Search by Constraint Propagation

Thierry Martinez, François Fages, Sylvain Soliman

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Thierry Martinez François Fages Sylvain Soliman
Inria
Thierry.Martinez@inria.fr Francois.Fages@inria.fr Sylvain.Soliman@inria.fr

Abstract

Constraint programming is traditionally viewed as the combination of two components: a constraint model and a search procedure. In this paper we show that tree search procedures can be fully internalized in the constraint model with a fixed enumeration strategy. This approach has several advantages: 1) it makes search strategies declarative, and modeled as constraint satisfaction problems; 2) it makes it possible to express search strategies in existing front-end modeling languages supporting reified constraints without any extension; 3) it opens up constraint propagation algorithms to search constraints and to the implementation of novel search procedures based on constraint propagation. We illustrate this approach with a Horn clause extension of the MiniZinc modeling language and the modeling in this language of a variety of search procedures, including dynamic symmetry breaking procedures and limited discrepancy search, as constraint satisfaction problems. We show that this generality does not come with a significant overhead, and can in fact exhibit exponential speedups over procedural implementations, thanks to the propagation of the search constraints..

Keywords modeling languages, search, constraint programming, Horn clauses

1. Introduction

Constraint programming is traditionally presented as the combination of two components: a constraint model and a search procedure [21]. Front-end modeling languages are designed for solving problems using constraint programming solvers, thus either rely on a fixed strategy (e.g. Essence [8]), or contain special features for specifying the search strategy for the constraint solvers (e.g. Zinc [15]). The modeling language Zinc, and its implementation MiniZinc [1], succeeded in becoming a common input format across many solvers in the Constraint Programming community. In Zinc, the search procedure is specified through special annotations that are dedicated to the constraint solver [17] and ignored by the other solvers.

In this paper, we show that a completely different approach for specifying search is possible, by internalizing the search procedure in the constraint model with a fixed enumeration strategy. In principle, transforming search procedures into constraint satisfaction problems presents several advantages:

1. it makes search strategies declarative, and modeled as constraint satisfaction problems;
2. it makes it possible to express search strategies in existing front-end modeling languages without any extension;
3. it opens up constraint propagation algorithms to search constraints and to the implementation of novel search procedures based on constraint propagation.

The idea of this transformation is to associate to each choice point a reified constraint with an auxiliary model variable for representing that choice (e.g. value enumeration, domain splitting or any constraint). The search heuristic can then be specified simply by the enumeration strategy for the choice variables. This approach is not limited to static search procedures in which all choice points are precisely known statically, but can accommodate dynamic search strategies, such as dichotomic or interval splitting search [20] for example. In constraint programming, dynamic search procedures rely on the values of indexicals (domain size, minimum value, etc.). They are expressed in the framework presented here by extending the enumeration strategy with annotations that assign the values of indexicals to auxiliary model variables. Static search procedures do not rely on the values of indexicals and their encodings do not need any specific support on the solver-side. The encoding of dynamic search procedures can be run through simple additions in the solvers for providing the capability to query the values of indexicals.

In this paper, to make concrete the presentation of the transformation, we consider the Zinc modeling language and introduce ClpZinc [2], a language extending Zinc with the ability to describe new relations by Horn clauses. The choice of Horn clauses as a specification language for search procedures is guided by Horn clauses being the smallest language with the addition of constraint to the store as primitive and closed by conjunction and disjunction (for expressing choices), and with a general form of recursion. Given a constraint system $\mathcal{X}$ (e.g. finite domains) and the Herbrand constraint system $\mathcal{H}$, we describe a partial evaluation procedure to transform any terminating Horn$(\mathcal{X}+\mathcal{H})$ goal to an and/or tree with constraints over $\mathcal{X}$.

Plan

In Section 2, we introduce the encoding of search strategies into constraints on the Korf's Square Packing example, showing that a couple of arithmetic constraints encode Helmut Simonis and Barry O'Sullivan's search strategy for this example, that is one of the best known search strategies. In Section 3, we generalize this approach by encoding any search strategy described as an

[1] `http://www.minizinc.org/`

[2] The Clp2Zinc compiler that transforms ClpZinc models into MiniZinc is available for download, together with patches for Choco, JaCoP, SICStus, Gecode and or-tools: `http://lifeware.inria.fr/~tmartine/clp2zinc/`

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and/or tree into reified constraints. In Section 4, we introduce the ClpZinc language for describing search strategies and describe the transformation of search procedures from Horn($\mathcal{X} + \mathcal{H}$) to and/or trees over $\mathcal{X}$. In the subsequent sections, we evaluate this approach on benchmarks of models with specific search strategies, namely: Korf's Square Packing problem in Section 5, limited discrepancy search in Section 6.1 and symmetry breaking during search in Section 6.2. In Section 7, we show how it is possible to go beyond tree search procedures by using a simple mechanism of annotations for global store, and specify optimization procedures such as Branch-and-Bound. In Section 8, we compare our approach with Search Combinators, which are another proposal for describing search strategies in Zinc. In Section 9, we conclude on some new perspectives.

2. Korf's Square Packing Strategy Revisited

In this section, we consider Korf's Optimal Rectangle Square Packing problem [13], *i.e.*, given an integer $n \geq 1$, find an enclosing rectangle of smallest area containing n squares from sizes 1×1, 2×2, up to $n \times n$, without overlap. Helmut Simonis and Barry O'Sullivan proposed a complex dynamic search strategy for that problem in [20], that they have implemented for a specific Constraint Logic Programming solver (SICStus prolog).

Their model involves three global constraints: a geometric non-overlapping constraint over the squares, and two redundant constraints that express in each dimension that the sum of lengths of squares placed at every abscissa (resp. ordinate) should not exceed the height (resp. the width) of the enclosing rectangle. This is expressed by two "cumulative" constraints that are usually used in resource allocation and scheduling [5].

$$\begin{cases} \text{non-overlap}(\{(x_i, y_i, i \times i) \mid 1 \leq i \leq n\}) \\ \text{cumulative}(\forall 1 \leq x \leq w, \sum\{i \mid x_i \leq x \leq x_i + i\} \leq h) \\ \text{cumulative}(\forall 1 \leq y \leq h, \sum\{i \mid y_i \leq y \leq y_i + i\} \leq w) \end{cases}$$

This model that relates abscissas x_i with the height h and ordinates y_i with the width w gives surprisingly good results experimentally with a strategy that first fully enumerates in one dimension, *e.g.* x_i, and then fully enumerates in the other one, y_i. The enumeration strategy that Simonis and O'Sullivan proposed first makes a coarse-grained placement that splits each dimension into intervals up to one third of the square widths, from the biggest square to the smallest one and, then, the placement is refined by dichotomy, still from the biggest square to the smallest one. We show that these two enumeration strategies, interval splitting and dichotomy, can be expressed as arithmetic constraints.

Let x be a finite-domain variable and let s be the size of intervals which the domain should be split into. Interval splitting on x can be reduced to an Euclidean division with denominator s.

$$x = s \times q + r$$

r is a finite-domain variable left unbound between 0 and $s - 1$, and the quotient q is a finite-domain variable that selects one of the intervals of size s where x should lay in. Therefore, interval splitting equivalents to the value enumeration of q.

Now, let x be a finite-domain variable, $0 \leq x < 2^d$, to be enumerated by dichotomy. Dichotomy is equivalent to the enumeration of the values of the bits in the writing of x in base 2, from the most significant bit to the less significant one.

$$x = \sum_{0 \leq k < d} x^{(k)} 2^k$$

$x^{(k)}$ are variables of domain $\{0, 1\}$: dichotomy equivalents to the value enumeration of the d variables $x^{(d-1)}$, then $x^{(d-2)}$, etc. up to $x^{(0)}$. It is worth noticing that the number of introduced variables is only logarithmic in the size of the domain.

We will see in the next section that this transformation of search strategies into arithmetic constraints plus simple value enumeration can be formalized generally for every tree search strategy. We will see in section 5 that this does not come with significant overhead.

3. Compiling And/Or Trees into Reified Constraints

In this section, we consider and/or trees with leaves with constraints over a domain X. and/or trees are widely used in constraint programming systems for representing search strategies: they can be described by constraint logic programming [12], list monads in functional settings [3], search combinators [19], etc.

Definition 1. *An* and/or tree *over a constraint domain $\mathcal{X}$ is either*

- *a constraint c over $\mathcal{X}$,*
- *a conjunction $t \wedge t'$ where t and t' are two and/or trees,*
- *a disjunction $t \vee t'$ where t and t' are two and/or trees.*

Sometimes, by abuse of notation, search strategies are defined directly on a search tree, whereas they are actually traversals of an and/or tree that result in a given search tree. In other words, a search tree is the result of applying a search strategy to a given instance.

Definition 2. *A* search tree *over a constraint domain $\mathcal{X}$ is either*

- *a constraint c over $\mathcal{X}$,*
- *a node $(c, t \vee t')$ where c is a constraint over $\mathcal{X}$, and t and t' are two search trees.*

Let c be a constraint and t be a search tree. We denote $c \wedge t$ the search tree defined either by $c \wedge c'$ if $t = c'$, or by $(c \wedge c', t_0 \vee t_1)$ if $t = (c', t_0 \vee t_1)$.

Now, we define the usual transformation from and/or-trees to search trees, as done for instance by the CSLD resolution. For practical reasons we define this transformation inductively and therefore describe it from sequences of and/or-trees to search trees.

Definition 3. *The interpretation function s from sequences of and/or-trees to search trees is described as follows, by induction over the sum of the heights of the trees in the sequence.*

- $s(\epsilon) = \top$,
- $s(c \cdot \kappa) = c \wedge s(\kappa)$
- $s((t \wedge t') \cdot \kappa) = s(t \cdot t' \cdot \kappa)$
- $s((t \vee t') \cdot \kappa) = (\top, s(t \cdot \kappa) \vee s(t' \cdot \kappa))$

Definition 4. *Given a CSP($\mathcal{X}$) model $\mathcal{M}$, two search trees t and t' over $\mathcal{X}$ are* equivalent w.r.t. $\mathcal{M}$ *if:*

- *t and t' have the same structure;*
- *for all path π from the root to a node in t, let c be the conjunction of the constraints labelling the nodes along π projected over the variables of $\mathcal{M}$ and c' be the corresponding constraint for the same path π in t'. We have $\mathcal{M} \models c \Leftrightarrow c'$*

We shall denote this property by: $t \equiv_{\mathcal{M}} t'$.

Given a CSP($\mathcal{X}$) model $\mathcal{M}$ and a tree search strategy represented by an and/or tree t, the generation of Zinc code proceeds by assigning an additional model variable to every or-node in the tree t, and by emitting search annotations that fix the enumeration strategy for these additional variables in a way compatible with the traversal ordering.

In Figure 2, the variable `X1` is assigned to the root node, with the domain `0..5` corresponding to the arity of the node. As shown in the Zinc model generated for Example 1, each constraint labeling the leaves under this or-node appears in the model guarded by an

implication checking for a particular value of `X1`. Therefore, when the search annotation `int_search` enumerates the possible values of `X1`, these guarded constraints are successively enabled for exploring the different branches of the tree.

More generally, the transformation presented in this paper can be seen as a constructive proof for the following theorem. We call *fixed enumeration strategies* the search strategies that are reduced to a sequence of variables selected in a fixed order and enumerated with the increasing value selection (`indomain_min`).

Theorem 1. *For every pair $(\mathcal{M}, t)$ where $\mathcal{M}$ is a CSP model and t a tree search strategy described as an and/or tree, there exists a model $\mathcal{M}_t$ and a fixed enumeration strategy t' such that $s(t) \equiv_{\mathcal{M}+\mathcal{M}_t} s(t')$.*

Proof. First, let us remark that in $\mathcal{M} + \mathcal{M}_t$, the variables and the constraints of $\mathcal{M}$ are left unchanged; only additional model variables accompanied with additional constraints are introduced in $\mathcal{M}_t$.

Let us assume that we have a function ℓ that maps each or-node n of t to a model variable $\ell(n) \in V(\mathcal{M}_t)$, such that for every pair n_1, n_2 of nodes of t, if $\ell(n_1) = \ell(n_2)$, either $n_1 = n_2$ or the lowest common ancestor of n_1 and n_2 is an or-node.

Each constraint c that appears as a leaf of t is translated as a constraint in $\mathcal{M}_t$. Let $n_1, \ldots, n_k$ denote the or-nodes that are traversed by the path π from the root of t to the leaf c and, for every $1 \leq i \leq k$, let p_i be the rank of the branch taken by π at node n_i. We adopt the convention that branches are numbered from left to right and that the left-most branch has rank 0. Then the following constraint is posted in the MiniZinc model, for translating the leaf c:

$$\texttt{constraint } \ell(n_1)\texttt{=}p_1 \texttt{ /\ } \ldots \texttt{ /\ } \ell(n_k)\texttt{=}p_k \texttt{ -> } c\texttt{;}$$

Let $X \in V(\mathcal{M}_t)$ be one of the variables that label or-nodes. The domain of X will be $0..\max\{\mathrm{w}(n) - 1 \mid \ell(n) = X\}$ where $\mathrm{w}(n)$ denotes the width of the or-node n (*i.e.*, the number of branches issued from n). For every or-node n_k such that $\ell(n_k) = X$ that does not reach this maximum, the following additional constraint is posted, where $(n_i, p_i)_i$ denotes the or-path to n_k as above:

$$\texttt{constraint } \ell(n_1)\texttt{=}p_1 \texttt{ /\ } \ldots \texttt{ /\ } \ell(n_{k-1})\texttt{=}p_{k-1} \texttt{ -> } \ell(n_k)\texttt{<}\,\mathrm{w}(k)\texttt{;}$$

To prevent enumerating on a variable X in branches where X does not occur, the following constraint imposes a fixed value to X on these branches.

$$\texttt{constraint } \texttt{(} \bigwedge_{\substack{(n_i,p_i)_i \\ \ell(n_k)=X}} \ell(n_1)\texttt{!=}p_1 \texttt{ \/ } \ldots \texttt{ \/ } \ell(n_k)\texttt{!=}p_k\texttt{)} \texttt{ -> } X\texttt{=0;}$$

We should now establish the connection between the enumeration of the variables that label the or-nodes and the exploration of the and/or tree.

Search annotations have to be emitted to fix the enumeration strategy for the variables $V(\mathcal{M}_t)$. MiniZinc search annotations have a depth-first semantics. To reproduce the semantics of $(\mathcal{M}, t)$, it is thus sufficient in t' to emit the annotations that select the variables in the order where their corresponding nodes are encountered following the traversal of t. The value selection strategy fixes the order in which the sub-branches are explored. For t', it can thus be reduced to the value selection `indomain_min` that selects the left-most branch, by switching the values if necessary. We thus have that, by construction, $(\mathcal{M} + \mathcal{M}_t, t')$ explores the same search tree as $(\mathcal{M}, t)$. □

Note that in order for the resulting Zinc program to have the intended semantics, we assume that the input and/or tree meets the following conditions:

1. all the variables that appear in constraints are finite-domain variables;
2. all lists are well-formed, in particular the tail of every non-empty list is a list and cannot be a variable since such a variable would be a finite-domain variable according to the previous condition (this ensures that lists can be expanded into Zinc array literals);

The following optimization is not mandatory but has been measured to give significant performance improvements, by reducing the number of generated constraints.

Proposition 1. *In the particular case where a constraint c occurs under an or-node n_k (possibly separated with some and-nodes) and when $\neg c$ occurs in all other branches of n_k, the constraints corresponding to the leaves c and $\neg c$ are logically equivalent to the following constraint.*

$$\texttt{constraint}$$
$$\ell(n_1)\texttt{=}p_1 \texttt{ /\ } \ldots \texttt{ /\ } \ell(n_{k-1})\texttt{=}p_{k-1} \texttt{ -> (}\ell(n_k)\texttt{=}p_k \texttt{ <-> } c\texttt{);}$$

That is to say, the implication on the leaf c may be replaced by an equivalence and the constraints corresponding to the leaves $\neg c$ are not posted.

This simplification can be seen in the Zinc code generated for Examples 2 and 3.

4. Extending Zinc with Horn Clauses

4.1 The Language ClpZinc

We propose to use Horn clauses to specify search strategies in Zinc. More precisely, given a constraint system $\mathcal{X}$ (e.g. finite domain constraints) and a CSP model with constraints in $\mathcal{X}$, we consider search procedures that are expressible as the traversal of an and/or tree with constraints over $\mathcal{X}$, *i.e.* an and/or tree where every leaf is either labeled by a constraint in $\mathcal{X}$ or, for dynamic search procedures, labeled by a query to indexicals. In addition, we consider the Prolog primitive constraint system, $\mathcal{H}$, *i.e.* Herbrand terms with unification. The choice of Herbrand terms for representing Zinc data structures makes the language look familiar to Prolog users and other constraint logic programming (CLP) systems. Similarly, we fix the strategy as depth-first and left-to-right.

The language ClpZinc is an extension of Zinc where the item `solve satisfy;` in models is replaced by a goal of the form "`:-` *goal*.", and where user-defined predicates are defined by Horn clauses of the form "$\texttt{p}(t_1,\ldots,t_n)$ `:-` *goal*.".

Example 1 (Labeling)**.** *The following ClpZinc model implements the search strategy that enumerates all possible values for a given variable in ascending order.*

```
var 0..5: x;
constraint x * x = x + x;

labeling(X, Min, Max) :-
   Min <= Max, (X = Min ; labeling(X, Min + 1, Max)).

:- labeling(x, 0, 5).
output [show(x)];
```

As shown in the following section, this ClpZinc model for the given goal of labeling `x` between 0 and 5, can be expanded to the following MiniZinc model:

```
var 0..5: x;
constraint x * x = x + x;
var 0..5: X1;
constraint X1 = 0 -> x = 0;
constraint X1 = 1 -> x = 1;
```

```
constraint X1 = 2 -> x = 2;
constraint X1 = 3 -> x = 3;
constraint X1 = 4 -> x = 4;
constraint X1 = 5 -> x = 5;
solve :: seq_search([
    int_search([X1], input_order, indomain_min, complete)
  ]) satisfy;
output [show(x)];
```

Definition 5. *A* ClpZinc goal *is either*

- *a constraint,*
- *a MiniZinc search annotation,*
- *a call to a user-defined predicate,*
- *the conjunction* (`A,B`) *or the disjunction* (`A;B`) *of two goals.*

A ClpZinc clause *is an item of the form* $p(t_1,\ldots,t_n)$ `:-` goal. *where* t_1 *and* t_n *are terms and* goal *is a ClpZinc goal. The goal part can be omitted: "*$p(t_1,\ldots,t_n)$*." is a shorthand for "*$p(t_1,\ldots,t_n)$ `:- true.`*".*

The search annotations of MiniZinc are accessible in goals in order to allow the composition of user-defined strategies with built-in ones. Terms are either logical variables (`X`, `Y`, `Max`, ...), numbers, or compound terms of the form $p(t_1,\ldots,t_n)$ where $t_1,\ldots,$ and t_n are terms. Model variables are a special case of compound terms, either atomic (`a`, `b`, ...) or array accessors (`x[I,J]`). Zinc arrays have been unified with Prolog-like lists to ease their enumeration in search strategies.

In Horn($\mathcal{X}+\mathcal{H}$), arithmetic differs from Prolog. Indeed, in accordance with the theory of CLP and unlike most Prolog systems, arithmetic is supposed to be contained in $\mathcal{X}$ and is distinguished from $\mathcal{H}$ terms, *e.g.*, "`1 + 1`" is undistinguishable from "`2`" and is not a $\mathcal{H}$ term. In ClpZinc, the different forms of unification, equality, and evaluation predicates that are encountered in Prolog systems (`=`, `#=`, `is`, ...) are thus all unified in a unique notion of equality, which is accessible either explicitly with the predicate `=`, or implicitly when predicate arguments in either $\mathcal{X}$ of $\mathcal{H}$ are unified.

Arithmetic expressions are also extended for accessing the indexicals of the model variables. For instance, the goal `M = min(X)` assumes that `X` is a model variable and unifies `M` with the currently known lower-bound of `X`. We consider the indexicals `min`, `max`, `card` and `dom_nth` (for retrieving the *n*th value in a variable domain). Concretely, an intermediary variable is introduced to receive the value of the indexical and search annotations are emitted for getting them with:

```
annotation indexical_min(var int: target, var int: x);
annotation indexical_max(var int: target, var int: x);
annotation indexical_card(var int: target, var int: x);
annotation indexical_dom_nth(var int: target, var int: x,
                             var int: n);
```

These annotations require to extend the solvers to communicate the indexicals. That is the only change made to the interface of the solvers.

Example 2 (Dichotomic search)**.** *The Zinc* `indomain_split` *value selection strategy can be implemented in ClpZinc using indexicals. The predicate* `dichotomy/3` *below expresses the bisection of a variable* `X` *that has the initial domain* `Min...Max`*. The bisection defined in the auxiliary predicate* `dichotomy/2` *is iterated* `Depth` $= \lceil \log_2 |X| \rceil$ *times to ensure that the domain is reduced to a value on every leaf.*

```
dichotomy(X, Min, Max) :-
   dichotomy(X, ceil(log(2, Max - Min + 1))).

dichotomy(X, Depth) :-
   Depth > 0,
   Middle = (min(X) + max(X)) div 2,
   (X <= Middle ; X > Middle),
   dichotomy(X, Depth - 1).
dichotomy(X, 0).

var 0..5: x;
:- dichotomy(x, 0, 5).
```

The MiniZinc model generated for the given goal is

```
var 0..5: x;
var 0..5: X3;  var 0..5: X5;  var 0..1: X7;
var 0..5: X4;  var 0..5: X6;  var 0..5: X2;
var 0..1: X8;  var 0..5: X1;  var 0..1: X9;
constraint X7 = 0 <-> x <= (X1 + X2) div 2;
constraint X8 = 0 <-> x <= (X3 + X4) div 2;
constraint X9 = 0 <-> x <= (X5 + X6) div 2;
solve :: seq_search([
    indexical_min(X1, x),
    indexical_max(X2, x),
    int_search([X7], input_order, indomain_min, complete),
    indexical_min(X3, x),
    indexical_max(X4, x),
    int_search([X8], input_order, indomain_min, complete),
    indexical_min(X5, x),
    indexical_max(X6, x),
    int_search([X9], input_order, indomain_min, complete)
  ]) satisfy;
```

The next example shows a partial search strategy that is not available using the usual MiniZinc search annotations. This is the interval slitting strategy introduced in [20] for solving Korf's packing problem [13] by making a preliminary coarse-grained filtering of the variable domains.

Example 3 (Interval splitting)**.** *The* `interval_splitting/4` *predicate, defined below, expresses the splitting of the domain of* `X` *into intervals of width* `Step`*.* `X` *is supposed to have the initial domain* `Min...Max`*.*

```
interval_splitting(X, Step, Min, Max) :-
   Min + Step <= Max, NextX = min(X) + Step,
   (
      X < NextX
   ;
      X >= NextX,
      interval_splitting(X, Step, Min + Step, Max)
   ).
interval_splitting(X, Step, Min, Max) :-
   Min + Step > Max.
var 0..5: x;
:- interval_splitting(x, 2, 0, 5).
```

The corresponding MiniZinc model for the given goal is

```
var 0..5: x;
var 0..1: X3;  var 0..1: X4;  var 0..5: X2;
var 0..5: X1;
constraint X3 = 0 <-> x < X1 + 2;
constraint X3 = 1 -> (X4 = 0 <-> x < X2 + 2);
constraint X3 = 0 -> X4 = 0;
solve :: seq_search([
    indexical_min(X1, x),
    int_search([X3], input_order, indomain_min, complete),
    indexical_min(X2, x),
    int_search([X4], input_order, indomain_min, complete)
  ]) satisfy;
```

4.2 Partial Evaluation of ClpZinc into And/Or Trees

From now on, let us assume that the initial ClpZinc goals provided in the items "`:-` *goal*." of the ClpZinc models that we consider, always terminate. That hypothesis should hold even if $\mathcal{X}$ only resolves fully instantiated constraints, as is the case of the static partial evaluator. Verifying termination of logic programs is a classical topic for which many results have been obtained using type systems or abstract interpretation techniques [4]. The description of these techniques is however beyond the scope of this paper.

Given a constraint system $\mathcal{X}$, the partial evaluation of a Horn($\mathcal{X}+\mathcal{H}$) goal will lead to an and/or tree with constraints over $\mathcal{X}$. The partial evaluator resolves predicate calls, Herbrand constraints and fully instantiated arithmetic constraints, *i.e.*, arithmetic tests. Since, without loss of generality, we settled for a left-to-right evaluation of

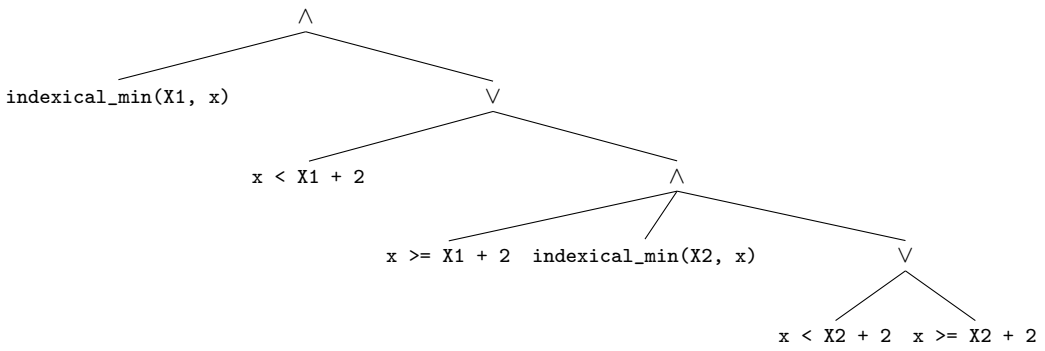


Figure 1. And/Or tree for Example 3 (Interval splitting)

Horn clauses, the and/or trees will be traversed in a similar left-to-right fashion in our examples, but any other traversal order can be treated similarly (see for example Figure 1 where the branches of the and-nodes should be executed from left to right). Note that the transformation itself does not follow a strict DFS and does not either rely on the classical CSLD resolution, which would lead to a pure or-tree. It is rather defined as a continuation-based generation of the and/or tree. The precise definition is given below, where s is the initially empty partial evaluation store built along the transformation and $\Box$ denotes a *hole* around which the context is built and labelled by such a store. The unfolding of predicate calls and resolution of arithmetic tests is left out for readability reasons.

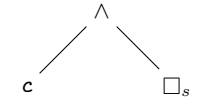
Definition 6. *The transformation* $[\![\bullet]\!]_s$: goal $\rightarrow$ search-tree*, in the partial evaluation store* $s \in \mathcal{H}$*, is defined inductively over the structure of the goal.*

$[\![true]\!]_s \longrightarrow \Box_s$

$[\![false]\!]_s \longrightarrow \bot$

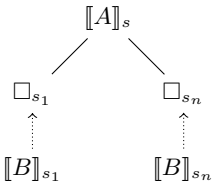
$[\![X = v]\!]_s \longrightarrow \Box_{s \wedge (X = v)}$

$[\![c]\!]_s \longrightarrow$



where c *is a constraint or a search annotation*

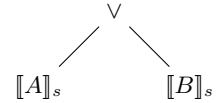
$[\![A, B]\!]_s \longrightarrow$



all $\Box_{s_i}$ *of* $[\![A]\!]_s$ *are filled with* $[\![B]\!]_{s_i}$
i.e., $[\![A]\!]_s[\forall i,\ [\![B]\!]_{s_i}/\Box_{s_i}]$

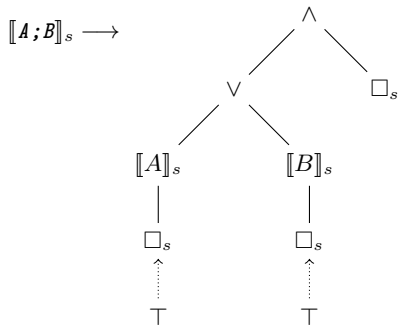
- *if* A *or* B *changes the store,* i.e., $\exists s' \neq s,\ \Box_{s'} \in [\![A]\!]_s$ *or* $[\![B]\!]_s$*:*

$[\![A ; B]\!]_s \longrightarrow$



- *if neither* A *nor* B *changes the store:*

$[\![A ; B]\!]_s \longrightarrow$



the leftmost leaf is $[\![A]\!]_s[\top/\Box_s]$ *and its sibling* $[\![B]\!]_s[\top/\Box_s]$

The associativity of the nodes is enforced during the transformation, allowing *n-ary* nodes.

4.3 Examples of Transformation

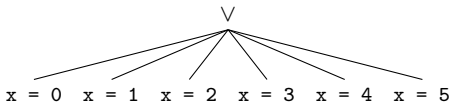


Figure 2. And/Or tree derived from Example 1 (Labeling)

As shown in Figure 2 for Example 1, or-nodes are flattened so that nested choices become a single large disjunction. And-nodes are similarly flattened into conjunctions. In the general case, the partial evaluation of the continuation may duplicate constraints with different partial instanciations. For instance, Figure 3 shows a simple example of duplication with partial instanciation of the bounding constraint `Min <= x, x <= Max`.

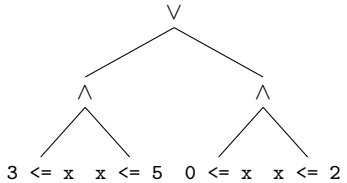


Figure 3. And/Or tree for the ClpZinc goal

```
var 0..5: x;
:- (Min=3, Max=5; Min=0, Max=2), Min <= x, x <= Max.
```

However, when the partial evaluation store is left unchanged by a choice (typically, when only constraints in $\mathcal{X}$ are involved), the continuation will remain undeveloped, as shown in Figure 4 for Example 2. The and/or tree is in logarithmic size with respect to the size of the domain whereas the fully expanded search tree would be in linear size.

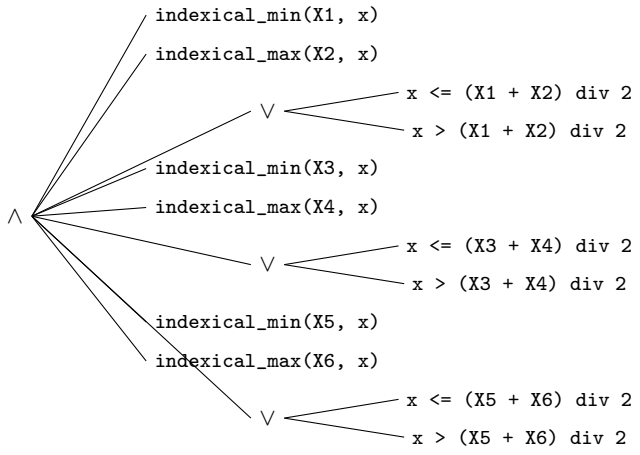


Figure 4. And/Or tree for Example 2 (Dichotomic search)

5. Computation Results on Korf's Square Packing Benchmark

In this section, we consider Korf's Optimal Rectangle Square Packing problem [13], *i.e.*, given an integer $n \geq 1$, find an enclosing rectangle of smallest area containing n squares from sizes 1×1, 2×2, up to $n \times n$, without overlap. Helmut Simonis and Barry O'Sullivan proposed a complex dynamic search strategy for that problem in [20], which is interesting to specify and evaluate in ClpZinc.

First, the model they consider for packing the nconsecutive squares in a rectangle of size `w` × `h` can be written in MiniZinc as follows. Since the 1×1 square can always be placed afterward if the area `w` × `h` is big enough, the model only considers the remaining $n - 1$ other squares. Two redundant `cumulative` constraints are introduced. The two last constraints break some symmetries by forcing the largest square to be in the bottom-left quadrant.

```
int: n;

constraint diffn(
   x,y,[i+1|i in 1..n-1],[i+1|i in 1..n-1]
);
constraint cumulative(
   x,[i+1|i in 1..n-1],[i+1|i in 1..n-1],h
);
constraint cumulative(
   y,[i+1|i in 1..n-1],[i+1|i in 1..n-1],w
);

constraint forall(i in 1..n-1)
   (x[i] <= w - i /\ y[i] <= h - i);
constraint x[n-1] <= (w - n + 2) div 2;
constraint y[n-1] <= (h + 1) div 2;
```

Second, the optimization procedure used in [20] enumerates all the possible sizes `w` × `h` for the enclosing rectangle by increasing area. This strategy can be internalized in the model by successively considering all the rectangles up to `max_size` × `max_size`, from the minimal area covered by the squares themselves ($\sum_{1 \leq i \leq n} i^2$) and with bounds on `w` and `h` that are described in [20]:

```
int: max_size;
array[1..n-1] of var 1..max_size: x;
array[1..n-1] of var 1..max_size: y;
var 0..max_size: w; var 0..max_size: h;
var 0..max_size * max_size: area;

constraint w * h = area /\ w <= h;
constraint sum([i*i | i in 1..n]) <= area;
constraint w >= 2 * n - 1
  \/ h >= (n * n + n -
        ((w + 1) div 2 - 1) * ((w + 1) div 2 - 1)
    - ((w + 1) div 2 - 1)) div 2;
```

Now, the search strategy of [20] firsts enumerates in x and then in y, considering in each dimension a preliminary interval splitting on the origins of the squares from sizes $n \times n$ to 7×7, and then a dichotomic search on the origins, still by considering the biggest square first. This search strategy is implemented in ClpZinc by enumerating first on `area` and `w` to find the rectangle of smallest area first. It is worth noticing that we can combine the user-defined interval splitting strategy defined in Example 3 with the built-in dichotomic search (`indomain_split`).

```
interval_splitting_list(L, S, Stop) :-
   (S <= Stop ; S > Stop, L = []).
interval_splitting_list([H | T], S, Stop) :-
   S > Stop,
   interval_splitting(
      H, max(1, (S * 3) div 10) + 1, 0, max_size
   ),
   interval_splitting_list(T, S - 1, Stop).

:- int_search(
      [area, w], input_order, indomain_min, complete
   ),
   reverse(x, RXs), interval_splitting_list(RXs, n, 6),
   int_search(
      RXs, input_order, indomain_split, complete
   ),
   reverse(y, RYs), interval_splitting_list(RYs, n, 0),
   int_search(
      RYs, input_order, indomain_split, complete
   ).
```

where `reverse` is implemented as usual with Horn clauses.

This strategy can be compared to the use of the dichotomic search only, on each dimension, from the biggest to the smallest square, relying on the native `indomain_split` of MiniZinc. This is indeed a good candidate for the best strategy that can be easily written in MiniZinc without the help of ClpZinc.

```
solve :: seq_search([
      int_search(
         [area, w], input_order, indomain_min, complete
      ),
      int_search(
         [x[n-i] | i in 1..n-1] ++ [y[n-i] | i in 1..n-1],
         input_order, indomain_split, complete
      )
   ]) satisfy;
```

To measure the overhead of ClpZinc, we also include the version of dichotomic search relying on the user-defined predicate of Example 2.

```
:- int_search(
      [area, w], input_order, indomain_min, complete
   ),
   reverse(x, Rx), dichotomy_list(Rx, 0, max_size),
   reverse(y, Ry), dichotomy_list(Ry, 0, max_size).
```

Table 1 shows the results of the native dichotomic search procedure in MiniZinc, of the user-defined dichotomic and interval-splitting search procedure in ClpZinc, solved using either Choco or SICStus solvers, and of the original SICStus-Prolog program of [20], all of them running on Intel®Xeon®CPU E5-1620 0 @ 3.60GHz machines. As shown in Table 1, the overhead introduced

n	Choco 3			SICStus		number of generated Zinc constraints
	dichotomic `indomain_split`	dichotomic ClpZinc	interval split then dichotomic ClpZinc	interval split then dichotomic ClpZinc	interval split then dichotomic Original	
16	9232	14402	853	710	340	1199
17	16321	21643	982	450	250	1249
18	422116	570407	7978	9400	4850	1299
19	785080	1051418	6984	11710	4310	1349
20			12572	17330	8970	1573
21			42892	88310	32370	1619
22			208632	303810	153860	1859
23			1340816	2104020	999020	1913
24			2312933	3433410	1481910	1959
25			29201522		10662860	3039
26			142702128		62179600	3109

Table 1. Solving times in ms for Korf's problem for strategies implemented in ClpZinc with Choco 3 and SICStus as solvers, compared to the original SICStus program. For $16 \leq n \leq 19$, `max_size`=80; For $20 \leq n \leq 24$, `max_size`=100; For $25 \leq n \leq 26$, `max_size`=150.

by the reification of the search procedure is quite reasonable, averaging a two-fold slowdown of the program. On the other hand, the reified search enables the encoding of the interval splitting strategy that induces a crucial increase in performance comparable to the results obtained in [20]. Note that, though the number of generated constraints is much larger than in the original model, it mostly depends on `max_size` and appears to have little impact on the observed two-fold overhead, which remains quite constant.

That table also shows that the specification in ClpZinc of the dichotomic and interval-splitting search strategy makes it readily available in a variety of solvers for which its implementation was not trivial. The implementation in Choco is the most efficient, followed by SICStus-Prolog, probably due to differences in the implementation of reified constraints.

6. LDS and SBDS as Strategy Transformers in ClpZinc

Since and/or trees are first-class terms in ClpZinc, they can be arguments of ClpZinc predicates to define search strategy transformers. In this section, we illustrate this possibility with the modeling of Limited Discrepancy Search (LDS) [11] and Symmetry Breaking During Search (SBDS) [9] as strategy transformers for labeling or dichotomic search for instance. This technique is closely related to the monadic approach of strategy transformers presented in [18]. The main difference, outside of purely syntactic choices, is that the monadic transformations described in [18] heavily rely on laziness to not expand the trees, whereas in ClpZinc, in order to finally compile towards a CSP, we fully meta-interpret, and therefore expand, the search trees, with some possible benefits thanks to the propagation of search constraints.

6.1 Limited Discrepancy Search

LDS can be modeled very simply in ClpZinc using meta-interpretation. Basically the and/or tree is developed but the right turns are counted at the same time, by increment when going in the right branch of an *or* and by addition of the two branches when going through an *and*:

```
lds(true, L).
lds((A ; B), L) :-
   domain(L0, 0, 1024), domain(D, 0, 1),
   (  D = 0, lds(A, L0)
   ;  D = 1, lds(B, L0)),
   L = D + L0.
lds((A, B), L) :-
   domain(L0, 0, 1024), domain(L1, 0, 1024),
   lds(A, L0), lds(B, L1),
   L = L0 + L1.
lds(B, L) :- builtin(B), B, L = 0.
lds(H, L) :- clause(H, B), lds(B, L).
```

Interestingly, since right turns are counted at the constraint level, the propagation of search constraints may actively reduce the search space, whereas a classical procedural implementation of LDS limits the number of right turns by generate-and-test. The following example demonstrates an *exponential speed-up* thanks to this propagation with respect to a procedural implementation of LDS.

```
var 0..1: x;
var 0..1: y;
array[0..n] of var 0..1: a;

:- int_search(a, input_order, indomain_min, complete),
   lds(((x = 0; x = 1), (y = 0; y = 1)), 0), x != y.
```

Whereas a procedural implementation would explore the 2^n possible assignments for `a` before detecting that the model is unsatisfiable within the reduced search space, the inconsistency is immediately detected in the MiniZinc model generated by ClpZinc.

```
var 0..1: x;
var 0..1: y;
array[0..n] of var 0..1: a;
constraint x = 0;
constraint y = 0;
constraint x != y;
solve :: seq_search([
    int_search(a, input_order, indomain_min, complete)
  ]) satisfy;
n = 1000;
```

6.2 Symmetry Breaking During Search

Symmetry Breaking During Search [2, 9] is a general method that transforms a search tree so as to remove symmetric branches from enumeration. Each time the search backtracks from enumerating solutions with a given search constraint c, the other search branch considers $\neg c$ and also all the symmetric constraints $\sigma(\neg c)$ for symmetries σ compatible with search constraints already posted. This schema is implemented in the predicate below, supposing a predicate `cut_symmetry` that adds the symmetric negations for a given constraint.

```
sbds(top, _).
sbds(or(A, B), Path) :-
   (  A = constraint(C, A0),
      (  C, sbds(A, [C | Path])
      ;  cut_symmetry(C, Path), sbds(B, Path))
   ;  A \= constraint(_, _),
```

```
      (sbds(A, Path) ; sbds(B, Path))).
sbds(constraint(C, T), Path) :- C, sbds(T, [C | Path]).
:- search_tree(labeling_list(queens, 1, n), T),
   sbds(T, []).
```

The predicate `search_tree` constructs the search tree associated with the and/or tree of a goal by meta-interpretation.

7. Beyond Tree Search Strategies

Some search strategies require to iterate a search tree several times with a memory passed from one branch to another. That is typically the case for optimization methods like branch-and-bound where the best score reached up to now is remembered from one iteration to another of the underlying search strategy, or for shaving, where one step of propagation is performed and undone in order to select the best one. In languages like Prolog, such methods are implemented with the help of a global state, most commonly stored within the fact database (with `assert` and `retract`). We propose two additional annotations for search in MiniZinc to handle global state.

```
annotation store(var bool: c, string: id,
                 array[int] of var int: src);
annotation retrieve(string: id,
                 array[int] of var int: target);
```

The semantics of `store(cond, id, source)` is to remember, if `cond` is `true`, the current values of the sequence of variables `source` into the global state identified as `id`. The `store` annotation does nothing if `cond` is false, such that the assignation to `id` is skipped outside the computation branch that involves this assignation. The parameter `cond` does not appear in ClpZinc: it is implicitly fixed to the guard associated to the path leading to the node where the annotation appears in the and/or tree. The semantics of `retrieve(id, target)` is to assign the values previously remembered into the global state identified as `id` into the sequence of variables `target`.

As shown below, these two simple annotations allow the specification of branch-and-bound optimization in ClpZinc. Once again, for such strategy one might also use the native `maximize` annotation of MiniZinc, but as far as we know, more complex iterative procedures like shaving or enumerating solutions, using previously found ones in the search (whether to guide it or to limit it), cannot be natively written in MiniZinc.

```
maximize(G, S, Min, Max) :-
   domain(I, Min, Max + 1), domain(Best, Min, Max),
   domain(Fail, 0, 1),
   domain(A, 0, 1), domain(B, 0, 1), domain(C, 0, 1),
   (Fail = 0 -> A != B /\ B != C /\ A != C),
   store("bb_best", [Min, 0]),
   labeling(I, Min, Max + 1),
   retrieve("bb_best", [Best, Fail]),
   (  Fail = 0, store("bb_best", [Best, 1]),
      S > Best, G, store("bb_best", [S, 0]),
      labeling(A, 0, 1), labeling(B, 0, 1)
   ;  Fail = 1, I = Max + 1, S = Best, G).

minimize(G, S, Min, Max) :-
   domain(Dual, Min, Max), Dual = Max - S + Min,
   maximize(G, Dual, Min, Max).
```

Note that in order to make this branch-and-bound procedure possible, the gap between failures at the search and at the constraint level has to be bridged. Using the incompleteness of arc-consistency, the reified constraint imposing that `A`, `B` and `C` are all different allows us to fail at will in the success branches (`Fail = 0`) by labelling `A` and `B`. There is also an optimization in the above code where the upper bound on the score is used in the `Fail = 1` branch as some kind of *cut*: all attempts after the first failure will be immediately discarded, except the last one where appropriate values for variables will be rebuilt by running the goal `G` again.

s ::= `prune`
prunes the current search branch (*i.e.*, fails)

| `base_search(`*vars*`,`*var-select*`,`*domain-split*`)`
search annotation

| `let(`*x*`,`*initial-value*`,`*s*`)`
introduces a new variable (*i.e.*, global state) *x* in *s*

| `assign(`*x*`,`*new-value*`)`
changes the value of the global state *x*

| `post(`*c*`,`*s*`)`
posts the constraint *c* at every choice-point during *s*

| `ifthenelse(`*cond*`,`s_1`,`s_2`)`
substitutes in s_1 every subtree where *c* is false by s_2

| `and([`s_1`,`s_2`,...,`s_n`])`
performs s_1, s_2, up to s_n while they succeed, and fails otherwise

| `or([`s_1`,`s_2`,...,`s_n`])`
if s_1 succeeds then succeeds, otherwise if s_2 succeeds, then succeeds, otherwise...

| `portfolio([`s_1`,`s_2`,...,`s_n`])`
performs s_1, s_2, up to s_n until one of them is exhaustive (*i.e.*, does not perform `prune`)

| `restart([`*cond*`,`*s*`])`
restarts *s* as long as *cond* holds

Table 2. Primitive search combinators.

8. Encoding Search Combinators

Search combinators [19] introduce a domain-specific language for modeling search. We recall the primitive constructions of this domain-specific language in table 8. Search combinators are known to go beyond the conjunctions and disjunctions of CLP goals, as it is mentioned in the related work section of [19]. However, we show that these combinators are straightforward to express using Horn clauses through meta-interpretation, thus can be expressed through the above transformation as labelling in pure CSPs. The only restriction is on `restart`: since the unfolding of Horn clauses into search trees should terminate, the number of iteration should be bounded statically. We only illustrate some encodings.

The `prune` combinator is encoded as a failure at constraint level, as in section 7.

```
prune :-
    store(prune, 1),
    domain([A, B, C], 0, 1),
    A != B, B != C, A != C.
```

The global state `prune` allows `portfolio` to implement the exhaustiveness check: by definition, the search tree is exhaustive if no call to `prune` are performed during its exploration.

```
portfolio([]) :-
    prune.
portfolio([H | T]) :-
    retrieve(prune, Former_prune),
    store(prune, 0),
    H,
    retrieve(prune, New_prune),
    store(prune, Former_prune),
    (
        New_prune = 0
    ;
        New_prune = 1,
        portfolio(T)
    ).
```

`let` and `assign` rely on a global state. `new_atom` is a primitive that generates a fresh Herbrand function symbol.

```
let(X, V, S) :-
    new_atom(X),
    store(X, V),
    S.

assign(X, V) :-
    store(X, V).
```

`ifthenelse` and `post` interleaves the exploration of the search tree with the test of the condition by meta-interpretation. We illustrate this meta-interpretation on `post`.

```
post(C, (A, B)) :-
    post(C, A),
    post(C, B).

post(C, (A ; B)) :-
    (
        C, A
    ;
        C, B
    ).

post(C, B) :-
    builtin(B),
    C,
    B.
```

Finally, note that even if the `first_fail` variable selection strategy (or a similar one) is not available as a built-in in the underlying solver, it is once again quite straightforward to implement in ClpZinc. Remark that, due to the lack of access to indexicals, such an encoding is not possibles using search combinators which are limited to what `base_search` makes available. The `first_fail` variable selection strategy selects the variable with the smallest domain among the variables that are not already instantiated (*i.e.*, whose domain is not reduced to a singleton). If all variables are instantiated, the predicate returns by convention the first variable, so that there is no failure when iterating this predicate once per model variable, which ensures complete instantiation.

```
first_fail(Vars, X) :-
   select(X, Vars, Other_vars),
   CardX = card(X),
   CardX > 1,
   check_card_greater_than(Other_vars, CardX).

first_fail(Vars, X) :-
   fully_instantiated(Vars),
   Vars = [H | _].

check_card_greater_than([], _).

check_card_greater_than([H | T], Min) :-
   CardH = card(H),
   (CardH < Min -> CardH = 1),
   check_card_greater_than(T, Min).

fully_instantiated([]).

fully_instantiated([H | T]) :-
   card(H) = 1,
   fully_instantiated(T).
```

9. Conclusion

We have shown that tree search procedures, such as for instance heuristic labeling, dichotomy, interval-splitting, limited discrepancy search, and dynamic symmetry breaking during search, can be internalized in a constraint model through reified constraints. On the complex dynamic strategy used for solving Korf's benchmark for square packing, we have shown that the implementation overhead is limited to a factor 2, and can be measured on different CSP solvers without particular support for search. We have also shown with an example that the propagation of search constraints can in fact exhibit an exponential speed up, compared to a classical procedural implementation of the search strategy.

This has been demonstrated by realizing an extension of MiniZinc with Horn clauses. Annotations for indexicals have also been added to MiniZinc for defining dynamic search strategies. Furthermore, by adding annotations for storing intermediate values during search, we have shown that this approach can be generalized to non tree search procedures, such as branch-and-bound optimization.

It is worth noting that the conversion of search into constraints opens up a whole field of challenges for CSP solvers with limited built-in search strategies. For instance, Korf's packing problem with the complex strategy of [20] can now be proposed for the MiniZinc contest, since any constraint solver implementing the indexical *min* can in principle solve it. Therefore, the two sentences of [20] stating that this packing problem "nicely tests the generality of a search method" and is a "more attractive benchmark for placement problems than the perfect square" can now apply to compare a broad range of CSP solvers. Furthermore, work on complex problems with dedicated heuristics can now become more independent of a particular solver through the modeling of the search strategy in our approach. We have added the needed indexicals to the FlatZinc parser of some solvers (Choco [7], JaCoP [14], SICStus [1], Gecode [6], or-tools [10]) and encourage all solver developers to so in order to tackle these new challenging problems for the MiniZinc community.

Finally, as a perspective for future work, the reification of choice point constraints in our scheme is in principle compatible with lazy clause generation techniques [16] and the learning of nogood by using a SAT solver. Such a combination of modeling search by constraints and learning constraints during search is however quite intriguing and will be the matter of future work.

Acknowledgments

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