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Abstract: The Mellin transform is an efficient tool to determine the behavior of a function at the neighbourhood of a point, in particular when the function admits a series expansion. This report aims at collecting some results related to this transform which turn out to be very useful when dealing with the behavior of the solution to the acoustic wave equation. We first recall the definition and some basic properties of the Mellin transform. Next, we explicit the relation between the behavior of a function in the the neighbourhood of the origin and the domain of analyticity of its Mellin transform.

Key-words: Mellin transform, Weighted Sobolev spaces, Singularity theory, Analyticity

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La Transformée de Mellin

Résumé : La transformée de Mellin est un des outils de la théorie des singularités qui permet de déterminer le comportement d'une fonction au voisinage d'un point. L'objectif de ce rapport est de rassembler dans un même document un certain nombre de résultats couramment utilisés et qui s'avèrent utiles quand on veut justifier le développement en série de la solution régulière de l'équation des ondes acoustiques. Nous rappelons d'abord la définition et quelques propriétés élémentaires de la transformation de Mellin. Puis nous explicitons le lien entre le comportement au voisinage de l'origine d'une fonction et le domaine d'analycité de sa transformée de Mellin.

Mots-clés : Transformée de Mellin, Espaces de Sobolev à poids, Théorie des singularités, Analycité

Introduction

The numerical simulation of transient waves in media including very small scatterers is an ongoing problem which deserves a particular attention since it is involved in many applications covering a wide range of applications like medical imaging which tries to detect very small tumors or aeronautics which uses gas turbine combustors including plenty of very small apertures generating scattering problems. It is well known that it is very difficult to determine the characteristics of an object whose dimensions are very small against the smallest wavelength. Numerical methods have been developed in the past but they mainly tackle the problem with harmonic waves. To the best of our knowledge, the case of time-dependent problems has not been considered before Vanessa Mattesi PhD thesis [3]. Obviously, the case of one single scatterer can be tackled by performing a direct numerical simulation but to provide both stability and accuracy to the calculations, a very high computational burden has to be spent. In particular, refinement of the grid in the proximity of the scatterer is mandatory and this implies high computational costs. That is why people develop approximate models which are based on asymptotic analysis to justify approximate problems which have lower computational rates. Regarding time-dependent problems, the main idea consists in replacing the small scatterer by a point source which amplitude is of a same order of magnitude than the scatterer size. By this way, the numerical simulation is performed as if there is no scatterer. That is a very good point because there is no longer necessary to refine the mesh in the vicinity of the scatterer.
Scattering problems involve two kinds of wave fields which differentiate into near and far fields. There is thus a need to account for the distance to the scatterer and two asymptotic expansions can be defined, each of them being justified in its own domain. Then, the construction of an approximate problem goes through the connection of the two expansions and the corresponding analysis is known as the technique of matched asymptotic expansions. In [3], it has been claimed that there exists a sequence $u_{m,n}$ depending on the variables r and t and a sequence $Z_{m,n}$ depending on the angular variables θ and ϕ such that the regular solution u of the acoustic wave equation :

$$\Delta u(\mathbf{x},t) = \frac{1}{c^2}\partial_t^2 u(\mathbf{x},t), \tag{1}$$

can be expanded as :

$$u(\mathbf{x},t) = \sum_{n=0}^{N} \sum_{m=-n}^{n} u_{m,n}(r,t) \times Z_{m,n}(\theta,\varphi) + \mathrm{u}^N(\mathbf{x},t). \tag{2}$$

There is then a need in proving that u^N actually satisfies :

$$\begin{cases} \max\limits_{t\leq T}|\mathrm{u}^N(\mathbf{x},t)| = \underset{r\to 0}{O}(r^{N+1}), \\ \max\limits_{t\leq T}|\partial_r \mathrm{u}^N(\mathbf{x},t)| = \underset{r\to 0}{O}(r^{N}). \end{cases} \tag{3}$$

We are thus faced to the question of analyzing a wave field given by a series and in that case, it is known that the Mellin transform turns out to be an efficient tool. This report aims at gathering useful properties of Mellin transforms that will be involved in a upcoming paper justifying (3).

1 Definition of the Mellin transform

Let $\lambda \in \mathbb{C}$ be the Mellin variable. It is a complex number defined as :

$$\lambda = \beta + i\xi \text{ with } \beta \in \mathbb{R} \text{ and } \xi \in \mathbb{R}. \tag{4}$$

Let $\mathcal{D}(]0,+\infty[)$ be the space of functions which have a compact support into $]0,+\infty[$, that is :

$$\mathcal{D}(]0,+\infty[) \;=\; \Big\{v :]0,+\infty[\longrightarrow \mathbb{R} :\; v(r)=0 \text{ out of } [r_-,r_+] \text{ with } 0<r_-<r_+\Big\}. \tag{5}$$

For any $v \in \mathcal{D}(]0,+\infty[)$, the Mellin transform is defined for every $\lambda \in \mathbb{C}$ by :

$$(\mathcal{M}v)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} r^{-\lambda} v(r) \frac{dr}{r}. \tag{6}$$

For any $\lambda = \beta + i\xi \in \mathbb{C}$ and any positive real r, we have $r^\lambda = r^\beta \; \exp(i\xi \ln(r))$.
The Mellin transform is connected to the Fourier transform which is defined for any $u \in \mathcal{D}(]-\infty,+\infty[)$ and $\xi \in \mathbb{R}$ by :

$$(\mathcal{F}u)(\xi) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x)\; \exp(-i\xi x)\, dx. \tag{7}$$

We indeed have for any $v \in \mathcal{D}(]0,+\infty[)$ and $\lambda = \beta + i\xi$:

$$(\mathcal{M}v)(\lambda) \;=\; (\mathcal{F}f_v)(\xi), \quad \text{with } f_v(x) = v(e^x)e^{-\beta x}. \tag{8}$$

The previous identity can be obtained by applying the change of variable $x = \ln(r)$ into the Mellin transform.

2 Functional space

From a density argument, the Mellin transform can be set in the space L^1_β, with $\beta \in \mathbb{R}$, which is a weighted space of functions $v\; :]0,+\infty[\,\to\, \mathbb{C}$ equipped with the norm

$$\|v\|_{L^1_\beta} = \int_0^{+\infty} r^{-\beta} |v(r)| \frac{dr}{r}. \tag{9}$$

Then for any $v \in L^1_\beta$, the Mellin transform $\mathcal{M}v$ is no more defined on the whole space $\mathbb{C}$ but on the straight line $\mathbb{C}_\beta$ with :

$$\mathbb{C}_\beta \;:=\; \Big\{\lambda \in \mathbb{C} : \mathcal{R}e(\lambda) = \beta\Big\}, \tag{10}$$

see Fig. 1. We then get :

Proposition 1. *Let $\beta \in \mathbb{R}$. For any $v \in L^1_\beta$, we have*

$$|(\mathcal{M}v)(\beta + i\xi)| \le \frac{1}{\sqrt{2\pi}} \|v\|_{L^1_\beta}, \quad \forall \xi \in \mathbb{R}. \tag{11}$$

Proof. First, $|r^{-i\xi}| = 1$. Then,

$$\left| \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} r^{-\beta - i\xi} v(r) \frac{dr}{r} \right| \le \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} |r^{-i\xi}|\; r^{-\beta}\; |v(r)| \frac{dr}{r} = \frac{\|v\|_{L^1_\beta}}{\sqrt{2\pi}}, \tag{12}$$

which completes the proof. □

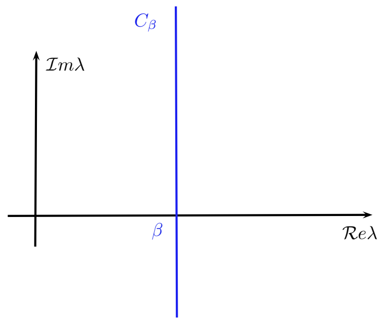

Figure 1: The imaginary straight line $\mathbb{C}_\beta$

Following the Parseval-Fourier identity, we obtain :

$$\forall u \in L^2(\mathbb{R}), \quad \mathcal{F}u \in L^2(\mathbb{R}) \quad \text{and} \quad \|u\|_{L^2(\mathbb{R})} = \|\mathcal{F}u\|_{L^2(\mathbb{R})}, \tag{13}$$

which shows that the Mellin transform defines an isomorphism from the weighted space K^0_β equipped with the norm :

$$\|v\|_{K^0_\beta} = \left(\int_0^{+\infty} r^{-2\beta} \big|v(r)\big|^2 \frac{dr}{r} \right)^{1/2}, \tag{14}$$

and the space $\widehat{K}^0_\beta$ of functions $\mathbb{C}_\beta \to \mathbb{C}$ equipped with the norm :

$$\|\omega\|_{\widehat{K}^0_\beta} = \left(\int_{-\infty}^{+\infty} \big|\omega(\beta + i\xi)\big|^2 d\xi \right)^{1/2}. \tag{15}$$

Proposition 2. *Let $\beta \in \mathbb{R}$. For any $v \in K^0_\beta$, we have $\mathcal{M}v \in \widehat{K}^0_\beta$ and*

$$\|v\|_{K^0_\beta} = \|\mathcal{M}v\|_{\widehat{K}^0_\beta}. \tag{16}$$

Proof. The result is a consequence of Parseval-Fourier theorem

$$\|\mathcal{M}v\|_{\widehat{K}^0_\beta} = \|\mathcal{F}f_v\|_{L^2(\mathbb{R})} = \|f_v\|_{L^2(\mathbb{R})} = \|v\|_{K^0_\beta}, \tag{17}$$

with $f_v(x) = e^{-\beta x} v(e^x)$. □

For more information on the Kondratiev spaces, we refer to as [2], [1] et [4].

3 Properties of the Mellin Transform

The Fourier transform modifies differential operators into multiplications. Regarding the Mellin transform, differential operators are changed into operators $\cdot^{\{\ell\}}$ defined by :

$$v^{\{\ell\}}(r) = \big(r d_r\big)^\ell v(r). \tag{18}$$

It is thus natural to introduce the Kondratiev spaces :

$$K_\beta^p \;=\; \Big\{ v : \mathbb{R}^+ \to \mathbb{C} \;\Big|\; v^{\{\ell\}} \in K_\beta^0, \quad \forall \ell \le p \Big\}, \tag{19}$$

where $\beta \in \mathbb{R}$ and p is into $\mathbb{N}$ and is associated to a weight. These spaces are equipped with the hilbertian norm :

$$\|v\|_{K_\beta^p} \;=\; \Big(\sum_{\ell=0}^{p} \big\|v^{\{\ell\}}\big\|^2_{K_\beta^0} \Big)^{\frac{1}{2}}. \tag{20}$$

We then have :

Property 1. *For any $\ell \in \mathbb{N}$ and $v \in \mathcal{D}(]0,+\infty[)$, the following identity is satisfied for all $\lambda \in \mathbb{C}$:*

$$(\mathcal{M}v^{\{\ell\}})(\lambda) = \lambda^\ell (\mathcal{M}v)(\lambda). \tag{21}$$

Proof. For any $\ell = 1$, we obtain by integrating by parts :

$$(\mathcal{M}v^{\{1\}})(\lambda) = \int_0^{+\infty} r^{-\lambda}(r d_r) v(r) \frac{dr}{r} = -\int_0^{+\infty} d_r(r^{-\lambda}) v(r) dr, \tag{22}$$

because $v \in \mathcal{D}(]0,+\infty[)$. Then, using that $dr\ (r^{-\lambda}) = -\lambda\, r^{-\lambda-1}$ we get :

$$(\mathcal{M}v^{\{1\}})(\lambda) = \lambda \int_0^{+\infty} r^{-\lambda} v(r) \frac{dr}{r} = \lambda(\mathcal{M}v)(\lambda). \tag{23}$$

For any $\ell > 1$, it is sufficient to apply a reasoning by induction. □

Now let us introduce the norm that is employed for $\omega \in \widehat{K}_\beta^p$:

$$\|\omega\|_{L^2(\mathbb{C}_\beta)} = \Big(\int_{-\infty}^{+\infty} \big|\omega(\beta + i\xi)\big|^2 d\xi \Big)^{\frac{1}{2}}. \tag{24}$$

The Hilbert space $\widehat{K}_\beta^p$ is equipped with the norm

$$\|\omega\|_{\widehat{K}_\beta^p} = \left(\sum_{\ell=0}^{p} \|\omega^{\{\ell\}}\|^2_{L^2(\mathbb{C}_\beta)} \right)^{1/2}. \tag{25}$$

Since $\mathcal{D}(]0,+\infty[)$ is dense into K_β^p, the following result holds.

Corollary 1. *For any $v \in K_\beta^p$, we have $v^{\{p\}} \in K_\beta^0$ and for almost all $\lambda \in \mathbb{C}_\beta$*

$$(\mathcal{M}v^{\{p\}})(\lambda) \;=\; \lambda^p \big(\mathcal{M}v\big)(\lambda). \tag{26}$$

Corollary 2. *The Mellin transform satisfies a Parseval identity reading as :*

$$\|\mathcal{M}u\|_{\widehat{K}_\beta^p} = \|u\|_{K_\beta^p}. \tag{27}$$

A multiplication by r^q inside the physical plane corresponds to a translation inside the complex plane.

Proposition 3. *Let $\beta \in \mathbb{R}$ and $v_q : \mathbb{R}_+ \longrightarrow \mathbb{C}$ be defined by $v_q(r) = r^q v(r)$, with $q \in \mathbb{R}$. If $v \in K^0_\beta$, then :*

$$v_q \in K^0_{\beta+q}, \tag{28}$$

and for any $\lambda \in \mathbb{C}_{q+\beta}$:

$$(\mathcal{M}v_q)(\lambda) \ = \ (\mathcal{M}v)(\lambda - q). \tag{29}$$

Proof. For any $q \in \mathbb{R}$, we have :

$$(\mathcal{M}v_q)(\lambda) \ = \ \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} r^{-\lambda+q} v(r) \frac{dr}{r} \ = \ (\mathcal{M}v)(\lambda - q), \tag{30}$$

which completes the proof. □

The multiplication by $\ln(r)$ corresponds to a differentiation with respect to the Mellin variable (see Prop. 4) related to the complex line $\mathbb{C}_\beta$.

Proposition 4. *Let $\beta \in \mathbb{R}$. For any $v \in L^1_\beta$, we introduce $v_{\ln}(r) := \ln(r)v(r)$. If $v_{\ln} \in L^1_\beta$, for all $\lambda \in \mathbb{C}_\beta$ we have :*

$$(\mathcal{M}v_{\ln})(\lambda) = -d_\lambda(\mathcal{M}v)(\lambda). \tag{31}$$

Proof. By definition $\big($see (6)$\big)$, we have :

$$(\mathcal{M}v_{\ln})(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} r^{-\lambda} \ln(r) v(r) \frac{dr}{r}. \tag{32}$$

We then define $g(\lambda, r) = r^{-\lambda} \dfrac{v(r)}{r}$ and we observe that :

$$\partial_\lambda g(\lambda, r) = -\ln(r) r^{-\lambda} \frac{v(r)}{r} \text{ and } \Big|\partial_\lambda g(\lambda, r)\Big| \leq \Big|\ln(r) r^{-\beta} \frac{v(r)}{r}\Big| \tag{33}$$

It follows from the Lebesgues theorem that :

$$(\mathcal{M}v_{\ln})(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} -d_\lambda (r^{-\lambda} \frac{v(r)}{r}) dr = -d_\lambda(\mathcal{M}v)(\lambda). \tag{34}$$

This ends the proof. □

The inverse Mellin transform can be deduced from the inverse Fourier transform as follows :

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\xi) \exp(ix\xi)\, d\xi. \tag{35}$$

We indeed have for any $\omega \in \widehat{K}^p_\beta$ and $\lambda = \beta + i\xi$:

$$(\mathcal{M}^{-1}_\beta \omega)(r) = (\mathcal{F}^{-1} g_\omega)(x), \quad \text{with } g_\omega(\xi) = e^{\beta x} \omega(\beta + i\xi). \tag{36}$$

We get this result by handling the changes of variable $x = \ln r$ and $\beta + i\xi = \lambda$ into the expression of the inverse Mellin transform.

Definition 1. *The inverse Mellin transform $\mathcal{M}^{-1}_\beta : \widehat{K}^p_\beta \longrightarrow K^p_\beta$ can be computed by using the formula :*

$$(\mathcal{M}^{-1}_\beta \omega)(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{C}_\beta} r^\lambda \omega(\lambda) \frac{d\lambda}{i}. \tag{37}$$

Remark 1. *The integration is perfomed inside the complex plane and the straight-line $\mathbb{C}_\beta$ is oriented upwards. This means that :*

$$(\mathcal{M}_\beta^{-1}\omega)(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} r^{\beta+i\xi}\omega(\beta+i\xi)d\xi. \tag{38}$$

Proposition 5. *The inverse Mellin transform satisfies the following items :*

i) For any $v \in \mathcal{D}(]0,+\infty[)$ and any $\beta \in \mathbb{R}$, we have :

$$\big(\mathcal{M}_\beta^{-1}(\mathcal{M}v)\big)(r) = v(r). \tag{39}$$

ii) For any real β and any $v \in K_\beta^0$, we have :

$$\big(\mathcal{M}_\beta^{-1}(\mathcal{M}v)\big)(r) = v(r). \tag{40}$$

iii) For any real β and any $\omega \in \widehat{K}_\beta^0$, we have :

$$\big(\mathcal{M}(\mathcal{M}_\beta^{-1}\omega)\big)(\lambda) = \omega(\lambda). \tag{41}$$

Proof. i) We correlate any $v \in \mathcal{D}(]0,+\infty[)$ with $f_v(x) = v(e^x)e^{-\beta x}$. We then have :

$$\mathcal{F}^{-1}(\mathcal{F}f_v)(x) = f_v(x). \tag{42}$$

According to (8), $(\mathcal{M}v)(\lambda) = (\mathcal{F}f_v)(\xi)$. It then follows that :

$$v(e^x)e^{-\beta x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi}(\mathcal{M}v)(\beta+i\xi)d\xi. \tag{43}$$

We now introduce the auxiliary variable $r = e^x$ and we have :

$$v(r)r^{-\beta} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} r^{i\xi}(\mathcal{M}v)(\beta+i\xi)d\xi, \tag{44}$$

which completes the proof.

ii) is a consequence of the density of $\mathcal{D}(]0,+\infty[)$ into K_β^0.

iii) For any $\omega \in \widehat{K}_\beta^0$, we define $g_\omega(\xi) = e^{\beta x}\omega(\beta+i\xi)$. We then have :

$$\mathcal{F}(\mathcal{F}^{-1}g_\omega)(\xi) = g_\omega(\xi). \tag{45}$$

According to (36), $(\mathcal{M}_\beta^{-1}\omega)(r) = (\mathcal{F}^{-1}g_\omega)(x)$. It then follows that :

$$e^{\beta x}\omega(\beta+i\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\xi}(\mathcal{M}_\beta^{-1}\omega)(e^x)dx. \tag{46}$$

Now if $r = e^x$ we get :

$$\omega(\beta+i\xi) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} r^{-\beta-i\xi}(\mathcal{M}_\beta^{-1}\omega)(r)\frac{dr}{r}. \tag{47}$$

At last, by introducing $\beta + i\xi = \lambda$ we obtain the expected result. $\square$

Corollary 3. *If $u \in K^1_\beta$, then we have :*

$$\big|u(r)\big| \;\leq\; \frac{\|u\|_{K^1_\beta}\, r^\beta}{2^{1/2}(1+\beta^2)^{1/4}}. \tag{48}$$

Proof. From *ii)* of the proposition 5, we get :

$$u(r) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} r^{\beta+i\xi}(\mathcal{M}u)(\beta+i\xi)d\xi. \tag{49}$$

Since $|r^{i\xi}| = 1$, we have

$$\big|u(r)\big| \leq \frac{r^\beta}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Big|(\mathcal{M}u)(\beta+i\xi)\Big|d\xi. \tag{50}$$

To prove that the right hand side is bounded we use a Cauchy Schwartz inequality

$$\big|u(r)\big| \leq \frac{r^\beta}{\sqrt{2\pi}}\Big(\int_{-\infty}^{+\infty}\frac{d\xi}{1+\beta^2+\xi^2}\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{+\infty}\Big(1+\beta^2+\xi^2\Big)\Big|(\mathcal{M}u)(\beta+i\xi)\Big|^2 d\xi\Big)^{\frac{1}{2}}. \tag{51}$$

By applying the Mellin-Parseval identity, $\big($see (27)$\big)$, we get :

$$\big|u(r)\big| \;\leq\; \frac{r^\beta}{\sqrt{2\pi}}\Big(\int_{-\infty}^{+\infty}\frac{1}{1+\beta^2+\xi^2}d\xi\Big)^{\frac{1}{2}}\|u^{\{1\}}\|_{K^0_\beta}, \tag{52}$$

and we can conclude remarking that : $\displaystyle\int_{-\infty}^{+\infty}\frac{d\xi}{1+\beta^2+\xi^2} \;=\; \frac{\pi}{\sqrt{1+\beta^2}}.$ ☐

Remark 2. *We deduce from* (50)*, if $u \in K^0_\beta \cap L^1(\mathbb{C}_\beta)$, that*

$$|u(r)| \leq \frac{r^\beta}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}|\widehat{u}(\beta+i\xi)|d\xi. \tag{53}$$

4 Analyticity and weighted space

The behavior of a function at the origin can be deduced from the study of the analytic continuation of its Mellin transform. We give here some arguments which allow to perform this study most of the time.

4.1 Case of a strip

Let us remark that for any $v \in \mathcal{D}(]0,+\infty[)$, the Mellin transform is defined in the whole complex plane. On the contrary, the Mellin transform of a function of K^0_β is only defined in $\mathbb{C}_\beta$. In that case, the question of analyticity is no longer relevant. Hence, we introduce the space :

$$K^p_{[\beta_1,\beta_2]} \;=\; K^p_{\beta_1}\cap K^p_{\beta_2}, \tag{54}$$

which usefulness becomes obvious after proving the following propositions.

Proposition 6. *Let p be a positive integer. Let β_1 and β_2 be two real numbers such that $\beta_1 < \beta_2$. If $v \in K^p_{[\beta_1,\beta_2]}$, then for any $\beta \in [\beta_1,\beta_2]$ the function v belongs to K^p_β.*

Proof. The proposition is a consequence of :

$$\|u\|^2_{K^0_\beta} \leq \|u\|^2_{K^0_{\beta_1}} + \|u\|^2_{K^0_{\beta_2}}, \tag{55}$$

which can be obtained once observing that :

$$\|u\|^2_{K^0_\beta} = \int_0^1 r^{-2\beta}|u(r)|^2 \frac{dr}{r} + \int_1^{+\infty} r^{-2\beta}|u(r)|^2 \frac{dr}{r}, \tag{56}$$

and that $r^{-2\beta}$ is upper bounded by $r^{-2\beta_2}$ on $[0,1]$ and by $r^{-2\beta_1}$ on $[1,+\infty[$. We then get :

$$\begin{aligned} \|u\|^2_{K^0_\beta} &\leq \int_0^1 r^{-2\beta_2}|u(r)|^2 \frac{dr}{r} + \int_1^{+\infty} r^{-2\beta_1}|u(r)|^2 \frac{dr}{r} \\ &\leq \int_0^{+\infty} r^{-2\beta_2}|u(r)|^2 \frac{dr}{r} + \int_0^{+\infty} r^{-2\beta_1}|u(r)|^2 \frac{dr}{r}. \end{aligned} \tag{57}$$

The proof is then completed. □

Proposition 7. *Let β_1 and β_2 be two real numbers such that $\beta_1 < \beta_2$. If $v \in K^0_{[\beta_1,\beta_2]}$ then for any $\beta \in]\beta_1, \beta_2[$ the function v belongs to L^1_β, (see (9)).*

Proof. We split the norm of L^1_β into :

$$\|u\|_{L^1_\beta} = \int_0^1 r^{-\beta}|u(r)| \frac{dr}{r} + \int_1^{+\infty} r^{-\beta}|u(r)| \frac{dr}{r}. \tag{58}$$

The first term can be upper bounded thanks to the Cauchy-Schwarz inequality :

$$\left\{ \begin{aligned} \int_0^1 r^{-\beta}|u(r)| \frac{dr}{r} &\leq \Big(\int_0^1 r^{2\beta_2 - 2\beta} \frac{dr}{r} \Big)^{\frac{1}{2}} \Big(\int_0^1 r^{-2\beta_2}|u(r)|^2 \frac{dr}{r} \Big)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{1}{2\beta_2 - 2\beta}} \|u\|_{K^0_{\beta_2}}. \end{aligned} \right. \tag{59}$$

In the same way, we get for the second term :

$$\int_1^{+\infty} r^{-\beta}|u(r)| \frac{dr}{r} \leq \sqrt{\frac{1}{2\beta - 2\beta_1}} \|u\|_{K^0_{\beta_1}}. \tag{60}$$

We then obtain that $u \in L^1_\beta$ because :

$$\|u\|_{L^1_\beta} \leq \sqrt{\frac{1}{2\beta - 2\beta_1}} \|u\|_{K^0_{\beta_1}} + \sqrt{\frac{1}{2\beta_2 - 2\beta}} \|u\|_{K^0_{\beta_2}}. \tag{61}$$

This ends the proof. □

Proposition 8. *Let β_1 and β_2 be two real numbers such that $\beta_1 < \beta_2$. If $v \in K^0_{[\beta_1,\beta_2]}$ then for any $\beta \in]\beta_1, \beta_2[$ the function $r \mapsto \ln r \; v(r)$ belongs to L^1_β.*

Proof. The proof looks like the one of proposition 7. We use the Cauchy-Schwarz inequality to obtain :

$$\left\{ \begin{array}{rcl} \displaystyle\int_0^1 |\ln(r)|\; r^{-\beta} |u(r)| \frac{dr}{r} & \leq & \displaystyle\Big(\int_0^1 |\ln(r)|^2\; r^{2\beta_2 - 2\beta} \frac{dr}{r}\Big)^{\frac{1}{2}} \Big(\int_0^1 r^{-2\beta_2} |u(r)|^2 \frac{dr}{r}\Big)^{\frac{1}{2}} \\ & \leq & \displaystyle\frac{1}{2(\beta_2 - \beta)^{3/2}} \|u\|_{K^0_{\beta_2}}. \end{array} \right. \tag{62}$$

We also have :

$$\int_1^{+\infty} |\ln(r)|\; r^{-\beta} |u(r)| \frac{dr}{r} \leq \frac{1}{2(\beta - \beta_1)^{3/2}} \|u\|_{K^0_{\beta_1}}. \tag{63}$$

This implies that :

$$\int_0^{+\infty} |\ln(r)|\; r^{-\beta} |u(r)| \frac{dr}{r} \leq \frac{1}{2(\beta_2 - \beta)^{3/2}} \|u\|_{K^0_{\beta_2}} + \frac{1}{2(\beta - \beta_1)^{3/2}} \|u\|_{K^0_{\beta_1}}, \tag{64}$$

and the proof is over. □

Theorem 1. *Let β_1 and β_2 be two real numbers such that $\beta_1 < \beta_2$. If $v \in K^0_{[\beta_1,\beta_2]}$, then the Mellin transform of v is analytical in $\mathbb{C}_{]\beta_1,\beta_2[}$ which is defined by :*

$$\mathbb{C}_{]\beta_1,\beta_2[} := \{\lambda \in \mathbb{C} : \beta_1 < \beta < \beta_2\}. \tag{65}$$

Proof. It is sufficient to prove that $\mathcal{M}v$ is differentiable in $\mathbb{C}_{]\beta_1',\beta_2'[}$ with respect to the complex variable for any β_1' et β_2' such that $\beta_1 < \beta_1' < \beta_2' < \beta_2$. The conclusion comes from the Lebesgue interchanging theorem for which we have to check the hypotheses i) - iv).

(i) The Mellin transform of v is given by the formula :

$$\mathcal{M}v(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} g(\lambda, r) dr, \quad \text{with } g(\lambda, r) = r^{-\lambda} \frac{v(r)}{r}. \tag{66}$$

(ii) We observe that for almost $r \in]0, +\infty[$ the function $\lambda \mapsto g(\lambda, r)$ is continuous.

(iii) The differential of this function is continuous and defined by :

$$\frac{dg(\lambda, r)}{d\lambda}(\lambda) \;=\; -\ln(r)\; r^{-\lambda} \frac{v(r)}{r}. \tag{67}$$

(iv) We have the following estimate :

$$|\ln(r)\; g(\lambda, r)| \leq \varphi_1(r), \tag{68}$$

with $\varphi_1(r) = 1_{\{0<r<1\}} |\ln(r) g(\beta_2', r)| + 1_{\{1<r\}} |\ln(r) g(\beta_1', r)|$. The function φ_1 belongs to $L^1(]0, +\infty[)$ because

$$\|\varphi_1\|_{L^1(]0,+\infty[)} \leq \|w\|_{L^1_{\beta_1'}} + \|w\|_{L^1_{\beta_2'}}, \tag{69}$$

with $w(r) = \ln(r)\; v(r)$. According to proposition 8, the right-hand-side is bounded independently of β.

□

4.2 Case of the half-space

This section deals with some properties of the Mellin transform of functions $v \; :]0,+\infty[\to \mathbb{C}$ strongly vanishing at infinity, that is : $\exists r_\star > 0$, $v(r) = 0$, $\forall r > r_\star$.

Proposition 9. *Let β_0 be a real number. If $v \in K^0_{\beta_0}$ and $v(r) = 0$ for any $r > r_\star$ then for any $\beta < \beta_0$,*

$$v \in K^0_\beta \quad and \quad \|v\|_{K^0_\beta} \leq r_\star^{\beta_0-\beta}\|v\|_{K^0_{\beta_0}}. \tag{70}$$

Proof. We use the following estimate :

$$\begin{aligned}\|v\|^2_{K^0_\beta} = \int_0^{+\infty} r^{-2\beta}\frac{|v(r)|^2}{r}dr &\leq \int_0^{+\infty} r^{2\beta_0-2\beta}r^{-2\beta_0}\frac{|v(r)|^2}{r}dr \\ &\leq r_\star^{2\beta_0-2\beta}\|v\|^2_{K^0_{\beta_0}},\end{aligned}$$

which concludes the proof. □

Proposition 10. *Let $\beta_0 \in \mathbb{R}$. If $v \in K^0_{\beta_0}$ and $v(r) = 0$ for any $r > r_\star$, then for any $\beta < \beta_0$,*

$$v \in L^1_\beta \quad and \quad \|v\|_{L^1_\beta} \leq \frac{r_\star^{\beta_0-\beta}}{\sqrt{2(\beta_0-\beta)}}\|v\|_{K^0_{\beta_0}}. \tag{71}$$

Proof. According to the Cauchy-Schwarz inequality, we have :

$$\begin{aligned}\|v\|_{L^1_\beta} &= \int_0^{+\infty} r^{-\beta}\frac{|v(r)|}{r}dr = \int_0^{+\infty} r^{\beta_0-\beta}r^{-\beta_0}\frac{|v(r)|}{r}dr \\ &\leq \left(\int_0^{+\infty} r^{2\beta_0-2\beta}dr\right)^{1/2}\|v\|_{K^0_{\beta_0}},\end{aligned}$$

and the proof is completed. □

Theorem 2. *Let $\beta_0 \in \mathbb{R}$. If $v \in K^0_{\beta_0}$ and $v(r) = 0$ for any $r > r_\star$, then $\mathcal{M}v$ is analytical in $\mathbb{C}_{]-\infty,\beta_0[}$ and :*

$$|(\mathcal{M}v)(\lambda)| \leq \frac{r_\star^{\beta_0-\beta}}{2\sqrt{\pi(\beta_0-\beta)}}\|v\|_{K^0_\beta}. \tag{72}$$

Proof. Since $v \in K^0_{\beta_0}$ and $v(r) = 0$ in the neighbourhood of infinity, Prop.9 gives rise to $v \in K^0_\beta$ for any $\beta < \beta_0$. Following Theorem 1, $\mathcal{M}v$ is analytical in $\mathbb{C}_{]\beta,\beta_0[}$ and thus in $\mathbb{C}_{]-\infty,\beta_0[} := \{\lambda \in \mathbb{C} : \mathcal{R}e(\lambda) < \beta_0\}$. By definition of the Mellin transform and since $v(r) = 0$ for any $r > r_\star$, we have :

$$(\mathcal{M}v)(\lambda) \; = \; \frac{1}{\sqrt{2\pi}}\int_0^{r_\star} r^{-\lambda}\; v(r)\; \frac{dr}{r}. \tag{73}$$

The Cauchy-Schwarz inequality implies then that :

$$\begin{aligned}\Big|(\mathcal{M}v)(\lambda)\Big| &\leq \frac{1}{\sqrt{2\pi}}\left(\int_0^{r_\star} r^{2(\beta_0-\beta)}\;\frac{dr}{r}\right)^{1/2}\left(\int_0^{r_\star} r^{-2\beta_0}\;|v(r)|^2\;\frac{dr}{r}\right)^{1/2}, \\ &\leq \frac{r_\star^{(\beta_0-\beta)}}{2\sqrt{\pi(\beta_0-\beta)}}\;\Big\|v\Big\|_{K^0_{\beta_0}}.\end{aligned} \tag{74}$$

The proof is then over. □

By observing that $(\mathcal{M}v^{\{p\}})(\lambda) = \lambda^p(\mathcal{M}v)(\lambda)$, corollary 1 implies that.

Corollary 4. *Let p be an integer and β_0 be a real. For any $v \in K^p_{\beta_0}$ such that $v(r) = 0$ for any $r > r_\star > 0$, the function $v \in K^p_\beta$ for any $\beta \le \beta_0$. The Mellin transform is analytical in the half-space $\mathbb{C}_{]-\infty,\beta_0[}$ and for $\mathcal{R}e(\lambda) = \beta < \beta_0$ we have :*

$$|\lambda^p(\mathcal{M}u)(\lambda)| \;\le\; \frac{r_\star^{\beta_0-\beta}}{2\sqrt{\pi(\beta_0-\beta)}}\,\|u\|_{K^p_{\beta_0}}. \tag{75}$$

5 Fundamental theorem of the theory of singularities

Under suitable hypothesis, the following theorem relates the domain of analyticity of a function to its behavior at the origin. Roughly speaking, if the Mellin transform is analytic on the band $\mathbb{C}_{]\beta_0,\beta_2[}$, then the function is a $\underset{r\to 0}{O}(r^\beta)$, for all $\beta \in]\beta_0, \beta_2[$.

Theorem 3. *Let $\beta_0 < \beta_1 < \beta_2$ be three real numbers. If $v \in K^0_{\beta_1}$ such that :*

i) the Mellin transform $\mathcal{M}v : \mathbb{C}_{\beta_1} \to \mathbb{C}$ has an analytical continuation $\widehat{v}$ in $\mathbb{C}_{]\beta_0,\beta_2[}$;

ii) there exists a real continuous function $g :]\beta_0, \beta_2[\longrightarrow \mathbb{R}^+$ such that for any $\lambda = \beta + i\xi \in \mathbb{C}_{]\beta_0,\beta_2[}$ with $|\xi| \ge 1$, $|\xi^2\,\widehat{v}(\lambda)| \;\le\; g(\beta)$;

then we have $v \in K^1_{\beta_1'}$, $\forall \beta_1' \in]\beta_0, \beta_2[$.

Remark 3. *The last result reveals that $v = \underset{r\to 0}{O}(r^{\beta_1'})$, for all $\beta_1' \in]\beta_0, \beta_2[$, see corollary 3. Thanks to* (50)*, we have*

$$|v(r)| \le \frac{r^\beta}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |\widehat{v}(\beta + i\xi)| d\xi. \tag{76}$$

Proof. The proof is divided into two steps which read as follows :

1. show that $\widehat{v} \in \widehat{K}^1_{\beta_1'}$, $\forall \beta_1' \in]\beta_0, \beta_2[$;
2. prove that $\lambda \mapsto \widehat{v}(\lambda)$ is the Mellin transform of v for any $\lambda \in \mathbb{C}$ such that $\mathcal{R}e(\lambda) \in]\beta_0; \beta_2[$;

Let us begin with proving that $\widehat{v} \in \widehat{K}^1_{\beta_1'}$ with $\beta_1' \in]\beta_0, \beta_2[$. We then have :

$$\|\widehat{v}\|^2_{\widehat{K}^1_{\beta_1'}} = \int_{|\xi|<1} \left(|\lambda|^2 + 1\right) |\widehat{v}(\beta_1' + i\xi)|^2 d\xi + \int_{|\xi|>1} \left(|\lambda|^2 + 1\right) |\widehat{v}(\beta_1' + i\xi)|^2 d\xi. \tag{77}$$

From i), we deduce that $\widehat{v}$ is locally bounded :

$$\int_{|\xi|<1} \left(|\lambda|^2 + 1\right) |\widehat{v}(\beta_1' + i\xi)|^2 d\xi < +\infty. \tag{78}$$

From ii), we deduce that :

$$\int_{|\xi|>1} \left(|\lambda|^2 + 1\right) |\widehat{v}(\beta_1' + i\xi)|^2 d\xi \le \int_{|\xi|>1} \left(|\lambda|^2 + 1\right) \frac{g(\beta)}{|\xi|^4} d\xi < +\infty. \tag{79}$$

This ends the proof of item 1). Regarding ii) assessment, we observe that : $v = \mathcal{M}^{-1}_{\beta_1}\widehat{v}$ $\big($because $v \in K^1_{\beta_1}$ and $\widehat{v}(\beta_1 + i\xi) = \mathcal{M}_{\beta_1}v(\lambda)\big)$. Then we integrate the holomorphic function

$f(\lambda) = \dfrac{r^\lambda}{\sqrt{2\pi i}}\widehat{v}(\lambda)$ on a suitable closed path denoted by γ_{ξ_0} and Cauchy theorem implies that the corresponding integrand vanishes.
Let $\xi_0 > 0$. We define the path γ_{ξ_0} as follows. We restrict ourselves to the case where $\beta_1' > \beta_1$ knowing that the additional configuration can be handled in the same way. If γ_{i,ξ_0} stands for the oriented graph of the maps $s \mapsto (x_i(s), y_i(s))$, we set $\gamma_{\xi_0} = \gamma_{1,\xi_0} + \gamma_{2,\xi_0} + \gamma_{3,\xi_0} + \gamma_{4,\xi_0}$ where :

$$\left\{\begin{array}{llll} x_1(s) = \beta_1 + s & \text{and} \quad y_1(s) = -\xi_0 & \text{with} & s \in [0, \beta_1' - \beta_1], \\ x_2(s) = \beta_1' & \text{and} \quad y_2(s) = -\xi_0 + s & \text{with} & s \in [0, 2\xi_0], \\ x_3(s) = \beta_1' - s & \text{and} \quad y_3(s) = \xi_0 & \text{with} & s \in [0, \beta_1' - \beta_1], \\ x_4(s) = \beta_1 & \text{and} \quad y_4(s) = \xi_0 - s & \text{with} & s \in [0, 2\xi_0]. \end{array}\right. \tag{80}$$

See Fig. 2 depicting γ_{ξ_0}.

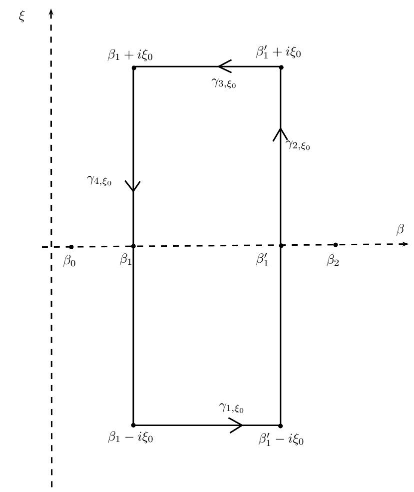

Figure 2: Path γ_{ξ_0}

We then compute the four integrands :

$$\left\{\begin{array}{l} \displaystyle\int_{\gamma_{1,\xi_0}} f(\lambda)\, d\lambda = \frac{1}{\sqrt{2\pi i}} \int_{\beta_1}^{\beta_1'} r^{\beta - i\xi_0}\ \widehat{v}(\beta - i\xi_0)\, d\beta, \\ \displaystyle\int_{\gamma_{2,\xi_0}} f(\lambda)\, d\lambda = \frac{1}{\sqrt{2\pi}} \int_{-\xi_0}^{\xi_0} r^{\beta_1' + i\xi}\ \widehat{v}(\beta_1' + i\xi)\, d\xi, \\ \displaystyle\int_{\gamma_{3,\xi_0}} f(\lambda)\, d\lambda = \frac{-1}{\sqrt{2\pi i}} \int_{\beta_1}^{\beta_1'} r^{\beta + i\xi_0}\ \widehat{v}(\beta + i\xi_0)\, d\beta, \\ \displaystyle\int_{\gamma_{4,\xi_0}} f(\lambda)\, d\lambda = \frac{-1}{\sqrt{2\pi}} \int_{-\xi_0}^{\xi_0} r^{\beta_1 + i\xi}\ \widehat{v}(\beta_1 + i\xi)\, d\xi. \end{array}\right. \tag{81}$$

Regarding the two integrands which are defined on γ_{1,ξ_0} and γ_{3,ξ_0}, we prove that they are converging towards 0. According to ii) and (80), we have for $\xi_0 > 1$:

$$\left\{ \begin{aligned} &\Big| \int_{\gamma_{1,\xi_0}} f(\lambda)\, d\lambda \Big| \le \int_{\beta_1}^{\beta_1'} \Big| \frac{r^{\beta - i\xi_0}}{\sqrt{2\pi} i} \Big| \frac{\alpha}{\xi_0^2} d\beta = \int_{\beta_1}^{\beta_1'} \frac{r^\beta}{\sqrt{2\pi}} \frac{\alpha}{\xi_0^2} d\beta, \\ &\Big| \int_{\gamma_{3,\xi_0}} f(\lambda)\, d\lambda \Big| \le \int_{\beta_1}^{\beta_1'} \Big| \frac{r^{\beta + i\xi_0}}{\sqrt{2\pi} i} \Big| \frac{\alpha}{\xi_0^2} d\beta = \int_{\beta_1}^{\beta_1'} \frac{r^\beta}{\sqrt{2\pi}} \frac{\alpha}{\xi_0^2} d\beta, \end{aligned} \right. \tag{82}$$

with

$$\alpha = \max_{\beta \in [\beta_1, \beta_1']} g(\beta). \tag{83}$$

This implies that :

$$\left\{ \begin{aligned} &\lim_{\xi_0 \to +\infty} \Big| \int_{\gamma_{1,\xi_0}} f(\lambda)\, d\lambda \Big| = 0, \\ &\lim_{\xi_0 \to +\infty} \Big| \int_{\gamma_{3,\xi_0}} f(\lambda)\, d\lambda \Big| = 0. \end{aligned} \right. \tag{84}$$

Likewise, by accounting for the behavior of $\widehat{v}$ when ξ becomes large $\big(|\widehat{v}(\lambda)| \le \dfrac{\alpha}{\xi^2}\big)$, we get :

$$\left\{ \begin{aligned} &\lim_{\xi_0 \to +\infty} \int_{\gamma_{2,\xi_0}} f(\lambda)\, d\lambda = (\mathcal{M}_{\beta_1'}^{-1} \widehat{v})(r), \\ &\lim_{\xi_0 \to +\infty} \int_{\gamma_{4,\xi_0}} f(\lambda)\, d\lambda = -(\mathcal{M}_{\beta_1}^{-1} \widehat{v})(r). \end{aligned} \right. \tag{85}$$

We then use Cauchy theorem :

$$\int_{\gamma_{\xi_0}} f(\lambda)\, d\lambda = 0, \tag{86}$$

and we finally obtain by letting ξ_0 to $+\infty$:

$$(\mathcal{M}_{\beta_1'}^{-1} \widehat{v})(r) = (\mathcal{M}_{\beta_1}^{-1} \widehat{v})(r). \tag{87}$$

We then deduce that :

$$(\mathcal{M}_{\beta_1'}^{-1} \widehat{v})(r) = v(r), \tag{88}$$

which shows that $\xi \mapsto \widehat{v}(\beta_1' + i\xi)$ is the Mellin transform of v for any $\beta_1' \in]\beta_1, \beta_2[$. It follows that $v \in K_{\beta_1'}^1$ since $\widehat{v} \in \widehat{K}_{\beta_1'}^1$. □

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