Families of solutions of order nine to the NLS equation with sixteen parameters

Pierre Gaillard, Mickaël Gastineau

Families of solutions of order nine to the NLS equation with sixteen parameters.

+Pierre Gaillard, × Mickaël Gastineau
+ Institut de Mathématiques de Bourgogne,
9 Av. Alain Savary, Dijon, France : Dijon, France :
e-mail: Pierre.Gaillard@u-bourgogne.fr,
× ASD, IMCCE-CNRS UMR8028, Observatoire de Paris, UPMC,
77 Av. Denfert-Rochereau, 75014 Paris, France :
e-mail: gastineau@imcce.fr

April 26, 2015

Abstract

We construct new deformations of the Peregrine breather (P_9) of order 9 with 16 real parameters. With this method, we obtain explicitly new families of quasi-rational solutions to the NLS equation in terms of a product of an exponential depending on t by a ratio of two polynomials of degree 90 in x and t; when all the parameters are equal to 0, we recover the classical P_9 breather. We construct new patterns of different types of rogue waves as triangular configurations of 45 peaks as well as rings and concentric rings.

PACS : 35Q55, 37K10.
Keywords : NLS equation, Peregrine breather, rogue waves.

1 Introduction

After the works of the precursors, Zakharov and Shabat in 1972 [1, 2], a considerable number of studies were carried out by a multiplicity of authors. The first expressions of the quasi-rational solutions were given by Peregrine in 1983 [3]. Akhmediev, Eleonski and Kulagin constructed the first higher order analogue of the Peregrine breather [4, 5] in 1986. Akhmediev et al. [6, 7], constructed other families of order 3 and 4, using the Darboux transformations.

It has been found in [10] solutions for the order N (for determinants of order 2N) to the NLS equation depending on $2N-2$ real parameters.

With this method, we construct explicitly here new quasi rational solutions to the NLS equation for the order 9 depending on sixteen parameters. When all the parameters are equal to zero, we recover the famous (analogue) Peregrine breather P_9. We obtain new deformations at order 9 with 16 real parameters of the P_9 breather as a ratio of two polynomials of degree 90 in x and t by an exponential depending on t. Because of the length of these expressions, we cannot present them here; we only present patterns depending on the choices of the parameters and make the analysis of the evolution of the solutions.

2 Determinant representation of solutions to NLS equation

We recall briefly the results obtained in [11, 12] that we use in the following to construct deformations of the P_9 breather, solutions to the NLS equation.

Theorem 2.1 *The function v defined by*

$$v(x,t)=\frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})}e^{(2it-i\varphi)}$$

is a quasi-rational solution of the NLS equation

$$iv_t+v_{xx}+2|v|^2v=0,$$

where

$$\begin{array}{l}
n_{j1}=f_{j,1}(x,t,0),\\
n_{jk}=\frac{\partial^{2k-2}f_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
n_{jN+1}=f_{j,N+1}(x,t,0),\\
n_{jN+k}=\frac{\partial^{2k-2}f_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
d_{j1}=g_{j,1}(x,t,0),\\
d_{jk}=\frac{\partial^{2k-2}g_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
d_{jN+1}=g_{j,N+1}(x,t,0),\\
d_{jN+k}=\frac{\partial^{2k-2}g_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
2\le k\le N,\ 1\le j\le 2N
\end{array}$$

The functions f and g are defined for $1\le k\le N$ by :

$$\begin{array}{l}
f_{4j+1,k}=\gamma_k^{4j-1}\sin A_k,\\
f_{4j+2,k}=\gamma_k^{4j}\cos A_k,\\
f_{4j+3,k}=-\gamma_k^{4j+1}\sin A_k,\\
f_{4j+4,k}=-\gamma_k^{4j+2}\cos A_k,\\
f_{4j+1,N+k}=\gamma_k^{2N-4j-2}\cos A_{N+k},\\
f_{4j+2,N+k}=-\gamma_k^{2N-4j-3}\sin A_{N+k},\\
f_{4j+3,N+k}=-\gamma_k^{2N-4j-4}\cos A_{N+k},\\
f_{4j+4,k}=\gamma_k^{2N-4j-5}\sin A_{N+k},\\
g_{4j+1,k}=\gamma_k^{4j-1}\sin B_k,\\
g_{4j+2,k}=\gamma_k^{4j}\cos B_k,\\
g_{4j+3,k}=-\gamma_k^{4j+1}\sin B_k,\\
g_{4j+4,k}=-\gamma_k^{4j+2}\cos B_k,\\
g_{4j+1,N+k}=\gamma_k^{2N-4j-2}\cos B_{N+k},\\
g_{4j+2,N+k}=-\gamma_k^{2N-4j-3}\sin B_{N+k},\\
g_{4j+3,N+k}=-\gamma_k^{2N-4j-4}\cos B_{N+k},\\
g_{4j+4,N+k}=\gamma_k^{2N-4j-5}\sin B_{N+k},
\end{array} \qquad (1)$$

The arguments A_ν and B_ν of these functions are given for $1 \leq \nu \leq 2N$ by

$$\begin{array}{l} A_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\ B_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2. \end{array}$$

The terms κ_ν, δ_ν, γ_ν are defined by $1 \leq \nu \leq 2N$

$$\begin{array}{l} \kappa_j = 2\sqrt{1-\lambda_j^2},\ \delta_j = \kappa_j \lambda_j, \\ \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}},\ \kappa_{N+j} = \kappa_j, \\ \delta_{N+j} = -\delta_j,\ \gamma_{N+j} = 1/\gamma_j, \\ 1 \leq j \leq N, \end{array} \tag{2}$$

where λ_j are given for $1 \leq j \leq N$ by :

$$\lambda_j = 1 - 2j^2\epsilon^2,\ \lambda_{N+j} = -\lambda_j. \tag{3}$$

The terms $x_{r,\nu}$ $(r = 3, 1)$ are defined for $1 \leq \nu \leq 2N$ by :

$$x_{r,\nu} = (r-1)\ln \frac{\gamma_\nu - i}{\gamma_\nu + i}. \tag{4}$$

The parameters e_ν are given by

$$\begin{array}{l} e_j = i\sum_{k=1}^{N-1} \tilde{a}_j \epsilon^{2k+1} j^{2k+1} \\ - \sum_{k=1}^{N-1} \tilde{b}_j \epsilon^{2k+1} j^{2k+1}, \\ e_{N+j} = i\sum_{k=1}^{N-1} \tilde{a}_j \epsilon^{2k+1} j^{2k+1} \\ + \sum_{k=1}^{N-1} \tilde{b}_j \epsilon^{2k+1} j^{2k+1}, \\ 1 \leq j \leq N, \end{array} \tag{5}$$

3 Deformations of the P_9 breather with sixteen parameters

We have constructed in [17, 16, 19, 18, 20, 22, 14] and in a series of articles on the archive hal, solutions for the cases from $N = 1$ until $N = 8$ with $2N - 2$ parameters.
Here we make the study for the order nine. We don't give the analytic expression of the solution of NLS equation of order 9 with sixteen parameters because of the length of the expression. For simplicity, we denote

$$\begin{array}{l} d_3 := \det((n_{jk})_{j,k\in[1,2N]}) \\ d_1 := \det((d_{jk})_{j,k\in[1,2N]}). \end{array}$$

The number of terms of the polynomials of the numerator $d3$ and denominator $d1$ of the solutions are shown in the table below (Table 1) when other a_i and b_i are set to 0.

N=9	number of terms
$d_3(a_1, b_1, x, t)$	184554
$d_1(a_1, b_1, x, t)$	94332
$d_3(a_2, b_2, x, t)$	72174
$d_1(a_2, b_2, x, t)$	36894
$d_3(a_3, b_3, x, t)$	39813
$d_1(a_3, b_3, x, t)$	20347

Table 1: Number of terms for the polynomials d_3 and d_1 of the solutions of the NLS equation.

We construct figures to show deformations of the ninth Peregrine breather. The computations were done using the computer algebra systems Maple and TRIP [26]. For example, the computations of the determinants of $Q(x,t)$ take only 3 hours for the figure 1 using the parallel kernel of the software TRIP on a workstation with 6 cores.
We get different types of symmetries in the plots in the (x,t) plane. We give some examples of this fact in the following.

3.1 Peregrine breather of order 9

If we choose $\tilde{a}_i = \tilde{b}_i = 0$ for $1 \leq i \leq 8$, we obtain the classical Peregrine breather

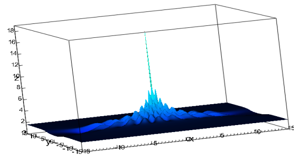

Figure 1: Solution to NLS, N=9, all parameters equal to 0, Peregrine breather P_9.

In the following, we give different deformations of this P_9 breather depending on the choices of the parameters. The triangular patterns have already been explained for the orders until $N = 7$ in [27]. The ring patterns, and the classification of the solutions to the NLS equation were already given in [28] using numerical methods until order $N = 6$. In that paper, the cases of order 7, 8, 9 and 10 given in table page 9 have been extrapolated. It was also pointed out that the number of peaks in the different figures for $N = 9$ is $N(N+1)/2 = 45$.
The results here are obtained from explicitly exact solutions.

3.2 Variation of parameters

With other choices of parameters, we obtain all types of configurations : triangles and multiple concentric rings configurations with a maximum of 45 peaks.

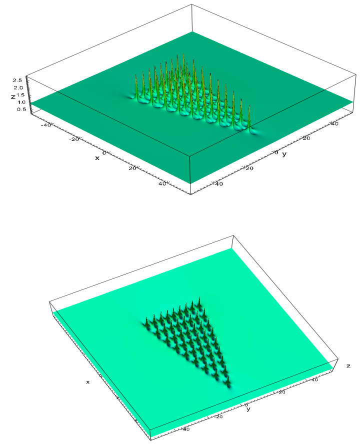

Figure 2: Solution to NLS, N=9, $\tilde{a}_1 = 10^3$: triangle with 45 peaks; in bottom, sight of top.

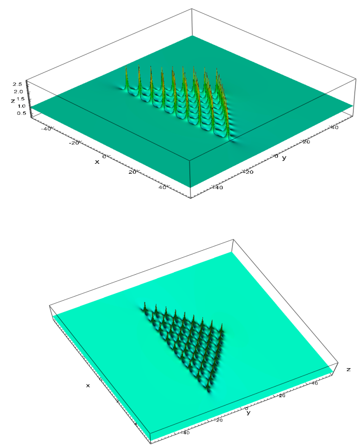

Figure 3: Solution to NLS, N=9, $\tilde{b}_1 = 10^3$: triangle with 45 peaks; in bottom, sight of top.

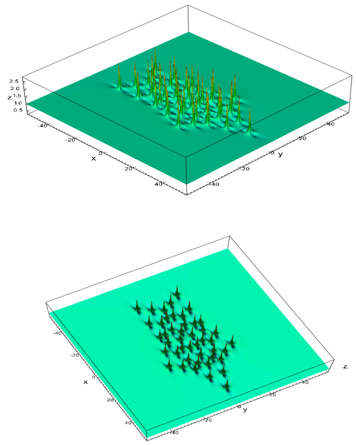

Figure 4: Solution to NLS, N=9, $\tilde{a}_2 = 10^5$: 6 rings without a peak in the center; in bottom, sight of top.

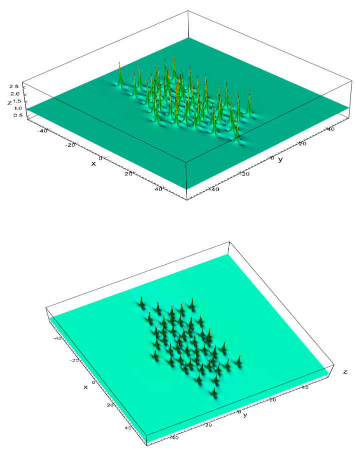

Figure 5: Solution to NLS, N=9, $\tilde{b}_2 = 10^5$: 6 rings without a peak in the center; in bottom, sight of top.

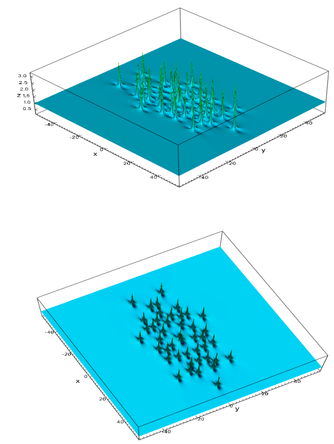

Figure 6: Solution to NLS, N=9, $\tilde{a}_3 = 10^7$: 4 rings with in the center P_2; in bottom, sight of top.

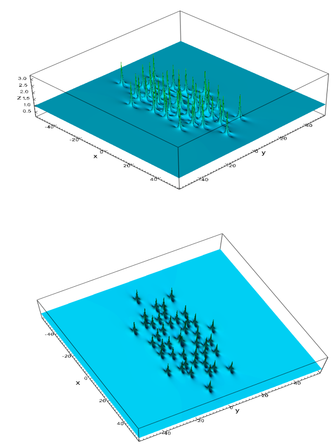

Figure 7: Solution to NLS, N=9, $\tilde{b}_3 = 10^7$: 4 rings with in the center P_2; in bottom, sight of top.

Similar figures were given in [28], and in all these figures, we have $N(N+1)/2 = 45$ peaks as it was already extrapolated in the table page 9 of that article. In the same paper, it was al-

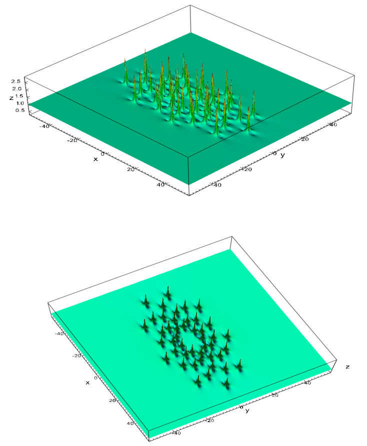

Figure 8: Solution to NLS, N=9, $\tilde{a}_4 = 10^9$: 5 rings without a central peak; in bottom, sight of top.

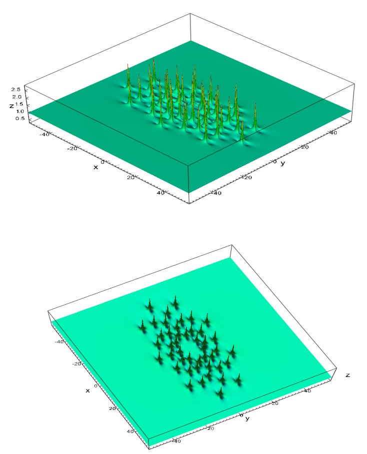

Figure 9: Solution to NLS, N=9, $\tilde{b}_4 = 10^9$: 5 rings without a central peak; in bottom, sight of top.

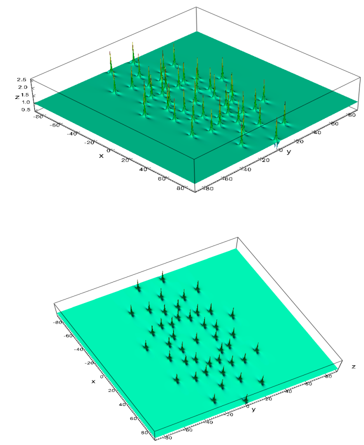

Figure 10: Solution to NLS, N=9, $\tilde{a}_5 = 10^{15}$: 4 rings with in the center one peak; in bottom, sight of top.

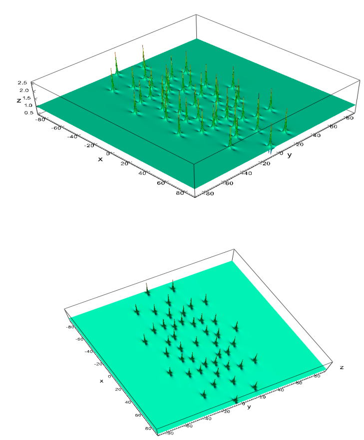

Figure 11: Solution to NLS, N=9, $\tilde{b}_5 = 10^{15}$: 4 rings with in the center one peak; in bottom, sight of top.

ready conjectured that in the case of one ring, the ring has $2N-1=17$ peaks surrounding the $P_{N-2}=P_7$ breather. It can be noted that when a more convenient normalization of the NLS equation is used, these rings can be transformed in circles.

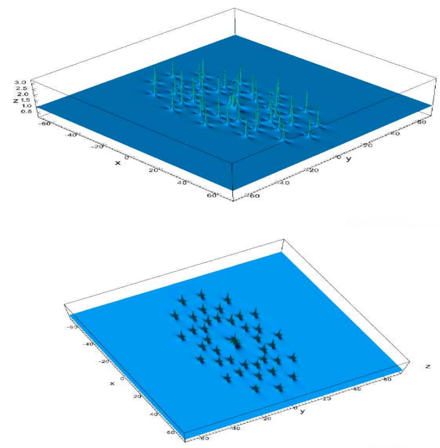

Figure 12: Solution to NLS, N=9, $\tilde{a}_6 = 10^{15}$: 3 rings with in the center the Peregrine breather of order 3; in bottom, sight of top.

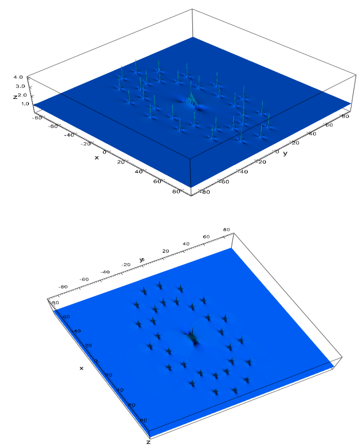

Figure 14: Solution to NLS, N=9, $\tilde{a}_7 = 10^{20}$: two rings with in the center the Peregrine breather of order 5; in bottom, sight of top.

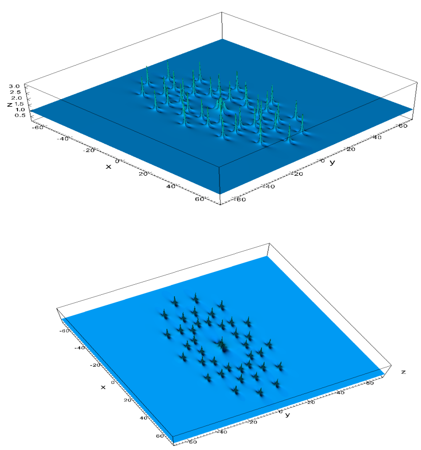

Figure 13: Solution to NLS, N=9, $\tilde{b}_6 = 10^{15}$: 3 rings with in the center the Peregrine breather of order 3; in bottom, sight of top.

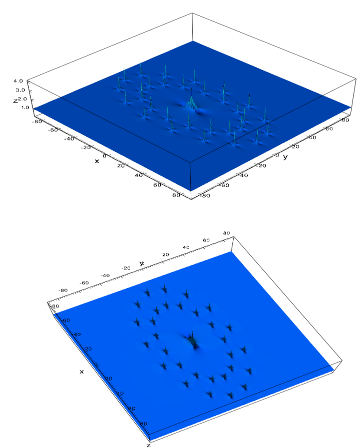

Figure 15: Solution to NLS, N=9, $\tilde{b}_7 = 10^{20}$: two rings with in the center the Peregrine breather of order 5; in bottom, sight of top.

4 Conclusion

In the present paper we have constructed explicitly solutions to the NLS equation of order 9 with 16 real parameters. The explicit representation in terms of polynomials in x and t is too large to

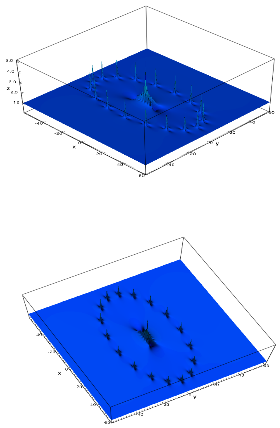

Figure 16: Solution to NLS, N=9, $\tilde{a}_8 = 10^{20}$: one ring with in the center the Peregrine breather of order 7; in bottom, sight of top.

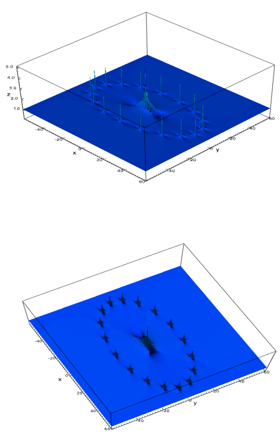

Figure 17: Solution to NLS, N=9, $\tilde{b}_8 = 10^{20}$: one ring with in the center the Peregrine breather of order 7; in bottom, sight of top.

be published here.
In the case of the variation of one parameter, we obtain different types of configurations with a maximum of 45 peaks.
By different choices of these parameters, we obtained new patterns in the $(x;t)$ plane; we recognized ring shape as already observed in the case of deformations depending on two parameters [12, 10]. We get new triangular shapes and multiple concentric rings.
Many applications in nonlinear optics and hydrodynamics have been given recently : we can mention in particular the works of Chabchoub et al. [29] or Kibler et al. [30].
These explicit solutions of order 9 and their deformations with eighteen parameters are presented for the first time to our knowledge. These deformations of the ninth Peregrine breather give a better understanding of the phenomenon of appearance of rogue waves and their asymptotic behavior.
The study of the solutions to the NLS equation has been done until order $N = 6$ by Akhmediev et al. in [28] and extrapolated until order $N = 10$. Present work gives explicitly exact solutions to the NLS equation at order 9; it verifies the conjectured classification given in [28].
It would be important to continue this study to try to classify solutions to the NLS equation in the general case of order N $(N > 10)$.

References

[1] V. E. Zakharov, Stability of periodic waves of finite amplitude on a surface of a deep fluid, J. Appl. Tech. Phys, V. **9**, 86-94, (1968)

[2] V. E. Zakharov, A.B. Shabat Exact theory of two dimensional self focusing and one dimensinal self

modulation of waves in nonlinear media, Sov. Phys. JETP, V. **34**, 62-69, (1972)

[3] D. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, J. Austral. Math. Soc. Ser. B, V. **25**, 16-43, (1983)

[4] N. Akhmediev, V. Eleonski, N. Kulagin, Generation of periodic trains of picosecond pulses in an optical fiber : exact solutions, Sov. Phys. J.E.T.P., V. **62**, 894-899, (1985).

[5] N. Akhmediev, V. Eleonski, N. Kulagin, Exact first order solutions of the nonlinear Schrödinger equation, Th. Math. Phys., V. **72**, N. 2, 183-196, (1987)

[6] N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Rogue waves and rational solutions of nonlinear Schrödinger equation, Physical Review E, V. **80**, N. 026601, (2009)

[7] N. Akhmediev, A. Ankiewicz, P.A. Clarkson, Rogue waves, rational solutions, the patterns of their zeros and integral relations, J. Phys. A : Math. Theor., V. **43**, 122002-1-9, (2010)

[8] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, Max-Planck-Institut für Mathematik, MPI 02-31, V. **161**, 2002

[9] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, Lett. Math. Phys., V. **68**, 77-90, 2004

[10] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, J. Phys. A : Meth. Theor., V. **44**, 1-15, (2011)

[11] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, Journal of Math. Sciences : Advances and Applications, V. **13**, N. 2, 71-153, (2012)

[12] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, Jour. Of Math. Phys., V. **54**, 013504-1-32, (2013)

[13] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, Jour. Of Mod. Phys., V. **4**, N. 4, 246-266, 2013

[14] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, Jour. Of Math., V. **2013**, 1-111, 2013

[15] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, Jour. Of Theor. And Appl. Phys., V. **7**, N. **45**, 1-6, 2013

[16] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, J. Math. Phys., V. **54**, 073519-1-21, 2013

[17] P. Gaillard, Deformations of third order Peregrine breather solutions

of the NLS equation with four parameters, Phys. Rev. E, V. **88**, 042903-1-9, 2013

[18] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, Physica Scripta, V. **89**, 015004-1-7, 2014

[19] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, Commun. Theor. Phys., V. **61**, 365-369, 2014

[20] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, Jour. Of Phys. : conferences series, V. **482**, 012016-1-7, 2014

[21] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, Jour. Of Math. Phys., V. **5**, 093506-1-12, 2014

[22] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, Jour. Of Phys. : conferences Series, **574**, 012031-1-5, 2015

[23] P. Gaillard, Tenth Peregrine breather solution of the NLS, Ann. of Phys., **355**, 293-298, 2015

[24] P. Gaillard, Other 2N-2 parameters solutions to the NLS equation and 2N+1 highest amplitude of the modulus of the N-th order AP breather, J. Phys. A: Math. Theor., V. **48**, 145203-1-23, 2015

[25] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, Adv. Res., **4**, 346-364, 2015

[26] M. Gastineau, J. Laskar, TRIP 1.3.8 : TRIP Reference manual, IMCCE, Paris Observatory, 2014, http://www.imcce.fr/trip/

[27] D. Kedziora, A. Ankiewicz, N. Akhmediev, Triangular rogue wave cascades, Phys. Review E, V. **86**, 056602-1-9, 2012.

[28] D. J. Kedziora, A. Ankiewicz, N. Akhmediev, Classifying the hierarchy of the nonlinear Schrödinger equation rogue waves solutions, Phys. Review E, V. **88**, 013207-1-12, 2013

[29] A. Chabchoub, H. Hoffmann, M. Onorato, N. Akhmediev, Super rogue waves : observation of a higher-order breather in water waves, Phys. Review X, V. **2**, 1-6, (2012).

[30] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev, J.M. Dudley, The Peregrine soliton in nonlinear fibre optics, Nat, Phys, V. 6, 790795, (2010)