theorem 1.1

Proposition 2.5. *Let $T^{(1)}, \cdots, T^{(m)}$ be wandering portraits such that $\alpha(T^{(i)})$ have disjoint forward orbits. Then*

$$\sum_{1\leq i\leq m} (v(T^{(i)}) - 2) \leq d - 2. \tag{2.5}$$

Proof. Let $\epsilon_0 > 0$ be smaller than any $\delta_{T^{(i)}}$, for $1 \leq i \leq m$, as stated in the Lemma 2.4. Firstly, applying Lemma 2.4 to the case $T = T^{(1)}$, $k = 1$ and $\epsilon = \epsilon_0$, we obtain a critical value sector $S_1(T^{(1)}_{n_{\epsilon_0,1}})$ and

$$\epsilon := l(S_1(T^{(1)}_{n_{\epsilon_0,1}})) < \epsilon_0 < l(S_2(T^{(1)}_{n_{\epsilon_0,1}})). \tag{2.6}$$

Let $n_{k,i} := n_{\epsilon,k}(T^{(i)})$, for $1 \leq i \leq m$, $1 \leq k \leq v(T^{(i)})$. By the definition of $n_{k,i}$ and orbits of $\alpha(T^{(i)})$ disjoint in the condition, it is easy to see that

$$n_{k_1,i} \neq n_{k_2,j} \neq n_{\epsilon_0,1} \quad \text{and} \quad \alpha(T^{(i)}_{n_{i,k_1}}) \neq \alpha(T^{(j)}_{n_{j,k_2}}) \neq \alpha(T^{(1)}_{n_{\epsilon_0,1}}), \tag{2.7}$$

for $1 \leq i, j \leq m$ and $(i, k_1) \neq (j, k_2)$, $1 \leq k_1 \leq v(T^{(i)}), 1 \leq k_2 \leq v(T^{(j)})$. By Lemma 2.4 again, we obtain $N := \sum_{1\leq i\leq m}(v(T^{(i)})-2)$ critical value sectors, denoted by $S_{\tau(k,i)}(T^{(i)}_{n_{k,i}})$, and we have

$$l(S_{\tau(k,i)}(T^{(i)}_{n_{k,i}})) < \epsilon < l(S_{k+1}(T^{(i)}_{n_{k,i}})), \quad 1 \leq \tau(k,i) \leq k. \tag{2.8}$$

By (2.7) and Lemma 2.1, for any distinct two of the $N+1$ critical value sectors $S_1(T^{(1)}_{n_{\epsilon_0},1})$ and $S_{\tau(k,i)}(T^{(i)}_{n_{k,i}})$, they are neither disjoint or one contains the other.

We claim that the latter case can not happen. If not, suppose $S_{\tau(k_1,i_1)}(T^{(i_1)}_{n_{k_1,i_1}})$ are contained in $S_{\tau(k_2,i_2)}(T^{(i_2)}_{n_{k_2,i_2}})$. By Lemma 2.1, we have

$$S_{k_1+1}(T^{(i_1)}_{n_{k_1,i_1}}) \subset S_{\tau(k_2,i_2)}(T^{(i_2)}_{n_{k_2,i_2}}) \quad \text{and} \quad l(S_{k_1+1}(T^{(i_1)}_{n_{k_1,i_1}})) < l(S_{\tau(k_2,i_2)}(T^{(i_2)}_{n_{k_2,i_2}})).$$

This contradicts (2.8). If one of them is $S_1(T^{(1)}_{n_{\epsilon_0},1})$, similarly by (2.6), it is impossible.

Thus the $N+1$ critical values sectors are pairwise disjoint and each of them contains at least one critical value. Since it is known that, for degree d polynomials, there exist at most $d-1$ critical values. So $N+1 \leq d-1$. The proof is completed. □

Proof of Theorem 1.1. The theorem follows immediately by Propositions 2.5. □

Actually the result in this section can extended to polynomials with Julia set connected or not connected. We omit the details. See Appendix A in [Ki02].

Corollary 2.6. *Let f be a polynomial with the Julia set J_f locally connected. Then the number of grand orbits of wandering branched points is finite.*

3. Regulated arcs

According to Fatou and Sullivan, every bounded Fatou components of polynomials must eventually be mapped to the immediate basin of attraction of an attracting periodic point, or to an attracting petal of a parabolic periodic point, or to a periodic Siegel disk[Mi06][Su83]. We refer to these cases simply as *hyperbolic*, *parabolic* and *Siegel* cases.

For any two points $x, y \in K_f$ there usually exist more than one arc γ in K_f connecting x and y. In the following, we will give the definition of *internal ray* and *regulated arc* in K_f and show how to choose a canonical embedded arc between any two points in the filled Julia set. Under certain condition, such arc is unique (See Lemma 3.4).

3.1. **Extended rays**

Now consider the polynomial f with J_f locally connected. We have,

Lemma 3.1 (Bounded Fatou components are Jordan domains)**.** *For any bounded Fatou component U, ∂U is a Jordan curve.*

Proof. Since J_f is locally connected, then ∂U is locally connected. Consider the Riemann map: $\Phi_U : \mathbb{D} \to U$, it extends continuously to $\mathbb{D}$ by Carathéodory Theorem. Therefore, ∂U is the curve $\Phi_U(S^1)$. If $\Phi_U|S^1$ is not injective. Then there exists $t < t'$ in S^1 with $\Phi_U(t) = \Phi_U(t')$. The two rays $\Phi_U([0,1]e^{2\pi i t})$ and $\Phi_U([0,1]e^{2\pi i t'})$ will bound a domain U', which contains subset of the Julia set $\Phi_U(\{e^{2\pi i\eta} : t < \eta < t'\}$. Since J_f is the boundary of infinity attracting domain Ω_f, some points in U' will escape to infinity. This contradicts the Maximum Value Principle. □

Given any bounded Fatou component U, pick a point $c(U)$ in U as *center point* and a Riemann map $\varphi_U : U \to \mathbb{D}$ with $\varphi_U(c(U)) = 0$. Then extend it to a homeomorphism $\varphi_U : \overline{U} \to \overline{\mathbb{D}}$ by Carathédory Theorem.

An arc in $\overline{U}$ of the form $\varphi_U^{-1}\{re^{i\theta} : 0 \le r \le 1\}$ is called a *internal ray* of U with angle θ. All these internal rays meet at the center point $c(U)$. Each ray has a well defined landing point in the boundary of U. Conversely, for any point z in the boundary of U, there exists an unique internal ray of U landing at z. We denote this internal ray by $R_U(z)$. For any $\theta \in \mathbb{R}/\mathbb{Z}$, if $\alpha(\theta) = z \in \partial U$, define the *extended ray*

$$\widehat{R}_U(\theta) := R(\theta) \bigcup R_U(z).$$

3.2. **Components of $J_f \setminus \{x\}$ are arcwise connected**

Recall that a topological space X is said to be *arcwise connected* provided that there is a topological embedding of $[0,1]$ into X (called *arc*) joining any two given distinct points. If $p \in X$, then X is said to be *locally arcwise connected* resp. *locally connected* at p, provided that every neighborhood of p contains an arcwise connected neighborhood resp. connected neighborhood of p. The space X is said to be *locally arcwise connected* resp. *locally connected*, provided that X is locally arcwise connected resp. locally connected at every point. We have the following well-know result.

Lemma 3.2. *If a compact metric space X is locally connected, then it is locally arcwise connected.*

It follows directly by the Lemma 17.17 and Lemma 17.18 in [Mi06].

Corollary 3.3. *If a compact metric space X is connected and locally connected, then it is arcwise connected. Moreover, every connected component of $X \setminus \{x\}$ is arcwise connected for any x in X.*

Proof. Fix $p \in X$, define Y as follows

$$Y = \{p\} \bigcup \{x \in X : \text{there is an arc in } X \text{ joining } p \text{ and } x\}$$

Obviously, $Y \neq \emptyset$. Since X is locally arcwise connected by Lemma 3.2. A simple argument show that both Y and $X \setminus Y$ are open in X. Thus, since $Y \neq \emptyset$ and X is connected, we must have $Y = X$. So X is arcwise connected.

Let C be a connected component of $X \setminus \{x\}$, then C is open in X. Indeed, since X is locally connected, every z in C has a sufficiently small connected neighborhood W_z avoiding x, thus $W_z \subseteq C$.

Since X is locally arcwise connected by Lemma 3.2, C is locally arcwise connected as well. Then one can show that C is arcwise connected in exactly the same way as above. □

Hence all Julia sets and filled Julia sets discussed in this paper are locally arcwise connected and arcwise connected.

3.3. **Uniqueness of regulated arc**

An arc γ in K_f is called to be *regulated* if it joins two distinct points in J_f and for any bounded Fatou component U, the intersection $\gamma \bigcap \overline{U}$ is an empty set or a point or exactly two internal rays.

Lemma 3.4 (Uniqueness of regulated arc)**.** *For any two distinct points x, y in J_f, there exists only one regulated arc in K_f joining x and y.*

Proof. Let $\eta(t) : [0,1] \to K_f$ be the arc joining x and y with $\eta(0) = x$ and $\eta(1) = y$. For any Fatou component U whose closure intersects the arc η, set $x_U = \inf_{0 \le t \le 1}\{t : \eta(t) \in \overline{U}\}$, i.e., the first time η meets $\overline{U}$, and $y_U = \sup_{0 \le t \le 1}\{t : \eta(t) \in \overline{U}\}$, i.e., the last time η meets

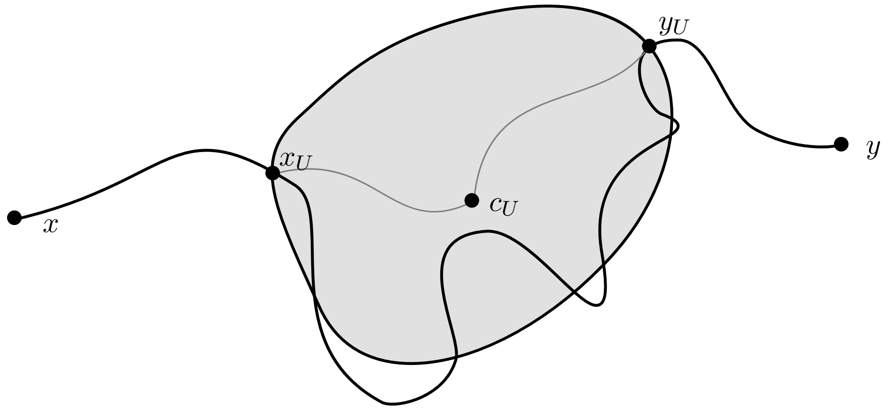

FIGURE 3. Constructing regulated arc

$\overline{U}$. If $x_U \neq y_U$. Then we replace the segment $\eta((x_U, y_U))$ starting at $\eta(x_U)$ ending at $\eta(y_U)$ by the internal rays $R_U(\eta(x_U)$ and $R_U(\eta(y_U))$, updating $\eta = \eta[0, x_U] \cup R_U(\eta(x_U)) \cup R_U(\eta(y_U)) \cup \eta[y_U, 1]$. After doing these processes for countable many Fatou components, we obtain a regulated arc η connecting x and y as required.

For the uniqueness, if η' is the other one. Then $\mathbb{C} \setminus \eta \cup \eta'$ consists of several disjoint connected components. Let W be one of the bounded component in K_f. Then W is a Jordan domain and $\partial W \subseteq \eta \cup \eta'$. Applying the Maximum value Principle, W belongs to the Fatou set. Let U be the Fatou component containing W. Thus $\overline{W} \subseteq \overline{U}$. Since $(\eta \cup \eta') \bigcap \overline{U}$ consists at most four internal rays and all of the internal rays hit only at the center point $c(U)$. It is impossible for them to bounded a domain W, a contradiction. $\square$

The regulated arc is denoted by $[x, y]$. The open arc (x, y) is defined by $[x, y] \setminus \{x, y\}$, and similarly the semi-open arc $[x, y)$ and $(x, y]$.

3.4. **Quasi-buried regulated arc**

A regulated arc γ is called *quasi-buried* if the intersection between γ and the closure of any bounded Fatou component is either empty or exactly one point. Obviously if $K_f = J_f$, every regulated arc is quasi-buried. But if $K_f \neq J_f$, does there exist quasi-buried arc? We conjecture that for some special locally connected J_f such regulated arc exists.

Similarly as the quadratic case, for high degree polynomials, we still define β *fixed point* as the landing point of external ray $R(0)$. It can be a branched point with at most $d-1$ external rays landing at.

Let $E' := \bigcup_{i \geq 0} \{f^{-i}(\beta)\}$, i.e., the preimages of β fixed points. Set E be the union of E' and branched points in J_f. If J_f is a segment, then $E' = E$. We know that E' is dense in J_f [Mi06] and thus E is dense in J_f. Moreover, we have the following,

Lemma 3.5 (Denseness of E in quasi-buried arcs)**.** *Let $I := [x, y]$ be a quasi-buried regulated arc in K_f. Then E is dense in I.*

Proof. Let p be any point in $I \setminus \{x, y\}$. Since J_f is locally arcwise connected by Lemma 3.2, we can choose sufficiently small arcwise connected neighborhood W_p in J_f such that

$$W_p \bigcap \{x, y\} = \emptyset \quad \text{and} \quad p \in W_p \bigcap I \Subset I. \tag{3.1}$$

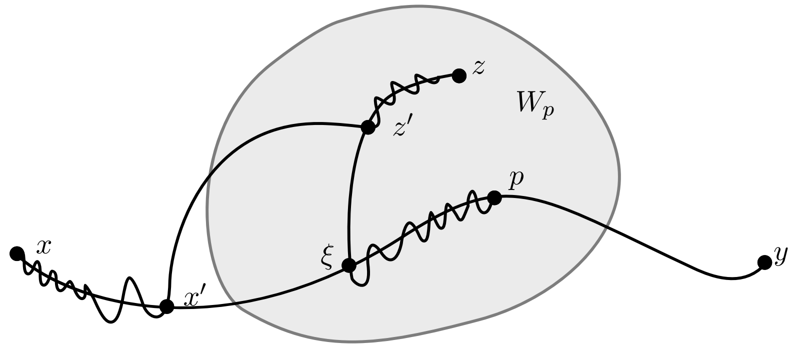

FIGURE 4. illustrating the proof of lemma 3.5

See figure 4. By the denseness of E in J_f, $W_p \bigcap E$ is not empty. Choose a point z in $W_p \bigcap E$. If z is in I, then we are done. If not, there exists an arc γ_{zp} in W_p joining z and p, because W_p is arcwise connected.

Let ξ be the point at which γ_{zp} meets I at the first time. Then ξ belongs to $I \setminus \{x, y\}$ by (3.1). Let $\gamma_{z\xi}$ be the subarc of γ_{zp} joining z and ξ. It follows that the three arcs $\gamma_{z\xi}$, $[x, \xi]$ and $[y, \xi]$, meeting at ξ, form a "Y" shape.

We are left to show that ξ is a branched point. Due to the Theorem 6.6 in [Mc95], we only have to proof that $K_f \setminus \{\xi\}$ has at least three connected components. Actually we have the following.

Claim that x, y and z lie in distinct connected components of $K_f \setminus \{\xi\}$.

Proof. If not, suppose x, z in the same component C. By Corollary 3.3, C is arcwise connected, thus there exists an arc $\gamma_{xz}(t)$ in C joining x and z with $\gamma_{xz}(0) = x$ and $\gamma_{xz}(1) = y$. Set

$$t_x := \sup_{0 \le t \le 1}\{t : \gamma_{xz}(t) \in [x, \xi]\} \quad \text{and} \quad t_z := \inf_{0 \le t \le 1}\{t : \gamma_{xz}(t) \in \gamma_{z\xi}\}.$$

Denote by $x' = \gamma_{xz}(t_x)$ and $z' = \gamma_{xz}(t_z)$. Note that x', z' are contained in $[x, \xi)$ and $\gamma_{z\xi} \setminus \{\xi\}$ respectively. Let $\gamma_{x'z'}$ be the subarc in γ_{xz} joining x' and z'. It follows that $\eta := \gamma_{x'z'} \cup [z', \xi] \cup [x', \xi]$ bounds a Jordan domain V. By the Maximum Value Principle, V must be contained in some Fatou component U. Then $[x', \xi] \subseteq \partial U$. This contradicts the definition that $I \bigcap \overline{U}$ is either empty or only one point. A same argument show that y, z and x, y cannot lie in the same component of $K_f \setminus \{\xi\}$. The claim is completed. □

Thus ξ is a branched point. The proof is completed. □

4. The topological polynomial F

The regulated arcs in K_f may not be preserved by the dynamic of f. In this section, we will construct a nice topological polynomial F by modifying f in each bounded Fatou set. F will coincide with f on the basin of infinity and the Julia set J_f. The above difficulty can be most conveniently overcome by investigating F instead of f. Since we only interest in the Julia set and the combination of external angles. These changes make no essentially differences.

4.1. **Branched covering map**

Let X and Y be domains in $\overline{\mathbb{C}}$, $g : X \to Y$ be a continuous map. Then g is called a *branched covering map* if we can write it locally as the map $z \mapsto z^n$ for some $n \in \mathbb{N}$ after orientation-preserving homeomorphic changes of coordinates in domain and range. More precisely, we require that for each point $q \in Y$ and any preimage p in $g^{-1}(q)$ there exists $n \in \mathbb{N}$, open neighborhoods U of p and V of q, open neighborhoods U' and V' of $0 \in \mathbb{C}$ and orientation-preserving homeomorphisms $\phi : U \to U'$ and $\psi : V \to V'$ with $\phi(p) = 0$ and $\psi(q) = 0$ such that

$$(\psi \circ g \circ \phi^{-1})(z) = z^n \tag{4.1}$$

for all $z \in U'$.

The integer $\deg_g(p) := n \geq 1$ is uniquely determined by g and p and called the *local degree* of g at p. A point $c \in \overline{\mathbb{C}}$ with $\deg_g(c) \geq 2$ is called a *critical point* of g and its image $g(c)$ *critical value*. Moreover, g is an open and surjective mapping. If the set of all critical points only consists of finite isolated points, then g is finite-to-one, i.e., every point has finitely many preimages under g. More precisely, if $\deg(g)$ is the topological degree of g, then

$$\sum_{p \in g^{-1}(q)} \deg_g(p) = \deg(g)$$

for every $q \in Y$. A branched covering with no critical point is called *unbranched covering*. A branched covering map $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called *topological polynomial* if $g^{-1}(\infty) = \infty$, that is , ∞ is a fixed point with local degree $\deg(g)$.

4.2. **From polynomial f to topological polynomial F**

For polynomial f, a bounded Fatou component is called *critical Fatou component* if it contains critical point of f. Its image is *critical value Fatou component*. Given a bounded Fatou component U, f maps U to Fatou component U' holomorphic. $f|_{\partial U} : \partial U \to \partial U'$ is an unbranched covering map with degree $\deg(f|_U)$.

Recall that $\varphi_U : U \to \mathbb{D}$ $c(U) \mapsto 0$ is a conformal parameterization . Set

$$\varphi_{UU'} := \varphi_{U'} \circ f \circ \varphi_U^{-1}|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \partial\mathbb{D}.$$

Now we extend $\varphi_{UU'}$ to be

$$\varphi_{UU'} : \overline{\mathbb{D}} \to \overline{\mathbb{D}} \qquad re^{2\pi i\theta} \mapsto r\varphi_{UU'}(e^{2\pi i\theta}).$$

One can check that $\varphi_{UU'}$ is a branched covering. Define $F_U := \varphi_{U'}^{-1} \circ \varphi_{UU'} \circ \varphi_U : \overline{U} \to \overline{U'}$ by the following communicate diagram,

$$\begin{array}{ccc} (U, c(U)) & \xrightarrow{F_U} & (U', c(U')) \\ \psi_U \Big\downarrow & & \Big\downarrow \psi_{U'} \\ (\mathbb{D}, 0) & \xrightarrow{\psi_{UU'}} & (\mathbb{D}, 0). \end{array}$$

By the construction, F_U satisfies

- $F_U|_{\partial U} = f|_{\partial U}$.
- F_U sends $c(U)$ to $c(U')$.
- F_U is a branched covering with degree $\deg(f|_U)$ and the critical point can only be $c(U)$.
- F_U sends internal rays to internal rays, more precisely, $F_U(R_U(z)) = R_{U'}(f(z))$.

Now we define the topological polynomial $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$,

$$F(z) := \begin{cases} F_U(z) & \text{If } z \text{ in some bounded Fatou component } U, \\ f(z) & \text{Otherwise.} \end{cases} \tag{4.2}$$

Evidently, F takes the same value as f in the Julia set and the basin of infinity. Furthermore, we have the following.

4.3. Properties of the topological polynomial F

Lemma 4.1. (1) *F is continuous.*

(2) *F is a branched covering map.*

(3) *For any $x \neq y \in J_f$, $[F(x), F(y)] \subseteq F([x, y])$.*

(4) *$F(\widehat{R}_U(\theta)) = \widehat{R}_{U'}(\sigma_d(\theta))$, where $U' = F(U)$, for any extended ray $\widehat{R}_U(\theta)$.*

Proof. (1) We only have to show that, for any $z \in J_f$, F is continuous at z. Let $\{z_k\}$ be an arbitrary sequence such that $z_k \to z$ as $k \to \infty$. We continue the discussion into three cases,

• If $\{z_k\} \subseteq \overline{\Omega}_f$. Since $F|_{\overline{\Omega}_f} = f$ and f is continuous, then $F(z_k) \to F(z)$ as $k \to \infty$.

• If $\{z_k\}$ are contained in $\mathbb{C} \setminus \overline{\Omega}_f$. Let $\{U_k\}$ be a sequence of bounded Fatou components such that $z_k \in U_k$ and $\mathcal{U} := \{U_k : k \geq 1\}$. If $\#\mathcal{U} < \infty$, since F is continuous in any Fatou component, we $f(z_k) \to f(z)$ as $k \to \infty$. If $\#\mathcal{U} = \infty$, since J_f is locally connected, the diameter of Fatou component $F(U_k)$ converges to zero as $k \to \infty$ (See for example Lemma 19.5 in [Mi06]). Thus,

$$\begin{aligned} |F(z_k) - F(z)| &\leq |F(z_k) - f(z_k)| + |f(z_k) - F(z)| \\ &\leq \operatorname{diam} F(U_k) + |f(z_k) - f(z)| \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

• In other cases, decompose $\{z_k\}$ into two subsequence $\{z_{k_i}\}$, contained in $\overline{\Omega}_f$, and $\{z_{k_i'}\}$ in Fatou set. By the former arguments, both of the image of the two subsequence converge to $F(z)$ as $k \to \infty$. So $F(z_k) \to 0$ as $k \to \infty$.

Thus F is continuous.

(2) Let $\mathrm{Crit}(F)$ to be the union of critical points of f in J_f and the center of critical Fatou components.

Firstly, claim that $F : \overline{\mathbb{C}} \setminus F^{-1}(F(\mathrm{Crit}(F))) \to \overline{\mathbb{C}} \setminus F(\mathrm{Crit}(F))$ is an unbranched covering. We only have to show that F is locally homeomorphic on $\overline{\mathbb{C}} \setminus \mathrm{Crit}(F)$. For any z in some Fatou component U, It follows by the construction of F_U. For any $z \in J_f \setminus \mathrm{Crit}(F)$, choose a sufficiently small neighborhood W_z such that

• f on W_z is injective,

• $F|_{W_z \bigcap U}$ is injective for any critical Fatou components,

• $f(U) \neq f(U')$ for any distinct Fatou component U and U' which intersect W_z.

By the definition of F, We know that $F|_{W_z}$ is injective. Therefore, $F|_{W_z}$ is a homeomorphism by the domain invariance theorem. The claim follows.

Secondly, consider point z in the finite set $\mathrm{Crit}(F)$. Let W be sufficiently small topological disk around $F(z)$ and

$$\phi : W \to \mathbb{D} \quad F(z) \mapsto 0$$

the topological parameterization. Let W' be one of the component $F^{-1}W$ containing z. Since $F : W' \setminus \{z\} \to W \setminus \{F(z)\}$ is an unbranched covering by the claim. The Riemann Hurwitz formula implies W' is a topological disk around z. Denote by $\delta := \deg(F|_{W' \setminus \{z\}})$.

Consider the following communicate diagram,

$$\begin{array}{ccc} W' - \{z\} & \xrightarrow{\psi} & \mathbb{D} - \{0\} \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle z\mapsto z^{\delta}} \\ W - \{F(z)\} & \xrightarrow{\phi} & \mathbb{D} - \{0\} \end{array}$$

where ψ is a homeomorphism obtained by Lifting ϕ through F and $z \mapsto z^{\delta}$. Set $\psi(z) = 0$. Thus F satisfies (4.1) at z.

Therefore, F is a branched covering. The critical points set is $\mathrm{Crit}(F)$.

(3) $F([x, y])$, consisting of internal rays, is a curve connecting $F(x)$ and $F(y)$. There exists a regulated arc $\gamma \subseteq F([x, y])$ joining $F(x)$ and $F(y)$. By Lemma 3.4, $\gamma = [F(x), F(y)]$.

(4) Let $z \in \partial U$ to be the landing point of $R(\theta)$. Then $F(z) = \alpha(\sigma_d(\theta)) \in \partial U'$. Since F_U maps internal ray $R_U(z)$ to internal ray $R_{U'}(F(z))$ and $F(R(\theta)) = R(\sigma_d(\theta))$. Thus $F(\widehat{R}_U(\theta)) = \widehat{R}_{U'}(\sigma_d(\theta))$. The proof is completed. $\square$

5. Partitions induced by critical portraits

In this section our objective is to divide the plane into several simple connected domains by external rays and extended rays. These rays land at $\mathrm{Crit}(F)$ and collide together after F. The restriction of F on each pieces is homeomorphic.

5.1. Supporting arguments resp. rays

Following [Po93], we give the definition of supporting arguments resp. supporting rays. Let U be a Fatou component and $p \in \partial U$ with total k rays $R(\theta_1), \cdots, R(\theta_k)$ landing at. These rays, numbered in counterclockwise cyclic order, divide the plane into k sectors. Suppose U belong to the sector bounded by $R(\theta_1)$ and $R(\theta_2)$. The argument θ_1 resp. the ray R_{θ_1} is called the *left supporting argument* resp. *left supporting ray* of the Fatou component U. We can also define the *right supporting arguments* resp. *right supporting rays* in analogous way. If only one ray lands at p, then the two supporting rays coincide.

Lemma 5.1. *For any U and $p \in \partial U$, the left resp. right supporting ray of U at p exists and is unique. Let $R(\theta)$ be a ray land at p, then $R(\theta)$ is the left resp. right supporting ray of U at p if and only if $F(R(\theta))$ is the left resp. right supporting ray of $F(U)$ at $F(p)$*

Proof. Firstly, there are at least one and at most finite many rays landing at p by [DH84] and Theorem 1.1. Thus it exists and is unique by definition.

Let $R(\theta')$ be the right (left) supporting ray of U at p. $L_{\theta\theta'} := R(\theta) \bigcup \{p\} \bigcup R(\theta')$ bounds a domain V containing U. The map $F|_V$ is locally homeomorphic at p. So $F(R(\theta))$ and $F(R(\theta'))$ are rays supporting $F(U)$. Since F preserves the orientation. $F(R(\theta)), F(U)$ and $F(R(\theta'))$ are in the same cyclic order around $F(p)$ as $R(\theta), U$ and $R(\theta')$ around p. Thus the lemma follows. $\square$

5.2. Definition of critical portraits

Firstly we define $\Theta(c)$, $\Theta(U)$ resp. $\mathcal{R}(c)$, $\mathcal{R}(U)$, for critical point c in J_f and critical Fatou component U by the following way.

• For any critical point $c \in J_f$, we set

$$\Theta(c) := \{\theta_1, \cdots, \theta_{\deg_F(c)}\} \text{ and } \mathcal{R}(c) := \{R(\theta_1), \cdots, R(\theta_{\deg_F(c)})\}$$

such that the total $\deg_F(c)$ external rays meet at c and F maps them onto exactly one external ray.

• For any strictly pre-periodic Fatou component U, we denote by

$$\Theta(U) := \{\theta_1, \cdots, \theta_{\deg_{F|_U}}\} \text{ and } \mathcal{R}(U) := \{\widehat{R}_U(\theta_1), \cdots, \widehat{R}_U(\theta_{\deg(F|_U)})\}$$

such that the $\deg(F|_U)$ external rays $R(\theta_i)$ support U and collide onto one after F. Clearly, by Lemma 5.1, they are supporting U in the same direction.

• For any critical Fatou component cycle $U_0, \cdots, U_{p-1}$ with $F^i(U_0) = U_i$, $U_p := U_0$, it can only be attracting or parabolic [Mi06]. Let $U_{k_0}, \cdots, U_{k_l}$, $0 \le k_0 < \cdots < k_l \le p-1$, be critical with degree $n_0, \cdots, n_l$ respectively.

Firstly, For $1 \le i \le p$, choose (z_i, θ_i), $z_i \in \partial U_i$ and $R(\theta_i)$ landing at z_i, such that $F^i(z_0) = z_i$, $F^p(z_p) = z_p$, $F^i(R(\theta_0)) = R(\theta_i)$ and $R(\theta_p)$ supporting U_p at z_p. Since $F^p : \partial U_0 \to \partial U_0$ is $\delta := n_0 \cdots n_l$ to 1 branched covering, there exist $\delta - 1$ distinct choices of z_p. By Lemma 5.1, all the p external rays supports the Fatou cycle in the same direction.

Secondly, for critical Fatou component U_{k_i}, $0 \le i \le l$, $\Theta(U_{k_i})$ is the set of n_i angles of external rays, which are supporting U_{k_i} and lie in the preimages of $R(\theta_{k_i+1})$, and $\mathcal{R}(U_{k_i})$ is the collection of n_i extended rays of U_{k_i} with angles in $\Theta(U_{k_i})$.

After finishing the choice of $\Theta(U_{k_i})$ and $\mathcal{R}(U_{k_i})$ in critical Fatou cycle, we now state the following lemma by adopting the same notations as above,

Lemma 5.2. *If $z, z' \in \partial U_0$ have the same itinerary respect to $\mathcal{R}(U_{k_0}), \cdots, \mathcal{R}(U_{k_l})$, then $z = z'$.*

Proof. Consider the covering $F^p : \partial U_0 \to \partial U_0$. There are δ preimages of z_p in ∂U_0. These points cut ∂U_0 into open segments $\gamma_0, \cdots, \gamma_{\delta-1}$, numbered in positive cyclic order which starts at z_0. Denote by

$$[s_0, \cdots, s_l] := s_0 n_1 \cdots n_l + s_1 n_2 \cdots n_l + \cdots + s_{l-1} n_l + s_l,$$

where $0 \le s_0 \le n_0 - 1, \cdots, 0 \le s_l \le n_l - 1$.

Let $\gamma_{k_i,0}, \cdots, \gamma_{k_i,n_i-1}$ be the segments of $\partial U_{k_i} \setminus \bigcup_{\theta \in \Theta(U_{k_i})} \alpha(\theta)$, numbered in positive cyclic order which starts at z_{k_i}. Then F maps $\gamma_{k_i,j}$ onto $\partial U_{k_i+1} \setminus \{z_{k_i+1}\}$ one to one.

By the construction above, we can see that $\xi \in \gamma_{[s_0,\cdots,s_l]}$ if and only if $F^{k_i}(\xi) \in \gamma_{k_i s_i}$ for $0 \le i \le l$. Hence by the condition, $\{F^{jp}(z), F^{jp}(z')\}$, for arbitrary $j \ge 0$, are always contained in one segment of $\gamma_0, \cdots, \gamma_{\delta-1}$. Now we show that it is impossible.

Let $\gamma_{zz'}$ be the component of $\partial U_0 \setminus \{z, z'\}$ contained in some segment γ_j. Since F^p is expanding on ∂U_0. There must exist a minimal positive s such that $F^{sp}(\gamma_{zz'})$ can not lie in one of $\gamma_0, \cdots, \gamma_{\delta-1}$. Let $F^{(s-1)p}(\gamma_{zz'}) \subseteq \gamma_{i_0}$. Since $F^p|_{\gamma_{i_0}}$ covers $\partial U_0 \setminus \{z_p\}$ by sticking the two endpoints into z_p, which is the common boundary of γ_j and $\gamma_{(j+1) \bmod \delta}$ for some $0 \le j \le \delta - 1$. Thus $F^{sp}(z)$ and $F^{sp}(z')$ must be in distinct segments. The proof is completed. □

It is easy to see that all the $\mathcal{R}(c)$ and $\mathcal{R}(U)$ defined above are in star shape with a critical point in the center.

Lemma 5.3 (Properties of $\mathcal{R}(c)$ and $\mathcal{R}(U)$)**.** (1) *$\mathcal{R}(c) \bigcap \mathcal{R}(c') = \emptyset$, for distinct critical points c, c' in J_f.*

(2) *If $\mathcal{R}(c) \bigcap \mathcal{R}(U) \neq \emptyset$, then $c \in \partial U$ and the intersection is exactly either a point $\{c\}$ or one ray together with the landing point c. The latter happens if and only if $\Theta(c) \bigcap \Theta(U) \neq \emptyset$.*

(3) *If* $\mathcal{R}(U)\bigcap\mathcal{R}(U')\neq\emptyset$*, for distinct critical Fatou component* U,U'*, then the intersection is exactly either a point* $\{p\}:=\partial U\bigcap\partial U'$ *or one ray together with the landing point* p*. The latter happens if and only if* $\Theta(U)\bigcap\Theta(U')\neq\emptyset$*.*

Proof. By definition, (1) and (2) follow immediately.

(3) Since for any two distinct Fatou component U,U', the intersection $\overline{U}\bigcap\overline{U'}$ is at most one point. $\mathcal{R}(U)\bigcap\mathcal{R}(U')\neq\emptyset$ implies $\overline{U}\bigcap\overline{U'}:=\{p\}$. If $\Theta(U)\bigcap\Theta(U')\neq\emptyset$, then the latter case happens. Otherwise, we have $\mathcal{R}(U)\bigcap\mathcal{R}(U')=\{p\}$. □

In $\mathbb{R}/\mathbb{Z}$, let $\mathcal{A}:=\{\Theta(c_1),\cdots,\Theta(c_m),\Theta(U_1),\cdots,\Theta(U_n)\}$. For any $\Theta\in\mathcal{A}$, let

$$\widehat{\Theta}:=\bigcup\{\Theta':\exists\text{ a chain }\ \Theta_0:=\Theta,\cdots,\Theta_k:=\Theta'\text{ in }\mathcal{A}\text{ such that }\Theta_i\bigcap\Theta_{i+1}\neq\emptyset\}.$$

The collections $\widehat{\mathcal{A}}:=\{\widehat{\Theta}_1,\cdots,\widehat{\Theta}_N\}$ are called *critical portrait* of F. One can check that the following conditions are satisfied.

(1) $\sum_{1\leq i\leq N}(\#\,\widehat{\Theta}_i-1)=d-1$.

(2) $\widehat{\Theta}_1,\cdots,\widehat{\Theta}_N$ are *pairwise unlinked*, that is, for each $i\neq j$ the sets $\widehat{\Theta}_i$ and $\widehat{\Theta}_j$ are contained in disjoint sub-intervals of $\mathbb{R}/\mathbb{Z}$.

(3) σ_d sends $\widehat{\Theta}_i$ onto exactly one argument.

5.3. Critical diagram associated to $\widehat{\mathcal{A}}$

Given critical portrait $\widehat{\mathcal{A}}$, one can construct a *critical diagram* $\mathcal{D}\subseteq\overline{\mathbb{D}}$ as follows. See figure 5.

Start with the unit circle $\mathbb{R}/\mathbb{Z}$, for each $\widehat{\Theta}_i$, mark all of the points $e^{2\pi i\theta}$ with $\theta\in\widehat{\Theta}_i$. Let $\widehat{z}_i$ be the center of gravity of the marked points, and join each of these points $e^{2\pi i\theta}$ to $\widehat{z}_i$ by a straight line segment l_θ. Then we obtain a closed set $D_i:=\bigcup l_\theta$ in the unit disk. It follows easily by Conditions (2) that distinct D_i and D_j will not cross each other. Let $\mathcal{D}:=\bigcup_{1\leq i\leq d}D_i$ be critical diagram associated to $\widehat{\mathcal{A}}$.

The Condition (1) implies that $\mathbb{D}\setminus\mathcal{D}$ are d simply connected domains $W_1,\cdots,W_d$. Denote by I_i the interior of $\overline{W}_i\bigcap\partial\mathbb{D}$. Then $\{I_i\}_{1\leq i\leq d}$ is a partition of $\mathbb{R}/\mathbb{Z}$, each elements of which consists of finite open intervals with total length $1/d$ by Condition (3).

5.4. Partition in the dynamic plane

Let $\mathcal{L}:=\{\mathcal{R}(c_1),\cdots,\mathcal{R}(c_m),\ \mathcal{R}(U_1),\cdots,\ \mathcal{R}(U_n)\}$. For any $\mathcal{R}\in\mathcal{L}$, set $\widehat{\mathcal{R}}:=\bigcup\{\mathcal{R}'$: there exists a chain $\mathcal{R}_0:=\mathcal{R},\cdots,\mathcal{R}_k:=\mathcal{R}'$ in $\mathcal{L}$ such that, for $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$, the latter case in Lemma 5.3(3) happens$\}$.

By Lemma 5.3, each $\widehat{\mathcal{R}}$ corresponds to a $\widehat{\Theta}$, characterized by the property that $R(\theta)$ is in $\widehat{\mathcal{R}}$ if and only if $\theta\in\widehat{\Theta}$.

Lemma 5.4 (Properties of $\widehat{R}$)**.** (1) $T:=\widehat{\mathcal{R}}\bigcap K_f$ *is a tree. Namely, any* $z,z'\in T\bigcap J_f$ *can be joined by a regulated arc in* T*. Moreover, the branching points in the tree must be critical points in* J_f *or* $c(U)$ *in critical Fatou component* U*.*

(2) *Suppose* $R(\theta_1),\cdots,R(\theta_l)$ *be all the external rays in* $\widehat{\mathcal{R}}$*, numbered in counter-clockwise order. Let* $L_{\theta_i\theta_{i+1}}:=R(\theta_i)\bigcup R(\theta_{i+1})\bigcup[\alpha(\theta_i),\alpha(\theta_{i+1})]$, $1\leq i\leq l$, $\theta_{l+1}:=\theta_1$*. Then* $L_{\theta_i\theta_{i+1}}$ *cuts the plane into two domains* Y, Y'*. Let* Y *be the one disjoint with* $R(\theta_j),1\leq j\leq l$*. Then for any* $x,y\in Y\bigcap J_f$, $[x,y]\subseteq\overline{Y}$ *and* $F|_{[x,y]\bigcap\partial Y}$ *is one-to-one.*

(3) *The image* $F(L_{\theta_i\theta_{i+1}})$ *has only three types:*

- *Type I: one ray union the landing point,*
- *Type II: one extended ray union the landing point,*
- *Type III: two internal rays and one external ray, which looks like "Y".*

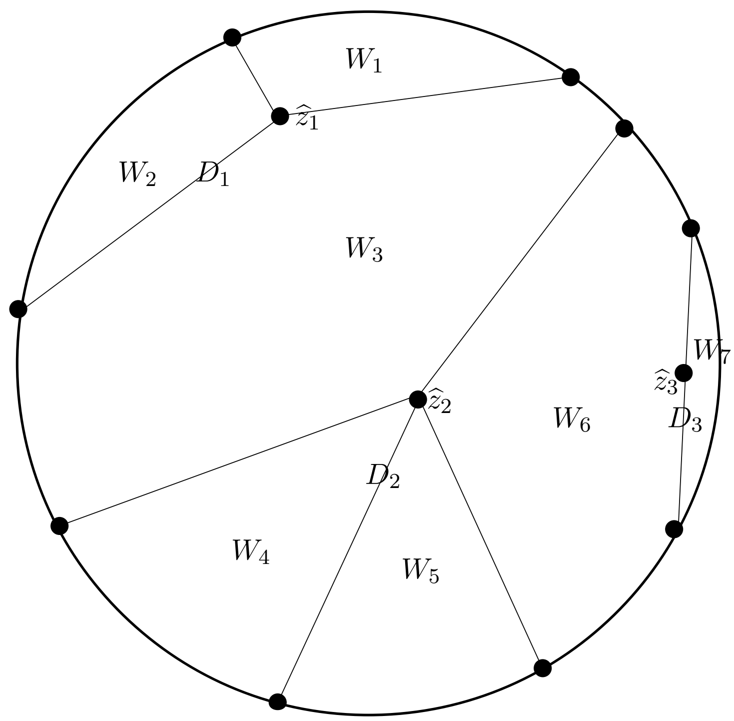

FIGURE 5. An example of critical diagram $\mathcal{D}$

(4) *For another* $\widehat{\mathcal{R}}'$*, if* $\widehat{\mathcal{R}} \cap \widehat{\mathcal{R}}' \neq \emptyset$*, then the intersection is a point.*

Proof. (1) By the construction of $\widehat{\mathcal{R}}$, it is clear that $[z, z'] \subseteq T$ if $z, z' \in T \bigcap J_f$. The lemma 3.4 implies that regulated arcs cannot form a loop in K_f. Thus T is a tree. Branched point z in Fatou component U is obviously a critical point $c(U)$. If z is in J_f, there are at least three critical Fatou component U_i such that $z \in \mathcal{R}(U_i) \subseteq \widehat{\mathcal{R}}$, $i \in \{1, 2, 3\}$. If z is not critical, $\mathcal{R}(U_i)$ share a common external ray which landing at z. Since one ray supports at most two Fatou components. It is impossible.

(2) Consider $[\alpha(\theta_i), \alpha(\theta_{i+1})]$. It has only three possibilities

(2.1) $\alpha(\theta_i) = \alpha(\theta_{i+1})$, then $[\alpha(\theta_i), \alpha(\theta_{i+1})]$ is degenerated.

(2.2) $[\alpha(\theta_i), \alpha(\theta_{i+1})] \subseteq \overline{U}$ passes through one critical Fatou component, consisting of two internal rays.

(2.3) $[\alpha(\theta_i), \alpha(\theta_{i+1})]$ passes through two critical Fatou component U and U', consisting of four internal rays.

In fact, if $[\alpha(\theta_1), \alpha(\theta_2)]$ passes through more than two critical Fatou component. Let U be one of them with $U \bigcap \{\alpha(\theta_i), \alpha(\theta_{i+1})\} = \emptyset$. Then the supporting properties imply that there exists a external ray in $\mathcal{R}(U)$ contained in Y, impossible.

Assume $[x, y] \setminus Y \neq \emptyset$, otherwise, (2) follows. Let $\gamma(t) := [x, y]$ with $\gamma(0) = x$ and $\gamma(1) = y$. Set $t_1 := \inf_{0 \le t \le 1}\{t : \gamma(t) \in \mathbb{C} \setminus Y\}$ and $t_2 := \sup_{0 \le t \le 1}\{t : \gamma(t) \in \mathbb{C} \setminus Y\}$. Then $\gamma(t_i) \in \partial Y$, $i \in \{0, 1\}$ and so $[\gamma(t_0), \gamma(t_1)] \subseteq L_{\theta_i \theta_{i+1}}$. We have $[x, y] \bigcap \partial Y = [\gamma(t_0), \gamma(t_1)]$ and $[x, y] = [x, \gamma(t_0)] \bigcup [\gamma(t_0), \gamma(t_1)] \bigcup [\gamma(t_1), y]$. Thus $[x, y] \subseteq \overline{Y}$.

Now we have to show that $F|_{[\gamma(t_0), \gamma(t_1)]}$ is one-to-one. Note that $[\gamma(t_0), \gamma(t_1)]$ consists exactly several internal rays.

In case (2.2), at least one of $R(\theta_i)$, $R(\theta_{i+1})$ is supporting U, because $\mathcal{R}(U) \subseteq \widehat{R}$. So $[\gamma(t_0), \gamma(t_1)] \bigcap \overline{U}$ is either a point or one internal ray. Thus $F|_{[\gamma(t_0),\gamma(t_1)]}$ is one-to-one immediately.

In case (2.3), let $\{p\} = \overline{U} \bigcap \overline{U'}$. We have $R(\theta_i)$, $R(\theta_{i+1})$ supporting U, U' respectively. Otherwise, there exists a ray in $\mathcal{R}(U)$ or $\mathcal{R}(U')$ landing at p contained in Y, impossible. Thus the intersection between $[\gamma(t_0), \gamma(t_1)]$ and $\overline{U}$ resp. $\overline{U}'$ is at most one internal ray.

We are only left to consider the case $[\gamma(t_0), \gamma(t_1)] = [c(U), c(U')]$. Suppose $F|_{[c(U),c(U')]}$ is not one-to-one. Then $F([p, c(U)]) = F([p, c(U')])$, thus p is a critical point. There exists at least a external ray in $\mathcal{R}(p)$ contained in Y. Otherwise, consider the section S of $\mathbb{C} \setminus \mathcal{R}(p)$ containing U, U', $F|_S$ is locally homeomorphic at p, thus it can not paster $[p, c(U)]$ and $[p, c(U)]$ together. This contradicts the choice of $R(\theta_i)$ and $R(\theta_{i+1})$. Therefore $F|_{[c(U),c(U')]}$ is one-to-one.

(3) By the discussion in (2), it follows easily that $F(L_{\theta_i\theta_{i+1}})$ is in Type I, Type II or Type III if and only if $[\alpha(\theta_i), \alpha(\theta_{i+1}]$ is in case (2.1), (2.2) and (2.3), respectively.

(4) It holds directly by the definition and Lemma 5.3. □

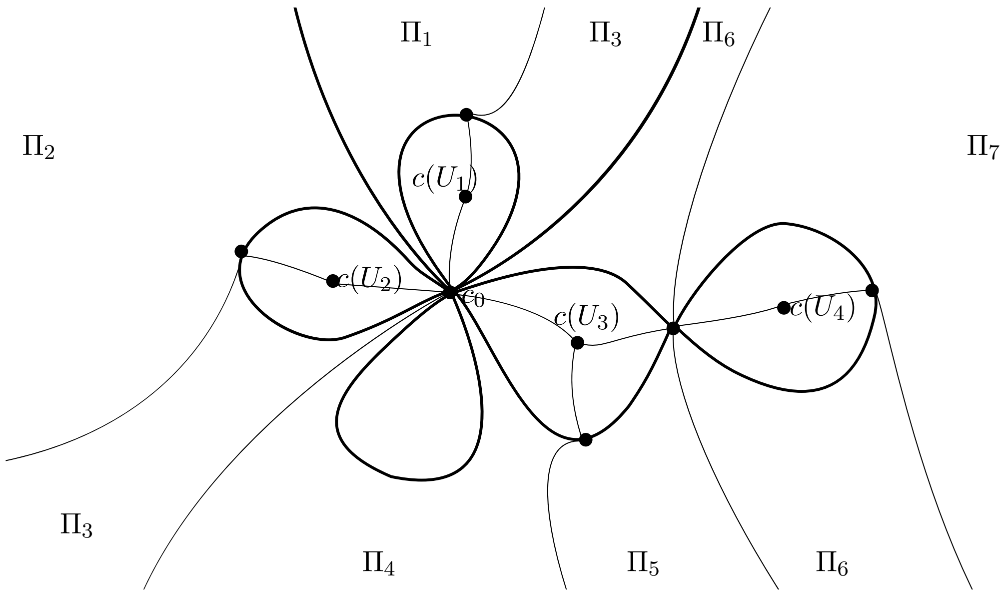

FIGURE 6. An example of partition corresponding to critical diagram in figure 5. Here the critical Fatou components are U_1, U_2, U_3 and U_4. There exists a critical points c_0 with $\deg_F(c_0) = 2$.

Let $\widehat{\mathcal{L}} := \{\widehat{\mathcal{R}}_1, \cdots, \widehat{\mathcal{R}}_N\}$. For simplification, the elements are numbered in such fine order that $\widehat{R}_i$ consists of (extended) rays with their arguments in $\widehat{\Theta}_i$. Let $P := \mathbb{C} \setminus \bigcup_{1\le i\le N} \widehat{\mathcal{R}}_i$ consists of finite unbounded pieces $P_1, \cdots, P_s$.

Consider the critical diagram $\mathcal{D}$. Given W_i, suppose it is bounded by $\bigcup_{1\le j\le k_i}(l_{\theta_j} \cup l_{\theta'_j})$ with $\theta_j, \theta'_j \in \widehat{\Theta}_j$ and $l_{\theta_j} \cup l_{\theta'_j} \subseteq D_j$. Then, in the dynamic plane, $L_{\theta_j\theta'_j}$, $1 \le j \le k_i$, in Lemma 5.4 (2) are well defined. As in Lemma 5.4 (2), let $Y_j \bigcup Y'_j := \mathbb{C} \setminus L_{\theta_j\theta'_j}$ where Y'_j be the component disjoint with $\bigcup_{\theta\in I_i} R(\theta)$.

Now we define the *partition* $\{\,\Pi_i\,\}_{1\le i\le d}$ of the dynamical plane by setting

$$\Pi_i := \mathbb{C} \setminus \bigcup_{1\le j\le k_i} \overline{Y'_j}.$$

We have

• $P = \bigcup_{1\le i\le d} \Pi_i$ and $\Pi_i \bigcap \Pi_j = \emptyset$ if $i \neq j$.

• each Π_i, maybe not a domain, consists of finite pieces P_j and $\partial\Pi_i$ are the union of several (extended) rays.

• there is an one-to-one correspondence between $\{\,I_i\,\}_{1\le i\le d}$ and $\{\,\Pi_i\,\}_{1\le i\le d}$ by the property that $\theta \in I_i$ if and only if $R(\theta) \subseteq \Pi_i$. See figure 5 and figure 6.

Based on the topological argument principle, we shall prove the following,

Proposition 5.5. *The restriction of F on each Π_i is homeomorphic.*

Proof. Recall $G_f : \mathbb{C} \to [0, \infty]$ is the Green's function which vanishes precisely on K_f and $G_t := \{z \in \mathbb{C} : G(z) < t\}$ a simply connected domain. Set $Q_t = G_t \bigcap \Pi_i$, which is bounded by edges in two types,

• The segments of the equipotential cure $G_f(z) = t$ which lies in $\overline{\Pi}_i$. Each one corresponds to an arc in I_i. We denote by Γ_i the union of all these segments.

• The segments of $L_{\theta_j\theta'_j}, 1 \le j \le k_i$ satisfying the potential inequality $G_f(z) \le t$.

Each segments in Γ_i is mapped to equipotential curve $\gamma_{dt} := \{z \in \mathbb{C} : G(z) = dt\}$ locally homeomorphic. Since F pastes the segments of the latter two types together as in Lemma 5.4 (3). It follows that γ_{dt} is covered by Γ_i at least once. We know that $F|_{\gamma_t} : \gamma_t \to \gamma_{dt}$ is d to 1 and γ_t is the union of Γ_i, $1 \le i \le d$, with their interiors disjoint. Thus $F(\Gamma_i)$ covers γ_d exactly once.

Let z_0 be any point of $\mathbb{C}$ which does not belong to the image $F(\partial Q_t)$. By the Topological Argument Principle, the number of solutions to the equation $F(z) = z_0$ with $z \in Q_t$, counted with multiplicity, is equal to the winding number of $F(\partial Q_t)$ around z_0. By the arguments above, it is not hard to check that this winding number is $+1$ for z_0 in $G_{dt} \setminus \bigcup_{1\le j\le k_i} F(L_{\theta_j\theta'_j})$ and zeros for z_0 in $\mathbb{C} \setminus \overline{G}_{dt}$. So $F|_{Q_t}$ is one-to-one. By the arbitrariness of t, F on Π_i is homeomorphic.

□

5.5. Regulated arcs in the partition

Lemma 5.6. *For any distinct $x, y \in \Pi_i \bigcap J_f$, the regulated arc $[x,\, y]$ is contained in $\overline{\Pi}_i$. Moreover,*

$$F : [x, y] \to [F(x), F(y)] \text{ is homeomorphic.} \tag{5.1}$$

Proof. We adopt the notations as before. For $1 \le j \le k_i$, $x, y \in Y_j$. Then the Lemma 5.4 (2) gives $[x, y] \subseteq \overline{Y}_j$. Thus $[x, y] \subseteq \bigcap_{1\le j\le k_i} \overline{Y}_j = \overline{\Pi}_i$.

Consider the set

$$X := \{z \in F([x, y]) : \text{ there exist } z_1 \neq z_2 \in [x, y] \text{ such that } F(z_1) = F(z_2) = z\}.$$

Since $F|_{\Pi_i}$ is one-to-one by Proposition 5.5, $X \subseteq F([x, y] \bigcap \partial\Pi_i)$.

We claim that $X \subseteq F([x, y] \bigcap \partial\Pi_i \bigcap J_f)$. If not, let $z \in X \bigcap U$ for some bounded Fatou component U. Then there exists two distinct $z_j \in U_j$ such that $F(z_j) = z$. Firstly, If $U_1 = U_2$, then U_1 must be critical. z_1 and z_2 are contained in two internal rays of $\mathcal{R}(U_1)$. It is impossible by Lemma 5.4 (2.2). If $U_1 \neq U_2$, consider the branched covering $F : U_j \to U$. The image $F(U_j \bigcap \Pi_i)$ is either U or $U \setminus R$ for some internal ray. In both of the cases we have

$$F(U_1 \cap \Pi_i) \bigcap F(U_2 \cap \Pi_i) \neq \emptyset.$$

This contradicts the fact that F is one-to-one on Π_i in Proposition 5.5. The claim follows.

Since $[x,y]\bigcap\partial\Pi_i\bigcap J_f$ is finite, then X is finite as well. This means $F([x,y])$ has only finite many self-intersection points. If $X\neq\emptyset$, then we can easily obtain a loop in $F([x,y])$, consisting of regulated arcs by Lemma 4.1 (3). Lemma 3.4 gives a contradiction. Thus we have $X=\emptyset$. Therefore, $F:[x,y]\to[F(x),F(y)]$ is homeomorphic.

□

6. Proof of the main theorem

In this section we aim to prove the main theorem, applying the tools prepared in the previous sections.

6.1. No wandering regulated arcs

Proposition 6.1. *For any regulated arc $[x,y]$ in K_f, there exist two integer $m\neq n\geq 0$ such that $F^m[x,y]\bigcap F^n[x,y]\neq\emptyset$.*

Proof. For any critical point, if $[x,y]$ is mapped onto it twice, then of course we are done. So, by iterated $[x,y]$ suitable times, we can assume $f^k|_{[x,y]}$ is homeomorphic. We continue the analysis by distinguishing the regulated arc into two case.

- $[x,y]$ is quasi-buried, i.e., $\#[x,y]\bigcap\overline{U}\leq 1$, for any bounded Fatou component U.
- there exists a bounded Fatou component U such that $\#[x,y]\bigcap\overline{U}\geq 2$.

In the first case, $[x,y]\subseteq J_f$. Recall that E is the union of branched points and preimages of β fixed points in J_f. By Lemma 3.5, E is dense in $[x,y]$. If some (pre-)periodic point lies in $[x,y]$, we are done. Then $E\bigcap[x,y]$ contains infinitely many wandering branched points. Since the number of grand orbits of wandering branched point is finite by Corollary 2.6. So there is at least a branched point z such that its grand orbit intersects $[x,y]$ infinitely many times. Choose any two distinct $z_1,z_2\in[x,y]$ in the grand orbit. Then we have $f^m(z_1)=f^n(z_2)$ for some $m,n\geq 0$. Therefore $f^m[x,y]\bigcap f^n[x,y]\neq\emptyset$. Since $f^m|_{[x,y]}$ and $f^n|_{[x,y]}$ is injective. We must have $m\neq n$.

In the second case, let $[x',y']:=[x,y]\bigcap\overline{U}$, consisting of two internal rays, particularly containing $c(U)$. By Sullivan's no wandering Fatou components, U will eventually be periodic. Then $c(U)\in[x',y']$ is pre-periodic. So there exists $m\neq n$ such that $f^m[x',y']\bigcap f^n[x',y']\neq\emptyset$. The proof is completed. □

6.2. Quasi-buried case

Proposition 6.2. *Let $\{\Pi_i\}_{1\leq i\leq d}$ be the partition of $\mathbb{C}$ induced by the critical portrait of f. Let $[x,y]$ be quasi-regulated arc in K_f. If x,y have the same itinerary respect to $\{\Pi_i\}_{1\leq i\leq d}$, then $x=y$.*

Proof. We argue by contradiction and suppose $x\neq y$. Denote by $z_n:=F^n(z)$ for any $z\in\mathbb{C}$. By Lemma 5.6, for any $m\geq 0,n\geq 1$,

$$F^n:[x_m,y_m]\to[x_{m+n},y_{m+n}]\text{ is homeomorphic.}\tag{6.1}$$

Firstly, we claim that there exist $M\neq N\geq 0$ and ξ such that

- $\xi\in[x_M,y_M]\bigcap[x_N,y_N]$,
- The orbit of ξ is disjoint with the finite set $X:=\bigcup_{1\leq i\leq d}(\partial\Pi_i\cap J_f)$.

Proof of Claim. Consider the set

$$Y:=\{z\in[x,y]:\text{ there exist }m,n\geq 0\text{ and }z'\neq z\in[x,y]\text{ such that }F^m(z)=F^n(z')\}.$$

Since there is no wandering regulated arc by Proposition 6.1, Y is dense in $[x, y]$.

For any $z \in Y$, there exist $m \neq n \geq 0$ such that $z_m \in [x_m, y_m] \bigcap [x_n, y_n]$. If the orbit $\{z_i\}_{i\geq 0}$ never hit X, we are done. If $z_{n_0} \in X$ and the orbit $\{z_{n_0+i}\}_{i\geq 0}$ is infinite, then there exists a large number N_0 such that the orbit $\{z_{N_0+i}\}_{i\geq 0}$ avoids the finite points X. Let $M = N_0 + m, N = N_0 + n$ and $\xi = z_{m+N_0}$, we are done.

Otherwise, we can suppose that all Y are eventually iterated to $X_0 \subseteq X$ and points in X_0 are (pre-)periodic. Then there exist a periodic point w with period p and infinite many points in Y iterated to w. Thus we have (z', n') and (z'', n''), $z' \neq z'' \in Y$, such that $F^{n'}(z') = F^{n''}(z'') = w$ and $n' = n'' \bmod p$. Let $n'' = n' + kp$, $k > 0$. Then $F^{n''}(z') = F^{n''}(z'') = w$, which contradicts (6.1). The claim follows. $\square$

For simplicity we write $[x, y] = [x_M, y_M]$. Let $\xi \in [x, y] \bigcap [x_N, y_N]$, $N \geq 1$, such that its orbit never hits the boundary of the partition $\{\Pi_i\}_{1\leq i\leq d}$. Let

$$H := [x, y] \bigcup [x_N, y_N] \bigcup [x_{2N}, y_{2N}] \bigcup \cdots . \tag{6.2}$$

Then,

• For any $\zeta, \eta \in H$, $[\zeta, \eta] \subseteq H$. Indeed, suppose $\zeta \in [x_{n_1N}, y_{n_1N}]$ and $\eta \in [x_{n_2N}, y_{n_2N}]$ with integers $n_1 \leq n_2$. Then the path

$$\gamma_{\zeta\eta} := [\zeta, \xi_{n_1N}] \bigcup [\xi_{n_1N}, \xi_{(n_1+1)N}] \bigcup \cdots \bigcup [\xi_{n_2N}, \eta]$$

joins ζ and η. By the uniqueness of regulated arc in Lemma 3.4, It follows that $[\zeta, \eta] \subseteq \gamma_{\zeta\eta}$. Since $[\zeta, \xi_{n_1N}] \subseteq [x_{n_1N}, y_{n_2N}]$, $[\xi_{kN}, \xi_{(k+1)N}] \subseteq [x_{kN}, y_{(k+1)N}]$ and $[\zeta_{n_2N}, \eta] \subseteq [x_{n_2N}, y_{n_2N}]$, then $\gamma_{\zeta\eta} \subseteq H$. Thus $[\zeta, \eta] \subseteq H$.

• For any $n \geq 0$, if $F^n(\xi) \in \Pi_{i(n)}$, then $F^n(H) \subseteq \overline{\Pi}_{i(n)}$. Indeed, since ξ is never mapped into $\bigcup_{1\leq i\leq d} \partial\Pi_i$, such $\Pi_{i(n)}$ exists. We claim that $x_{kN}, \xi_{kN}, \xi_{(k+1)N}, y_{kN}$ have the same itinerary respect to $\{\Pi_i\}$. Since $\xi, \xi_N \in [x, y]$, then $\xi_{kN}, \xi_{(k+1)N} \in [x_{kN}, y_{kN}]$. By Lemma 5.6 and (5.1), $[x_{kN+j}, y_{kN+j}]$ must be contained in some $\overline{\Pi}_{j(n)}$ for any j. In particularly, we have $F^j(x_{kN}), F^j(\xi_{kN}), F^j(\xi_{(k+1)N}), F^j(y_{kN}) \in \Pi_{j(n)}$. By the arbitrariness of j, the claim follows. Therefore we obtain a sequence $\xi, \xi_N, \xi_{2N}, \cdots, \xi_{kN}, x_{kN}, y_{kN}$, which have the same itinerary. Thus if $F^n(\xi) \in \Pi_i$, $F^n[x_{kN}, y_{kN}] \subseteq \overline{\Pi}_i$. By the arbitrariness of k, it follows that $F^n(H) \subseteq \overline{\Pi}_i$.

• For any $n \geq 0$, $F^n|_H$ is homeomorphism and $F^N(H) \subseteq H$. The latter follows immediately by definition. For the former, if not, there exists a minimal number $n_0 \geq 0$ such that we have $\zeta \neq \eta \in F^{n_0}(H)$ with $F(\zeta) = F(\eta)$. By the above conclusions, we see that $[\zeta, \eta] \subseteq F^{n_0}(H)$ and is contained in some $\overline{\Pi}_{i(n_0)}$. Since $[\zeta, \eta]$ is quasi-buried, there exists $[\zeta^{(i)}, \eta^{(i)}] \subseteq [x, y]$ with $\zeta^{(i)}, \eta^{(i)} \in \Pi_{i(n_0)}$ such that $\zeta^{(i)} \to \zeta$, $\eta^{(i)} \to \eta$ as $i \to \infty$. Then $F|_{[\zeta^{(i)}, \eta^{(i)}]}$ is one-to-one by Lemma 5.6. Thus $F[\zeta, \eta]$ is a loop. It is impossible by Lemma 3.4.

Now we pay attention to the two regulated arc $[\xi, \xi_N]$ and $[\xi_N, \xi_{2N}]$. Both of them are contained in H. Their relations are in one of the following five possibilities. See figure 7.

(1) $[\xi, \xi_N] \bigcap [\xi_N, \xi_{2N}] = \{\xi_N\}$.
(2) $[\xi, \xi_N]$=$[\xi_N, \xi_{2N}]$.
(3) $[\xi_N, \xi_{2N}] \subset [\xi, \xi_N]$.
(4) $[\xi, \xi_N] \subset [\xi_N, \xi_{2N}]$.
(5) $[\xi, \xi_N] \bigcap [\xi_N, \xi_{2N}] = [\eta, \xi_N]$ for some $\eta \in (\xi, \xi_N)$.

We will show that all of them are impossible and so the proof is completed.

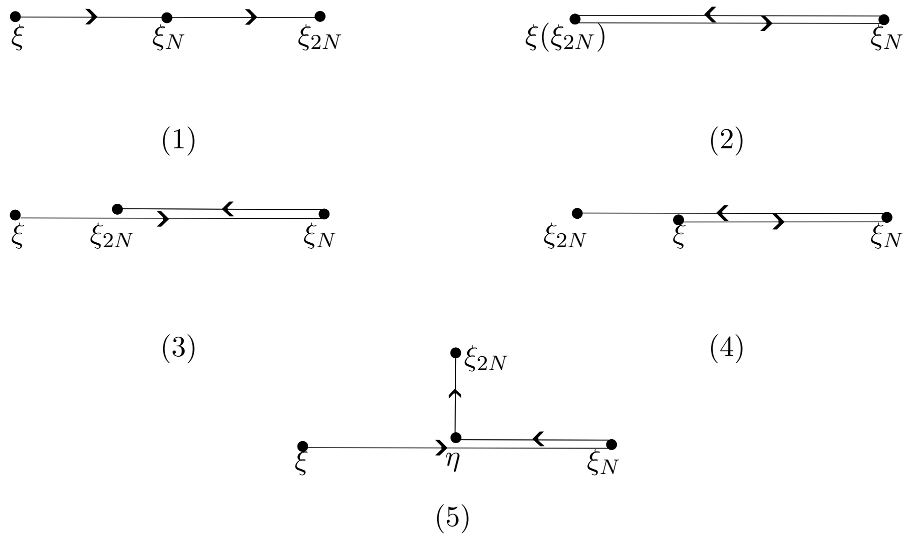

FIGURE 7. Relations of $[\xi, \xi_N]$ and $[\xi_N, \xi_{2N}]$

For case (1), we have $[\xi, \xi_{2N}] = [\xi, \xi_N] \bigcup [\xi_N, \xi_{2N}] \subseteq H$. Then $F|_{[\xi,\xi_{2N}]}$ is homeomorphic. Note that $F^N[\xi, \xi_N] = [\xi_N, \xi_{2N}]$. It follows that $[\xi_{2N}, \xi_{3N}] \bigcap [\xi_N, \xi_{2N}] = \{\xi_{2N}\}$. We also have $[\xi_{2N}, \xi_{3N}] \bigcap [\xi, \xi_N] = \emptyset$. Otherwise, the three arcs $[\xi, \xi_N] \bigcup [\xi_N, \xi_{2N}] \bigcup [\xi_{2N}, \xi_{3N}]$ would form a loop. By induction, it follows that $[\xi_{nN}, \xi_{(n+1)N}] \bigcap [\xi, \xi_{nN}] = \{\xi_{nN}\}$ for $n \geq 0$. Then (ξ, ξ_N) is a wandering regulated arc of F^N. By Proposition 6.1, it is impossible.

Case (2) can not happen. Indeed, otherwise $F^N : [\xi, \xi_N] \to [\xi, \xi_N]$ is homeomorphic. Choose any subarc I in $[\xi, \xi_N]$ such that $F^N(I) \bigcap I = \emptyset$. Then I is a wandering regulated arc of F^N.

For case (3), choose an arbitrary subarc I in (ξ, ξ_{2N}). Then $F^N(I) \subseteq (\xi_N, \xi_{2N})$. Since $F^N : [\xi, \xi_N] \to [\xi_N, \xi_{2N}]$ is homeomorphic and $[\xi_N, \xi_{2N}] \subset [\xi, \xi_N]$, I is a wandering regulated arc of F^N, a contradiction.

For case (4), by the intermediate value theorem, there is a fixed point $\nu \in (\xi, \xi_N)$ of F^N. Then $[\nu, \xi] \subset [\nu, \xi_{2N}]$ and the map $F^{2N} : [\nu, \xi] \to [\nu, \xi_{2N}]$ is homeomorphic. Let $\xi_{-2N} \in [\nu, \xi]$ such that $F^{2N}(\xi_{-2N}) = \xi$. Then $[\xi_{-2N}, \xi] \bigcap [\xi, \xi_{2N}] = \{\xi\}$. Similar to case (1), it is impossible.

For case (5), let $\eta_{-N} \in [\xi, \xi_N]$ with $F^N(\eta_{-N}) = \eta$. We distinguish three possibilities to analyze.

(5.1) $\eta_{-N} \in (\xi, \eta)$. Then $\eta_N \in (\eta, \xi_{2N})$. Therefore $[\eta_{-N}, \eta] \bigcap [\eta, \eta_N] = \{\eta\}$. By case (1) again, it is impossible.

(5.2) $\eta_{-N} = \eta$. Then η is a fixed point of F^N. We claim that there exist $\nu \in (\eta, \xi)$ and $n_0 \geq 3$ such that $F^{n_0 N}[\eta, \nu] \subseteq [\eta, \xi]$. Indeed, since $F^{3N}[\eta, \xi] = [\eta, \xi_{3N}]$ and $F^N|_H$ is injective, hence $[\eta, \xi_{3N}] \bigcap ([\eta, \xi_N] \cup [\eta, \xi_{2N}]) = \{\eta\}$. If $[\eta, \xi_{3N}] \bigcap [\eta, \xi] \neq \{\eta\}$, the claim follows. Otherwise, continue the process to $[\eta, \xi_{3N}] \cdots$, until $[\eta, \xi_{kN} \bigcap [\eta, \xi] \neq \{\eta\}$. Otherwise, we obtain an infinity sequence $\{(\eta, \xi_{kN}]\}_{k \geq 0}$ which are pairwise disjoint. This contradict Proposition 6.1. Hence the claim follows.

Choose $\nu' \in (\eta, \nu)$ such that $\nu'_{n_0 N} \neq \nu'$. If $\nu'_{n_0 N} \in (\eta, \nu')$, similarly in case (3), it is impossible. If $\nu'_{n_0 N} \in (\nu, \xi)$, let $\nu'_{-n_0 N} \in (\eta, \nu')$ be the preimage of $F^{n_0 N}|_{[\eta, \nu']}$, then similar in case (1), $(\nu'_{-n_0 N}, \nu')$ is a wandering regulated arc of $F^{n_0 N}$. It is impossible.

(5.3) $\eta_{-N} \in (\xi_N, \eta)$. Applying intermediate value theorem to $F : [\eta_{-N}, \eta] \to [\eta, \eta_N]$, we obtain a fixed point $\nu \in (\eta_{-N}, \eta)$. Since $[\nu, \xi_N] \bigcap [\nu, \xi_{2N}] = \{\nu\}$. So this is the case (5.2), impossible. The proof is completed. $\square$

6.3. **General cases**

Proposition 6.3. *Let $\{\Pi_i\}_{1 \le i \le d}$ be the partition of $\mathbb{C}$ induced by the critical portrait of f. If $x, y \in J_f$ have the same itinerary respect to $\{\Pi_i\}_{1 \le i \le d}$, then either $x = y$ or x, y are in the boundary of a Fatou component, which is mapped to a siegel disk.*

Proof. Suppose $x \neq y$. Consider the regulated arc $[x, y]$. Let

$$\mathcal{U} := \{U : U \text{ is a Fatou component such that } U \bigcap [x, y] \neq \emptyset\}.$$

Then $[x, y] \backslash \bigcup_{U \in \mathcal{U}} U$ consists of several disjoint quasi-buried regulated arcs. By Proposition 6.2, each such arcs is a single point.

Firstly, $\mathcal{U}$ is finite. If not, since there is no wandering Fatou components and the number of periodic Fatou components is finite. Infinite many elements in $\mathcal{U}$ will eventually be mapped onto a periodic one. This contradicts (6.1).

Secondly, any $U \in \mathcal{U}$ is mapped to a siegel disk. If not, let $(x', y') = U \bigcap [x, y]$. If there exists $N \ge 0$ such that orbits of x'_N and y'_N avoid the finite set $X := \bigcup_{1 \le i \le d} (J_f \bigcap \partial \Pi_i)$, then x'_N and y'_N have the same itinerary. Lemma 5.2 gives $x'_N = y'_N$. This contradicts (6.1). Thus there exist N and a periodic point $\xi \in X$ such that $x'_N = \xi$ or $y'_N = \xi$. Suppose $x'_N = \xi$. Let $\xi \in \Theta(U_0)$ and p the period of ξ. Then F^p fixes x'_N and iterates y'_N to at least two distinct segments of $\partial U_0 \setminus \Theta(U_0)$. By properties of supporting rays, x_n, y_n must be separated by $\Theta(U_0)$ for some n, a contradiction.

Finally, $\mathcal{U}$ consists of only one Fatou component. If not, let $U \neq U' \in \mathcal{U}$. Let M, N be integers such that $F^M(U) = F^{M+N}(U), F^M(U') = F^{M+N}(U')$. Then

$$F^{M+N}[c(U), c(U')] = F^M[c(U), c(U')].$$

By Lemma 3.4, $\xi := \partial F^M(U) \bigcap F^M[c(U), c(U')]$ is periodic. Since $F^N|_{\partial F^M(U)}$ conjugates a irrational rotation. Thus ξ can not be periodic, a contradiction. The proof is completed. $\square$

Proof of Theorem 1.2. The theorem follows immediately by Propositions 6.3. $\square$

7. Application to core entropy

Consider a quadratic polynomial family $\mathcal{F} := \{f_c = z^2 + c : f_c$ has no Siegel disks and J_{f_c} is locally connected $\}$. As an application of Theorem 1.2, we shall prove the monotonicity of core entropy.

7.1. **Characteristic arc I_c**

In order to introduce a partial order on $\mathcal{F}$, we need the following definition of *characteristic arc I_c*.

$(C1)$ If f_c has a parabolic or attracting Fatou cycle of period $p \ge 1$. Then there exists a unique point z in the boundary of critical value Fatou component U such that $f_c^p(z) = z$. Let S be the sector containing U and bounded by supporting rays of U at z. Then set $I_c := \overline{\{\theta \in \mathbb{R}/\mathbb{Z} : R(\theta) \subseteq S\}}$. Obviously, $I_c = \mathbb{R}/\mathbb{Z}$ if and only if exactly one ray lands at z.

$(C2)$ In other cases, we have $c \in J_{f_c}$. Then there is a unique sector S based at c containing critical point 0. Set $I_c := \overline{\mathbb{R}/\mathbb{Z} \setminus \{\theta \in \mathbb{R}/\mathbb{Z} : R(\theta) \subseteq S\}}$. Evidently, I_c is a single angle if and only if only one ray lands at critical value c.

For any $f_c, f_{c'}$ in $\mathcal{F}$, we say $f_c \prec f_{c'}$ if and only if $I_c \supseteq I_{c'}$.

If $I_c \neq \mathbb{R}/Z$, denote by $[\eta_c, \xi_c] := I_c$. Let $I'_c \bigcup I''_c := \sigma_2^{-1}(I_c)$ with $I'_c := [\eta'_c, \xi'_c]$ and $I''_c := [\eta''_c, \xi''_c]$, where $\{\eta'_c, \eta''_c\} := \sigma_2^{-1}(\eta_c)$ and $\{\xi'_c, \xi''_c\} := \sigma_2^{-1}(\xi_c)$. The above $[\bullet, \bullet]$ are measured in positive cyclic order and we distinguish it from the notation of regulated arc by acting on distinct categories. Evidently, I'_c and I''_c are symmetric respect to origin with length $|I'_c| = |I''_c| = \frac{1}{2}|I_c|$.

Lemma 7.1 (Properties of characteristic arc)**.** *For any $f_c \in \mathcal{F}$, then*

(1) *If f_c is in case $(C2)$, then*

(1.1) *The rays $R(\eta'_c), R(\eta''_c), R(\xi'_c), R(\xi''_c)$ land at critical point 0.*

(1.2) *If I_c is not a single point, let S'_c resp. S''_c be the sectors bounded by $R(\eta'_c)$ and $R(\xi'_c)$ resp. $R(\eta''_c)$ and $R(\xi''_c)$ and S_c the sectors bounded by $R(\eta_c)$ and $R(\xi_c)$ avoiding the critical point. Then $(S'_c \bigcup S''_c) \bigcap S_c = \emptyset$ and f maps S'_c resp. S''_c conformally onto S_c. Denote by $H_c := S'_c \bigcup S''_c$.*

(2) *If f_c is in case $(C1)$ and $I_c \neq \mathbb{R}/\mathbb{Z}$, then*

(2.1) *$L_{\eta_c\xi_c}$ separates critical point 0 and critical value c. Recall $L_{\eta_c\xi_c} := R(\eta_c) \bigcup R(\xi_c) \bigcup \{z\}$. Therefore, $|I_c| < \frac{1}{2}$.*

(2.2) *$R(\eta'_c)$ and $R(\xi''_c)$ resp. $R(\eta''_c)$ and $R(\xi'_c)$ land together at z' resp. z'' with $\{z', z''\} := f^{-1}(z)$.*

(2.3) *Let S_c be the sectors bounded by $R(\eta_c)$ and $R(\xi_c)$ avoiding the critical point and H_c the domain bounded by $L_{\eta'_c\xi''_c}$ and $L_{\eta''_c\xi'_c}$, then $H_c \bigcap S_c = \emptyset$ and $f : H_c \to S_c$ is a branched covering of degree two.*

(3) *For any $f_c, f_{c'} \in \mathcal{F}$, if $f_c \prec f_{c'}$, then $I'_{c'} \bigcup I''_{c'} \subseteq I'_c \bigcup I''_c$.*

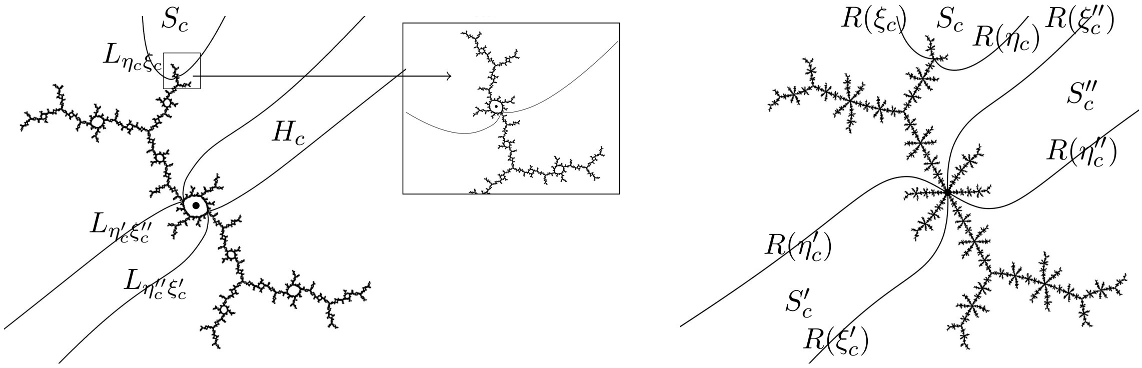

FIGURE 8. illustrating the proof of Lemma 7.1, Left: case (C1). Right: case (C2)

Proof. (1) Since both $R(\eta_c)$, $R(\xi_c)$ land at critical value c. Then, at the critical point 0, there exist preimages, rays $R(\eta'_c), R(\eta''_c), R(\xi'_c), R(\xi''_c)$. If I_c is a single point, we have $\eta'_c = \xi'_c, \eta''_c = \xi''_c$. If I_c is not a single point, consider the sectors S'_c and S''_c based at 0. By lemma 2.1 $(S'_c \bigcup S''_c) \bigcap S_c = \emptyset$. Since $f|_{S'_c}$ resp. $f|_{S''_c}$ is conformal. Thus $l(f(S'_c)) = 2l(S'_c) = 2l(S''_c)$. Note that $l(\mathbb{C} \setminus \overline{S_c}) > l(S'_c) + l(S''_c)$. It follows that

$f(S'_c) = f(S''_c) = S_c$.

(2) Let p be the period of the critical value Fatou component U and $z_0 := z, z_1 := f(z), \cdots, z_p := f^p(z)$ with $z_p = z_0$. Since this orbit is disjoint with critical point, we can set L_{z_i} the preimage of $f^{-(p-i)}L_{\eta_c\xi_c}$ at each z_i. Obviously, $L_{z_0} = L_{z_p}$, because both of them support Fatou component U. Let S_{z_i} be one of the components $\mathbb{C} \setminus L_{z_i}$ containing 0 and S'_{z_i} the other.

For (2.1), suppose L_{z_0} does not separate 0 and c, then S_{z_0} contains both of them. For $i = p-1$, By Lemma 2.2 (4), the sector map σ_2 must send the critical sector $S_{z_{p-1}}$ to critical value sector S_{z_p}, and thus $\sigma_2(S'_{z_{p-1}}) = S'_{z_p}$. We have $l(S'_{z_{p-1}}) = \frac{1}{2}l(S'_{z_0})$. Claim $L_{z_{p-1}}$ cannot separate 0 and c. Otherwise, using Lemma 2.1 and properties of supporting rays, we have $S'_{z_{p-1}} \supset S'_{z_0}$, thus $l(S'_{z_{p-1}}) > l(S'_{z_0})$, impossible. For $i = p-2, \cdots, 0$, the same argument as above implies $l(S'_{z_i}) = \frac{1}{2}l(S'_{z_{i+1}})$ and S_{z_i} contains both 0 and c. Thus $l(S'_{z_0}) = \frac{1}{2^p}l(S'_{z_0})$, a contradiction.

For (2.2), since z is not a critical value. We have two $z' \neq z''$ preimages of z. We discuss by contradiction and assume $R(\eta'_c), R(\xi'_c)$ resp. $R(\eta''_c), R(\xi''_c)$ land at z' resp. z''. Then consider the sector $S'_c := \bigcup_{\theta \in I'_c} R(\theta)$ resp. $S''_c := \bigcup_{\theta \in I''_c} R(\theta)$. We have $\sigma_2(S'_c) = \sigma_2(S''_c) = S'_{z_0}$. Since $l(S'_c) = l(S''_c) = l(I'_c) = l(I''_c) < \frac{1}{2}$, by Lemma 2.2, $f|_{S''_c}, f|_{S''_c}$ are conformal. Therefore, the image S'_{z_0} cannot contain critical value c. This contradicts (2.1).

For (2.3), note that both of $L_{\eta'_c\xi''_c}$ and $L_{\eta''_c\xi'_c}$ support the critical Fatou component and are symmetry respect the original. Then the fact $|I_c| > |I'_c| = |I''_c|$ implies $H_c \bigcap S_c = \emptyset$.

For (3), one can easily check it by definition. □

7.2. **Dynamic of biaccessible angles**

Given $f_c \in \mathcal{F}$, an angle θ in $\mathbb{R}/\mathbb{Z}$ is called to be *biaccessible*, if there exists $\theta' \neq \theta$ such that $R(\theta)$ and $R(\theta')$ landing together. Evidently, if θ is biaccessible, then the preimages $\sigma_2^{-1}(\theta)$ are biaccessible. Inversely, if θ is biaccessible and $\alpha(\theta)$ is not the critical point, then $\sigma_2(\theta)$ is also biaccessible. Denote by $Acc(f_c)$ the set of all biaccessible angles of f_c. Then if $I_c = \mathbb{R}/\mathbb{Z}$, $Acc(f_c) = \emptyset$ by lemma 5.2.

Lemma 7.2. *Let $I_c \neq \mathbb{R}/\mathbb{Z}$ and not a single angle. Let θ be a biaccessible angle of f_c and the orbit of the landing point $\zeta_0 := \alpha(\theta)$ avoid critical point 0. Then there exists a $N \geq 0$ such that the orbit of $\zeta_N := f^N(\zeta_0)$ is disjoint with H_c, where H_c is defined in Lemma* 7.1 (1.2)(2.3). *Therefore, there exists $\vartheta \neq \theta_N := \sigma_2^N(\theta)$ such that, for any $\nu \in (\eta_c, \xi_c)$, ϑ and θ_N have the same itinerary respect to $\mathbb{R}/\mathbb{Z} \setminus \sigma_2^{-1}(\nu)$.*

Proof. Let $\theta' \neq \theta$ with $\alpha(\theta') = \alpha(\theta) = \zeta_0$. Since ζ_0 will never meet the critical point. For $n \geq 0$, $L_{\theta_n\theta'_n} = f^n(L_{\theta\theta'})$ bounds two sectors S_{ζ_n} and S'_{ζ_n}, where we assume S_{ζ_n} is the one containing 0.

Firstly, there exists a $N \geq 0$ such that $L_{\theta_N\theta'_N}$ separates 0 and c. If not, for each $n \geq 0$, σ_2 must send S_{ζ_n} to $S_{\zeta_{n+1}}$ and S'_{ζ_n} to $S'_{\zeta_{n+1}}$, therefore, $l(S'_{\zeta_{n+1}}) = 2l(S'_{\zeta_n})$ by Lemma 2.2 (2), (3) and (4). It follows that $l(S'_{\zeta_n}) \to \infty$ as $n \to \infty$, impossible.

By Lemma 7.1 (1.2)(2.3), points in H_c will be mapped to S_c. Thus we only have to show that $\zeta_n \notin S_c$, $n \geq N$. Claim $l(S'_{\zeta_n}) \geq l(S_c)$. If not, let $n_0 > N$ be first integer such that $l(S'_{\zeta_{n_0}}) < l(S_c)$. Thus $S'_{\zeta_{n_0}}$ must be a critical value sector. This means $S'_{\zeta_{n_0}} \supseteq S_c$ or $S'_{\zeta_{n_0}} \supseteq \mathbb{C}\setminus\overline{S_c}$, both of which imply $l(S'_{\zeta_{n_0}}) \geq l(S_c)$, a contradiction. Therefore, $\zeta_n \notin S_c, n \geq N$. □

7.3. Monotonicity of core-entropy

Proof of Theorem 1.4. If $I_c = \mathbb{R}/\mathbb{Z}$, $Acc(f_c) = \emptyset$.

If $\#I_c = 1$, then $I_c = I_{c'}$, hence $I'_c = I'_{c'}$ and $I''_c = I''_{c'}$. By Theorem 1.2, $Acc(f_c) = Acc(f_{c'})$.

In other cases, we have either $I_c = I_{c'}$ or $I_{c'} \subsetneq I_c$.

If $I_{c'} \subsetneq I_c$. We can assume $\eta_{c'} \in (\eta_c, \xi_c)$. For any $\theta \in Acc(f_c)$, if the orbit of $\alpha(\theta)$ is disjoint with critical point 0, by Lemma 7.2, there exist N and $\theta' \neq \theta_N$ such that θ_N and ϑ have the same itinerary respect to partition $\mathbb{R}/\mathbb{Z} \setminus \sigma_2^{-1}(\eta_{c'})$. By theorem 1.2, in the dynamic plane of $f_{c'}$, external rays with arguments θ_N, ϑ land together. Thus $\theta \in Acc(f_{c'})$. If $\alpha(\theta)$ is iterated to 0, then critical point is not periodic. Evidently, the above N and ϑ exist as well.

If $I_c = I_{c'}$. For any $\theta \in Acc(f_c)$, if $\alpha(\eta_c)$ is not periodic, by the same argument as above, such ϑ and N exist. If $\alpha(\eta_c)$ is periodic. If the orbit of $\alpha(\theta)$ avoids $\alpha(\eta_c)$, then such ϑ and N exist. If $\alpha(\theta)$ is mapped to $\alpha(\eta_c)$. Then θ is iterated into $\{\eta_c, \xi_c\} \subseteq Acc(f_{c'})$. Thus $\theta \in Acc(f_{c'})$. The proof is completed. $\square$

References

[BFH92] B.Bielefield, Y.Fisher and J.H.Hubbard, The classifiction of critically preperiodic polynomials as dynamical systems, *J.Amer.Math.Soc.* 5(1992), 721-762.

[Bl05] A.Blokh, Necessary conditions for the existence of wandering triangles for cubic laminations, *Discrete Contin. Dyn. Syst.* 13(1)2005, 13-34.

[BL02] A.Blokh and G.Levin, Growing trees, laminations and the dynamics on the Julia set, *Ergod. Th. and Dynam. Sys.*, 22(2002), 63-97.

[BO08] A.Blokh and L.Oversteegen, Wandering gaps for weakly hyperbolic polynomials. *Complex Dynamics: Families and Friends.* Ed. D.Schleicher A.K.Peters, Wellesley, MA, 2008,139-168.

[Do95] A.Douady, Topological entropy of unimodal maps: monotonicity for quadratic polynomials, in *Real and complex dynamical systems (Hillerød,1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* 464(1995) 65-87.

[DH84] A.Douady and J.H.Hubbard[1984-85] Étude Dynamique des Polynômes Complexes I and II, *Publ. Math. Orsay.*

[Ga13] Y.Gao, Dynatomic curves and core entropy of polynomials, PhD thesis, Université d'Angers, 2013.

[GM93] L.R.Goldberg and J.Milnor, Fixed points of polynomial maps II: Fixed point portraits, *Ann. Scient. École Norm. Sup., 4^e série*, 26(1993),51-98.

[Ju13] W.Jung, Core entropy and biaccessibility of quadratic polynomials, available at arXiv:1401.4792[math.DS].

[Ki02] J.Kiwi, Wandering orbit portraits, *Trans. Amer. Math. Soc.*, 354(2002), 1473-1485.

[Ki04] J.Kiwi, Real laminations and the topological dynamics of complex polynomials, *Advances in Mathematics* 184(2004), 207-267.

[Ki05] J.Kiwi, Combinatorial continuity in complex polynomial dynamics, *Proc. London Math. Soc.* 91(2005), no.3, 215-248.

[Le98] G.Levin, On backward stability of holomorphic dynamical systems, *Fundamenta Mathematicae*, 158(1998), 97-107.

[Li07] T.Li,*A monotonicity conjecture for the entropy of Hubbard trees*, PhD Thesis, SUNY Stony Brook, 2007.

[Mc95] C.T.McMullen, *Complex Dynamics and Renormalization*, Annals of Mathematics Studies 135, Princeton University Press 1995.

[Mi00] J.Milnor, *Periodic orbits, externals rays and the Mandelbrot set: An expository account*, in "Géomé Complexe et Systèmes Dynamiques, Colloque en l'honneur d'Adrien Douady", Astérisque, 261(2000), 277-333.

[Mi06] J.Milnor, *Dynamics in one complex variable*, 3rd ed., Princeton Univ. Press, Princeton, NJ, 2006.

[NaJ92] Sam B.Nadler and Jr., *Continuum theory. An introduction*, Monographs and Textbooks in Pure and Applied Mathematics, 158, Marcel Dekker, Inc., New York, NY (1992).

[Po93] A.Poirier, On postcritically finite polynomials, part I: critical portraits, IMS preprint #5, Stony Brook (1993).

[Po93] A.Poirier, On post critically finite polynomials, part II: Hubbard trees, IMS preprint #7, Stony Brook (1993).

[Ru87] W.Rudin,*Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987.

[Su83] D.Sullivan, *Conformal Dynamical Systems*, Springer Lecture Notes in Mathematics,1007(1983) 725-752.

[Th85] W.Thurston, The combinatorics of iterated rational maps (1985). *Complex Dynamics: families and Friends.* Ed. D.Schleicher. A. K. Peters, Wellesley, MA, 2008, pp.3-137.

[Th14] W.Thurston, Entropy in dimension one, in *Frontiers in Complex Dynamics: In Celebration of John Milonor's 80th Birthday,* edited by A.Bonifant, M.Lyubich, and S.Sutherland, Princeton,2014.

[Ti13] G.Tiozzo, Topological entropy of quadratic polynomials and dimension of sections of the Mandelbrot set, available at arXiv:1305.3542[math.DS].

[Ti14] G.Tiozzo, Continuity of core entropy of quadratic polynomials, available at arXiv:1409.3511[math.DS].

[Ts00] M.Tsujii, A simple proof for monotonicity of entropy in the quadratic family, *Ergod. Th. and Dynam. Sys.*, 20(2000), no.3, 925-933.

[Za00] S.Zakeri, Biaccessibility in quadratic Julia sets, *Ergod. Th. and Dynam. Sys.*, 20(2000), no.6, 1859-1883.

[Zd00] A.Zdunik, On biaccessible points in Julia sets of polynomials, *Fund. Math.* 163(2000), no.3, 277-286.

Jinsong Zeng, Département de Mathématiques, Université d'Angers, Angers, 49000, France
E-mail address: `mczjs@qq.com`