ISS of multistable systems with delays: application to droop-controlled inverter-based microgrids

Denis Efimov, Romeo Ortega and Johannes Schiffer

***Abstract*—Motivated by the problem of phase-locking in droop-controlled inverter-based microgrids with delays, the recently developed theory of input-to-state stability (ISS) for multistable systems is extended to the case of multistable systems with delayed dynamics. Sufficient conditions for ISS of delayed systems are presented using Lyapunov-Razumikhin functions. It is shown that ISS multistable systems are robust with respect to delays in a feedback. The derived theory is applied to two examples. First, the ISS property is established for the model of a nonlinear pendulum and delay-dependent robustness conditions are derived. Second, it is shown that, under certain assumptions, the problem of phase-locking analysis in droop-controlled inverter-based microgrids with delays can be reduced to the stability investigation of the nonlinear pendulum. For this case, corresponding delay-dependent conditions for asymptotic phase-locking are given.**

I. Introduction

The increasing penetration of renewable distributed generation (DG) units at the low and medium voltage levels has a strong impact on the power system structure [9], [38], [8]. This fact requires new control and operation strategies to ensure a reliable and efficient electrical power supply [9], [11]. An emerging concept to address these challenges is the microgrid [17], [14], [9]. A microgrid is a locally controllable subset of a larger electrical network. It is composed of several DG units, storage devices and loads.

Typically, most DG units in an AC microgrid are connected to the network via AC inverters [11]. Under ideal conditions, an inverter-based DG unit can be modeled as an ideal controllable voltage source [18], [26]. Furthermore, a popular control scheme to operate inverter-based DG units with the purpose to achieve frequency synchronization and power sharing in the network is droop control [5], [13]. Conditions for stability in droop-controlled microgrids with inverters modeled as ideal controllable voltage sources have been derived, e.g., in [30], [28], [21].

In general, inverter-based microgrids operated with droop control have several equilibria [30], [28]. Thus they are multistable systems. Stability analysis [3], [7], [37], [20], [22], [24], [25], [27], [31] and robust stability analysis [1], [2], [4], [6], [35] for this class of systems is rather complicated. Recently, the ISS theory [33] has been extended to multistable systems in [2] (see also [15] for discussion on ISS property with respect to an unbounded set).

Furthermore, in a practical setup, the droop control scheme is applied to an inverter by means of digital discrete time control. Besides clock drifts, see, e.g., [29], digital control usually introduces time delays [16], [19], [23]. According to [23], the main reasons for this are 1) sampling of control variables, 2) calculation time of the digital controller and 3) generation of the pulse-width-modulation. We refer the reader to, e.g., [23] for further details. To the best of the authors' knowledge this fact has yet not been considered in previous analysis of droop-controlled microgrids.

Motivated by the abovementioned phenomenon, the main contribution of the present paper is to extend the recently derived ISS framework for multistable systems [2] to multistable systems with delay. In particular, sufficient conditions for ISS of multistable systems in the presence of delays are given in terms of a Lyapunov-Razumikhin function. It is also shown that ISS multistable systems are robust with respect to feedback delays. This result is illustrated via the example of a nonlinear pendulum. Next, based on the established results, we provide a condition for asymptotic phase-locking in a microgrid composed of two droop-controlled inverters with delay. The analysis is conducted for a simplified inverter model derived under the assumptions of constant voltage amplitudes and ideal clocks, as well as negligible dynamics of the internal inverter filter and controllers. In that scenario, the delay merely affects the phase angle of the inverter output voltage. The stability results are illustrated by simulations.

II. Preliminaries

For an n-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold M without a boundary, let the map $f(x,d) : M \times \mathbb{R}^m \to T_x M$ be of class $\mathcal{C}^1$, and consider a nonlinear system of the following form:

$$\dot{x}(t) = f(x(t), d(t)), \tag{1}$$

where the state $x \in M$ and $d(t) \in \mathbb{R}^m$ (the input $d(\cdot)$ is a locally essentially bounded and measurable signal) for $t \geq 0$. We denote by $X(t, x_0; d)$ the uniquely defined solution of (1) at time t fulfilling $X(0, x_0; d) = x_0$. Together with (1) we will analyze its unperturbed version:

$$\dot{x}(t) = f(x(t), 0). \tag{2}$$

Denis Efimov is with with Non-A team @ Inria, Parc Scientifique de la Haute Borne, 40 avenue Halley, 59650 Villeneuve d'Ascq, France, with CRIStAL (UMR-CNRS 9189), Ecole Centrale de Lille, Avenue Paul Langevin, 59651 Villeneuve d'Ascq, France and Department of Control Systems and Informatics, University ITMO, 49 avenue Kronverkskiy, 197101 Saint Petersburg, Russia.

Romeo Ortega is with Laboratoire des Signaux et Systémes, École Supérieure d'Electricité (SUPELEC), Gif-sur-Yvette 91192, France.

Johannes Schiffer is with Technische Universität Berlin, Einsteinufer 11, 10587 Berlin, Germany.

This work was supported in part by Région Nord-Pas de Calais, by the Government of Russian Federation (Grant 074-U01) and the Ministry of Education and Science of Russian Federation (Project 14.Z50.31.0031).

A set $S \subset M$ is invariant for the unperturbed system (2) if $X(t, x; 0) \in S$ for all $t \in \mathbb{R}$ and for all $x \in S$. Define the distance from a point $x \in M$ to the set $S \subset M$ as $|x|_S = \min_{a \in S} \delta(x, a)$, where the symbol $\delta(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M, $|x| = |x|_{\{0\}}$ for $x \in M$ or a usual euclidean norm of a vector $x \in \mathbb{R}^n$. For a signal $d : \mathbb{R} \to \mathbb{R}^m$ the essential supremum norm is defined as $\|d\|_\infty = \operatorname{ess\,sup}_{t \geq 0} |d(t)|$.

A. Decomposable sets

Let $\Lambda \subset M$ be a compact invariant set for (2).

Definition 1. [22] A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \dots, \Lambda_k$ such that

$$\Lambda = \bigcup_{i=1}^{k} \Lambda_i.$$

For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$W^s(\Lambda) = \{x \in M : |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to +\infty\},$$
$$W^u(\Lambda) = \{x \in M : |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to -\infty\}.$$

Define a relation on $\mathcal{W} \subset M$ and $\mathcal{D} \subset M$ by $\mathcal{W} \prec \mathcal{D}$ if $W^s(\mathcal{W}) \cap W^u(\mathcal{D}) \neq \oslash$.

Definition 2. [22] Let $\Lambda_1, \dots, \Lambda_k$ be a decomposition of Λ, then

1. An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \dots, i_r$ such that $\Lambda_{i_1} \prec \dots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
2. A 1-cycle is an index i such that $[W^u(\Lambda_i) \cap W^s(\Lambda_i)] - \Lambda_i \neq \oslash$.
3. A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

As we can conclude from Definition 2, existence of an r-cycle with $r \geq 2$ is equivalent to existence of a heteroclinic cycle for (2) [12]. Furthermore, existence of a 1-cycle implies existence of a homoclinic cycle for (2) [12].

Definition 3. The set $\mathcal{W}$ is called decomposable if it admits a finite decomposition without cycles, $\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, which form a filtration ordering of $\mathcal{W}$, as detailed in definitions 1 and 2.

B. Robustness notions

The following robustness notions for systems represented by (1) have been introduced in [2].

Definition 4. We say that the system (1) has the practical asymptotic gain (pAG) property if there exist $\eta \in \mathcal{K}_\infty$ and a non-negative real q such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t \to +\infty} |X(t, x; d)|_{\mathcal{W}} \leq \eta(\|d\|_\infty) + q.$$

If $q = 0$, then we say that the asymptotic gain (AG) property holds.

Definition 5. We say that the system (1) has the limit property (LIM) with respect to $\mathcal{W}$ if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t \geq 0} |X(t, x; d)|_{\mathcal{W}} \leq \mu(\|d\|_\infty).$$

Definition 6. We say that the system (1) has the practical global stability (pGS) property with respect to $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x \in M$ and measurable essentially bounded inputs $d(\cdot)$ the following holds for all $t \geq 0$:

$$|X(t, x; d)|_{\mathcal{W}} \leq q + \beta(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}).$$

It has been shown in [2] that to characterize pAG property in terms of Lyapunov functions the following notion is appropriate.

Definition 7. We say that a $\mathcal{C}^1$ function $V : M \to \mathbb{R}$ is a practical ISS-Lyapunov function for (1) if there exists $\mathcal{K}_\infty$ functions $\alpha_1, [\alpha_2], \alpha_3$ and γ, and scalar $q \geq 0$ [and $c \geq 0$] such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq [\alpha_2(|x|_{\mathcal{W}} + c)],$$

the function V is constant on each $\mathcal{W}_i$ and the following dissipation holds:

$$DV(x) f(x, d) \leq -\alpha_3(|x|_{\mathcal{W}}) + \gamma(|d|) + q.$$

If the latter inequality holds for $q = 0$, then V is said to be an ISS-Lyapunov function.

Notice that α_2 and c are in square brackets as their existence follows (without any additional assumptions) by standard continuity arguments.

The main result of [2] connecting these robust stability properties is stated below, it extends the results of [32], [34] obtained for connected sets.

Theorem 1. *Consider a nonlinear system as in (1) and let a compact invariant set containing all α- and ω-limit sets of (2) $\mathcal{W}$ be decomposable (in the sense of Definition 3). Then the following facts are equivalent.*

1. *The system admits an ISS Lyapunov function;*
2. *The system enjoys the AG property;*
3. *The system admits a practical ISS Lyapunov function;*
4. *The system enjoys the pAG property;*
5. *The system enjoys the LIM property and the pGS.*

A system in (1), for which this list of equivalent properties is satisfied, is called ISS with respect to the set $\mathcal{W}$ [2].

III. Multistable systems with delays

Let $\tau > 0$, for a function $d : [-\tau, +\infty) \to \mathbb{R}^m$ and $t \geq 0$ denote a function $d_t(\cdot) : [-\tau, 0] \to \mathbb{R}^m$ defined by $d_t(\theta) = d(t + \theta)$ for $\theta \in [-\tau, 0]$. Denote by $\mathcal{D}$ a set of bounded and piecewise continuous functions $d_t(\cdot) : [-\tau, 0] \to \mathbb{R}^m$. Consider a functional differential equation on an n-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold M without a boundary:

$$\dot{x}(t) = F(x_t, d_t),\ x_0 \in \mathcal{C}_\tau, \tag{3}$$

where the map $F : \mathcal{C}_\tau \times \mathcal{D} \to T_x M$ is of class $\mathcal{C}^1$ (we will denote a set of continuous functions $\xi : [-\tau, 0] \to M$ by $\mathcal{C}_\tau$), $x(t) \in M$ is the state, $x_t \in \mathcal{C}_\tau$ and $d_t \in \mathcal{D}$ for all $t \geq 0$. We denote by $X(t, x_0; d)$ the uniquely defined solution of (3) at time t fulfilling $X(\theta, x_0; d) = x_0(\theta)$ for all $\theta \in [-\tau, 0]$; $X_t^{x_0,d}(\theta) = X(t+\theta, x_0; d)$ for $\theta \in [-\tau, 0]$. Define as in [36]

$$|x_t| = \max_{\theta\in[-\tau,0]} |x(t+\theta)|,\ ||x||_{t_0} = \sup_{t\geq t_0} |x_t| = \sup_{t\geq t_0-\tau} |x(t)|.$$

Again, together with (3), we will analyze its unperturbed version:

$$\dot{x}(t) = F(x_t, 0). \tag{4}$$

A set $\mathcal{S} \subset \mathcal{C}_\tau$ is invariant for the unperturbed system (4) if $X_t^{x_0,0} \in \mathcal{S}$ for all $t \in \mathbb{R}_+$ and for all $x_0 \in \mathcal{S}$. Define the distance from a function $\xi \in \mathcal{C}_\tau$ to a set $\mathcal{S} \subset \mathcal{C}_\tau$ as $||\xi||_{\mathcal{S}} = \min_{\alpha\in\mathcal{S}} |\xi - \alpha|$.

Let $\mathcal{W} \subset M$ be a set, denote by $\overbrace{\mathcal{W}}$ a subset of $\overline{\mathcal{W}} = \{\xi \in \mathcal{C}_\tau : \xi(t) \in \mathcal{W}\ \forall t \in [-\tau, 0]\}$ such that if $\zeta \in \overbrace{\mathcal{W}}$ then $\zeta = X_\tau^{\xi,0}$ for $\xi \in \overline{\mathcal{W}}$. For stability analysis in time-delay systems it is required to define a distance to invariant sets in two spaces: in $\mathbb{R}^n$ with respect to the set $\mathcal{W}$ and in $\mathcal{C}_\tau$ with respect to corresponding invariant set $\overbrace{\mathcal{W}}$ (functions from $\mathcal{C}_\tau$ taking values in $\mathcal{W}$ and solutions of (3)). The following stability notions for (3) are considered in this work.

Definition 8. The system (3) has the pAG property with respect to the set $\mathcal{W}$ if there exist $\eta \in \mathcal{K}_\infty$ and a non-negative real q such that for all $x_0 \in \mathcal{C}_\tau$ and all bounded piecewise continuous inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t\to+\infty} |X(t, x_0; d)|_{\mathcal{W}} \leq \eta(\|d_t\|_0) + q.$$

If $q = 0$, then we say that the AG property holds.

This property can be equivalently stated as

$$\limsup_{t\to+\infty} ||X_t^{x_0,d}||_{\overbrace{\mathcal{W}}} \leq \eta(\|d_t\|_0) + q$$

and it implies that (a subset of) $\overbrace{\mathcal{W}}$ is invariant for (4) if $q = 0$.

Definition 9. The system (3) has the pGS property with respect to the set $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x_0 \in \mathcal{C}_\tau$ and all bounded piecewise continuous inputs $d(\cdot)$ the following holds for all $t \geq 0$:

$$|X(t, x_0; d)|_{\mathcal{W}} \leq q + \beta(\max\{||x_0||_{\overbrace{\mathcal{W}}}, \|d_t\|_0\}).$$

To characterize pAG and pGS properties for a time-delay system (3) the Lyapunov-Razumikhin approach is used in this work. Given a continuous function $x : [-\tau, +\infty) \to M$ with a $\mathcal{C}^1$ function $U : M \to \mathbb{R}$ denote $U(t) = U(x(t))$, if $x(t) = X(t, x_0; d)$ is a solution to (3) for some piecewise continuous $d : [-\tau, +\infty) \to \mathbb{R}^m$ and initial condition $x_0 \in \mathcal{C}_\tau$, then the upper right-hand side derivative of U along this solution is

$$D^+U(t) = \limsup_{h\to 0^+} \frac{U(t+h) - U(t)}{h}.$$

Definition 10. A $\mathcal{C}^1$ function $U : M \to \mathbb{R}$ is a practical ISS-Lyapunov-Razumikhin (ISS-LR) function for (3) if there exist $\mathcal{K}_\infty$ functions $\alpha_1, [\alpha_2], \alpha_4, \gamma$ and γ_U, $\gamma_U(s) < s$ for all $s > 0$, and scalar $q \geq 0$ [and $c \geq 0$] such that

$$\begin{gathered}\alpha_1(|x|_{\mathcal{W}}) \leq U(x) \leq [\alpha_2(|x|_{\mathcal{W}} + c)],\\ U(t) \geq \max\{\gamma_U(|U_t|), \gamma(|d_t|), q\}\ \Rightarrow\\ D^+U(t) \leq -\alpha_4[U(t)].\end{gathered}$$

If the latter inequality holds for $q = 0$, then U is said to be an ISS-LR function.

The following result can be stated connecting pAG, pGS properties and existence of an ISS-LR function.

Theorem 2. *Consider the system (3). Suppose there exists an ISS-LR function $U : M \to \mathbb{R}$ as in Definition 10. Then the system (3) admits the pAG property from Definition 8 with $\eta(s) = \alpha_1^{-1} \circ \gamma(s)$ and pGS property from Definition 9.*

Proof. The proof mainly follows the ideas of [36]. □

IV. ISS OF MULTISTABLE SYSTEMS WITH DELAYED PERTURBATIONS

In this section we consider the robustness of the system (1) with respect to a disturbance d, which is dependent on a delayed state. The analysis is conducted under the assumption that the system (1) is ISS with respect to a set $\mathcal{W}$.

A. Robustness analysis

If (1) is ISS with respect to the set $\mathcal{W}$, then by Theorem 1 there exists an ISS Lyapunov function V as in Definition 7. From the inequalities $\alpha_3[0.5\alpha_2^{-1} \circ V(x)] \leq \alpha_3(0.5[|x|_{\mathcal{W}} + c]) \leq \alpha_3(|x|_{\mathcal{W}}) + \alpha_3(c)$ we obtain

$$DV(x)f(x, d) \leq -\alpha_4[V(x)] + \gamma(|d|) + \tilde{q},$$

where $\alpha_4(s) = \alpha_3[0.5\alpha_2^{-1}(s)]$ and $\tilde{q} = q + \alpha_3(c)$.

Assume that the input d has two terms d_1 and d_2, and d_2 is a function of $x_t \in \mathcal{C}_\tau$ for some $\tau > 0$, *i.e.*:

$$d = d_1 + d_2,\ d_2 = g(x_t), \tag{5}$$

where g is a continuous function, $|g(x_t)| \leq \upsilon(|V_t|) + \upsilon_0$ for $\upsilon \in \mathcal{K}_\infty$ and $\upsilon_0 \geq 0$. Denote further for simplicity of notation $d = d_1$, then the system (1) is transformed to (3) with

$$F(x_t, d_t) = f(x(t), d + g(x_t)),$$

and

$$D^+V(t) \leq -\alpha_4(V(t)) + \gamma(2\upsilon(|V_t|) + 2\upsilon_0) + \gamma(2|d_t|) + \tilde{q}.$$

This estimate can be rewritten as follows:

$$V(t) \geq \max\{\hat{\gamma}_V(|V_t|), \hat{\gamma}(|d_t|), \hat{q}\} \Rightarrow$$
$$D^+V(t) \leq -0.5\alpha_4(V(t)),$$
$$\hat{\gamma}_V(s) = \alpha_4^{-1}[6\gamma(4\upsilon(s))],\ \hat{\gamma}(s) = \alpha_4^{-1}[6\gamma(2s)],$$
$$\hat{q} = \alpha_4^{-1}[6\tilde{q} + 6\gamma(4\upsilon_0)].$$

It is straightforward to see that if $\hat{\gamma}_V(s) < s$ for all $s > 0$, then V is an ISS-LR function for (1) with (5), and by Theorem 2 this system possesses pAG and pGS properties.

B. Illustration for a nonlinear pendulum

Consider a nonlinear pendulum:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\Omega^2 \sin(x_1) - \kappa x_2 + d, \end{aligned} \tag{6}$$

where the state $x = [x_1, x_2]$ takes values on the cylinder $M := \mathbb{S} \times \mathbb{R}$, $d(t) \in \mathbb{R}$ is an exogenous disturbance, and Ω, κ are constant positive parameters. The unperturbed system (6) admits a Hamiltonian $H(x) = 0.5x_2^2 + \Omega^2(1 - \cos(x_1))$ and $\dot{H} = x_2 d - \kappa x_2^2$. The unperturbed system (6) has two equilibria $[0, 0]$ and $[\pi, 0]$ (the former is attractive and the latter one is a saddle-point). Thus, $\mathcal{W} = \{[0, 0] \cup [\pi, 0]\}$ is a compact set containing all α- and ω-limit sets of (6) for $d = 0$. In addition, it is straightforward to check that $\mathcal{W}$ is decomposable in the sense of Definition 3.

Lemma 1. *The system (6) is ISS with respect to the set $\mathcal{W}$.*

Proof. Developing ideas of [4], the result follows from Theorem 1 considering a Lyapunov function candidate

$$V(x) = H(x) + \kappa\epsilon(1 - \cos(x_1)) + \epsilon x_2 \sin(x_1),$$

which admits derivative

$$\begin{aligned} \dot{V} &= -[\kappa - \epsilon\cos(x_1)]x_2^2 - \epsilon\Omega^2 \sin^2(x_1) \\ &\quad + \epsilon \sin(x_1) d + x_2 d \\ &\leq -0.5[\kappa - \epsilon]x_2^2 - 0.5\epsilon\Omega^2 \sin^2(x_1) \\ &\quad + 0.5[\epsilon\Omega^{-2} + \frac{1}{\kappa - \epsilon}]d^2 \end{aligned} \tag{7}$$

provided that $0 < \epsilon < \kappa$. □

Now consider a time-delay modification of (6):

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\Omega^2 \sin[x_1(t - \tau)] - \kappa x_2(t) + d(t), \end{aligned} \tag{8}$$

where $\tau > 0$ is a fixed delay. The unperturbed system (8) with $d(t) = 0$ has the same equilibria as (6), *i.e.* $[0, 0]$ and $[\pi, 0]$. The system (8) can be represented as follows:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\Omega^2 \sin[x_1(t)] - \kappa x_2(t) \\ &\quad + d(t) + \Omega^2\{\sin[x_1(t)] - \sin[x_1(t - \tau)]\}. \end{aligned}$$

By the Mean value theorem

$$|\sin[x_1(t)] - \sin[x_1(t - \tau)]| = |\cos[x_1(\phi)]x_2(\phi)\tau|$$
$$\leq |x_2(\phi)|\tau$$

for some $\phi \in [t - \tau, t]$. Thus, the system (8) can be analyzed as a perturbed nonlinear pendulum with part of the input d dependent on the delay. By taking the estimate derived for V in (7) we obtain for $\mu = \epsilon\Omega^{-2} + \frac{1}{\kappa - \epsilon}$:

$$\begin{aligned} D^+V(t) &\leq -0.5[\kappa - \epsilon]x_2^2 - 0.5\epsilon\Omega^2 \sin^2(x_1) \\ &\quad + \mu\Omega^4 x_2^2(\phi)\tau^2 + \mu d^2. \end{aligned}$$

It is straightforward to check that

$$V(x) \leq 0.5[1 + \epsilon]x_2^2 + 0.5\epsilon \sin^2(x_1) + 2[\Omega^2 + \kappa\epsilon],$$
$$x_2^2 \leq \frac{2}{1 - \epsilon}V(x) + \frac{\epsilon}{1 - \epsilon}$$

for $0 < \epsilon < \min\{1, \kappa\}$, then for $\rho = \min\{\frac{\kappa - \epsilon}{1 + \epsilon}, \Omega^2\}$

$$\begin{aligned} D^+V(t) &\leq -\rho\{V(t) - 2[\Omega^2 + \kappa\epsilon]\} \\ &\quad + \mu\Omega^4 x_2^2(\phi)\tau^2 + \mu d^2 \\ &\leq -\rho\{V(t) - 2[\Omega^2 + \kappa\epsilon]\} \\ &\quad + \frac{\mu\Omega^4}{1 - \epsilon}\tau^2[2V(\phi) + \epsilon] + \mu d^2. \end{aligned}$$

Therefore,

$$\begin{aligned} V(t) \geq \frac{6}{\rho}\max\{&2\frac{\mu\Omega^4}{1 - \epsilon}\tau^2|V_t|, 2\rho[\Omega^2 + \kappa\epsilon] \\ &+ \frac{\mu\Omega^4}{1 - \epsilon}\tau^2\epsilon, \mu d^2\} \Rightarrow \\ &D^+V(t) \leq -0.5\rho V(t) \end{aligned}$$

and V is an ISS-LR function for (8) provided that

$$\frac{12}{\rho}\frac{\mu\Omega^4}{1 - \epsilon}\tau^2 < 1. \tag{9}$$

The inequality (9) is a delay-dependent stability condition for (8), which is always satisfied for a sufficiently small delay τ. If we assume that $\max\{0, \frac{\kappa - \Omega^2}{1 + \Omega^2}\} < \epsilon < \min\{1, \kappa\}$, then $\min\{\frac{\kappa - \epsilon}{1 + \epsilon}, \Omega^2\} = \frac{\kappa - \epsilon}{1 + \epsilon}$ and the condition (9) can be rewritten as follows:

$$\tau^2 < \frac{1}{12\Omega^2}\frac{1 - \epsilon}{1 + \epsilon}\frac{1}{\epsilon(\kappa - \epsilon) + \Omega^2}.$$

Since the functions $\frac{1 - \epsilon}{1 + \epsilon}$ and $\frac{1}{\epsilon(\kappa - \epsilon) + \Omega^2}$ are decreasing for $\epsilon \in (\max\{0, \frac{\kappa - \Omega^2}{1 + \Omega^2}\}, \min\{1, \kappa\})$, selecting $\epsilon = \max\{0, \frac{\kappa - \Omega^2}{1 + \Omega^2}\} + \varepsilon$ for a sufficiently small $\varepsilon > 0$ optimizes the value of the admissible delay τ to

$$\tau^* = \frac{1}{2\Omega}\sqrt{\frac{1 - \epsilon}{1 + \epsilon}\frac{1/3}{\epsilon(\kappa - \epsilon) + \Omega^2}},$$

i.e. for any $\tau < \tau^*$ the system (8) admits V as an ISS-LR function.

V. Application to a microgrid composed of two droop-controlled inverters with delay

By following [28], under the assumption of constant voltage amplitudes, a lossless droop-controlled microgrid formed by two inverters with delay can be modeled as:

$$\begin{aligned} \dot{\theta}(t) &= \omega_1(t) - \omega_2(t), \\ \tau_{P_1}\dot{\omega}_1(t) &= -\omega_1(t) - k_{P_1}a_{12}\sin[\theta(t - \tau_{d_1})] + c_1 + d_1(t), \\ \tau_{P_2}\dot{\omega}_2(t) &= -\omega_2(t) + k_{P_2}a_{12}\sin[\theta(t - \tau_{d_2})] + c_2 + d_2(t), \end{aligned} \tag{10}$$

where $\theta(t) \in [0, 2\pi)$ is the phase difference in inverters, $\omega_1(t), \omega_2(t) \in \mathbb{R}$ are time-varying frequencies of the inverters; $\tau_{d_1} > 0$ and $\tau_{d_2} > 0$ are delays caused by the digital controls required to implement the droop controls; $\tau_{P_1} > 0$, $\tau_{P_2} > 0$, $k_{P_1} > 0$, $k_{P_2} > 0$, $a_{12} > 0$, c_1 and $c_2 = -\frac{k_{P_2}}{k_{P_1}} c_1$ are constant parameters, the disturbances $d_1(t)$ and $d_2(t)$ represent additional model uncertainties. We say that a solution of (10) is phase-locked if $\theta(t) = \theta_0$ is constant $\forall t \in \mathbb{R}_+$ for some $\theta_0 \in [0, 2\pi)$ [10]. If this property holds asymptotically, *i.e.*, for $t \to +\infty$, we speak about an asymptotic phase-locking.

For brevity of presentation, we impose the following restrictions on the values of parameters.

Assumption 1. *$\tau_{P_1} = \tau_{P_2} = \tau_P > 0$ and $\tau_{d_1} = \tau_{d_2} = \tau > 0$.*

Under this assumption, define the new coordinates:

$$x_1 = \theta,\ x_2 = \omega_1 - \omega_2,\ x_3 = \frac{k_{P_2}}{k_{P_1}}\omega_1 - \omega_2.$$

Then the system (10) can be rewritten as follows:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), && (11)\\ \tau_P \dot{x}_2(t) &= -x_2(t) - [k_{P_1} + k_{P_2}]a_{12}\sin[x_1(t-\tau)] \\ &\quad +[1 + \frac{k_{P_2}}{k_{P_1}}]c_1 + d_1 - d_2, && (12)\\ \tau_P \dot{x}_3(t) &= -x_3(t) + \frac{k_{P_2}}{k_{P_1}}d_1 - d_2. && (13)\end{aligned}$$

Thus, the system (10) is decomposed into two independent subsystems: (11), (12) and (13). The variable x_3 converges asymptotically to zero with the time constant τ_P if $d_1 = d_2 = 0$. Hence, asymptotically the frequencies ω_1 and ω_2 are locked. The dynamics (11), (12) have the form of (8) for $d = [1 + \frac{k_{P_2}}{k_{P_1}}]c_1 + d_1 - d_2$ and, as it has been established above, have pAG and pGS properties from definitions 8 and 9 respectively if condition (9) is satisfied, which for (11), (12) takes the form:

$$\tau^2 < \frac{\min\left\{\frac{\tau_P^{-1} - \epsilon}{1+\epsilon}, \frac{[k_{P_1}+k_{P_2}]a_{12}}{\tau_P}\right\}}{12\frac{[k_{P_1}+k_{P_2}]^2 a_{12}^2}{\tau_P^2(1-\epsilon)}\left[\frac{\epsilon}{[k_{P_1}+k_{P_2}]a_{12}} + \frac{1}{1-\tau_P\epsilon}\right]} \qquad (14)$$

for $0 < \epsilon < \min\{1, \tau_P^{-1}\}$. Therefore, for a sufficiently small delay τ the inverters may demonstrate a phase-locking behavior. According to [23], a good estimate of the overall delay introduced by the digital control is $\tau = 1.75 T_S$[1], where $T_S = 1/f_S$ and $f_S \in \mathbb{R}_{>0}$ is the switching frequency of the inverter. Since usually $f_S \in [5, 20]$ kHz [11], τ is reasonably small in most practical applications. Hence, we expect condition (14) to be satisfied for most practical choices of parameters τ_P, k_{P_1} and k_{P_2}.

The analysis is illustrated in a simulation example with the following set of parameters for the system (10): $\tau_P = 1$, $k_{P_1} = 10$, $k_{P_2} = 20$, $a_{12} = 0.1$, $c_1 = 0.2$ and $\tau = 0.05$. Condition (14) is satisfied for $\epsilon = 0.5\min\{1, \tau_P^{-1}\}$. The simulation results are shown in Fig. 1. The solid lines represent the state $(\theta, \omega_1, \omega_2)^T$ trajectories for the case $d_1(t) = d_2(t) = 0$, and the dashed lines correspond to $d_1(t) = 0.8\sin(t)$, $d_2(t) = 0.9\sin(2t)$. The phase-locking phenomenon is observed in these simulation results.

[1]The overall delay reduces to $\tau = 1.5T_S$ if no moving average function for the measurement is used [23].

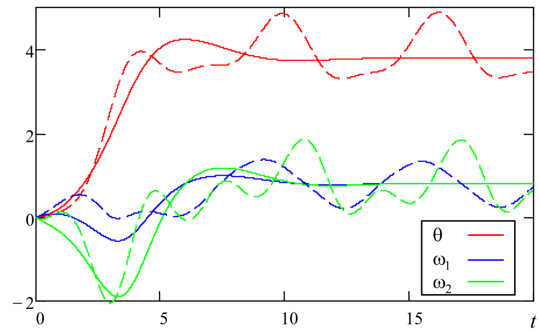

Figure 1. Simulation results for the system (10). The solid lines show the state trajectories for the case $d_1(t) = d_2(t) = 0$. The dashed lines correspond to the case $d_1(t) = 0.8\sin(t)$, $d_2(t) = 0.9\sin(2t)$.

VI. CONCLUSIONS

Sufficient conditions for ISS of multistable systems with delay have been derived. The conditions have been established using Lyapunov Razumikhin functions. The potential of the presented approach has been illustrated by providing several new robustness properties for a nonlinear pendulum with delay. Furthermore, it has been shown that phase-locking in a lossless droop-controlled microgrid formed by two inverters with delays can be analyzed based on the pendulum model. By exploiting this fact, a delay-dependent condition for ISS of such a microgrid has been presented.

Future work will consider an extension of the analysis to more complex inverter models with delays and, *e.g.*, time-varying voltages or internal filter and controllers.

REFERENCES

[1] D. Angeli. An almost global notion of input-to-state stability. *IEEE Trans. Automatic Control*, 49:866–874, 2004.

[2] D. Angeli and D. Efimov. On input-to-state stability with respect to decomposable invariant sets. In *Proc. 52nd IEEE Conference on Decision and Control*, Florence, 2013.

[3] D. Angeli, J.E. Ferrell, and E.D. Sontag. Detection of multistability, bifurcations and hysteresis in a large class of biological positive-feedback systems. *Proc. Natl. Acad. Sci. USA*, 101:1822–1827, 2004.

[4] D. Angeli and L. Praly. Stability robustness in the presence of exponentially unstable isolated equilibria. *IEEE Trans. Automatic Control*, 56:1582–1592, 2011.

[5] M.C. Chandorkar, D.M. Divan, and R. Adapa. Control of parallel connected inverters in standalone AC supply systems. *IEEE Transactions on Industry Applications*, 29(1):136 –143, jan/feb 1993.

[6] M. Chaves, T. Eissing, and F. Allgower. Bistable biological systems: A characterization through local compact input-to-state stability. *IEEE Trans. Automatic Control*, 45:87–100, 2008.

[7] D.V. Efimov and A.L. Fradkov. Oscillatority of nonlinear systems with static feedback. *SIAM Journal on Optimization and Control*, 48(2):618–640, 2009.

[8] Xi Fang, Satyajayant Misra, Guoliang Xue, and Dejun Yang. Smart grid - the new and improved power grid: a survey. *Communications Surveys & Tutorials, IEEE*, 14(4):944–980, 2012.

[9] H. Farhangi. The path of the smart grid. *IEEE Power and Energy Magazine*, 8(1):18 –28, january-february 2010.

[10] Alessio Franci, Antoine Chaillet, Elena Panteley, and Françoise Lamnabhi-Lagarrigue. Desynchronization and inhibition of Kuramoto oscillators by scalar mean-field feedback. *Mathematics of Control, Signals, and Systems*, 24(1-2):169–217, 2012.

[11] T.C. Green and M. Prodanovic. Control of inverter-based micro-grids. *Electric Power Systems Research*, Vol. 77(9):1204–1213, july 2007.

[12] J. Guckenheimer and P. Holmes. Structurally stable heteroclinic cycles. *Math. Proc. Camb. Phil. Soc.*, 103:189–192, 1988.

[13] J Guerrero, P Loh, Mukul Chandorkar, and T Lee. Advanced control architectures for intelligent microgrids – part I: Decentralized and hierarchical control. *IEEE Transactions on Industrial Electronics*, 60(4):1254–1262, 2013.

[14] N. Hatziargyriou, H. Asano, R. Iravani, and C. Marnay. Microgrids. *IEEE Power and Energy Magazine*, 5(4):78 –94, july-aug. 2007.

[15] Christopher M. Kellett, Fabian R. Wirth, and Peter M. Dower. Input-to-state stability, integral input-to-state stability, and unbounded level sets. In *Proc. 9th IFAC Symposium on Nonlinear Control Systems*, pages 38–43, Toulouse, 2013.

[16] Osman Kukrer. Discrete-time current control of voltage-fed three-phase PWM inverters. *IEEE Transactions on Power Electronics*, 11(2):260–269, 1996.

[17] R.H. Lasseter. Microgrids. In *IEEE Power Engineering Society Winter Meeting, 2002*, volume 1, pages 305 – 308 vol.1, 2002.

[18] J.A.P. Lopes, C.L. Moreira, and A.G. Madureira. Defining control strategies for microgrids islanded operation. *IEEE Transactions on Power Systems*, 21(2):916 – 924, may 2006.

[19] Dragan Maksimovic and Regan Zane. Small-signal discrete-time modeling of digitally controlled PWM converters. *IEEE Transactions on Power Electronics*, 22(6):2552–2556, 2007.

[20] P. Monzón and R. Potrie. Local and global aspects of almost global stability. In *Proc. 45th IEEE Conf. on Decision and Control*, pages 5120–5125, San Diego, USA, 2006.

[21] Ulrich Münz and Michael Metzger. Voltage and angle stability reserve of power systems with renewable generation. In *19th IFAC World Congress*, 2014.

[22] Z. Nitecki and M. Shub. Filtrations, decompositions, and explosions. *American Journal of Mathematics*, 97(4):1029–1047, 1975.

[23] Thomas Nussbaumer, Marcelo Lobo Heldwein, Guanghai Gong, Simon D Round, and Johann W Kolar. Comparison of prediction techniques to compensate time delays caused by digital control of a three-phase buck-type PWM rectifier system. *IEEE Transactions on Industrial Electronics*, 55(2):791–799, 2008.

[24] R. Rajaram, U. Vaidya, and M. Fardad. Connection between almost everywhere stability of an ode and advection pde. In *Proc. 46th IEEE Conf. Decision and Control*, pages 5880–5885, New Orleans, 2007.

[25] A. Rantzer. A dual to Lyapunov's stability theorem. *Syst. Control Lett.*, 42:161–168, 2001.

[26] Joan Rocabert, Alvaro Luna, Frede Blaabjerg, and Pedro Rodriguez. Control of power converters in AC microgrids. *IEEE Transactions on Power Electronics*, 27(11):4734–4749, Nov 2012.

[27] V.V. Rumyantsev and A.S. Oziraner. *Stability and stabilization of motion with respect to part of variables*. Nauka, Moscow, 1987. [in Russian].

[28] Johannes Schiffer, Romeo Ortega, Alessandro Astolfi, Jörg Raisch, and Tevfik Sezi. Conditions for stability of droop-controlled inverter-based microgrids. *Automatica*, 2014. Accepted.

[29] Johannes Schiffer, Romeo Ortega, Christian Hans, and Jörg Raisch. Droop-controlled inverters are robust to clock drifts. In *Submitted to ACC 2015*.

[30] J. W. Simpson-Porco, F. Dörfler, and F. Bullo. Synchronization and power sharing for droop-controlled inverters in islanded microgrids. *Automatica*, 49(9):2603 – 2611, 2013.

[31] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.

[32] E. D. Sontag and Y. Wang. On characterizations of input-to-state stability with respect to compact sets. In *Proc IFAC Non-Linear Control Systems Design Symposium, (NOLCOS '95)*, pages 226–231, Tahoe City, CA, 1995.

[33] E.D. Sontag. On the input-to-state stability property. *European J. Control*, 1:24–36, 1995.

[34] E.D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Trans. Autom. Control*, 41(9):1283–1294, 1996.

[35] G.-B. Stan and R. Sepulchre. Analysis of interconnected oscillators by dissipativity theory. *IEEE Trans. Automatic Control*, 52:256–270, 2007.

[36] Andrew R. Teel. Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Trans. Automat. Control*, 43(7):960–964, 1998.

[37] R. van Handel. Almost global stochastic stability. *SIAM J. Control and Optimization*, 45(4):1297–1313, 2006.

[38] Pravin P Varaiya, Felix F Wu, and Janusz W Bialek. Smart operation of smart grid: Risk-limiting dispatch. *Proceedings of the IEEE*, 99(1):40–57, 2011.