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Abstract

Topologies for data center networks have been proposed in the literature through various graph classes and operations. A common trait to most existing designs is that they enhance the symmetric properties of the underlying graphs. Indeed, symmetry is a desirable property for interconnection networks because it minimizes congestion problems and it allows each entity to run the same routing protocol. However, despite sharing similarities these topologies all come with their own routing protocol. Recently, generic routing schemes have been introduced which can be implemented for any interconnection networks. The performances of such universal routing schemes are intimately related to the *hyperbolicity* of the topology. Roughly, graph hyperbolicity is a metric parameter which measures how close is the shortest-path metric of a graph from a tree metric (the smaller gap the better). Motivated by the good performances in practice of these new routing schemes, we propose the first general study of the hyperbolicity of data center interconnection networks. Our findings are disappointingly negative: we prove that the hyperbolicity of most data center topologies scales linearly with their diameter, that it the worst-case possible for hyperbolicity. To obtain these results, we introduce original connection between hyperbolicity and the properties of the endomorphism monoid of a graph. In particular, our results extend to all vertex and edge-transitive graphs. Additional results are obtained for de Bruijn and Kautz graphs, grid-like graphs and networks from the so-called Cayley model.

1 Introduction

The network topologies that are used to interconnect the computing unit of large-scale facilities (e.g., super computers, data centers hosting cloud applications, etc.) are designed to optimize various constraints such as equipment cost, deployment time, capacity and bandwidth, routing functionalities, reliability to equipment failures, power consumption, etc. This large variety of (conflicting) criteria has yield numerous proposals of interconnection networks. See for instance [4, 29, 37, 49, 57] and [5, 35, 36, 48, 66] for most recent ones. A common feature of the proposed constructions is to design network topologies offering a high-level of *symmetries*. Indeed, it is easier to balanced the traffic load, and hence to minimize the congestion, on network topologies with a high-level of symmetry. Furthermore, it simplifies the initial wiring of the physical infrastructure and it ensures that each router node can run the protocol.

However, despite sharing properties, interconnection networks rely on specific routing algorithms that are optimized for each topologies. As a novel step toward efficient and topology agnostic routing schemes, the authors in [14–16] proposed to use greedy routing schemes based

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on an embedding of the topology into the hyperbolic space. This approach has been shown particularly efficient for Internet-like graphs [44,56] where routes with low stretch are obtained. One explanation of this good behavior is that Internet-like graphs have low *hyperbolicity* [24,34], a graph parameter providing sharp bounds on the worst stretch (or distortion) of the distances in a graph when it is embedded into an edge-weighted tree.

In this paper, we characterize or give upper and lower bounds on the hyperbolicity of a broad range of interconnection networks topologies. These results can be used to establish theoretical bounds on the worse case behavior of greedy routing schemes in these topologies.

Related work. The idea of a greedy routing scheme based on an embedding into the hyperbolic space as by introduced by Kleinberg in [44]. Since then, various authors explored further this approach [11, 17, 40, 56]). In particular, they showed that the graphs of the Autonomous Systems of the Internet embed better into a hyperbolic space than into an Euclidean space. Motivated by the above results, a recent paper [62] proved that the over-delay for such routing schemes, or equivalently the maximum stretch of the routing paths w.r.t. the shortest-paths, depends on a graph parameter called *hyperbolicity*. This is a metric parameter introduced by Gromov in the context of automatic groups [34] and then extended to more general metric spaces [28]. Especially, graph hyperbolicity provides sharp bounds on the distortion of the distances in a graph when it is embedded into an edge-weighted tree. On the algorithmic side, there are approximation algorithms for problems related to distances in graphs —like diameter and radius computation [21], and minimum ball covering [22]— whose approximation constant depends on the hyperbolicity. Sometimes the approximation factor is a universal constant but the algorithm relies on a data-structure whose size is proportional to the hyperbolicity of the network topology [46]. Geometric routing schemes in [11, 17, 40, 56] do not make exception and so have a stretch proportional to the hyperbolicity.

There have been measurements to confirm that complex networks such as the graphs of the Autonomous Systems of the Internet, social networks and phylogenetic networks all have a small hyperbolicity. We refer to [1, 2, 6, 24, 43] for the most important studies in this area. However, we are not informed of any such a study for the data center networks. In this paper, we aim to fill in this gap through a theoretical study of their underlying graphs.

Our contributions. In an attempt to confront with the diversity of interconnection network topologies that have been proposed, we relate hyperbolicity with a few graph properties that are frequently encountered in these topologies. Indeed, we do not aim to provide a —long and non-exhaustive— listing of unrelated results for each network, but rather to exhibit a small number of their characteristics that are strongly related with their metric invariants. In particular, we relate hyperbolicity with the symmetries of a graph.

We prove in Section 4 that for graphs whose center is a k-dominating set for some small value of k, the hyperbolicity scales linearly with the diameter. This class of graphs comprises all vertex and edge-transitive graphs. As a result, any interconnection network whose topology is based on a Cayley graph has large hyperbolicity[1]. We prove in addition that similar results hold for graphs admitting an *endomorphism* for which the minimum distance between any node and its symmetric image is large. For other symmetric networks such as de Bruijn, Kautz and grid-like graphs, we apply different techniques that are based on their shortest-paths distribution so

[1] Independently from this work, the authors in [7] proved that for any vertex-transitive graph, the hyperbolicity scales linearly with the diameter. However, their proof relies on another definition of hyperbolicity, and it is unclear whether the proof can be extended to other graph classes. By contrast, our proof yields a tighter lower-bound for hyperbolicity, and it relies on a much simpler and more general argument (*i.e.*, see Theorem 23). Especially, it also applies to edge-transitive graphs.

that we can prove in Section 3 that they also have a large hyperbolicity. All these results are summarized in Table 1.

Last, we extend our results in Section 5 to heterogeneous interconnection network that have been proposed for data center networks by relating hyperbolicity with several graph operations. Most of the operations that we take account were introduced in the Cayley model of [66] in order to enhance some desirable properties of data center networks.

Our main message is that existing designs in the literature yield graphs with the highest possible value for the hyperbolicity —w.r.t. their diameter—. On the negative side, it means that any greedy routing scheme whose maximum stretch depends on the hyperbolicity is not scalable enough to cope with large data centers. But on a more positive side, it also implies that any routing scheme that relies on a data-structure with size proportional to the hyperbolicity solely requires *sublogarithmic* space in the number of servers. Indeed, it is well-known that the interconnection networks of data centers often have a diameter that is logarithmic or sublogarithmic in their size.

We start this paper providing useful notations and definitions in Section 2, and we conclude it in Section 6 with an open question about the relationship between network congestion and graph hyperbolicity.

2 Preliminaries

We refer to [12, 30] for the usual graph terminology. Graphs in this study are finite, simple (hence, without loop nor multiple edges), connected and unweighted.

2.1 Metric graph theory

Given a connected graph $G = (V, E)$, the *distance* between any two vertices $u, v \in V$ is defined as the minimum number of edges on a uv-path. We will denote it by $\mathrm{d}_G(u, v)$, or by $\mathrm{d}(u, v)$ whenever G is clear from the context.

For any subset $S \subseteq V$, the *eccentricity* of vertex $u \in S$, denoted $\mathrm{ecc}_G(u, S)$, is defined as the maximum distance between u and any other vertex $v \in S$. The *radius* of S is defined as $\mathrm{rad}_G(S) = \min_{u \in S} \mathrm{ecc}_G(u, S)$, while the *diameter* of S is defined as $\mathrm{diam}_G(S) = \max_{u \in S} \mathrm{ecc}_G(u, S)$. Observe that it holds that $\mathrm{rad}_G(S) \leq \mathrm{diam}_G(S) \leq 2 \cdot \mathrm{rad}_G(S)$. In particular, for any vertex $u \in V$, we have $\mathrm{ecc}(u) = \mathrm{ecc}_G(u, V)$, $\mathrm{rad}(G) = \mathrm{rad}_G(V)$ and $\mathrm{diam}(G) = \mathrm{diam}_G(V)$.

The *center* $\mathcal{C}(G)$ of the graph is the subset of all vertices with minimum eccentricity $rad(G)$.

Last, we define graph hyperbolicity as follows.

Definition 1 (4-points Condition, [34]). Let G be a connected graph.

For every 4-tuple u, v, x, y of G, we define $\delta(u, v, x, y)$ as half of the difference between the two largest sums amongst

$$S_1 = \mathrm{d}(u, v) + \mathrm{d}(x, y),\ S_2 = \mathrm{d}(u, x) + \mathrm{d}(v, y), \text{ and } S_3 = \mathrm{d}(u, y) + \mathrm{d}(v, x).$$

The graph hyperbolicity, denoted by $\delta(G)$, is equal to $\max_{u,v,x,y} \delta(u, v, x, y)$.

Moreover, we say that G is δ-*hyperbolic*, for every $\delta \geq \delta(G)$.

Other definitions exist for the hyperbolicity, but they are pairwise equivalent up to a constant-factor (*e.g.*, see [34] for details). So far, the hyperbolicity of a few graph classes has been characterized such as: random graphs [19,51,52], chordal graphs [13], k-chordal graphs [64],

Name	Degree max.	Diameter	Order	δ	Proof
de Bruijn graph, $UB(d,D)$	$2d$	D	d^D	$\frac{1}{2}\left\lfloor\frac{D}{2}\right\rfloor \leq \delta \leq \left\lfloor\frac{D}{2}\right\rfloor$	Prop. 5
Kautz graph, $UK(d,D)$	$2d$	D	$d^D(d+1)$	$\left\lfloor\frac{D}{4}\right\rfloor + \left\lfloor\frac{D \pmod 4}{3}\right\rfloor \leq \delta \leq \left\lfloor\frac{D}{2}\right\rfloor$	Prop. 7
Shuffle exchange, $SE(n)$	3	$2n-1$	2^n	$\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor \leq \delta \leq n$	Prop. 9
(n,m)-grid	4	$n+m-2$	nm	$\min\{n,m\}-1$	Cor. 15
d-dimensional grid of size s	$2d$	$d(s-1)$	s^d	$(s-1)\left\lfloor\frac{d}{2}\right\rfloor$	Cor. 16
Triangular (n,m)-grid	6	$n+m-2$	nm	$\frac{\min\{n,m\}-1}{2}$	Lem. 18
Hexagonal (n,m)-grid	6	$\begin{cases} n-1+\left\lceil\frac{m-1}{2}\right\rceil & \text{when } m \leq 2n-1 \\ m-1 & \text{otherwise} \end{cases}$	nm	$\frac{\min\{n,m\}-1}{2}$	Lem. 20
Cylinder (n,m)-grid	4	$\left\lfloor\frac{n}{2}\right\rfloor+m-1$	nm	$\min\left\{\left\lfloor\frac{n}{2}\right\rfloor, \frac{1}{2}\left(\left\lfloor\frac{n}{2}\right\rfloor+m\right)-\varepsilon\right\}, \varepsilon \in \left\{\frac{1}{2},1\right\}$	Lem. 22
Torus (n,m)-grid	4	$\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{m}{2}\right\rfloor$	nm	$\left\lfloor\frac{1}{2}\left(\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{m}{2}\right\rfloor\right)\right\rfloor-1 \leq \delta \leq \left\lfloor\frac{1}{2}\left(\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{m}{2}\right\rfloor\right)\right\rfloor$	Lem. 30
Gen. hypercube, $G(m_1,\ldots,m_r)$	$\sum_{i=1}^r m_i - r$	r	$\prod_{i=1}^r m_i$	$\left\lfloor\frac{r}{2}\right\rfloor$	Lem. 32
Cube Connected Cycle, $CCC(n)$	3	$2n-2+\max\left\{2,\left\lfloor\frac{n}{2}\right\rfloor\right\}$	$n2^n$	$n \leq \delta \leq n-1+\left\lfloor\frac{\max\{2,\lfloor\frac{n}{2}\rfloor\}}{2}\right\rfloor$	Lem. 34
BCube$_k(n)$	$\max\{n,k+1\}$	$2(k+1)$	$2^k(n+k+1)$	$k+1$	Lem. 36
Fat-Tree$_k$	k	6	$\frac{k^2}{4}(k+5)$	2	Lem. 38
Butterfly graph, $BF(n)$	4	$2n$	$2^n(n+1)$	n	Lem. 40
Wrapped Butterfly graph, $WBF(n)$	4	$n+\left\lfloor\frac{n}{2}\right\rfloor$	$2^n(n+1)$	$\left\lfloor\frac{n}{2}\right\rfloor \leq \delta \leq \left\lfloor\frac{1}{2}\left(n+\left\lfloor\frac{n}{2}\right\rfloor\right)\right\rfloor$	Lem. 40
k-ary n-fly	$2k$	$2n$	$k^n(n+1)$	n	Lem. 42
k-ary n-tree	$3k$	$2n$	$k^{n-1}(n+k)$	$n-1$	Lem. 44
d-ary tree grid, $MT(d,h)$	$d+1$	$4h$	$d^h\left(d^h+2\frac{d^h-1}{d-1}\right)$	$2h$	Lem. 46
Bubble-sort graph, $BS(n)$	$n-1$	$\binom{n}{2}$	$n!$	$\left\lfloor\frac{n(n-1)}{4}\right\rfloor$	Lem. 49
Transposition graph, $T(n)$	$\binom{n}{2}$	$n-1$	$n!$	$\frac{1}{2}\left\lfloor\frac{n-1}{2}\right\rfloor \leq \delta \leq \left\lfloor\frac{n-1}{2}\right\rfloor$	Lem. 51
Star graph, $S(n)$	$n-1$	$\left\lfloor\frac{3(n-1)}{2}\right\rfloor$	$n!$	$\left\lfloor\frac{1}{2}\left\lfloor\frac{3(n-1)}{2}\right\rfloor-\frac{1}{2}\right\rfloor \leq \delta \leq \left\lfloor\frac{1}{2}\left\lfloor\frac{3(n-1)}{2}\right\rfloor\right\rfloor$	Lem. 53

Table 1: Summary of results

outerplanar graphs [23] and other geometrical graph classes [21]. Lower and upper-bounds for the hyperbolicity are obtained in [60] using other graph invariants, and also in [23, 61] using graph decompositions. In particular, we will make use of the following upper-bound for hyperbolicity:

Lemma 2 ([24, 34]). *For every connected graph G, it holds that $\delta(G) \leq \left\lfloor \frac{\mathrm{diam}(G)}{2} \right\rfloor$.*

Therefore to prove that the hyperbolicity of a graph scales linearly with the diameter, it suffices to prove that one can *lower-bound* the hyperbolicity with the diameter —up to a constant-factor. Our proofs will make use of the notion of *isometric subgraphs*, the latter denoting a subgraph H of a graph G such that $\mathrm{d}_H(u,v) = \mathrm{d}_G(u,v)$ for any two vertices $u, v \in H$.

2.2 Algebraic graph theory

A graph *endomorphism* is a mapping σ from the vertex-set of a graph G to itself which preserves the adjacency relations, *i.e.*, $\forall \{u,v\} \in E(G)$ we have that $\{\sigma(u), \sigma(v)\} \in E(G)$.

We note that a graph endomorphism might fail to preserve the *non-adjacency* relations, but it does so if it is a graph *automorphism*, *i.e.*, a one-to-one endomorphism. In particular a graph endomorphism σ is called *idempotent* if $\forall u \in V(G)$ it holds that $\sigma^2(u) = u$, and in such a case it is an automorphism.

A graph is called *vertex-transitive* if $\forall u, v \in V(G)$, there is an automorphism σ such that $\sigma(u) = v$. Similarly, we call a graph *edge-transitive* if $\forall e = \{u,v\}, e' = \{u',v'\} \in E(G)$, there is an automorphism σ such that $\{\sigma(u), \sigma(v)\} = \{u', v'\}$.

Finally, let $(\Gamma, \cdot)$ be a group and let S be a generating set of Γ that is symmetric, *i.e.*, $S = S^{-1}$. The *Cayley graph* $G(\Gamma, S)$ of group Γ w.r.t. S has vertex-set Γ and edge-set $\{\{g, g \cdot s\} \mid g \in \Gamma, s \in S\}$. It is well-known that every Cayley graph is vertex-transitive [4].

3 Using the shortest-path distribution

We start lower-bounding the hyperbolicity of "simple" topologies for which the shortest-path distribution is well-known and characterized. Our proofs for grid-like graphs introduce a novel way to make use of the maximal shortest-paths in the study of graph hyperbolicity.

3.1 The fellow traveler property for graphs defined on an alphabet

As a warm up, we will lower-bound the hyperbolicity of some graph classes defined on alphabets, starting with the undirected de Bruijn graph.

Definition 3 ([9]). The undirected de Bruijn graph $UB(d,D)$ has vertex-set the words of length D taken over an alphabet Σ of size d. The 2-set $\{u,v\}$ is an edge of $UB(d,D)$ if and only if $u = u_{d-1}u_{d-2}\dots u_1u_0$ and $v = u_{d-2}\dots u_1u_0v_0$ for some letters $u_{d-1}, u_{d-2}, \dots, u_1, u_0, v_0 \in \Sigma$.

De Bruijn graphs have been extensively studied in the literature [8, 26, 29, 50]. In particular, $UB(d,D)$ has diameter D, maximum degree $2d$, and d^D vertices. Shortest-path routing and shortest-path distances in $UB(d,D)$ are characterized as follows.

Lemma 4 ([50]). *Let u, v be two words of length D taken over some alphabet Σ of size d, and write $u = u_L \cdot x \cdot u_R$ and $v = v_L \cdot x \cdot v_R$ so that $D - |x| + \min\{|u_L| + |v_R|, |v_L| + |u_R|\}$ is minimized. Then it holds that $\mathrm{d}_{UB(d,D)}(u,v) = D - |x| + \min\{|u_L| + |v_R|, |v_L| + |u_R|\}$.*

We say that a graph G falsifies the *k-fellow traveler property* if there are two shortest-paths $\mathcal{P}_1, \mathcal{P}_2$ with same endpoints $u, v \in V(G)$, and there are two vertices $x \in \mathcal{P}_1, y \in \mathcal{P}_2$ such that $\mathrm{d}_G(u,x) = \mathrm{d}_G(u,y)$ and $\mathrm{d}_G(x,y) > k$. By a straightforward calculation we obtain that in such a case $\delta(u,v,x,y) = \mathrm{d}_G(x,y)/2 > k/2$. So, we can lower-bound the hyperbolicity of G with the least k such that it *satisfies* the $2k$-fellow traveler property.

Proposition 5. *For any positive integers d and D, we have $\delta(UB(d,D)) \geq \frac{1}{2} \cdot \left\lfloor \frac{D}{2} \right\rfloor$.*

Proof. We prove that $UB(d,D)$ cannot satisfy the k-fellow traveler property for some range of k.

W.l.o.g. the vertices of $UB(d,D)$ are labeled with the words of length D taken over the alphabet $\Sigma = \{0, 1, \ldots, d-1\}$. Let $u = 0^D$, $v = 1^D$, $x = 0^{\lfloor D/2 \rfloor} \cdot 1^{\lceil D/2 \rceil}$, and $y = 1^{\lceil D/2 \rceil} \cdot 0^{\lfloor D/2 \rfloor}$. By Lemma 4 it comes that $\mathrm{d}(u,v) = D = \lceil D/2 \rceil + \lfloor D/2 \rfloor = \mathrm{d}(u,x) + \mathrm{d}(x,v) = \mathrm{d}(u,y) + \mathrm{d}(y,v)$. As a result, the graph $UB(d,D)$ cannot satisfy the k-fellow traveler property for $k < \mathrm{d}(x,y) = \lfloor D/2 \rfloor$ and so, $\delta(UB(d,D)) \geq \lfloor D/2 \rfloor / 2$. $\square$

A closely related graph classes that has been extensively studied in the literature is the undirected Kautz graph $UK(d,D)$ [9, 42]. This graph has diameter D, maximum degree $2d$, and $d^D(d+1)$ vertices. Furthermore, it can be checked that the Kautz graph $UK(d,D)$ is an induced subgraph of the de Bruijn graph $UB(d+1,D)$.

Definition 6 ([9, 42])**.** The undirected Kautz graph $UK(d,D)$ has vertex-set the words of length D taken over an alphabet Σ of size $d+1$ and satisfying that no two adjacent letters are equal. The 2-set $\{u,v\}$ is an edge of $UK(d,D)$ if and only if $u = u_{d-1}u_{d-2}\ldots u_1u_0$ and $v = u_{d-2}\ldots u_1u_0v_0$ for some letters $u_{d-1}, u_{d-2}, \ldots, u_1, u_0, v_0 \in \Sigma$.

Proposition 7. *For any positive integers d and D, we have $\delta(UK(d,D)) \geq \left\lfloor \frac{D}{4} \right\rfloor + \left\lfloor \frac{D \pmod 4}{3} \right\rfloor$.*

Proof. As for the proof of Proposition 5, we prove that $UK(d,D)$ cannot satisfy the k-fellow traveler property for some range of k.

W.l.o.g. the vertices of $UK(d,D)$ are labeled with the words of length D taken over the alphabet $\{0,1,2,\ldots,d\}$. Let $u = (01)^{\lfloor D/2 \rfloor} \cdot 0^{D \pmod 2}$, $v = (21)^{\lfloor D/2 \rfloor} \cdot 2^{D \pmod 2}$. By Lemma 4 we have that $\mathrm{d}_{UK(d,D)}(u,v) \geq \mathrm{d}_{UB(d+1,D)}(u,v) = D$ and so, $\mathrm{d}_{UK(d,D)}(u,v) = D$ because $diam(UK(d,D)) = D$. In particular, let $\mathcal{P}_1$ be the uv-shortest-path in $UK(d,D)$ that one obtains by applying "right shiftings" on u until one obtains vertex v *i.e.*,

$$\begin{aligned}\mathcal{P}_1 = &(01)^{\lfloor D/2 \rfloor} \cdot 0^{D \pmod 2} \to 1 \cdot (01)^{\lfloor D/2 \rfloor - 1} \cdot 0^{D \pmod 2} \cdot 2 \\ &\to (01)^{\lfloor D/2 \rfloor - 1} \cdot 0^{D \pmod 2} \cdot 21 \to \cdots \to (21)^{\lfloor D/2 \rfloor} \cdot 2^{D \pmod 2}\end{aligned}$$

Similarly, let $\mathcal{P}_2$ be the vu-shortest-path in $UK(d,D)$ that one obtains by applying "right shiftings" on v until one obtains vertex u. That is,

$$\begin{aligned}\mathcal{P}_2 = &(21)^{\lfloor D/2 \rfloor} \cdot 2^{D \pmod 2} \to 1 \cdot (21)^{\lfloor D/2 \rfloor - 1} \cdot 2^{D \pmod 2} \cdot 0 \\ &\to (21)^{\lfloor D/2 \rfloor - 1} \cdot 2^{D \pmod 2} \cdot 01 \to \ldots \to (01)^{\lfloor D/2 \rfloor} \cdot 0^{D \pmod 2}\end{aligned}$$

Let now

$$\begin{aligned} x = &(01)^{\lfloor \lfloor D/2 \rfloor /2 \rfloor} \cdot 0^{D \pmod 2} \cdot (21)^{\lceil \lfloor D/2 \rfloor /2 \rceil} \in \mathcal{P}_1 \\ \text{and } y = &1^{D \pmod 2} \cdot (21)^{\lceil \lfloor D/2 \rfloor /2 \rceil - (D \pmod 2)} \cdot 2^{D \pmod 2} \cdot (01)^{\lfloor \lfloor D/2 \rfloor /2 \rfloor} \cdot 0^{D \pmod 2} \in \mathcal{P}_2 \end{aligned}$$

be such that $\mathrm{d}(u,x) = \mathrm{d}(u,y)$.

The graph $UK(d,D)$ falsifies the k-fellow traveler property for all $k < \mathrm{d}_{UK(d,D)}(x,y)$, and we have by Lemma 4 that $\mathrm{d}_{UK(d,D)}(x,y) \geq \mathrm{d}_{UB(d+1,D)}(x,y) \geq 2(\lfloor D/4 \rfloor + \lfloor (D \pmod 4)/3 \rfloor)$.

As a result, it holds that $\delta(UK(d,D)) \geq \lfloor D/4 \rfloor + \lfloor (D \pmod 4)/3 \rfloor$. $\square$

We last define another topology that is related to the de Bruijn graph:

Definition 8 ([29]). The shuffle-exchange graph $SE(n)$ has vertex-set the binary words of length n. The 2-set $\{u, v\}$ is an edge of $SE(n)$ if and only if $u = u_{n-1}u_{n-2}\dots u_1u_0$ and: either $v = u_0u_{n-1}u_{n-2}\dots u_1$, or $v = u_{n-2}\dots u_1u_0u_{n-1}$, or $v = u_{n-1}u_{n-2}\dots u_1\bar{u_0}$, for some booleans $u_{n-1}, u_{n-2}, \dots, u_1, u_0$.

It was proved in [29] that the diameter of $SE(n)$ is $2n-1$, and that the pair of vertices 0^n and 1^n is a diametral pair. Furthermore, it can be checked that one can obtain the de Bruijn graph $UB(2, n-1)$ from $SE(n)$ as follows: for each edge $\{u_{n-1}u_{n-2}\dots u_1u_0, u_{n-1}u_{n-2}\dots u_1\bar{u_0}\}$, we contract the edge and we label $u_{n-1}u_{n-2}\dots u_1$ the resulting vertex.

Proposition 9. *For any positive integers n, we have $\delta\left(SE(n)\right) \geq \frac{1}{2} \cdot \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. As for the proof of Proposition 5, we prove that $SE(n)$ cannot satisfy the k-fellow traveler property for some range of k.

Let $u = 0^n$, $v = 1^n$ be a diametral pair of $SE(n)$, with $\mathrm{d}(u, v) = 2n-1$. Let $\mathcal{P}_1$ be the uv-shortest-path:

$$0^n \to 0^{n-1} \cdot 1 \to 1 \cdot 0^{n-1} \to 1 \cdot 0^{n-2} \cdot 1 \to 11 \cdot 0^{n-2} \to \dots \to 1^{n-1} \cdot 0 \to 1^n$$

Similarly, let $\mathcal{P}_2$ be the vu-shortest-path:

$$1^n \to 1^{n-1} \cdot 0 \to 0 \cdot 1^{n-1} \to 0 \cdot 1^{n-2} \cdot 0 \to 00 \cdot 1^{n-2} \to \dots \to 0^{n-1} \cdot 1 \to 0^n.$$

Finally, let $x = 1^{\lfloor n/2 \rfloor} \cdot 0^{\lceil n/2 \rceil} \in \mathcal{P}_1$, $y = 0^{\lceil n/2 \rceil - 1} \cdot 1^{\lfloor n/2 \rfloor} \cdot 0 \in \mathcal{P}_2$ be such that $\mathrm{d}(u, x) = \mathrm{d}(u, y)$. By using the contraction mapping from $SE(n)$ to $UB(2, n-1)$ one obtains that $\mathrm{d}_{UB(2,n-1)}(x', y') \leq \mathrm{d}_{SE(n)}(x, y)$ with $x' = 1^{\lfloor n/2 \rfloor} \cdot 0^{\lceil n/2 \rceil - 1}$, $y' = 0^{\lceil n/2 \rceil - 1} \cdot 1^{\lfloor n/2 \rfloor}$. As a result, we have by Lemma 4 that the shuffle-exchange graph falsifies the k-fellow traveler property for every $k < \mathrm{d}_{UB(2,n-1)}(x', y') = \left\lfloor \frac{n}{2} \right\rfloor$ and so, it holds that $\delta\left(SE(n)\right) \geq \frac{1}{2} \cdot \left\lfloor \frac{n}{2} \right\rfloor$. □

3.2 The maximal shortest-paths in grid-like topologies

In this section, we name grid-like graphs some slight variations of the 2-dimensional grid. As a reminder, an (n, m)-grid is the Cartesian product of the path P_n, with n vertices, with the path P_m, with m vertices. That is, the vertex-set is $\{0, \dots, n-1\} \times \{0, \dots, m-1\}$, and the edge-set is $\{\{(i, j), (i', j')\} \mid |i - i'| + |j - j'| = 1\}$. Grid-like networks are used for modeling interconnection networks and other computational applications. We now propose to compute their hyperbolicity. Our main tool in this section is the notion of *far-apart pairs*, first introduced in [53, 61]:

Definition 10 (Far-apart pair [53, 61]). Given $G = (V, E)$, the pair (u, v) is far-apart if for every $w \in V \setminus \{u, v\}$, $\mathrm{d}(w, u) + \mathrm{d}(u, v) > \mathrm{d}(w, v)$ and $\mathrm{d}(w, v) + \mathrm{d}(u, v) > \mathrm{d}(w, u)$.

The main motivation for introducing far-apart pairs was to speed-up the computation of hyperbolicity, via the following pre-processing method.

Lemma 11 ([53, 61]). *Let G be a connected graph. There exist two far-apart pairs (u, v) and (x, y) satisfying:*

- $\mathrm{d}_G(u, v) + \mathrm{d}_G(x, y) \geq \max\{\mathrm{d}_G(u, x) + \mathrm{d}_G(v, y), \mathrm{d}_G(u, y) + \mathrm{d}_G(v, x)\}$;
- $\delta(u, v, x, y) = \delta(G)$.

We here propose a novel application of this result in order to simplify proofs for the hyperbolicity of grid-like topologies.

Definition 12. The $(s_1, s_2, \ldots, s_d)$-grid is a graph with vertex set $\Pi_{i=1}^d \{0, \ldots, s_i - 1\}$ such that any two vertices $\langle u_1, u_2, \ldots, u_d \rangle, \langle v_1, v_2, \ldots, v_d \rangle$ are adjacent only if $\sum_{i=1}^d |u_i - v_i| = 1$.

Definition 13. The d-dimensional grid of size s if the $(s_1, s_2, \ldots, s_d)$-grid with $\forall i, s_i = s$.

Let us show how we can determine the hyperbolicity of the above graphs.

Proposition 14. *The $(s_1, s_2, \ldots, s_d)$-grid has hyperbolicity:*

$$h_d(s_1, s_2, \ldots, s_d) = \max_{\mathcal{E} \subseteq \{1,\ldots,d\}} \min \left\{ \sum_{i \in \mathcal{E}} s_i - 1, \sum_{i \notin \mathcal{E}} s_i - 1 \right\}.$$

Proof. The 2^{d-1} far-apart pairs of the grid are the diametral pairs $\{(\langle u_1, \ldots, u_d \rangle, \langle v_1, \ldots, v_d \rangle) \mid \forall i, \{u_i, v_i\} = \{0, s_i - 1\}\}$. Let $(\langle u_1, \ldots, u_d \rangle, \langle v_1, \ldots, v_d \rangle)$ and $(\langle x_1, \ldots, x_d \rangle, \langle y_1, \ldots, y_d \rangle)$ be two such pairs, denoted with $(\overrightarrow{u}, \overrightarrow{v})$ and $(\overrightarrow{x}, \overrightarrow{y})$ for short. Finally, let $D = \sum_i s_i - 1$ be the diameter of the grid and let $l = \sum_{i | u_i \neq x_i} s_i - 1$. Then it comes that we have:

$$\begin{aligned} S_1 &= \mathrm{d}(\overrightarrow{u}, \overrightarrow{v}) + \mathrm{d}(\overrightarrow{x}, \overrightarrow{y}) = 2D \\ S_2 &= \mathrm{d}(\overrightarrow{u}, \overrightarrow{x}) + \mathrm{d}(\overrightarrow{v}, \overrightarrow{y}) = 2l \\ S_3 &= \mathrm{d}(\overrightarrow{u}, \overrightarrow{y}) + \mathrm{d}(\overrightarrow{v}, \overrightarrow{x}) = 2(D - l). \end{aligned}$$

As a result, we have that $\delta(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{u}, \overrightarrow{v}) = \min\{l, D - l\}$ which is maximum for $l = h_d(s_1, s_2, \ldots, s_d)$. We conclude that $h_d(s_1, s_2, \ldots, s_d)$ is the hyperbolicity by Lemma 11. $\square$

We highlight two particular cases of Proposition 14 that were already known in the literature.

Corollary 15 ([24]). *The (n, m)-grid is $(\min\{n, m\} - 1)$-hyperbolic.*

Corollary 16 ([24]). *The d-dimensional grid of size s is $(s - 1) \cdot \left\lfloor \frac{d}{2} \right\rfloor$-hyperbolic.*

Similar results can be obtained for other grid-like graphs which can be found in the literature. We briefly prove some of these results before concluding this section.

Definition 17. The triangular (n, m)-grid is a supergraph of the (n, m)-grid with same vertex-set and with additional edges $\{(i, j), (i + 1, j + 1)\}$ for every $0 \leq i \leq n - 2$ and $0 \leq j \leq m - 2$.

An example of a triangular $(6, 7)$-grid is given in Figure 1a.

Lemma 18. *The triangular (n, m)-grid is $\frac{\min\{n,m\}-1}{2}$-hyperbolic.*

Proof. Let $u = (i_u, j_u)$ and $v = (i_v, j_v)$ be two vertices of the grid. We can assume w.l.o.g. that $i_u \geq i_v$. In such a case, either $j_u \geq j_v$ and so, $\mathrm{d}(u, v) = \max\{i_u - i_v, j_u - j_v\}$; or $j_u < j_v$ and so, $\mathrm{d}(u, v) = (i_u - i_v) + (j_v - j_u)$. We deduce from the above characterization that there is only one far-apart pair (u, v) such that $\mathrm{d}(u, v) \neq \max\{|i_u - i_v|, |j_u - j_v|\}$ namely, $u = (n - 1, 0)$ and $v = (0, m - 1)$ for which $\mathrm{d}(u, v) = n + m - 2$. Furthermore, for any other far-apart pair (x, y) we have either $\mathrm{d}(x, y) = n - 1$ or $\mathrm{d}(x, y) = m - 1$.

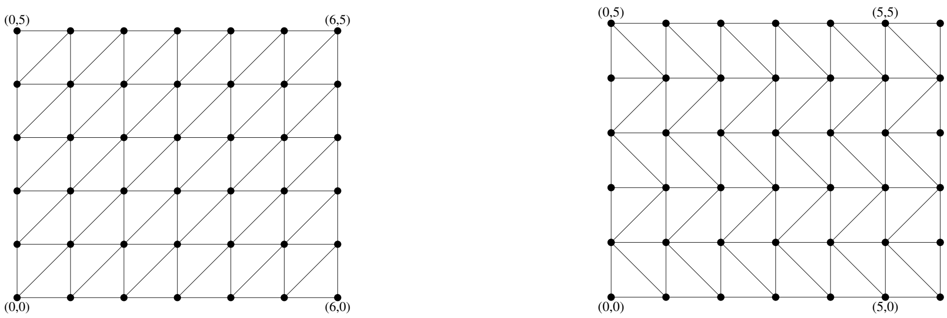
Let (u, v) and (x, y) be two far-apart pairs satisfying the conditions of the above Lemma 11. We assume w.l.o.g. that $\mathrm{d}(u, v) \geq \mathrm{d}(x, y)$, and we claim that $2\delta(u, v, x, y) \leq \min\{n, m\} - 1$. First, we have by [24] that $2\delta(u, v, x, y) \leq \min\{\mathrm{d}(u, v), \mathrm{d}(x, y)\} \leq \mathrm{d}(x, y)$. Note that $\mathrm{d}(x, y) = k \in \{n - 1, m - 1\}$ by the above characterization of the far-apart pairs in the grid. As a result, if $n = m$ then we are done because $\mathrm{d}(x, y) = \min\{n, m\} - 1$.

For the remaining of the proof, we will suppose that $n \neq m$ and $\mathrm{d}(x,y) = \max\{n,m\} - 1 = k$ (else we are done because $\mathrm{d}(x,y) = \min\{n,m\} - 1$). If $k = n-1$, it implies that $\mathrm{d}(u,v) \geq |i_u - i_v| = |i_x - i_y| = \mathrm{d}(x,y) = n-1$; else, it implies $\mathrm{d}(u,v) \geq |j_u - j_v| = |j_x - j_y| = \mathrm{d}(x,y) = m-1$. Therefore, we always have that $\max\{\mathrm{d}(u,x) + \mathrm{d}(v,y), \mathrm{d}(u,y) + \mathrm{d}(v,x)\} \geq 2k$. It follows by Lemma 11 that the hyperbolicity of the triangular grid is:

$$\begin{aligned}2\delta(u,v,x,y) &= \mathrm{d}(u,v) + \mathrm{d}(x,y) - \max\{\mathrm{d}(u,x) + \mathrm{d}(v,y), \mathrm{d}(u,y) + \mathrm{d}(v,x)\} \\ &\leq n + m - 2 + k - 2k = n + m - 2 - \max\{n-1, m-1\} = \min\{n,m\} - 1\end{aligned}$$

The bound is reached by setting $u = (n-1, 0)$, $v = (0, m-1)$, $x = (0,0)$, $y = (n-1, m-1)$. $\square$

In the example of Figure 1a, the hyperbolicity of the graph is given by the 4-tuple $u = (6,0)$, $v = (0,5)$, $x = (0,0)$, $y = (6,5)$.



(a) The triangular $(7,6)$-grid has hyperbolicity $\delta = \frac{5}{2} = \delta(u,v,x,y)$ with $u = (6,0)$, $v = (0,5)$, $x = (0,0)$, $y = (6,5)$.

(b) The hexagonal $(7,6)$-grid has hyperbolicity $\delta = \frac{5}{2} = \delta(u,v,x,y)$ with $u = (0,5)$, $v = (5,0)$, $x = (0,0)$, $y = (5,5)$.

Figure 1: Examples of grid-like graphs.

Definition 19. The hexagonal (n,m)-grid is a supergraph of the (n,m)-grid with same vertex-set and with additional edges $\{\{(i, m-2j-1), (i+1, m-2j-2)\} \mid 0 \leq i \leq n-2 \text{ and } 0 \leq j \leq \lfloor \frac{m}{2} \rfloor - 1\} \cup \{\{(i, m-2j-3), (i+1, m-2j-2)\} \mid 0 \leq i \leq n-2 \text{ and } 0 \leq j \leq \lfloor \frac{m-1}{2} \rfloor - 1\}$. The additional edges are called *diagonal* edges.

We refer to Figure 1b for an illustration.

Lemma 20. *The hexagonal (n,m)-grid is $\frac{\min\{n,m\}-1}{2}$-hyperbolic.*

Proof. We will first characterize the distances in the grid. Let $u = (i_u, j_u)$, $v = (i_v, j_v)$ be two vertices of the hexagonal grid. We can assume w.l.o.g. that $i_u \geq i_v$. We observe that in order to obtain an uv-shortest-path, it suffices to maximize the number of *diagonal* edges used in the path, that is $\min\{k, |i_u - i_v|\}$ with:

- $k = \lfloor |j_u - j_v|/2 \rfloor$ if both $j_u - j_v$ and $2\,[m - j_v \pmod 2] - 1$ have the same sign;
- $k = \lceil |j_u - j_v|/2 \rceil$ otherwise.

As a result we have that $\mathrm{d}(u,v) = |i_u - i_v| + |j_u - j_v| - \min\{k, |i_u - i_v|\}$ for some k depending on j_u and j_v, $k \in \{\lfloor |j_u - j_v|/2 \rfloor, \lceil |j_u - j_v|/2 \rceil\}$.

Suppose in addition that (u,v) is a far-apart pair. There are two cases. If $\mathrm{d}(u,v) = |j_u - j_v|$ then it is monotonically increasing with $|j_u - j_v|$ and so, $|j_u - j_v| = m - 1$. Else, $\mathrm{d}(u,v) = |i_u - i_v| + |j_u - j_v| - k$ for some k *only* depending on j_u and j_v, that is monotonically increasing with $|i_u - i_v|$ and so, $|i_u - i_v| = n - 1$.

Finally, let (u,v), (x,y) be two far-apart pairs satisfying the conditions of the above Lemma 11. We will prove that $2\delta(u,v,x,y) \leq \min\{n,m\} - 1$.

Case $\mathbf{m \leq n}$. If $\min\{\mathrm{d}(u,v), \mathrm{d}(x,y)\} \leq m-1$ then we are done because by [24] we have that $\delta(u,v,x,y) \leq \min\{\mathrm{d}(u,v), \mathrm{d}(x,y)\}/2 \leq (m-1)/2$. Else, we must have that $|i_u - i_v| = |i_x - i_y| = n-1$ and so, $\max\{\mathrm{d}(u,x)+\mathrm{d}(v,y), \mathrm{d}(u,y)+\mathrm{d}(v,x)\} \geq 2(n-1)$. Since we also have in such a case that $\mathrm{d}(u,v)+\mathrm{d}(x,y) \leq (n-1+\lceil(m-1)/2\rceil)+(n-1+\lfloor(m-1)/2\rfloor) = 2(n-1)+m-1$ then it follows once again that $\delta(u,v,x,y) \leq (m-1)/2$.

Case $\mathbf{m > n}$. There are three subcases to be considered.

- Suppose $\mathrm{d}(u,v) = |j_u - j_v| = m-1$, $\mathrm{d}(x,y) = |j_x - j_y| = m-1$. Then it comes that $\max\{\mathrm{d}(u,x)+\mathrm{d}(v,y), \mathrm{d}(u,y)+\mathrm{d}(v,x)\} \geq 2(m-1)$ and so, $\delta(u,v,x,y) = 0$.

- Suppose $\mathrm{d}(u,v) = j_u - j_v = m-1$ and $n-1+\lfloor(j_x-j_y)/2\rfloor \leq \mathrm{d}(x,y) \leq n-1+\lceil(j_x-j_y)/2\rceil$. Then it holds that $\mathrm{d}(u,y)+\mathrm{d}(v,x) \geq (j_u-j_y)+(j_x-j_v) = (j_u-j_x)+(j_x-j_y)+(j_x-j_v) = m-1+(j_x-j_y)$. As a result, we have that $2\delta(u,v,x,y) \leq (n-1+\lceil(j_x-j_y)/2\rceil+m-1)-(m-1+j_x-j_y) = n-1-\lfloor(j_x-j_y)/2\rfloor \leq n-1$.

- Else, we consider the smallest hexagonal grid of dimensions (n',m') for which there exists two far-apart pairs (u',v') and (x',y') that satisfy the conditions of the above Lemma 11 and such that $\delta(u',v',x',y') \geq \delta(u,v,x,y)$. We assume w.l.o.g. that $n' < m'$ and $\mathrm{d}(u',v') \neq |j_{u'} - j_{v'}|$, $\mathrm{d}(x',y') \neq |j_{x'} - j_{y'}|$ (otherwise we fall in one of the above cases). Note that it implies that $|i_{u'} - i_{v'}| = |i_{x'} - i_{y'}| = n'-1$ by our above characterization of the far-apart pairs.

  If the two far-apart pairs are $((0,0),(n'-1,m'-1))$ and $((0,m'-1),(n'-1,0))$, then we obtain by the computation that $2\delta(u',v',x',y') = n'-1+(n'-m') < n'-1 \leq n-1$.

  Else, by minimality of the subgrid there is some vertex in the 4-tuple, say u', which is contained amongst $\{(0,0),(n'-1,m'-1),(n'-1,0),(0,m'-1)\}$ and such that no other vertex $z \in \{v',x',y'\}$ satisfies that $j_{u'} = j_z$. By symmetry, we will assume that $u' \in \{(0,m'-1),(n'-1,m'-1)\}$. Then, using the above characterization of the distances in the hexagonal grid, it can be checked that for any $0 \leq i \leq n'-1$ and for any $0 \leq j \leq m'-2$, we have that:

$$\begin{aligned} &\mathrm{d}\left((n'-1,m'-2),(i,j)\right) = \mathrm{d}\left((n'-1,m'-1),(i,j)\right) - 1 \\ \text{and } &\mathrm{d}\left((1,m'-2),(i,j)\right) = \mathrm{d}\left((0,m'-1),(i,j)\right) - 1 \text{ unless } (i,j) = (0,m'-2) \end{aligned}$$

  Therefore, by the 4-point condition we have that $\delta(u',v',x',y') = \delta\left((n'-1,m'-2),v',x',y'\right)$ when $u' = (n'-1,m'-1)$. Furthermore, when $u' = (0,m'-1)$, we have $\delta(u',v',x',y') \leq \max\{\mathrm{d}\left((0,m'-1),(0,m'-2)\right), \delta\left((n'-1,m'-2),v',x',y'\right)\} \leq \max\{1, \delta\left((n'-1,m'-2),v',x',y'\right)\}$. In both cases, it contradicts the minimality of (n',m').

To conclude, let $l = \min\{n,m\}-1$. The upper-bound $l/2$ for the hyperbolicity is reached by setting $u = (0,m-1), v = (l,m-1-l), x = (0,m-1-l), y = (l,m-1)$. □

In the example of Figure 1b for an illustration, the hyperbolicity of the graph is given by the 4-tuple $u = (0,5)$, $v = (5,0)$, $x = (0,0)$, $y = (5,5)$.

Definition 21. The cylinder (n,m)-grid is the supergraph of the (n,m)-grid with the same vertex-set and with additional edge-set $\{\{(0,j),(n-1,j)\} \mid 0 \leq j \leq m-1\}$.

In particular, when $m=1$, then the cylinder (n,m)-grid is the n-cycle C_n.

Lemma 22. *The cylinder (n,m)-grid is*

$$\begin{cases} \left\lfloor \frac{n}{2} \right\rfloor\text{-hyperbolic} & \text{when } m > \left\lfloor \frac{n}{2} \right\rfloor \\ \left(\frac{\left\lfloor \frac{n}{2} \right\rfloor + m}{2} - 1 \right)\text{-hyperbolic} & \text{when } m \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } (n \text{ is odd or } \left\lceil \frac{n}{2} \right\rceil - m + 1 \text{ is odd}) \\ \left(\frac{\left\lfloor \frac{n}{2} \right\rfloor + m}{2} - \frac{1}{2} \right)\text{-hyperbolic} & \text{otherwise.} \end{cases}$$

Proof. Let $u = (i_u, j_u), v = (i_v, j_v)$ be two vertices of the grid. We have that $\mathrm{d}(u,v) = \min\{|i_u - i_v|, n - |i_u - i_v|\} + |j_u - j_v|$. As a result, the far-apart pairs of the cylinder (n,m)-grid are exactly the pairs $\{(i,0), (i + \lfloor n/2 \rfloor \pmod n), m-1)\}$, and the pairs $\{(i,0), (i + \lceil n/2 \rceil \pmod n), m-1)\}$, with $0 \leq i \leq n-1$. Equivalently, these are the pairs $\{(u',0), (v', m-1)\}$ with (u',v') an arbitrary far-apart pair of the n-cycle C_n.

Let (u,v) and (x,y) be two far-apart pairs of the cylinder (n,m)-grid satisfying the conditions of the above Lemma 11. Write $u = (u',0)$, $v = (v', m-1)$, $x = (x',0)$, $y = (y', m-1)$. Furthermore, let $S_1 = \mathrm{d}(u,v) + \mathrm{d}(x,y)$, $S_2 = \mathrm{d}(u,x) + \mathrm{d}(v,y)$, and $S_3 = \mathrm{d}(u,y) + \mathrm{d}(v,x)$. Similarly, let $S_1' = \mathrm{d}_{C_n}(u',v') + \mathrm{d}_{C_n}(x',y')$, $S_2' = \mathrm{d}_{C_n}(u',x') + \mathrm{d}_{C_n}(v',y')$, and $S_3' = \mathrm{d}_{C_n}(u',y') + \mathrm{d}_{C_n}(v',x')$. Note that we have that: $S_1' = 2\lfloor n/2 \rfloor = \max\{S_1', S_2', S_3'\}$; $S_1 = S_1' + 2(m-1) = 2(\lfloor n/2 \rfloor + m - 1)$, $S_2' = S_2$, and $S_3 = S_3' + 2(m-1)$.

There are two cases to be considered.

- Suppose that $m > \lfloor n/2 \rfloor$. We have that $\delta(u,v,x,y) \leq (S_1 - S_3)/2 \leq (S_1' - S_3')/2 \leq S_1'/2 \leq \lfloor n/2 \rfloor$. The bound is reached by setting $u' = y'$ and $v' = x'$.
- Suppose that $m \leq \lfloor n/2 \rfloor$. If $(u',v') = (y',x')$ then we obtain by the calculation that $\delta(u,v,x,y) = (m-1)/2$. Otherwise, we have that $S_2' + S_3' = n$ and hence $2\delta(u,v,x,y) = S_1' - \max\{S_3', S_2' - 2(m-1)\} = S_1' - \max\{S_3', (n - 2(m-1)) - S_3'\}$ is maximum when $\lfloor n/2 \rfloor - (m-1) \leq S_3' \leq \lceil n/2 \rceil - (m-1)$. In the following, let $\lceil n/2 \rceil - (m-1) = 2q + r$ with $0 \leq r \leq 1$. There are two subcases to be considered.

  (i) Assume that n is odd and let us set $u' = 0$, $v' = \lfloor n/2 \rfloor$, $x' = \lfloor n/2 \rfloor - q$, and $y' = n - q - r$. In such a case, we have that $S_3' = (q+r) + q = 2q + r$. As a result, $\delta(u,v,x,y) = (\lfloor n/2 \rfloor + m)/2 - 1$ and so, the above upper-bound is always reached when n is odd.

  (ii) Assume that n is even. Then, we have that $S_3' = 2\,\mathrm{d}(u,y)$, and so S_3' cannot be odd. It implies that the hyperbolicity is bounded from above by $n/4 + (m-1-r)/2$. We set $u' = 0$, $v' = n/2$, $x' = n/2 - q$, and $y' = n - q$. In such a case, $S_3' = 2q$, hence $(n - 2(m-1)) - S_3' = 4q + 2r - 2q = 2q + 2r$ and so, $\delta(u,v,x,y) = n/4 + (m-1-r)/2$ that is maximum.

□

4 The metric properties of the endomorphism monoid of a graph

Lower-bound methods for the hyperbolicity in Section 3 apply well to graphs for which the diametral paths are well-known and characterized. However, in most cases there is no good characterization of the shortest-path distribution of the graphs. There even exist interconnection networks topologies the diameter of which is still unknown [33, 45]. In a need of more robust methods, we introduce new lower-bounds on the hyperbolicity that are based on non-trivial symmetries of the graphs. For clarity, our results are presented separately from their applications to interconnection networks topologies.

4.1 Main results

We first introduce a very generic argument to obtain lower-bounds on the hyperbolicity that applies to highly symmetric graphs such as transitive graphs. The bounds obtained with this method are usually loose, but they are enough to prove that the hyperbolicity scales linearly with the diameter for most graphs we study in this work.

Theorem 23. *Let G be a connected graph, and let $k \geq 0$ be such that all vertices are at distance at most k from the center of G. Then, we have that $\delta(G) \geq \frac{1}{2} \cdot \left\lfloor \frac{diam(G)}{2} \right\rfloor - \frac{k}{2}$.*

Proof. Let $\mathcal{C}(G)$ be the center of G. By the hypothesis every node in G is at distance at most k from $\mathcal{C}(G)$, therefore $\mathrm{diam}_G(\mathcal{C}(G)) \geq \mathrm{diam}(G) - 2k$. Moreover, we have by [20] that $\mathrm{diam}_G(\mathcal{C}(G)) \leq 4\delta(G) + 1$. Consequently, it holds that $\delta(G) \geq \lfloor \mathrm{diam}_G(\mathcal{C}(G))/2 \rfloor /2 \geq \lfloor \mathrm{diam}(G)/2 \rfloor /2 - k/2$. □

Corollary 24. *Let G be a connected vertex-transitive graph. Then $\delta(G) \geq \frac{1}{2} \cdot \left\lfloor \frac{\mathrm{diam}(G)}{2} \right\rfloor$.*

Proof. Since G is vertex-transitive by the hypothesis, we have $\mathcal{C}(G) = V(G)$, and so we can apply Theorem 23 by setting $k = 0$. □

On the practical side, most of the interconnection networks topologies are based on a graph that is vertex-symmetric. This comprises hypercube-based networks [10], generalized Petersen graphs [27, 55], generalized Heawood graphs [38, 39] and Cayley graphs [4]. For some of these topologies such as the Pancake graph [33], a well-known Cayley graph, Corollary 24 is the best bound on the hyperbolicity we know so far.

Corollary 25. *Let G be a connected edge-transitive graph. Then $\delta(G) \geq \frac{1}{2} \cdot \left\lfloor \frac{\mathrm{diam}(G)}{2} \right\rfloor - \frac{1}{2}$.*

Proof. We first claim that the center $\mathcal{C}(G)$ is a dominating set. Indeed, let $u \in V(G)$ and $v \in \mathcal{C}(G)$, and let $x \in N_G(u)$ and $y \in N_G(v)$. Since G is edge-symmetric by the hypothesis, there exists an automorphism σ such that $\{\sigma(v), \sigma(y)\} = \{u, x\}$. Furthermore we have that $\sigma(v) \in \mathcal{C}(G)$ because σ is an automorphism and so, $\mathrm{d}_G(u, \mathcal{C}(G)) \leq \mathrm{d}_G(u, \sigma(v)) \leq 1$ which proves the claim. As a result, we can apply Theorem 23 by setting $k = 1$. □

Despite its wide applicability to interconnection networks, the above Corollaries 24 and 25 require graphs to have an automorphism group with constrained properties. A natural question is whether we can weaken the requirements by considering endomorphisms instead of automorphisms. To answer this question, we use weakly vertex-transitive graphs that have been defined in [31] in a similar fashion to vertex-transitive graphs. Namely, a graph G is weakly vertex-transitive if, for any two vertices $u, v \in V(G)$ there exists a graph *endomorphism* σ satisfying $\sigma(u) = v$. Unlike vertex-transitive graphs, the gap between hyperbolicity and diameter may be arbitrarily large for weakly vertex-transitive graphs. Indeed, on the one hand it was proved in [31] that bipartite graphs are weakly vertex-transitive. On the other hand, chordal bipartite graphs have bounded hyperbolicity because they have bounded chordality (see e.g. [64]), whereas they may have a diameter that is arbitrarily large. We now show that surprisingly, some lower-bounds on the hyperbolicity can still be deduced from graph endomorphisms.

Theorem 26. *Let G be a connected graph and σ be a graph endomorphism of G such that $\forall u \in V(G)$, we have that $\mathrm{d}_G(u, \sigma(u)) \geq l \geq 2$. Then it holds that $\delta(G) \geq \frac{1}{2} \cdot \left\lceil \frac{l}{2} \right\rceil$.*

Proof. We will consider a graph game which is a slight variation of the well-known 'Cop and Robber' game (*e.g.* see [3, 54, 59]). There are two players in this game that are playing alternatively on a (connected) graph, by moving along a path of length at most s, for some positive integer s. The first player to position herself on the graph is the Cop, and the second player is called the Robber. Last a graph is said *Cop-win* for this game if the Cop always has a winning-strategy *i.e.*, she can always reach the position of the Robber in a finite number of moves, and hence eventually catch the Robber. In [18] the authors proved that every connected graph G is Cop-win whenever $s \geq 4\delta(G)$. So, to prove the theorem we claim that it suffices to show that G is not Cop-win if $s \leq l-1$. Indeed, in such a case it holds that $4\delta(G) \geq l$, hence $2\delta(G) \geq l/2$ that implies $2\delta(G) \geq \lceil l/2 \rceil$ and so, $\delta(G) \geq \lceil l/2 \rceil /2$. Equivalently, we will exhibit a winning-strategy for the Robber in such a case.

We notice that if at each turn of the Cop the Robber is onto the image (by σ) of her current position, then it is a winning strategy for the Robber because by the hypothesis, both vertices are at distance at least l, and the maximum speed of the Cop is $l-1$. To achieve the result, we proceed as follows. First if the Cop picks vertex u as her initial position then the Robber starts the game at vertex $\sigma(u)$. Then, if the Cop moves along a path $(u = x_0, x_1, \ldots, x_i, \ldots, x_k = v)$, $k \leq l-1$, then the Robber moves along the path $(\sigma(u), \sigma(x_1), \ldots, \sigma(x_i), \ldots, \sigma(v))$ which exists because σ is a graph endomorphism. Such a move for the Robber is valid as long as $v \notin \{\sigma(u), \sigma(x_1), \ldots, \sigma(x_i), \ldots, \sigma(v)\}$, and that is always the case since $\sigma(x_i) = v$ would imply that $\mathrm{d}(x_i, \sigma(x_i)) \leq l-1$. $\square$

In practice, we will rely upon a stronger version of Theorem 26 so that we can obtain (almost) tight bounds on the hyperbolicity, but it requires stronger constrictions on the endomorphism.

Theorem 27. *Let G be a connected graph, σ be a graph endomorphism and l, l' be two non-negative integers such that $\forall u \in V(G)$, we have that $\mathrm{d}_G(u, \sigma(u)) \geq l$ and $\mathrm{d}_G(u, \sigma^2(u)) \leq l'$. Then it holds that $\delta(G) \geq \left\lfloor \frac{l}{2} \right\rfloor - \frac{l'}{2}$.*

Proof. Clearly, if $l \leq l'$ then we have that $\delta(G) \geq 0 \geq \lfloor l/2 \rfloor - l'/2$. Therefore, we will assume w.l.o.g. that $l \geq l'+1$. Let $u \in V(G)$ minimizing $\mathrm{d}_G(u, \sigma(u))$ and let v be on a $u\sigma(u)$-shortest-path such that $\mathrm{d}_G(u, v) = \lfloor \mathrm{d}_G(u, \sigma(u))/2 \rfloor$. Then we deduce from the endomorphism the following inequalities:

$$\begin{aligned}
S_1 &= \mathrm{d}(u, \sigma(u)) + \mathrm{d}(v, \sigma(v)) \geq 2 \cdot \mathrm{d}(u, \sigma(u)) \geq 2l; \\
S_2 &= \mathrm{d}(u, v) + \mathrm{d}(\sigma(u), \sigma(v)) \leq 2 \cdot \mathrm{d}(u, v) \leq 2 \left\lfloor \mathrm{d}(u, \sigma(u))/2 \right\rfloor; \\
S_3 &= \mathrm{d}(u, \sigma(v)) + \mathrm{d}(v, \sigma(u)) \leq \mathrm{d}(u, \sigma^2(u)) + \mathrm{d}(\sigma^2(u), \sigma(v)) + \mathrm{d}(u, \sigma(v)) \leq l' + 2 \cdot \mathrm{d}(u, \sigma(v)) \\
&\quad \leq 2 \left\lceil \mathrm{d}(u, \sigma(u))/2 \right\rceil + l' \leq \mathrm{d}(u, \sigma(u)) + 1 + l'.
\end{aligned}$$

In such a case, we have that $S_1 \geq \max\{S_2, S_3\}$ and as a result, we have that:

$$\delta(G) \geq \delta(u, v, \sigma(u), \sigma(v)) \geq \min\left(\left\lceil \frac{\mathrm{d}(u, \sigma(u))}{2} \right\rceil, \left\lfloor \frac{\mathrm{d}(u, \sigma(u))}{2} \right\rfloor - \frac{l'}{2} \right) \geq \left\lfloor \frac{l}{2} \right\rfloor - \frac{l'}{2}.$$

$\square$

Theorem 27 gives a lower-bound on the hyperbolicity that is sharp for almost every cycle. Indeed, let $\mathbb{Z}_n$ be the vertex set of the n-cycle C_n, and let σ be the automorphism mapping any vertex i to the vertex $i + \lfloor n/2 \rfloor \pmod{n}$. Applying Theorem 27 to σ, we obtain a lower-bound $\lfloor n/4 \rfloor$ for the hyperbolicity of even-length cycles, which is exact, and a lower-bound $\lfloor n/4 \rfloor - 1/2$ for odd-length cycles, which is exact when $n \equiv 1 \pmod 4$ and below $1/2$ of the true hyperbolicity when $n \equiv 3 \pmod 4$ [24].

We emphasize on the following consequence of Theorem 27.

Corollary 28. *Let G be a connected graph and σ be an idempotent endomorphism satisfying that $\forall u \in V(G)$ we have $\mathrm{d}(u, \sigma(u)) \geq l$. Then it holds that $\delta(G) \geq \lfloor \frac{l}{2} \rfloor$.*

Proof. By the hypothesis, the endomorphism σ is idempotent and so, we can apply Theorem 27 by setting $l' = 0$. □

4.2 Applications

We subsequently show the power of Theorems 26, 27 and Corollary 28 by applying them on a broad range of topologies that were studied in the literature. We will combine their lower-bound with a slight variation of the well-known upper-bound of Lemma 2. Indeed it is folklore that the hyperbolicity of a graph is the maximum hyperbolicity taken over all of its biconnected components and so, we have that $\delta(G) \leq \lfloor \mathrm{effdiam}(G)/2 \rfloor$, where the so-called *efficient diameter* $\mathrm{effdiam}(G)$ denotes the largest diameter amongst the biconnected components of the graph. By doing so, we will show that for most graphs in the literature, the above upper-bound is always reached —up to a small constant-factor—.

4.2.1 Torus

Let us first complete the results of the Section 3.2 with an additional grid-like topology.

Definition 29. The *torus* (n, m)-grid is a supergraph of the (n, m)-grid with additional edge-set $\{\{(0, j), (n-1, j)\} \mid 0 \leq j \leq m-1\} \cup \{\{(i, 0), (i, m-1)\} \mid 0 \leq i \leq n-1\}$.

Lemma 30. *Let $\delta_{n,m}$ be the hyperbolicity of the torus (n, m)-grid. We have that:*

$$\left\lfloor \frac{1}{2} \cdot \left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \right) \right\rfloor - 1 \leq \delta_{n,m} \leq \left\lfloor \frac{1}{2} \cdot \left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \right) \right\rfloor.$$

Proof. For any two vertices $u = (i_u, j_u), v = (i_v, j_v)$, we have that $\mathrm{d}(u, v) = \min\{|i_u - i_v|, n - |i_u - i_v|\} + \min\{|j_u - j_v|, m - |j_u - j_v|\}$. It implies that the diameter of the torus grid is $\lfloor n/2 \rfloor + \lfloor m/2 \rfloor$ and so, its hyperbolicity is bounded from above by $\lfloor (\lfloor n/2 \rfloor + \lfloor m/2 \rfloor)/2 \rfloor$ by Lemma 2. Finally, let σ be the automorphism of the torus grid which maps any vertex (i, j) to the vertex $(i + \lfloor n/2 \rfloor \pmod n, j + \lfloor m/2 \rfloor \pmod m)$. Since we have that $\mathrm{d}(u, \sigma(u)) = \lfloor n/2 \rfloor + \lfloor m/2 \rfloor$ and $\mathrm{d}(u, \sigma^2(u)) = (\lceil n/2 \rceil - \lfloor n/2 \rfloor) + (\lceil m/2 \rceil - \lfloor m/2 \rfloor) = 0$ for any vertex u, then it follows from Theorem 27 that the hyperbolicity of the torus grid is at least $\lfloor (\lfloor n/2 \rfloor + \lfloor m/2 \rfloor)/2 \rfloor - \lceil (\lceil n/2 \rceil + \lceil m/2 \rceil) - (\lfloor n/2 \rfloor + \lfloor m/2 \rfloor) \rceil /2 \geq \lfloor (\lfloor n/2 \rfloor + \lfloor m/2 \rfloor)/2 \rfloor - 1$. □

4.2.2 Hybercube-like networks

Definition 31 ([10, 37])**.** Let $m_1, m_2, \ldots, m_r$ be positive integers with $\forall i, m_i \geq 2$ and $r \geq 1$. The generalized hypercube $G(m_1, m_2, \ldots, m_r)$ has vertex-set $\{(x_1, x_2, \ldots, x_r) \mid \forall i, 0 \leq i \leq m_i - 1\}$, and two vertices $(x_1, x_2, \ldots, x_r), (y_1, y_2, \ldots, y_r)$ are adjacent in the graph if and only if their Hamming distance $\sum_i \mathbb{I}_{\{x_i = y_i\}}$ is equal to 1.

In particular, the k-ary hypercube $H_k(n)$ is the generalized hypercube $G(m_1, m_2, \ldots, m_n)$ with $\forall i, m_i = k$.

Lemma 32. $\delta(G(m_1, m_2, \ldots, m_r)) = \lfloor \frac{r}{2} \rfloor$.

Proof. The diameter of $G(m_1, m_2, \ldots, m_r)$ is r and so, $\delta(G(m_1, m_2, \ldots, m_r)) \leq \lfloor r/2 \rfloor$ by Lemma 2. To prove the lower-bound, we first make the observation that the binary hypercube $H_2(r)$ is an isometric subgraph of $G(m_1, m_2, \ldots, m_r)$. Let σ be the automorphism mapping any vertex $(x_1, x_2, \ldots, x_r) \in V(H_2(r))$ to its *complementary* vertex $(1 - x_1, 1 - x_2, \ldots, 1 - x_r)$. We have that for any vertex u, $\mathrm{d}_{H_2(r)}(u, \sigma(u)) = r$, and σ is idempotent. As a result, we conclude by Corollary 28 that $\delta(G(m_1, m_2, \ldots, m_r)) \geq \delta(H_2(r)) \geq \lfloor r/2 \rfloor$. □

Observe that the proof of Lemma 32 can also be done from the fact that the n-dimensional grid of size 2 is the hypercube $H_2(n)$, and so by Corollary 16 we get $\delta(H_2(r)) = \lfloor r/2 \rfloor$.

Definition 33 ([58]). The cube-connected-cycle $CCC(n)$ has vertex-set the pairs $\langle i, w \rangle$, for $0 \leq i \leq n-1$ and for w any binary word of length n; two vertices $\langle i, x_1 x_2 \ldots x_n \rangle$ and $\langle j, y_1 y_2 \ldots y_n \rangle$ are adjacent in the graph if, and only if either $i = j$, $x_i = 1 - y_i$ and $\forall k \neq i, x_k = y_k$, or $i = j+1 \pmod{n}$ and $\forall k, x_k = y_k$.

Lemma 34. $n \leq \delta(CCC(n)) \leq n - 1 + \left\lfloor \frac{\max\{2, \lfloor \frac{n}{2} \rfloor\}}{2} \right\rfloor$.

Proof. By [32], we have that $diam(CCC(n)) = 2n - 2 + \max\{2, \lfloor n/2 \rfloor\}$ and so, we have that $\delta(CCC(n)) \leq n - 1 + \lfloor (\max\{2, \lfloor n/2 \rfloor\})/2 \rfloor$ by Lemma 2. Furthermore, the mapping $\sigma : \langle i, w \rangle \to \langle i, \bar{w} \rangle$ is an idempotent endomorphism and it satisfies by [32] that for any vertex u, $\mathrm{d}(u, \sigma(u)) = 2n$. As a result, we conclude by Corollary 28 that $\delta(CCC(n)) \geq n$. $\square$

Definition 35 ([35]). Let $\mathbb{Z}_n^l$ be the set of words of length l over the alphabet $\{0, 1, \ldots, n-1\}$. The graph $\mathrm{BCube}_k(n)$ has vertex-set $\mathbb{Z}_n^{k+1} \cup \left(\{0, 1, \ldots, k\} \times \mathbb{Z}_n^k\right)$ and edge-set $\{\{\langle l, s_k s_{k-1} \ldots s_{l+1} s_{l-1} \ldots s_0 \rangle, \langle s_k s_{k-1} \ldots s_{l+1} s_l s_{l-1} \ldots s_0 \rangle\} \mid 0 \leq l \leq k \text{ and } \forall i, 0 \leq s_i \leq n-1\}$.

Lemma 36. $\delta(\mathrm{BCube}_k(n)) = k + 1$.

Proof. By [63] we have that $\mathrm{diam}(\mathrm{BCube}_k(n)) = 2(k+1)$ and so, we conclude that $\delta(\mathrm{BCube}_k(n)) \leq k + 1$ by Lemma 2. Then, let us assume that $n = 2$ because we have by [63] that $\mathrm{BCube}_k(2)$ is an isometric subgraph of $\mathrm{BCube}_k(n)$. We define the automorphism σ satisfying that for all binary word $w \in \mathbb{Z}_2^{k+1}$, $\sigma(w) = \bar{w}$, and for every pair $< l, w > \in \{0, 1, \ldots, k\} \times \mathbb{Z}_2^k$, we have that $\sigma(< l, w >) = < l, \bar{w} >$. By [35,63] we have that $\min_u \mathrm{d}(u, \sigma(u)) = 2(k+1)$ and so, by noticing that σ is idempotent we can conclude by Corollary 28 that $\delta(\mathrm{BCube}_k(n)) \geq \delta(\mathrm{BCube}_k(2)) \geq k + 1$. $\square$

4.2.3 Tree-like networks

Definition 37 ([5]). Let $k \geq 4$ be even. The Fat-Tree_k is a graph with vertex-set that is partitioned into a core layer labeled with $\{0\} \times \mathbb{Z}_{(k/2)^2}$, an aggregation layer labeled with $\{1\} \times \mathbb{Z}_k \times \mathbb{Z}_{k/2}$, an edge layer labeled with $\{2\} \times \mathbb{Z}_k \times \mathbb{Z}_{k/2}$, and finally a server layer labeled with $\{3\} \times \mathbb{Z}_k \times \mathbb{Z}_{(k/2)^2}$. Its edge-set can be defined as follows:

- For any $0 \leq q, r < k/2$ the vertex labeled $\langle 3, k, (k/2)q + r \rangle$ in the server layer is adjacent to the vertex labeled $\langle 2, k, q \rangle$ in the edge layer.
- For every $0 \leq i \leq k-1$ there is a complete join between the subsets of vertices $\{\langle 1, i, j \rangle \mid 0 \leq j \leq k/2 - 1\}$ and $\{\langle 2, i, j \rangle \mid 0 \leq j \leq k/2 - 1\}$.
- Last, for every $0 \leq i \leq (k/2)^2 - 1$ the vertex labeled $\langle 0, i \rangle$ in the core layer is adjacent to all the vertices labeled $\langle 0, j, i \pmod{k}/2 \rangle$ in the aggregation layer, with $0 \leq j \leq k-1$.

An example of a Fat-Tree_4 is given in Figure 2.

Lemma 38. $\delta(\text{Fat-Tree}_k) = 2$.

Proof. By construction, every vertex in the edge layer is a pending vertex, that is a vertex of degree one. As a result, it can be ignored for the computation of hyperbolicity because the hyperbolicity of a graph is equal to the maximum hyperbolicity taken over all its biconnected components. We thus have that the efficient diameter of Fat-Tree_k is 4, hence $\delta(\text{Fat-Tree}_k) \leq 2$.

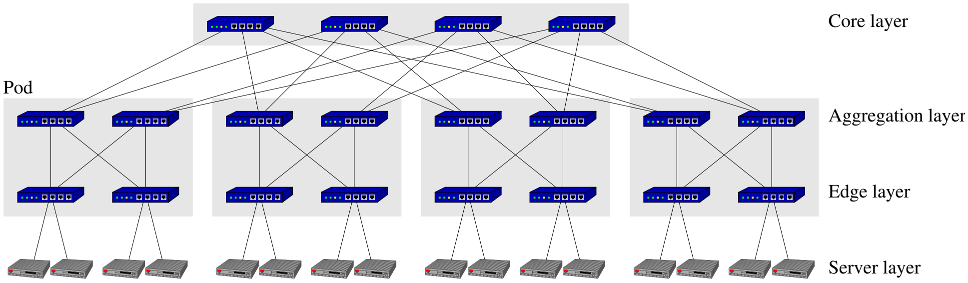


Figure 2: The graph Fat-Tree$_4$.

Furthermore, we also have by construction that Fat-Tree$_4$ is an isometric subgraph of Fat-Tree$_k$. So, let σ be the idempotent endomorphism of Fat-Tree$_4$ mapping: any vertex $\langle 0, i\rangle$ to the vertex $\langle 0, 3-i\rangle$ in the core layer; any vertex $\langle 1, i, j\rangle$ to the vertex $\langle 1, 3-i, 1-j\rangle$ in the aggregation layer, and in the same way any vertex $\langle 2, i, j\rangle$ to the vertex $\langle 2, 3-i, 1-j\rangle$ in the edge layer; last, any vertex $\langle 3, i, j\rangle$ to the vertex $\langle 3, 3-i, 3-j\rangle$ in the server layer. It can be hand-checked that $\min_u \mathrm{d}(u, \sigma(u)) = 4$ and so, we have by Corollary 28 that $\delta\,(\text{Fat-Tree}_k) \geq \delta\,(\text{Fat-Tree}_4) \geq 2$. $\square$

Definition 39 ([49]). The Butterfly graph $BF(n)$ is the graph with vertex-set $\{0, 1, \ldots, n\} \times \mathbb{Z}_2^n$ and with edge-set $\{\{\langle i, w\rangle, \langle i+1, w'\rangle\} \mid 0 \leq i \leq n-1 \text{ and } \forall j \neq i, w_j = w'_j\}$.

Lemma 40. $\delta\,(BF(n)) = n$.

Proof. Let w and w' be two binary words of length n and let i_1 and i_l be respectively the least and the largest index in which they differ. Then it can be checked that for any integer i we have that $\mathrm{d}_{BF(n)}(\langle i, w\rangle, \langle i, w'\rangle) = 2(i_l - i_1)$. As a result, the endomorphism σ mapping any vertex $\langle i, w\rangle$ to the vertex $\langle i, \bar{w}\rangle$ satisfies that $\min_u \mathrm{d}(u, \sigma(u)) = 2n$. Since we have that σ is idempotent then it follows from Corollary 28 that $\delta\,(BF(n)) \geq n$. Last, we also have that $\mathrm{diam}\,(BF(n)) = 2n$, hence $\delta\,(BF(n)) \leq n$ by Lemma 2. $\square$

In the literature, the edge-set of the Butterfly network is sometimes defined as $\{\{\langle i, w\rangle, \langle i+1 \pmod{n}, w'\rangle\} \mid 0 \leq i \leq n \text{ and } \forall j \neq i, w_j = w'_j\}$ [29], and this definition is also known as the wrapped Butterfly network. It modifies the diameter of the topology from $2n$ to $n + \lfloor n/2 \rfloor$, and the distance between any two vertices $\langle i, w\rangle, \langle i, \bar{w}\rangle$ from $2n$ to n. As a result, using the same arguments as for Lemma 40 we obtain for the wrapped Butterfly graph an upper-bound $\lfloor (n + \lfloor n/2 \rfloor)/2 \rfloor$ and a lower-bound $\lfloor n/2 \rfloor$ for its hyperbolicity.

Definition 41 ([57]). The k-ary n-fly is the graph with vertex-set $\{0, 1, \ldots, n\} \times \mathbb{Z}_k^n$ and with edge-set $\{\{\langle i, w\rangle, \langle i+1, w'\rangle\} \mid 0 \leq i \leq n-1 \text{ and } \forall j \neq i, w_j = w'_j\}$.

We can observe that the Butterfly graph $BF(n)$ is isomorphic to the 2-ary n-fly.

Lemma 42. *The k-ary n-fly is n-hyperbolic.*

Proof. By [57], the diameter of the k-ary n-fly is $2n$ and so, it has hyperbolicity bounded from above by n by Lemma 2. Moreover, by construction it contains the Butterfly graph $BF(n)$ as an isometric subgraph and so, it has hyperbolicity at least n by Lemma 40. $\square$

Definition 43 ([57]). The k-ary n-tree is the graph with vertex-set $\mathbb{Z}_k^n \cup \left(\{0, 1, \ldots, n-1\} \times \mathbb{Z}_k^{n-1}\right)$ and with edge-set $\{\{w \cdot b, \langle n-1, w\rangle\} \mid w \in \mathbb{Z}_k^{n-1}, b \in \mathbb{Z}_k\} \cup \{\{\langle i, w\rangle, \langle i+1, w'\rangle\} \mid 0 \leq i \leq n-2 \text{ and } \forall j \neq i, w_j = w'_j\}$.

Lemma 44. *The k-ary n-tree is $(n-1)$-hyperbolic.*

Proof. By construction, the biconnected components of the k-ary n-tree are composed of one single-vertex graph for each vertex $w \in \mathbb{Z}_k^n$, and of the k-ary $(n-1)$-fly. Since the hyperbolicity of the graph is equal to the maximum hyperbolicity taken over its biconnected components, then it follows from Lemma 42 that the k-ary n-tree is $(n-1)$-hyperbolic. $\square$

Definition 45 ([29]). The d-ary tree grid $MT(d,h)$ is a graph whose vertices are labeled with the pairs of words $< u, v >$ over an alphabet of size d and such that $\max\{|u|, |v|\} = h$. Any two vertices $\langle u, v\rangle$ and $\langle u', v'\rangle$ are adjacent in $MT(d,h)$ if and only if there is some letter λ such that: either $|u| = h$, $u = u'$ and $v = v' \cdot \lambda$; or $|v| = h$, $v = v'$ and $u' = u \cdot \lambda$.

Lemma 46. $\delta\left(MT(d,h)\right) = 2h$.

Proof. By [29] we have that $diam\left(MT(d,h)\right) = 4h$ and so, $\delta\left(MT(d,h)\right) \leq 2h$. Furthermore, we have that $MT(2,h)$ is an isometric subgraph of $MT(d,h)$ by construction. Let σ be the idempotent endomorphism of $MT(2,h)$ mapping any vertex $< u, v >$ to the vertex $\langle \bar{u}, \bar{v}\rangle$. By construction we have that for any vertex $\langle u, v\rangle \in V\left(MT(2,h)\right), \mathrm{d}_{MT(2,h)}(\langle u, v\rangle, \langle \bar{u}, \bar{v}\rangle) = 4h$ and so, we conclude by Corollary 28 that $\delta\left(MT(d,h)\right) \geq \delta\left(MT(2,h)\right) \geq 2h$. $\square$

4.2.4 Symmetric networks and Cayley graphs

Let $(\Gamma, \cdot)$ be a group and let S be a generating set of Γ that is symmetric, *i.e.*, $S = S^{-1}$. The *Cayley graph* $G(\Gamma, S)$ of group Γ w.r.t. S has vertex-set Γ and edge-set $\{\{g, g\cdot s\} \mid g \in \Gamma, s \in S\}$. It is well-known that every Cayley graph is vertex-transitive [4]. Furthermore, it has been shown (see for instance Exercise 2.4.14 in [29]) that the cube connected cycle $CCC(n)$ and the Butterfly graph $BF(n)$ are Cayley graphs.

Lemma 47. *Let $(\Gamma, \cdot)$ be a commutative group and S be a symmetric generating set. If $G(\Gamma, S)$ is not a clique, then we have that $\delta\left(G(\Gamma, S)\right) \geq \frac{1}{2}\left\lceil \frac{\mathrm{diam}(G(\Gamma,S))}{2}\right\rceil$.*

Proof. Let $id_\Gamma, g \in \Gamma$ be such that id_Γ is the neutral element of group Γ and $\mathrm{d}(id_\Gamma, g) = \mathrm{diam}\left(G(\Gamma, S)\right) = D > 1$. The mapping $\sigma : u \to g\cdot u$ is an endomorphism satisfying that $\forall u \in \Gamma, \mathrm{d}(u, \sigma(u)) = \mathrm{d}(id_\Gamma, u^{-1}\cdot g \cdot u) = \mathrm{d}(id_\Gamma, g) = D$. Therefore, we can conclude by Theorem 26 that $\delta\left(G(\Gamma, S)\right) \geq \lceil D/2\rceil/2$. $\square$

Definition 48 ([4]). The Bubble-sort graph $BS(n)$ has vertex-set the n-element permutations, that is $\{\phi_1\phi_2\ldots\phi_i\ldots\phi_n \mid \{\phi_1,\ldots,\phi_n\} = \{1,\ldots,n\}\}$. The edge-set of $BS(n)$ is $\{\{\phi_1\ldots\phi_i\phi_{i+1}\ldots\phi_n, \phi_1\ldots\phi_{i+1}\phi_i\ldots\phi_n\} \mid 1 \leq i \leq n-1\}$.

Lemma 49. $\delta(BS(n)) = \left\lfloor \frac{n(n-1)}{4}\right\rfloor$.

Proof. Let σ be the idempotent endomorphism mapping any vertex $\phi_1\phi_2\ldots\phi_i\ldots\phi_n$ to $\phi_n\ldots\phi_{n-i+1}\ldots\phi_2\phi_1$. By [4] all pairs $(u, \sigma(u))$ are diametral pairs and so, we can conclude by Corollary 28 that $\delta(BS(n)) \geq \lfloor\mathrm{diam}(BS(n))/2\rfloor$. As a result, we have by Lemma 2 that $\delta(BS(n)) = \lfloor\mathrm{diam}(BS(n))/2\rfloor$. Furthermore we have that $\mathrm{diam}(BS(n)) = \binom{n}{2}$ by [4]. $\square$

Definition 50 ([47]). The Transposition graph $T(n)$ has vertex-set the n-element permutations and edge-set $\{\{\phi_1\phi_2\ldots\phi_i\ldots\phi_j\ldots\phi_{n-1}\phi_n, \phi_1\phi_2\ldots\phi_j\ldots\phi_i\ldots\phi_{n-1}\phi_n\} \mid 1 \leq i, j \leq n\}$.

Lemma 51. $\frac{1}{2}\left\lceil\frac{n-1}{2}\right\rceil \leq \delta\left(T(n)\right) \leq \left\lfloor\frac{n-1}{2}\right\rfloor$.

Proof. By [47] the diameter of $T(n)$ is $n-1$ and so, $\delta(T(n)) \leq \lfloor (n-1)/2 \rfloor$ by Lemma 2. Moreover, let σ be the endomorphism mapping any vertex $\phi_1\phi_2\ldots\phi_i\ldots\phi_{n-1}\phi_n$ to $\phi_2\phi_3\ldots\phi_{i+1}\ldots\phi_n\phi_1$. Again by [47] all pairs $(u,\sigma(u))$ are diametral pairs and so, we can conclude by Theorem 26 that $\delta(S(n)) \geq \lceil n-1/2 \rceil /2$. □

Definition 52 ([4]). The star graph $S(n)$ has vertex-set the n-element permutations and edge-set $\{\{\phi_1\ldots\phi_{i-1}\phi_i\phi_{i+1}\ldots\phi_n, \phi_i\ldots\phi_{i-1}\phi_1\phi_{i+1}\ldots\phi_n\} \mid 2 \leq i \leq n\}$.

Lemma 53. $\left\lfloor \frac{1}{2}\left\lfloor \frac{3(n-1)}{2} \right\rfloor - \frac{1}{2} \right\rfloor \leq \delta(S(n)) \leq \left\lfloor \frac{1}{2}\left\lfloor \frac{3(n-1)}{2} \right\rfloor \right\rfloor$.

Proof. By [4] the diameter of $S(n)$ is $\lfloor 3(n-1)/2 \rfloor$ and so, $\delta(S(n)) \leq \lfloor \lfloor 3(n-1)/2 \rfloor /2 \rfloor$ by Lemma 2. Then, given $\phi = \phi_1\phi_2\ldots\phi_i\ldots\phi_{n-1}\phi_n$, let ψ be the unique n-element permutation satisfying that $\psi_{n-2j} = \phi_{n-2j-1}, \psi_{n-2j-1} = \phi_{n-2j}$ for every $0 \leq j \leq \lfloor (n-1)/2 \rfloor - 1$. Again by [4] we have that $\mathrm{d}(\psi,\phi) \geq \lfloor 3(n-1)/2 \rfloor - (n+1 \pmod 2) \geq \lfloor 3(n-1)/2 \rfloor - 1$. Moreover it can be checked that the mapping $\sigma : \psi \to \phi$ is an idempotent endomorphism of $S(n)$. Therefore, we have by Corollary 28 that $\delta(S(n)) \geq \lfloor \lfloor 3(n-1)/2 \rfloor /2 - 1/2 \rfloor$. □

5 Relations between hyperbolicity and some graph operations

Our results so far are heavily focused on the so-called *homogeneous* data center networks. By contrast, heterogeneous data centers are based on the composition of homogeneous data center topologies through graph operations. We survey a few of these operations so that we can study the impact that they may have on the hyperbolicity of the network.

5.1 Biswap operation and biswapped networks

Definition 54 ([65]). Let G be a graph. The biswapped graph $Bsw(G)$ has vertex set $\{0,1\} \times V(G) \times V(G)$. Two vertices (b,u,v) and (b',u',v') are adjacent if, and only if either $b = b'$, $u = u'$ and $\{v,v'\} \in E(G)$, or $b = \bar{b'} = 1-b'$, $u = v'$, and $u' = v$.

Lemma 55. *For any connected graph G, we have that $\delta(Bsw(G)) = \mathrm{diam}(G) + 1$.*

Proof. By [65] we have $\mathrm{diam}(Bsw(G)) = 2 \cdot \mathrm{diam}(G) + 2$ and so, $\delta(Bsw(G)) \leq \mathrm{diam}(G) + 1$ by Lemma 2. To prove the lower-bound, let $u,v \in V(G)$ be such that $\mathrm{diam}(G) = \mathrm{d}_G(u,v)$. We define $\overrightarrow{x_1} = (0,u,v)$, $\overrightarrow{x_2} = (0,v,u)$, $\overrightarrow{x_3} = (1,u,u)$ and $\overrightarrow{x_4} = (1,v,v)$. We deduce from [65] that:

$$\begin{aligned}
S_1 &= \mathrm{d}(\overrightarrow{x_1},\overrightarrow{x_2}) + \mathrm{d}(\overrightarrow{x_3},\overrightarrow{x_4}) = 2(2\,\mathrm{d}_G(u,v) + 2) \\
S_2 &= \mathrm{d}(\overrightarrow{x_1},\overrightarrow{x_3}) + \mathrm{d}(\overrightarrow{x_2},\overrightarrow{x_4}) = 2(\mathrm{d}_G(u,v) + 1) \\
S_3 &= \mathrm{d}(\overrightarrow{x_1},\overrightarrow{x_4}) + \mathrm{d}(\overrightarrow{x_2},\overrightarrow{x_3}) = 2(\mathrm{d}_G(u,v) + 1)
\end{aligned}$$

As a result, $\delta(Bsw(G)) \geq \delta(\overrightarrow{x_1},\overrightarrow{x_2},\overrightarrow{x_3},\overrightarrow{x_4}) = \mathrm{d}_G(u,v) + 1 = \mathrm{diam}(G) + 1$. □

What Lemma 55 proves is that the biswap operation always makes the hyperbolicity of the resulting graph scale with its diameter, *regardless of the topology that is used for the operation.*

5.2 Generic Cayley construction

Let us now consider the following transformation of a Hamiltonian graph, and the consequences of it on the hyperbolicity.

Lemma 56. *Let G be a Hamiltonian graph and c be a positive integer. We construct a graph G' from G by replacing every edge in some Hamilton cycle of G with a path of length c. Then it holds that $\delta(G') \geq \frac{1}{2}\left\lceil \frac{c-1}{2} \right\rceil$.*

Proof. Let P be a path of length c added by the construction, let x and y be the endpoints of P, and let P' be a xy-shortest-path in $G' \setminus (P \setminus \{x, y\})$. We have that the union of P with P' is an isometric cycle and so, it has length upper-bounded by $4 \cdot \delta(G') + 3$ by [25]. Moreover, we have that the length of P' is at least 2 because $\{x, y\}$ is an edge of G by the hypothesis. Thus it comes that the length of the cycle is at least $c + 2$ and so, $c \leq 4 \cdot \delta(G') + 1$. □

The Cayley model in [66] aims to apply the construction defined in Lemma 56 to some Hamiltonian graph G of order N, with $c = \Omega(\log N)$ and so that the diameter of the resulting graph G' is $O(\log N)$. Summarizing, we get.

Theorem 57. *Graphs in the Cayley model have hyperbolicity $\Theta(\log N)$, which scales linearly with their diameter.*

6 Conclusion

We proved in this work that the topologies of various interconnection networks have their hyperbolicity that scales linearly with their diameter. This property is inherent to any graph having desired properties for data centers such as a high-level of symmetry. Interestingly, symmetries are a common way to minimize network congestion whereas it was shown in [41], using a simplified model, that a bounded hyperbolicity might explain the congestion phenomenon observed in some real-life networks. Therefore, we let open whether a more general relationship between congestion and hyperbolicity can be determined.

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