

# Stationary Graph Signals using an Isometric Graph Translation

Benjamin Girault

▶ **To cite this version:**

Benjamin Girault. Stationary Graph Signals using an Isometric Graph Translation. 23rd European Signal Processing Conference (EUSIPCO), Aug 2015, Nice, France. pp.1531-1535, 10.1109/EUSIPCO.2015.7362637 . hal-01155902

**HAL Id: hal-01155902**
**https://inria.hal.science/hal-01155902**

Submitted on 4 Sep 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STATIONARY GRAPH SIGNALS USING AN ISOMETRIC GRAPH TRANSLATION

*Benjamin Girault*

École Normale Supérieure de Lyon, Université de Lyon
IXXI – UMR 5668 (CNRS – ENS Lyon – UCB Lyon 1 – Inria)

## ABSTRACT

In this communication, we extend the concept of stationary temporal signals to stationary graph signals. We introduce the concept of strict sense stationary and wide sense stationary graph signals as a statistical invariance through an isometric graph translation. Using these definitions, we propose a spectral characterisation of WSS graph signals allowing to study stationarity using only the spectral components of a graph signal.

Finally, we apply this characterisation a synthetic graph in order to study a few important stochastic graph signals. Also, using geographic data, we study weather readings on the graph of weather stations and show evidence of stationarity in the temperature readings.

***Index Terms—*** Signal processing on graphs, stationary signals

## 1. INTRODUCTION

The evolution of computing capabilities or techniques have produced an avalanche of data, and more importantly structured data. Social networks connecting people, technological networks connecting computers or smart devices, or geographic data are a few examples among many.

The emerging field of signal processing aims at tackling the problem of studying these data using the powerful approach of signal processing. The question at hand is then how to extend successes obtained by signal processing on temporal data or images to the less ordered, more complex, yet very rich graph structures. Recent successes include, but are not limited to, filtering [1], short-time Fourier transform and wavelets [2], empirical mode decomposition [3], or incertitude principle [4].

This communication aims at being part of this journey to a powerful signal processing on graphs. We propose a definition of stationary graph signals, a class of signal that proved to be crucial in the context of temporal data [5]. A spectral characterisation of those signals completes the definition. Finally we apply those to a few illustrative applications.

Work partially funded by Labex MILYON.

## 2. BACKGROUND

### 2.1. Stationary Time Signals

We denote $\mathbf{x}(t)$ a stochastic signal. Throughout this communication, variables written with a bold font are stochastic variables. Among the wide class of stochastic signals, one subclass is of particular interest: the stationary signals. The initial definition is that of *Strict-Sense Stationary* (SSS) signals verifying:

$$\forall t, \forall s, \quad \mathbf{x}(t) \overset{\mathrm{d}}{=} \mathbf{x}(s), \tag{1}$$

*i.e.* the probability density functions of $\mathbf{x}(t)$ and $\mathbf{x}(s)$ are equal.

This property is in practice impossible to verify since it involves the computation of all statistical moments. We usually are satisfied with a less strict definition named *Wide Sense Stationary* (WSS). A signal is WSS if its first two statistical moments $\eta_{\mathbf{x}}(t) = \mathbb{E}[\mathbf{x}(t)]$ and $R_{\mathbf{x}}(t,s) = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^*(s)]$ verify:

$$\eta_{\mathbf{x}}(t) = \eta_{\mathbf{x}} \tag{2}$$

$$R_{\mathbf{x}}(t,s) = \gamma_{\mathbf{x}}(t-s). \tag{3}$$

The first order property (2) is equivalent to the mean being the Direct Current (DC) component of the signal. In the second order property (3), the function $\gamma$ is called the *autocorrelation function* of the signal. In the context of WSS time series, $R$ is a Toeplitz matrix with $R_{ij} = \gamma(t_i - t_j)$.

WSS signals admit a harmonic decomposition, *i.e.* they can be written as the Fourier-Stieljes integral:

$$\mathbf{x}(t) = \int_{-\infty}^{\infty} e^{-\imath\omega t} d\mathbf{X}(\omega), \tag{4}$$

with $d\mathbf{X}(\omega)$ a probability measure on the spectral decomposition of the signal named the *spectral increments*. Stationarity is then characterised by a *double orthogonality* [5]: The exponential basis is orthogonal, and the spectral increments are orthogonal:

$$\int_{-\infty}^{\infty} e^{\imath\omega t} e^{-\imath\omega' t} dt = 2\pi\delta(\omega - \omega') \tag{5}$$

$$\mathbb{E}[d\mathbf{X}(\omega) d\mathbf{X}(\omega')^*] = 2\pi\delta(\omega - \omega')\Gamma(\omega) d\omega d\omega', \tag{6}$$

$\Gamma(\omega)$ being the Fourier transform of $\gamma(t)$, and called the *power spectrum density*. Stochastic signal analysis of $\mathbf{x}$ of-

ten involves the study of $\gamma(\tau)$ and $\Gamma(\omega)$ making these two functions crucial to the field [6].

Both definitions of stationarity naturally invoke the time shift operator. Let $T_\tau$ be the time shift operator acting on a signal $x$ as $T_\tau\{x\}(t) = x(t-\tau)$. The SSS property reduces to a statistical invariance of a stochastic signal with respect to $T_\tau$, while WSS is the invariance of the first two moments ($\eta_{\mathbf{x}} = \eta_{T_\tau\{\mathbf{x}\}}$ and $R_{\mathbf{x}} = R_{T_\tau\{\mathbf{x}\}}$).

In other words, given a time instant, the statistical properties of the signal and its translated version are statistically the same at that instant. Also, the samples of the signal at all time instants are statistically the same. For temporal signals those two properties are equivalent, but are worth noticing since only one of them will remain true for graph signals.

Moreover, given $e_\omega(t) = e^{\imath\omega t}$ the Fourier mode of (angular) frequency $\omega$, the expression of the time shift in the frequency domain is given by:

$$T_\tau\{e_\omega\}(t) = e^{\imath\tau\omega} e_\omega(t). \tag{7}$$

The Fourier modes are eigenvectors of the time shift operator. Therefore, the time shift behaves as a phase shifting operator in the frequency domain.

We refer the interested reader to [5] and [6] for details on stationarity.

### 2.2. Signal Processing on Graphs

We now introduce the field of signal processing on graphs. Let $\mathcal{G} = (V, E)$ be a *graph*, with $V$ the set of *vertices* and $E$ the set of *edges* $ij$. We suppose the graphs to be *symmetric*, *i.e.* $ij \in E$ if and only if $ji \in E$. Let $A$ be the *adjacency* matrix of $\mathcal{G}$, with $a_{ij}$ the *weight* of the edge $ij$. Let $D$ be the diagonal matrix of *degrees* $d_{ii} = \sum_j a_{ij} = d_i$. Since $\mathcal{G}$ is symmetric, $A$ is a symmetric matrix. We define then $L = D - A$ the *Laplacian matrix* of $\mathcal{G}$. $L$ is a semi-definite positive matrix. Let $L\chi_l = \lambda_l\chi_l$ be the eigendecomposition of $L$, with $0 = \lambda_0 \leq \lambda_1 \leq \dots$ [7]. We will suppose the graph to be connected such that $\lambda_1 > 0$ [7].

We call a *graph signal* a function assigning values to vertices: $X : V \to \mathbb{R}$ or $\mathbb{C}$. Most often, the *Fourier transform* is defined as the projection on the basis of eigenvectors of $L$ [2]. Let $F$ be the matrix of this Fourier transform. We have $L = F^*\Lambda F$, with $\Lambda = \mathrm{diag}(\lambda_0, \dots, \lambda_{N-1})$. Since $L$ is semi-definite positive, $\{\chi_l\}_l$ is orthogonal and the matrix $F$ is unitary. We have an equivalence of Parseval's identity: $\|X\|_2 = \|FX\|_2$, with $\|X\|_2^2 = \sum_i |X_i|^2$ the $l_2$-norm. We denote $\widehat{X} = FX$ the Fourier transform of a signal $X$.

Let $H$ be a linear operator on graph signals. We use the notation $H$ to denote both the operator and its matrix representation. Its action is that of $X \mapsto HX$, with $X$ represented as a column vector. We denote $\widehat{H} = FHF^*$ the expression of $H$ in the Fourier domain.

## 3. STATIONARITY FOR GRAPH SIGNALS

Stochastic analysis for temporal signals have produced very interesting results, and we naturally wish to leverage the full potential of the statistics of graph signals to analyse them. Naturally, we need definitions, concepts and associated characterisations for that purpose. As seen in the previous section, stationnarity is one of those important concepts and we propose to define its counterpart for graph signals.

For temporal signals, stationarity can be interpreted as the statistical invariance by time shifting. There exists in the literature two operators defining equivalents of the time shift for graph signals. The generalized translations in [2] define operators acting as generalized convolution by a delta centred on vertex $i$ whose expression in the Fourier domain is $\widehat{T_iX}(l) = \sqrt{N}\widehat{\delta}_i(l)\widehat{X}(l)$. These operators, when applied to very specific input signals, localise their input around vertex $i$. The authors of [1] proposed the graph shift as the matrix multiplication $X \mapsto AX$. The graph shift is then an operator diffusing a sample from one vertex to its neighbours according to the edge weights.

Unfortunately, neither operator is isometric with respect to the $l_2$-norm, *i.e.* $\|T_iX\|_2 \neq \|X\|_2$ and $\|AX\|_2 \neq \|X\|_2$. As opposed to the time shift, these two operators lack the mathematical comfort of isometry to define statistical invariance. In another publication, we proposed an alternative time-shift-like operator for graph signals which is isometric by design. After recalling its definition, we use it to propose a definition of stationary graph signals.

### 3.1. Graph Translation

In [8] we proposed the graph translation as a new operator defined by analogy to the time shift for temporal signals. To this end, the graph translation is defined as an isometric operator shifting the phase of each of the spectral components of the input, similarly to (7). More precisely, we define graph frequencies as:

$$\omega_l = \pi\sqrt{\lambda_l/\rho_\mathcal{G}}, \tag{8}$$

where $\rho_\mathcal{G}$ is an upper bound on the eigenvalues of $L$ [9]:

$$\rho_\mathcal{G} = \max_{i\in V} \sqrt{2d_i(d_i + \bar{d}_i)} \quad \text{with} \quad \bar{d}_i = \frac{\sum_{ij\in E} w_{ij}d_j}{d_i}.$$

The graph frequencies lie in $[0, \pi]$, and a small $\omega_l$ corresponds to a Fourier mode $\chi_l$ of lower frequency.

The graph translation is then defined as a phase shifting operator on the Fourier modes, *i.e.* such that it verifies $T_\mathcal{G}\chi_l = e^{\imath\omega_l}\chi_l$:

**Definition 1** (Graph Translation)**.** *The* graph translation *operator is defined using its matrix representation as:*

$$T_\mathcal{G} = \exp\left(\imath\pi\sqrt{L/\rho_\mathcal{G}}\right).$$

Details on the construction, justification of the formula, and comparison to the generalized translations and graph shift can be found in [8].

### 3.2. Definition of stationary signals

We now propose to define stationarity for graph signals as the statistical invariance to the graph translation:

**Definition 2** (Strict-Sense Stationary)**.** *A stochastic signal* $\mathbf{X}$ *on the graph* $\mathcal{G}$ *is* Strict-Sense Stationary *(SSS) if and only if:*

$$\mathbf{X} \overset{d}{=} T_{\mathcal{G}}\mathbf{X}. \tag{9}$$

Def. 2 is tractable thanks to the isometric nature of the graph translation. However, as for temporal signals, SSS is difficult to verify in practice, and we introduce a weaker, more practical definition of stationarity:

**Definition 3** (Wide-Sense Stationary)**.** *A stochastic signal* $\mathbf{X}$ *on the graph* $\mathcal{G}$ *is* Wide-Sense Stationary *(WSS) if and only if:*

$$\mathbb{E}[\mathbf{X}] = \mathbb{E}[T_{\mathcal{G}}\mathbf{X}] \tag{10}$$

$$\mathbb{E}[\mathbf{X}\mathbf{X}^*] = \mathbb{E}[(T_{\mathcal{G}}\mathbf{X})(T_{\mathcal{G}}\mathbf{X})^*]. \tag{11}$$

The invariance property is interpreted as follows: Given a vertex $i \in V$, the random variables $\mathbf{X}_i$ and $(T_{\mathcal{G}}\mathbf{X})_i$ are statistically equal (SSS) or have identical moments of the first and second orders (WSS). However, in contrast to temporal signals, given two different vertices $i$ and $j$, there is in general no equality in law between $\mathbf{X}_i$ and $\mathbf{X}_j$.

The definition of WSS signals is introduced in the vertex domain. As for temporal signals, we propose a dual characterisation of WSS signals in the spectral domain. We begin by the first moment. Let $\eta = \mathbb{E}[X]$ be the mean of the graph signal. Eq. 10 is then equivalent to $\eta = T_{\mathcal{G}}\eta$. In other words, $\eta$ is an eigenvector of $T_{\mathcal{G}}$ associated to the eigenvalue $1 = e^0 = e^{\omega_0}$. Since $\lambda_1 > 0$, $\eta$ is collinear to $\chi_0$: The vector of mean is collinear to the Fourier mode of frequency $\omega_0 = 0$. As for temporal signals, the mean of the signal is the equivalent of its DC component.

Next, we characterise the second moment. Let $R = \mathbb{E}[\mathbf{X}\mathbf{X}^*]$ be the autocorrelation matrix of the signal. Eq. 11 gives then:

$$R = \mathbb{E}[(T_{\mathcal{G}}\mathbf{X})(T_{\mathcal{G}}\mathbf{X})^*] = T_{\mathcal{G}}\mathbb{E}[\mathbf{X}\mathbf{X}^*]T_{\mathcal{G}}^* = T_{\mathcal{G}}RT_{\mathcal{G}}^*.$$

Let $S = \mathbb{E}[(F\mathbf{X})(F\mathbf{X})^*]$ be the autocorrelation matrix of the Fourier transform of the signal. By linearity of the Fourier transform, we obtain $S = FRF^*$ and $R = F^*SF$. The equality above becomes:

$$S = \widehat{T_{\mathcal{G}}}S\widehat{T_{\mathcal{G}}}^*. \tag{12}$$

Assuming that all eigenvalues of $L$ are distinct, then (12) is verified if and only if $S$ is diagonal. In other cases, $S$ diagonal is a sufficient, but not necessary, condition. We can now formally write the spectral characterisation of WSS signals:

**Proposition 1** (Second Moment Characterisation)**.** *The second moment of a graph signal is invariant through graph translation if its spectral components are uncorrelated (sufficient condition). This condition is also necessary if all graph frequencies are distinct.*

The eigenvalue uniqueness condition is actually verified on many real world graphs. *i.e.* when weights depend on measurements. This comes from the fact that a small random perturbation of weights will slightly change the eigenvalues making them easily unequal. On the contrary, synthetic graphs showing high regularity, such as cycles, or regular grids with unit weights have many multiple eigenvalues [7].

Prop. 1 is similar to the spectral characterisation of WSS time series (*resp.* WSS signals) where the spectral components (*resp.* the spectral increments) are uncorrelated. We have then a doubly orthogonal decomposition.

### 3.3. Discussion

We remark first that the operator $T_{\mathcal{G}}$ is in general a complex operator, meaning that if $\mathbf{X}$ is real-valued, then $T_{\mathcal{G}}\mathbf{X}$ is usually complex. For WSS signals, this is not an issue since Prop. 1 shows that only the correlation between spectral components matters: A signal can be both WSS and real, and both sides of (11) are real for WSS signals.

We consider now the SSS property in the particular case of real-valued signals. Let $\mathbf{X}$ be such a signal. Def. 2 and the Fourier transform give then:

$$\forall i, \forall x, \mathbb{P}[\mathbf{X}_i = x] = \mathbb{P}[(T_{\mathcal{G}}\mathbf{X})_i = x],$$

and by linear combination of the equalities above:

$$\begin{aligned} &\forall l, \forall \widehat{x}, \mathbb{P}[\widehat{\mathbf{X}}_l = \widehat{x}] = \mathbb{P}[(\widehat{T_{\mathcal{G}}\mathbf{X}})_l = \widehat{x}] \\ \Leftrightarrow \quad &\forall l, \forall \widehat{x}, \mathbb{P}[\widehat{\mathbf{X}}_l = \widehat{x}] = \mathbb{P}[e^{\imath\omega_l}\widehat{\mathbf{X}}_l = \widehat{x}]. \end{aligned}$$

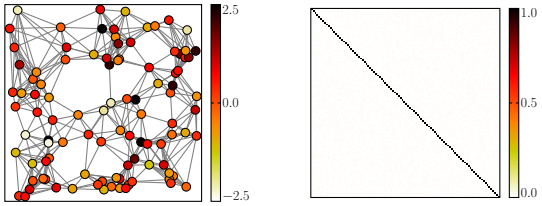
When $l \neq 0$, $e^{\imath\omega_l}\widehat{\mathbf{X}}_l$ is complex and $\widehat{\mathbf{X}}_l$ is real, such that the probabilities above are non zero only for $\widehat{x} = 0 = \widehat{\mathbf{X}}$. Therefore:

$$\forall l \neq 0, \mathbb{P}[\widehat{\mathbf{X}}_l = 0] = 1.$$

A real SSS signal is reduced to a DC component of random amplitude. This case corresponds to the deterministic relation $\mathbf{X} = T_{\mathcal{G}}\mathbf{X}$ since a DC component is invariant through the graph shift.

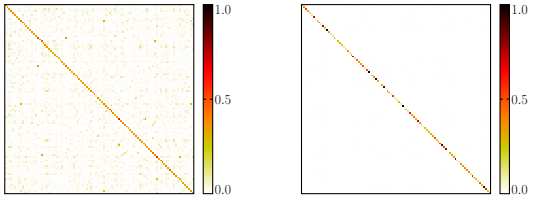
On the other hand, complex graph signals do not suffer from this constraint. This contrasts with temporal SSS signals that can be real without reducing to a DC component.

Another remark concerns the autocorrelation matrix $R$. We know that for temporal WSS signals, this matrix is Toeplitz. This is not the case for graph signals. This is to be expected since having such a structure would mean having an autocorrelation function $\gamma$ which is a function of the difference between vertex indices. In general such a difference does not make sense. For example, shifting all indices by one would leave the matrix $R$ unchanged. However, the graph




(a) A realisation. (b) Empirical $S$.

**Fig. 1**: White noise on a graph with $\sigma^2 = 1$ and zero mean. (a) shows a realisation of this white noise, and (b) the empirical spectral correlation matrix computed using 50k realisations.




(a) $R$. (b) $S$.

**Fig. 2**: Empirical second moments of a WSS signal with zero mean and non uniform spectral power with 50k realizations.

translation does not perform such a shift such that it is unlikely that the signal is invariant to both the graph translation and this vertex index shifting operation.

Next, the characterisation of the first order moment in the previous section is actually independent of the underlying graph structure. Indeed, $\chi_0$ being always a constant vector, the mean of a WSS signal is constant across vertices. However, several definitions of Fourier transform for graph signals exists, $F$ being one of them. There exists an alternative definition based on the *normalized Laplacian* $\mathcal{L} = D^{-1/2}LD^{-1/2}$ [2]. Both matrices share similar properties. In particular, they are both semi-definite positive. We denote $\psi_l$ the eigenvector associated to the eigenvalue $\mu_l$ of $\mathcal{L}$. Prop. 1 is still valid for the Fourier transform $\mathcal{F}$. However, and as opposed to $\chi_0$, $\psi_0$ is not constant and equals $(\sqrt{d_1}, \ldots, \sqrt{d_N})^T$. We denote $\mathcal{T}_\mathcal{G}$ the associated graph translation. We can then use this graph translation to obtain an alternative definition of stationarity. A connected graph verifies $\mu_1 > 0$ such that a WSS signal has a mean vector collinear to $\psi_0$. Also, SSS real-valued graph signals are now collinear to $\psi_0$, and not constant. Finally, results on the second moment can be adapted to $\mathcal{T}_\mathcal{G}$ directly.

The use of $F$ or $\mathcal{F}$ carries then a different notion of DC component, adapted or not to the local structural properties of the vertices. This accounts for the heterogeneity between vertices weighing differently in the graph structure. Using the Laplacian matrix or the normalized Laplacian matrix carries then different meanings, and using one or the other depends on the application at hand. The rest of this communication will use $F$ as the Fourier transform.

## 4. APPLICATIONS

In this section, we wish to apply the notion of stationary graph signals we introduced to concrete graph signals. First, using synthetic signals, and then to study a dataset and the stationarity of a real world signal.

### 4.1. Synthetic data

In this section, we study several simple stochastic signals. To that end, we use the graph of Fig. 1(a). This graph has 100 vertices randomly sampled in the 2 dimensional plane, each connected to its 8 nearest neighbours. The edge weights are defined by a Gaussian kernel of the Euclidean distance in the plane ($a_{ij} = \exp(-d(p_i, p_j))$).

We propose to define *white noise* on graphs as a stochastic signal having a flat power spectrum, *i.e.* with $S = \sigma^2 I_N$, with mean collinear to $\chi_0$. A realization of white noise with zero mean is shown on Fig. 1(a). Using Prop. 1, a white noise is a WSS signal. Since the Fourier matrix $F$ is unitary, we have also $R = \sigma^2 I_N$. Therefore, the samples on vertices are uncorrelated, and of equal variance.

We consider now a signal $\mathbf{X}$ with samples on the vertices independent and identically distributed (i.i.d.). Then, $R = \sigma^2 I_N = S$. $\mathbf{X}$ is therefore a white noise. The whole class of white noise is actually independent of the underlying graph structure. Indeed, the samples being uncorrelated, the edges play no role in explaining any correlation between them.

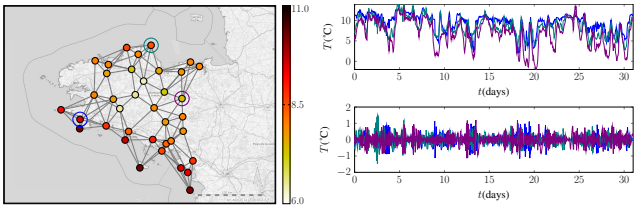
The third signal we consider is a WSS signal with non-constant spectral power, *i.e.* with $S$ diagonal but of non constant diagonal. Without loss of generality, we suppose the mean of the graph signal to be constant. Fig. 2 shows the empirical correlation matrices $R$ and $S$ obtained using 50k realisations. In general, such a signal has correlated samples, but in contrast to temporal signals, the second moment is in general not constant across vertices, *i.e.* the diagonal of $R$ is not constant. Therefore, the mere fact that the signal is WSS is not a guarantee that the samples have the same variance.

This is an illustration of the property that WSS depends on the underlying graph structure. Also, as soon as the signal shows some correlations between vertices, the edges of the graph contribute to explaining those correlations.

### 4.2. Real dataset

The French national meteorological service published in open access a dataset of hourly weather observations in Brittany for the month of January 2014[1]. From this dataset, we wish to study the temperature readings on the graph of ground weather stations. We define the graph of ground stations to be an image of the spatial distances between stations. Fig. 3(a) shows it. Each vertex (*i.e.* ground station) is connected to its five nearest neighbours, and the weights are a Gaussian

[1] https://www.data.gouv.fr/




(a) Mean temperature. (b) Temperature variations.

**Fig. 3**: Spatial and temporal variations of temperatures in Brittany in January 2014. (a) mean temperature of each station. (b) temporal variations of three stations circled in (a) on top, and first IMF of these time series below.

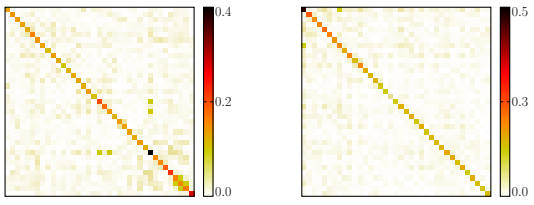
kernel (of parameter $10^{-5}$) of the spatial distance between ground stations. Note that these arbitrary parameters can be slightly changed without altering the results.

We wish to assess the stationarity of the temperature readings. Let $X_i(t)$ be the temperature at vertex $i$ at time $t$. Fig. 3(a) shows the mean temperature of each station. This mean is obviously not constant, hence not collinear to $\chi_0$. Therefore, we cannot model the readings $X_i(t)$ using a stationary graph signal of which the readings are realisations.

Fig. 3(b) shows the temperature variation over time of three ground stations. We observe that these time series have a significant trend, decomposed into seasonal variations and non-deterministic coarse scale variations. To attenuate this trend, we consider the first Intrinsic Mode Function (IMF) of the Empirical Mode Decomposition of each of the time series $X_i$ [10]. The first IMF goal is precisely to remove the trend of a time series. We denote the IMFs $Y_i$. We assume then that the graph signals $\{Y(t)\}_t$ are independent realisations of the stochastic graph signal $\mathbf{Y}$. Its first empirical moment is zero everywhere, since the time series $Y_i$ are centered.

Fig. 4 shows the empirical correlation matrices of $\mathbf{Y}$ and of its spectral decomposition $\widehat{\mathbf{Y}}$. The matrix $R_{\mathbf{Y}}$ shows some significant off-diagonal elements, suggesting significant spatial correlations between vertices. On the other hand, $S_{\mathbf{Y}}$ is close to a diagonal matrix, suggesting that $\mathbf{Y}$ has uncorrelated spectral component and is indeed a stationary graph signal.

We stress here that this assessment of stationarity is performed with respect to the graph structure we defined. This graph can be defined differently leading to another definition of Fourier transform, and eventually of stationarity.




(a) $|R_{\mathbf{Y}}|$. (b) $|S_{\mathbf{Y}}|$.

**Fig. 4**: Empirical spectral correlation matrices of $\mathbf{Y}$.

## 5. CONCLUSION

This communication introduces two definitions of stationary graph signals. In particular, our definition of WSS graph signals is simple yet rich enough to allow a tractable spectral characterisation. These definitions and their consequences share many properties with temporal signals, however, the more complex structural properties of graphs introduce fundamental differences with time, accounting for the heterogeneity of the structure.

Facing the still infancy of graph signal processing, our contribution brings in an important ingredient that will serve as the starting point of further development towards the constitution of an entire toolbox for analysing real-world measurements modelled as stochastic signals on graphs. A few examples in that toolbox include the study of more classes of stochastic signals, being stationary or not, optimal filtering, or the development of a stationarity test.

## REFERENCES

[1] A. Sandryhaila and J. M. F. Moura, "Discrete Signal Processing on Graphs," *IEEE Trans. on Sig. Pro.*, vol. 61, no. 7, pp. 1644–1656, 2013.

[2] D. I. Shuman et al., "The Emerging Field of Signal Processing on Graphs: Extending High-Dimensional Data Analysis to Networks and Other Irregular Domains.," *IEEE SP. Mag.*, vol. 30, no. 3, pp. 83–98, 2013.

[3] N. Tremblay, P. Borgnat, and P. Flandrin, "Graph empirical mode decomposition," in *Proceedings of EUSIPCO 2014*. IEEE, 2014, p. 2350–2354.

[4] A. Agaskar and Y. M. Lu, "A Spectral Graph Uncertainty Principle.," *IEEE Trans. on Inf. Th.*, vol. 59, no. 7, pp. 4338–4356, 2013.

[5] P. Flandrin, *Time-frequency/time-scale analysis*, Academic Press, Inc., 1999.

[6] A. Papoulis, *Probability, random variables, and stochastic processes*, McGraw-Hill, 3rd edition, 1991.

[7] A. E. Brouwer and W. H. Haemers, *Spectra of graphs*, Universitext. Springer, New York, 2012.

[8] B. Girault, P. Gonçalves, and E. Fleury, "Translation on Graphs: an Isometric Shift Operator," in preparation for submission to IEEE Signal Processing Letters.

[9] K. Ch. Das, "Extremal graph characterization from the bounds of the spectral radius of weighted graphs.," *App. Math. & Comp.*, vol. 217, no. 18, pp. 7420–7426, 2011.

[10] N. E. Huang et al., "The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis," *The Roy. Soc. of Lon. Proc. Series A. Math., Phys. and Eng. Sc.*, vol. 454, no. 1971, pp. 903–995, 1998.