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Abstract
This paper is focused on the numerical and computational issues associated to ocean-atmosphere coupling. It is shown that usual coupling methods do not provide the solution to the correct problem, but to an approaching one since they are equivalent to performing one single iteration of an iterative coupling method. The stability analysis of these ad-hoc methods is presented, and we motivate and propose the adaptation of a Schwarz domain decomposition method to ocean-atmosphere coupling to obtain a stable and consistent coupling method.

Keywords: Ocean-atmosphere coupling, Schwarz domain decomposition method, air-sea interface

1 Motivation and problem setting

The use of fully coupled ocean-atmosphere models has become widespread in recent years, in particular in the context of climate change assessment by IPCC[1] experts. Proper representation of air-sea interactions in such models cover a large range of issues: parameterization of atmospheric and oceanic boundary layers, estimation of air-sea fluxes, time-space numerical schemes, non conforming grids, coupling algorithms... Several coupling methods, whose precise contents, theoretical justification, and practical performances are often somewhat difficult to compare precisely, are presently used in actual applications. In this paper we aim at building a general framework to assess those coupling methods.

The first step is to introduce the coupling problem of interest. We symbolically describe the oceanic and atmospheric circulation models by partial differential operators $\mathcal{L}_{\mathrm{oce}}$ and $\mathcal{L}_{\mathrm{atm}}$ corresponding to the systems of equations solved by numerical models. The oceanic domain Ω_{oce} and the atmospheric domain Ω_{atm} have a common interface Γ (the air-sea interface). On

[1]Intergovernmental Panel on Climate Change

the computational domain $\Omega = \Omega_{\mathrm{atm}} \cup \Omega_{\mathrm{oce}}$ (with external boundaries $\partial\Omega_{\mathrm{atm}}^{\mathrm{ext}}$ and $\partial\Omega_{\mathrm{oce}}^{\mathrm{ext}}$), the integration over a time period $[0, \mathcal{T}]$ reads

$$\left\{ \begin{array}{lcll} \mathcal{L}_{\mathrm{atm}}\mathbf{U}^{\mathrm{a}} & = & f_{\mathrm{atm}} & \text{in } \Omega_{\mathrm{atm}} \times [0,\mathcal{T}], \\ \mathcal{B}_{\mathrm{atm}}\mathbf{U}^{\mathrm{a}} & = & g_{\mathrm{atm}} & \text{in } \partial\Omega_{\mathrm{atm}}^{\mathrm{ext}} \times [0,\mathcal{T}], \\ \mathcal{F}_{\mathrm{atm}}\mathbf{U}^{\mathrm{a}} & = & \mathbf{F}_{\mathrm{oa}}(\mathbf{U}^{\mathrm{o}}, \mathbf{U}^{\mathrm{a}}, \mathcal{R}) & \text{on } \Gamma \times [0,\mathcal{T}], \end{array} \right. \tag{1}$$

$$\left\{ \begin{array}{lcll} \mathcal{L}_{\mathrm{oce}}\mathbf{U}^{\mathrm{o}} & = & f_{\mathrm{oce}} & \text{in } \Omega_{\mathrm{oce}} \times [0,\mathcal{T}], \\ \mathcal{B}_{\mathrm{oce}}\mathbf{U}^{\mathrm{o}} & = & g_{\mathrm{oce}} & \text{in } \partial\Omega_{\mathrm{oce}}^{\mathrm{ext}} \times [0,\mathcal{T}], \\ \mathcal{F}_{\mathrm{oce}}\mathbf{U}^{\mathrm{o}} & = & \mathbf{F}_{\mathrm{oa}}(\mathbf{U}^{\mathrm{o}}, \mathbf{U}^{\mathrm{a}}, \mathcal{R}) & \text{on } \Gamma \times [0,\mathcal{T}], \end{array} \right. \tag{2}$$

with appropriate initial conditions, and boundary conditions provided through the boundary operators $\mathcal{B}_{\mathrm{oce}}$ and $\mathcal{B}_{\mathrm{atm}}$. In (1) and (2), $\mathbf{U}^{\mathrm{a}} = (\mathbf{u}_h^{\mathrm{a}}, T^{\mathrm{a}})^t$ and $\mathbf{U}^{\mathrm{o}} = (\mathbf{u}_h^{\mathrm{o}}, T^{\mathrm{o}})^t$ are the state variables with $\mathbf{u}_h$ the horizontal velocity and T the (potential) temperature, f_{atm} and f_{oce} are forcing terms. For the sake of simplicity we do not include here salinity and humidity in the formulation of the problem. $\mathbf{F}_{\mathrm{oa}}$ is a function allowing the computation of air-sea fluxes. This function, generally based on the atmospheric surface layer similarity theory [11], depends on $\mathbf{U}^{\mathrm{a}}$ and $\mathbf{U}^{\mathrm{o}}$ in the vicinity of the air-sea interface, and on a set of non-turbulent radiative fluxes $\mathcal{R}$. In (1)-(2), the interface operators are defined as

$$\mathcal{F}_{\mathrm{atm}}\bullet = \rho^{\mathrm{a}}\mathbf{K}^{\mathrm{a}}\partial_z\bullet, \qquad \mathcal{F}_{\mathrm{oce}}\bullet = \rho^{\mathrm{o}}\mathbf{K}^{\mathrm{o}}\partial_z\bullet,$$

where z is positive upward, ρ^{a} and ρ^{o} are the densities of the fluids, and

$$\mathbf{K}^{\mathrm{a}} = \begin{pmatrix} \mathrm{K}_{\mathrm{m}}^{\mathrm{a}} \\ \mathrm{K}_{\mathrm{m}}^{\mathrm{a}} \\ c_p^{\mathrm{a}}\mathrm{K}_{\mathrm{t}}^{\mathrm{a}} \end{pmatrix}, \qquad \mathbf{K}^{\mathrm{o}} = \begin{pmatrix} \mathrm{K}_{\mathrm{m}}^{\mathrm{o}} \\ \mathrm{K}_{\mathrm{m}}^{\mathrm{o}} \\ c_p^{\mathrm{o}}\mathrm{K}_{\mathrm{t}}^{\mathrm{o}} \end{pmatrix}.$$

We note $\mathrm{K}_{\mathrm{m}}^{\mathrm{a}}$ and $\mathrm{K}_{\mathrm{m}}^{\mathrm{o}}$ the eddy viscosities, and $\mathrm{K}_{\mathrm{t}}^{\mathrm{a}}$ and $\mathrm{K}_{\mathrm{t}}^{\mathrm{o}}$ the eddy diffusivities. The c_p terms correspond to the specific heat of the fluid.

In forced mode, $\mathbf{U}^{\mathrm{o}}$ (resp. $\mathbf{U}^{\mathrm{a}}$ and $\mathcal{R}$) in (1) (resp. (2)) is provided offline by existing satellite-based or reanalysis products. In coupled mode, both models are run either simultaneously or successively on the same time interval. In this case, the consistency required at the air-sea interface is the continuity of momentum[2] and net heat fluxes

$$\begin{array}{rcccll} \rho^{\mathrm{a}}\mathrm{K}_{\mathrm{m}}^{\mathrm{a}}\partial_z\mathbf{u}_h^{\mathrm{a}} & = & \rho^{\mathrm{o}}\mathrm{K}_{\mathrm{m}}^{\mathrm{o}}\partial_z\mathbf{u}_h^{\mathrm{o}} & = & \boldsymbol{\tau} & \text{on } \Gamma \times [0,\mathcal{T}] \\ \rho^{\mathrm{a}}c_p^{\mathrm{a}}\mathrm{K}_{\mathrm{t}}^{\mathrm{a}}\partial_z T^{\mathrm{a}} & = & \rho^{\mathrm{o}}c_p^{\mathrm{o}}\mathrm{K}_{\mathrm{t}}^{\mathrm{o}}\partial_z T^{\mathrm{o}} & = & Q_{\mathrm{net}} & \text{on } \Gamma \times [0,\mathcal{T}] \\ & & Q_{\mathrm{net}} & = & \mathcal{R} + Q_{\mathrm{S}} & \end{array} \tag{3}$$

where the surface wind stress $\boldsymbol{\tau}$ and the sensible heat flux Q_{S} (and consequently the net heat flux Q_{net}) are computed using the function $\mathbf{F}_{\mathrm{oa}} = (\boldsymbol{\tau}, Q_{\mathrm{net}})^t$ previously introduced. These turbulent air-sea fluxes are given by a parameterization of the atmospheric surface layer. They usually take the form

$$\boldsymbol{\tau} = \rho^{\mathrm{a}}C_D\|\Delta\mathrm{U}\|\Delta\mathrm{U}, \qquad Q_{\mathrm{S}} = \rho^{\mathrm{a}}c_p^{\mathrm{a}}C_H\|\Delta\mathrm{U}\|\Delta T, \tag{4}$$

where C_D and C_H are exchange coefficients that depend on surface roughness and local stability. $\Delta\mathrm{U}$ (resp. $\Delta\mathrm{T}$) correspond to the velocity (resp. temperature) jump across the air-sea interface which is defined, in a bulk way, as the region between the lowest vertical level in the atmospheric model and the shallowest vertical level in the oceanic model.

[2] Here, we assume wind-wave equilibrium, i.e. the atmospheric momentum flux to the wave field is immediately transferred to the ocean through wave breaking.

At this point we have defined all the necessary notations to formulate the coupled problem:

Find $\mathbf{U}^{\mathrm{a}}$ and $\mathbf{U}^{\mathrm{o}}$ that satisfy

$$\left\{\begin{array}{rcll} \mathcal{L}_{\mathrm{atm}}\mathbf{U}^{\mathrm{a}} & = & f_{\mathrm{atm}} & \text{in } \Omega_{\mathrm{atm}} \times [0,\mathcal{T}] \\ \mathcal{B}_{\mathrm{atm}}\mathbf{U}^{\mathrm{a}} & = & g_{\mathrm{atm}} & \text{in } \partial\Omega_{\mathrm{atm}}^{\mathrm{ext}} \times [0,\mathcal{T}] \\ \mathcal{L}_{\mathrm{oce}}\mathbf{U}^{\mathrm{o}} & = & f_{\mathrm{oce}} & \text{in } \Omega_{\mathrm{oce}} \times [0,\mathcal{T}] \\ \mathcal{B}_{\mathrm{oce}}\mathbf{U}^{\mathrm{o}} & = & g_{\mathrm{oce}} & \text{in } \partial\Omega_{\mathrm{oce}}^{\mathrm{ext}} \times [0,\mathcal{T}] \\ \mathcal{F}_{\mathrm{atm}}\mathbf{U}^{\mathrm{a}} & = & \mathcal{F}_{\mathrm{oce}}\mathbf{U}^{\mathrm{o}} = \mathbf{F}_{\mathrm{oa}}(\mathbf{U}^{\mathrm{o}},\mathbf{U}^{\mathrm{a}},\mathcal{R}) & \text{on } \Gamma \times [0,\mathcal{T}] \end{array}\right. \tag{5}$$

for given initial and boundary conditions. It is worth mentioning that [16] proved the existence of a global weak solution to problem (5) with interface conditions given by a linearized version of (4) (i.e. with $\|\Delta \mathrm{U}\| = \mathbf{v}_{\mathrm{ref}}$, and constant values for C_D and C_H), considering that $\mathcal{L}_{\mathrm{atm}}$ and $\mathcal{L}_{\mathrm{oce}}$ are the linearized primitive equations of the atmosphere and the ocean.

In this context, the first aim of this paper is to provide a tentative classification of existing coupling methods to solve (5) (Sec. 2). Then we discuss the main numerical and physical issues associated to these methods (Sec. 3), and introduce a framework to cure those issues (Sec. 4). Some final remarks on current and future work are given in Sec. 5.

2 A classification of usual coupling methods

Multiphysics coupling has led to a wide range of mathematical and numerical methods (see [10] for a review). Usual strategies for ocean-atmosphere coupling can be found for instance in [4] mostly for long-term integrations, and in [1] and [18] for regional short-term high-resolution studies. A first algorithmic approach is based on the exchange of averaged fluxes between the models (usually referred to as *asynchronous* coupling) whereas a second one deals with instantaneous fluxes (referred to as *synchronous* coupling). In this section, we expose the key differences between the two strategies.

2.1 Asynchronous coupling by time windows (averaged fluxes)

Asynchronous coupling is the strategy used in most climate models of the IPCC. For this method, the total simulation time $[0,\mathcal{T}]$ is split into M smallest time windows $[t_i, t_{i+1}]$, i.e. $[0,\mathcal{T}] = \cup_{i=1}^{M}[t_i, t_{i+1}]$. The length of those time windows are typically between 2 hours and 1 day depending on applications and on the need to resolve the diurnal cycle. On a given time window, the models only exchange time-averaged quantities. This has the advantage of requiring few communications between models and to ensure proper flux conservation. Noting $\langle . \rangle_i$ a temporal average over time window $[t_i, t_{i+1}]$, the coupling algorithm reads

$$\begin{array}{l} \left\{\begin{array}{rcll} \mathcal{L}_{\mathrm{atm}}\mathbf{U}^{\mathrm{a}} & = & f_{\mathrm{atm}} & \text{in } \Omega_{\mathrm{atm}} \times [t_i, t_{i+1}] \\ \mathcal{F}_{\mathrm{atm}}\mathbf{U}^{\mathrm{a}} & = & \mathbf{F}_{\mathrm{oa}}(\langle \mathbf{U}^{\mathrm{o}} \rangle_{i-1}, \mathbf{U}^{\mathrm{a}}, \mathcal{R}) & \text{on } \Gamma \times [t_i, t_{i+1}] \end{array}\right. \\ \text{then} \\ \left\{\begin{array}{rcll} \mathcal{L}_{\mathrm{oce}}\mathbf{U}^{\mathrm{o}} & = & f_{\mathrm{oce}} & \text{in } \Omega_{\mathrm{oce}} \times [t_i, t_{i+1}] \\ \mathcal{F}_{\mathrm{oce}}\mathbf{U}^{\mathrm{o}} & = & \langle \mathcal{F}_{\mathrm{atm}}\mathbf{U}^{\mathrm{a}} \rangle_i & \text{on } \Gamma \times [t_i, t_{i+1}] \end{array}\right. \end{array} \tag{6}$$

First the atmospheric model is advanced from t_i to t_{i+1} using the averaged ocean state computed on the previous time window. Then the fluxes used to force the atmospheric model are averaged and applied to the oceanic model on the same time interval (the fluxes are generally piecewise constant on each time window). This methodology ensures that over a time interval $[t_i, t_{i+1}]$ both models are forced by exactly the same mean fluxes. Indeed, in both models the time integral of surface fluxes over the simulation equals $\langle \mathcal{F}_{\rm atm} \mathbf{U}^{\rm a} \rangle_{[0,\mathcal{T}]}$. However the solution of algorithm (6) is not rigorously solution of the original problem (5) because there is a synchronicity issue: the modification of the oceanic state $\mathbf{U}^{\rm o}$ on $[t_i, t_{i+1}]$ is not provided to the atmospheric component on the proper time interval $[t_i, t_{i+1}]$ but on $[t_{i+1}, t_{i+2}]$.

2.2 Synchronous coupling at the time-step (instantaneous fluxes)

In the natural world, ocean and atmosphere continuously exchange fluxes on scales ranging from global to micro scales. Therefore proper coupling frequency between numerical models should be as small as possible, typically the largest time step between the oceanic and the atmospheric ones. In this regard, a relevant algorithm would consist in exchanging instantaneous fluxes. If the oceanic time step Δt_o is such that $\Delta t_o = N\Delta t_a$, the corresponding algorithm reads

$$
\begin{aligned}
&\left\{ \begin{array}{lcll} \mathcal{L}_{\rm atm}\mathbf{U}^{\rm a} &=& f_{\rm atm} & \text{in } \Omega_{\rm atm} \times [t_i, t_i + N\Delta t_a] \\ \mathcal{F}_{\rm atm}\mathbf{U}^{\rm a} &=& \mathbf{F}_{\rm oa}(\mathbf{U}^{\rm o}(t_i), \mathbf{U}^{\rm a}(t), \mathcal{R}(t)) & \text{on } \Gamma \times [t_i, t_i + N\Delta t_a] \end{array} \right. \\
&\left\{ \begin{array}{lcll} \mathcal{L}_{\rm oce}\mathbf{U}^{\rm o} &=& f_{\rm oce} & \text{in } \Omega_{\rm oce} \times [t_i, t_i + \Delta t_o] \\ \mathcal{F}_{\rm oce}\mathbf{U}^{\rm o} &=& \mathbf{F}_{\rm oa}(\mathbf{U}^{\rm o}(t_i), \mathbf{U}^{\rm a}(t_i), \mathcal{R}(t_i)) & \text{on } \Gamma \times [t_i, t_i + \Delta t_o] \end{array} \right.
\end{aligned}
\tag{7}
$$

The oceanic and atmospheric components are integrated forward for a time period corresponding to Δt_o (i.e. $N\Delta t_a$). Data exchange of instantaneous values is then performed and model integration continues for another Δt_o time period . This process is repeated until the final simulation time. In (7) the oceanic component receives instantaneous values from the atmosphere but integrated values can also be considered between t_i and $t_i + N\Delta t_a$ to avoid aliasing errors. Both choices raise either conservation, aliasing or synchronization problems. In addition, algorithm (7) is difficult to implement efficiently from computational and numerical view points. Code communications are extremely frequent and time integration schemes must be carefully considered for consistent interfacial conditions. At first glance, (7) may appear as a solution of the full problem (5). This is however formally true only in the limit $\Delta t_o, \Delta t_a \to 0$. Indeed, in all numerical models, vertical diffusion is treated implicitly-in-time; air-sea fluxes are thus required at time $t_i + \Delta t$ rather than t_i as in (7). Because of this synchronicity issue, we show in Sec. 3 that the numerical implementation of the synchronous coupling can be unstable. See also [6] for an analysis of air-sea coupling algorithms in a finite element framework with $\Delta t_o = \Delta t_a$.

3 Numerical and physical considerations

Usual multiphysics coupling problem generally assume that all scales are resolved by the numerical models. In the case of the ocean-atmosphere problem the presence of physical parameterizations gives rise to physics-dynamics coupling issues. This section is meant to illustrate the delicacies in the coupling procedure that are specific to the air-sea coupling problem.

3.1 Physics-dynamics consistency

The computation of air-sea fluxes is based on a parameterization scheme to account for unresolved processes. The fluxes exchanged by the models are thus not the result of a discrete derivative in the vicinity of the air-sea interface but are given by atmospheric surface layer parameterizations based on the so-called bulk aerodynamic formulae. Bulk formulations are symbolically represented as part of the function $\mathbf{F}_{\rm oa}$ in (5). They are defined and calibrated semi-empirically using measurements averaged in time over about one hour or more, and for a restricted range of stability values [11] (air-sea fluxes are very uncertain under weak and strong wind conditions). There is little knowledge and observations of air-sea fluxes at high temporal frequencies (see discussion in [7], Sec. 2) and the sign of air-sea fluxes is uncertain for time scales less than 10 minutes. In addition, physical processes associated with the wavy boundary layer which are dominant at high frequency are excluded from the current air-sea flux estimation schemes used in global and regional climate models. Therefore, we consider mean hourly fluxes preferable when using bulk formulations (see [11] for a discussion). An internal time-scale $\Delta t_{\rm phys}$ is thus assumed for the parameterization to be valid, and $\Delta t_{\rm phys}$ is generally larger than the model dynamical time-step $\Delta t_{\rm dyn}$[3]. For this reason, a time-averaging procedure is adopted in algorithm (6). We consider that algorithm (7) is relevant only if physical processes predominant on short time-scales are explicitly addressed in the flux computation. But such approaches are under development and are not mature enough to be routinely introduced in state-of-the-art global and regional climate models.

3.2 Stability analysis of numerical interface conditions

We showed in Sec. 2 that usual coupling methods used in ocean-atmosphere models are not entirely satisfactory with respect to consistency, conservation or synchronization. We will show now that, for some parameter values of the vertical parabolic Courant number and vertical resolution, these methods can be numerically unstable. To do so, we consider the one dimensional diffusion problem

$$\left\{\begin{array}{rcll} \partial_t q - \partial_z(\nu(z)\partial_z q) &=& 0 & \text{in } \mathbb{R}\times[0,\mathcal{T}], \\ q(z,t) &\to& 0 & z\to\pm\infty \\ q(z,0) &=& q_0(z) & \forall z\in\mathbb{R}. \end{array}\right. \tag{8}$$

where $\nu(z) = \left\{\begin{array}{ll} \nu_1, & z\in\mathbb{R}^- \\ \nu_2, & z\in\mathbb{R}^+ \end{array}\right.$. Each half of the domain $\Omega = \mathbb{R}$ is solved separately with boundary conditions containing information from the other. Problem (8) is meant to be quite representative of the vertical turbulent mixing occurring at the air-sea interface. Indeed it is generally assumed that the turbulent terms $\langle w'q'\rangle$ arising from the Reynolds decomposition can be parameterized in terms of the resolved quantities $\langle q\rangle$ as $\langle w'q'\rangle \sim -\nu(z)\partial_z\langle q\rangle$ (e.g. [17]).

3.2.1 Synchronous coupling with classical Dirichlet-Neumann conditions

By analogy to the ocean-atmosphere coupling problem, we consider that the computation in $\mathbb{R}^+$ (i.e. the atmosphere) uses Dirichlet data while the computation in $\mathbb{R}^-$ (i.e. the ocean) uses Neumann data. We assume that the coupling algorithm is the synchronous method previously described and that a backward Euler scheme is used. The corresponding algorithm reads

$$q_j^{(-,n+1)} - q_j^{(-,n)} = \sigma^{(-)}\left(q_{j+1}^{(-,n+1)} - 2q_j^{(-,n+1)} + q_{j-1}^{(-,n+1)}\right),\ j<0 \tag{9}$$

$$q_{0^-}^{(-,n+1)} - q_{0^-}^{(-,n)} = \Delta t\left[\mathcal{F}^{+,n} - \mathcal{F}^{-,n}\right]/(\Delta z_1/2) \tag{10}$$

$$\mathcal{F}^{+,n} = \nu_2\left[q_1^{(+,n)} - q_{0^+}^{(+,n)}\right]/\Delta z_2 \quad \text{and} \quad \mathcal{F}^{-,n} = \nu_1\left[q_{0^-}^{(-,n+1)} - q_{-1}^{(-,n+1)}\right]/\Delta z_1 \tag{11}$$

$$q_j^{(+,n+1)} - q_j^{(+,n)} = \sigma^{(+)}\left(q_{j+1}^{(+,n+1)} - 2q_j^{(+,n+1)} + q_{j-1}^{(+,n+1)}\right),\ j>0 \tag{12}$$

$$q_{0^+}^{(+,n+1)} = q_{0^-}^{(-,n)} \tag{13}$$

[3] by choosing a predefined constant value for $\Delta t_{\rm phys}$ we also ensure that the air-sea flux estimation is independent from $\Delta t_{\rm dyn}$, which is not the case with algorithm (7).

with $q^{(-,n)}$ (resp. $q^{(+,n)}$) the solution on $\mathbb{R}^-$ (resp. $\mathbb{R}^+$) at time $n\Delta t$, $\sigma^{(-)} = \nu_1 \Delta t/\Delta z_1^2$ (resp. $\sigma^{(+)} = \nu_2 \Delta t/\Delta z_2^2$) the parabolic Courant numbers, and where (11) is discretized consistently with [9] for instance. Following [9], we can investigate the stability properties by assuming the variable arrangement as in Fig. 1a, and a solution of the form

$$q_j^n = \begin{cases} q_j^{(-,n)} = \mathcal{A}^n \mathcal{K}_-^j & j = 0^-, -1, -2, ... \\ q_j^{(+,n)} = \mathcal{A}^{n-1} \mathcal{K}_+^j & j = 0^+, 1, 2, ... \end{cases}$$

The corresponding discretization is stable for $|\mathcal{A}| \leq 1$. Thanks to (9) and (12) it is possible to find expressions $\mathcal{K}_\pm$ that satisfy the boundary conditions at infinity. Then it can be shown that the amplification factor $\mathcal{A}$ satisfies the equation

$$\sqrt{1+2/s^-} + \frac{\Delta z_2}{\Delta z_1} \mathcal{A}^{-2} \left(\sqrt{1+2/s^+} - 1 \right) = 0, \quad \text{with } s^\pm = \frac{1-\mathcal{A}^{-1}}{2\sigma^{(\pm)}}.$$

The asymptotic solutions to this equation are

- For $\Delta z_2 \ll \Delta z_1$, $\mathcal{A} \approx \left(1 + 4\sigma^{(-)}\right)^{-1}$, i.e. the scheme is stable.
- For $\Delta z_1 \ll \Delta z_2$, $\mathcal{A}^{-1} \approx 0$, i.e. the scheme is unstable.
- For $\sigma^{(\pm)} \gg 1$, the stability limit is $0 < \Delta z_2/\Delta z_1 < \sqrt{\sigma^{(-)}/\sigma^{(+)}}$, i.e. $\sqrt{\nu_1/\nu_2} > 1$.

In the context of ocean-atmosphere coupling we have $\Delta z_2 = \mathcal{O}(30\text{m})$, $\Delta z_1 = \mathcal{O}(0.5\text{m})$, and $\sqrt{\nu_1/\nu_2} = \mathcal{O}(1/10)$, meaning that algorithm (9-13) would be unstable even if the equations for each subdomain are integrated using an unconditionally stable time-integration scheme (note however that inverting the Neumann and Dirichlet conditions would lead to a stable scheme). Moreover we expect additional stability issues when different time-steps are considered in each subdomain.

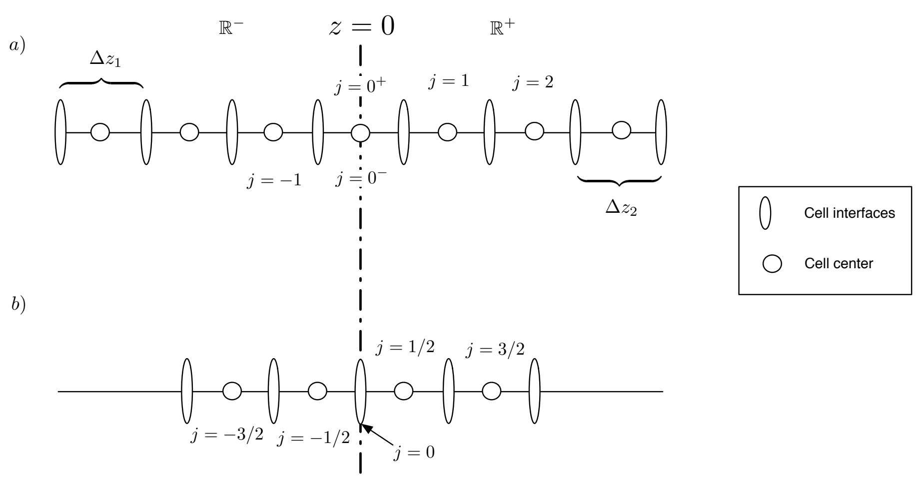

Figure 1: Arrangement of variables for the algorithm with Dirichlet-Neumann conditions (a) and with bulk interface conditions (b).

3.2.2 Synchronous coupling with bulk interface conditions

Considering that $\alpha = C_D \|\Delta U\|$ in (4) and the variable arrangement described in Fig. 1b), the algorithm with interface conditions representative of air-sea coupling is

$$\begin{array}{rcll} q_{j+\frac{1}{2}}^{(+,n+1)} - q_{j+\frac{1}{2}}^{(+,n)} &=& \Delta t \left(\mathcal{G}_{j+1}^{n+1} - \mathcal{G}_{j}^{n+1} \right) / \Delta z_2, & j \geq 0 \\ \mathcal{G}_0^{n+1} &=& \alpha \left(q_{\frac{1}{2}}^{(+,n)} - q_{-\frac{1}{2}}^{(-,n)} \right) & \\ q_{j+\frac{1}{2}}^{(-,n+1)} - q_{j+\frac{1}{2}}^{(-,n)} &=& \Delta t \left(\mathcal{Q}_{j+1}^{n+1} - \mathcal{Q}_{j}^{n+1} \right) / \Delta z_1, & j < 0 \\ \mathcal{Q}_0^{n+1} &=& (\rho_2/\rho_1) \mathcal{G}_0^{n+1} = (\rho_2/\rho_1) \alpha \left(q_{\frac{1}{2}}^{(+,n)} - q_{-\frac{1}{2}}^{(-,n)} \right) & \end{array} \tag{14}$$

where interfacial fluxes are $\mathcal{G}_j = \nu_2 \dfrac{q_{j+\frac{1}{2}}^{(+)} - q_{j-\frac{1}{2}}^{(+)}}{\Delta z_2}$ $(j \geq 1)$ and $\mathcal{Q}_j = \nu_1 \dfrac{q_{j+\frac{1}{2}}^{(-)} - q_{j-\frac{1}{2}}^{(-)}}{\Delta z_1}$ $(j \leq -1)$. Note that in (14), the interface conditions are expressed in term of dynamic viscosity (i.e. the kinematic viscosity multiplied by density) as in the context of OA coupling, so that we satisfy the compatibility condition (3). Note also that in (14) the surface boundary condition in the atmospheric component is assumed to be treated explicitly in time (an implicit treatment would lead to $\alpha \left(q_{\frac{1}{2}}^{(+,n+1)} - q_{-\frac{1}{2}}^{(-,n)} \right)$ in the second equation) because this is the only way to ensure both conservation and the fact that the two models can be run in parallel. An implicit treatment would impose the atmospheric model to advance first from n to $n+1$ to provide the flux so that the oceanic model can be advanced to time $n+1$. As in paragraph 3.2.1, let study the stability by assuming a solution of the form

$$q_{j+\frac{1}{2}}^{n} = \begin{cases} q_{j+\frac{1}{2}}^{(-,n)} = \mathcal{A}^n \mathcal{K}_-^j & j = -1, -2, -3, ... \\ q_{j+\frac{1}{2}}^{(+,n)} = \mathcal{A}^n \mathcal{K}_+^j & j = 0, 1, 2, 3, ... \end{cases}$$

In this case, after some algebra, we get two equations for the amplification factor, one for each subdomain

- For $z \geq 0$

$$|\mathcal{A}| = \frac{\beta}{\sigma^{(+)}} \left| \frac{\sqrt{s^-(2+s^-)} - s^-}{(s^+ + \sqrt{s^+(2+s^+)})(1 + s^- - \sqrt{s^-(2+s^-)})} \right| \leq \frac{\beta}{\sigma^{(+)}} \tag{15}$$

- For $z \leq 0$

$$|\mathcal{A}| = \frac{r\beta}{\sigma^{(-)}} \left| \frac{1 - \sqrt{1 + 2/s^{(-)}}}{1 + \sqrt{1 + 2/s^{(-)}}} \right| \leq \frac{r\beta}{\sigma^{(-)}} \tag{16}$$

with $\beta = \alpha \Delta t / \Delta z_2$ and $r = \rho_2 \Delta z_2 / \rho_1 \Delta z_1$. Those constraints are relatively intuitive because they express the fact that we require the simulation to be stable in each subdomain to get a stable coupling. In the atmosphere a conservative way of expressing the stability limit is

$$\beta \leq \sigma^{(+)} \qquad \Rightarrow \qquad \alpha \leq \frac{\nu_2}{\Delta z_2}$$

and in the ocean

$$r\beta \leq \sigma^{(-)} \qquad \Rightarrow \qquad \alpha \leq \frac{\nu_1}{\Delta z_1} \frac{\rho_1}{\rho_2}$$

In general, those constraints are easily satisfied in the oceanic component but not necessarily in the atmospheric component. In this case it would be necessary to treat the interface condition implicitly in time, thus leading to the aforementioned delicacies for the proper conservation and synchronisation between both models.

4 Global-in-time Schwarz method

The Schwarz-like domain decomposition methods (see [8] for a review) are widely used for coupling problems with different physics and/or different numerical treatments. Originally introduced for stationary problems, those methods have been recently extended to time-dependent problems to provide a global-in-time Schwarz method, a.k.a. Schwarz waveform relaxation (e.g. [2, 3]). The idea is to separate the original problem on $\Omega = \Omega_{\mathrm{atm}} \cup \Omega_{\mathrm{oce}}$ into subproblems on Ω_{atm} and Ω_{oce}, which can be solved separately. An iterative process is then applied to achieve convergence to the solution of the original problem. The main drawback of this approach is its iterative nature, which increases the computational cost of coupling, especially when convergence is slow (note that there is currently an active research aiming at optimizing the convergence speed of Schwarz-like methods; see discussion in Sec. 5). Using the notations introduced previously, the iterative algorithm on a time window $[t_i, t_{i+1}]$ can be written as follows (for a given initial condition at $t = t_i$) :

$$\text{Iterate until convergence}$$
$$\left\{\begin{array}{lcll} \mathcal{L}_{\mathrm{atm}}\mathbf{U}_k^{\mathrm{a}} & = & f_{\mathrm{atm}}, & \text{in } \Omega_{\mathrm{atm}} \times [t_i, t_{i+1}] \\ \mathcal{F}_{\mathrm{atm}}\mathbf{U}_k^{\mathrm{a}} & = & \mathbf{F}_{\mathrm{oa}}(\mathbf{U}_{k-1}^{\mathrm{o}}, \mathbf{U}_k^{\mathrm{a}}, \mathcal{R}_k), & \text{on } \Gamma \times [t_i, t_{i+1}] \end{array}\right.$$
$$\left\{\begin{array}{lcll} \mathcal{L}_{\mathrm{oce}}\mathbf{U}_k^{\mathrm{o}} & = & f_{\mathrm{oce}}, & \text{in } \Omega_{\mathrm{oce}} \times [t_i, t_{i+1}] \\ \mathcal{F}_{\mathrm{oce}}\mathbf{U}_k^{\mathrm{o}} & = & \mathcal{F}_{\mathrm{atm}}\mathbf{U}_k^{\mathrm{a}}, & \text{on } \Gamma \times [t_i, t_{i+1}] \end{array}\right. \tag{17}$$

where the subscripts k denote the iteration number. The first guess $\mathbf{U}_{k=0}^{\mathrm{o}}$ on $\Gamma \times [t_i, t_{i+1}]$ is generally taken from the converged solution on the previous time window $[t_{i-1}, t_i]$. The two models, at each iteration, are run successively: this is the so called *alternating* form of the algorithm. If the condition $\mathcal{F}_{\mathrm{oce}}\mathbf{U}_k^{\mathrm{o}} = \mathcal{F}_{\mathrm{atm}}\mathbf{U}_k^{\mathrm{a}}$ is replaced by $\mathcal{F}_{\mathrm{oce}}\mathbf{U}_k^{\mathrm{o}} = \mathcal{F}_{\mathrm{atm}}\mathbf{U}_{k-1}^{\mathrm{a}}$ both models can be run in parallel over the whole time window $[t_i, t_{i+1}]$: this is the *parallel* form of the algorithm. When convergence is reached, this algorithm gives the exact solution to (5). Note that with algorithm (17) the solution (but not the convergence rate) is independent of the size of time window $[t_i, t_{i+1}]$, as opposed to the asynchronous coupling method (due to the fact that this latter method performs only one iteration of the Schwarz algorithm).

However, for physical constraints on high-frequency treatment mentioned in Sec. 3, algorithm (17) should be modified to include time-averaging of the quantities near the air-sea interface as in (6):

$$\text{Iterate until convergence}$$
$$\left\{\begin{array}{lcll} \mathcal{L}_{\mathrm{atm}}\mathbf{U}_k^{\mathrm{a}} & = & f_{\mathrm{atm}}, & \text{in } \Omega_{\mathrm{atm}} \times [t_i, t_{i+1}] \\ \mathcal{F}_{\mathrm{atm}}\mathbf{U}_k^{\mathrm{a}} & = & \mathbf{F}_{\mathrm{oa}}\left(\left\langle \mathbf{U}_{k-1}^{\mathrm{o}} \right\rangle_i, \mathbf{U}_k^{\mathrm{a}}, \mathcal{R}_k\right), & \text{on } \Gamma \times [t_i, t_{i+1}] \end{array}\right.$$
$$\left\{\begin{array}{lcll} \mathcal{L}_{\mathrm{oce}}\mathbf{U}_k^{\mathrm{o}} & = & f_{\mathrm{oce}}, & \text{in } \Omega_{\mathrm{oce}} \times [t_i, t_{i+1}] \\ \mathcal{F}_{\mathrm{oce}}\mathbf{U}_k^{\mathrm{o}} & = & \langle \mathcal{F}_{\mathrm{atm}}\mathbf{U}_k^{\mathrm{a}} \rangle_i, & \text{on } \Gamma \times [t_i, t_{i+1}] \end{array}\right. \tag{18}$$

Note that the stability problems mentioned in Sec. 3 are no longer present when using the Schwarz method because there is no synchronicity problem anymore. Moreover it is also possible to treat implicitly the surface boundary condition in the atmospheric model without compromising conservation properties. That is why, as will be emphasized in the final section, we think that such a method is a very convenient framework for ocean-atmosphere coupling.

5 Summary and future work

We have studied in this paper the key role of the coupling algorithm in the design of an atmospheric and oceanic coupled model. We showed that coupling methods used in regional and global climate models do not provide the exact solution to the ocean-atmosphere coupling problem (5), but an approximation. We introduce a natural and non-intrusive method to handle this problem and we motivate its relevance. This method, called Global-in-Time Schwarz Method, is based on an iterative process. It can easily be shown that the usual asynchronous method (6) corresponds to only one iteration of a Schwarz algorithm. Similarly, the synchronous coupling method (7) corresponds to one iteration of a local-in-time Schwarz algorithm [5] and can be formally unstable for parameter values typical of ocean-atmosphere coupled simulations. The fact that this method has an unstable behavior does not necessarily lead to systematic blowups in realistic models essentially because other processes like diffusion and viscosity help mitigate the problem. We also emphasized the peculiarities of ocean-atmosphere coupling, in particular the necessary consistency between the physics (i.e. the fact that the parameterization schemes have their own internal time-scale) and the numerics.

Besides the theoretical work presented in this paper, numerical experiments using a mesoscale atmospheric model (WRF[4]) coupled with a regional oceanic model (ROMS[5]) for a realistic simulation of a tropical cyclone have been carried out in [15]. Ensemble simulations have been designed by perturbations of the coupling frequency and the initial conditions. One ensemble has been integrated using the Global-in-Time Schwarz Method and an other using the asynchronous method. The results show significant differences in terms of ensemble spread (with respect to the cyclone trajectory and intensity), the Schwarz iterative coupling method leading to a reduced spread. This suggests that part of the model sensitivity to perturbed parameters can be attributed to inaccuracies in the coupling method. Specifically, coupling inconsistencies can spuriously increase the physical stochasticity of simulated atmospheric and oceanic events (materialized here by the ensemble spread). For our particular case, three iterations of the Schwarz method are sufficient to improve the coupled solutions with respect to the ensemble spread. Additional numerical tests in the coupled LMDZ-NEMO global climate model (developed at LSCE[6]) are currently under investigation to assess the impact on long-term simulations.

More generally, it remains to be investigated how robust is the convergence w.r.t. the model formulation, particularly the boundary layer parameterizations at the air-sea interface. In this regard, the Schwarz algorithm can also be used as a diagnostic tool to assess consistency between atmospheric and oceanic boundary layer parameterizations: two schemes could be recognized as *compatible* if they lead to a converging Schwarz algorithm. Indeed, if it is not the case, this would probably highlight a problem in the physical approximations and mathematical formulation of the models. The building blocks have been recently introduced in [12, 13, 14]. We argue that this work at a fundamental level would be beneficial whatever the coupling method used in practice. We will also put effort in the near future on the design of idealized coupled cases to unambiguously assess the benefits of the iterative coupling and some aspects of the physical suitability of a given parameterization (e.g. by testing the convergence of the Schwarz approach).

[4]Weather Research and Forecasting model — `http://www.wrf-model.org`
[5]Regional Oceanic Modeling System — `http://www.myroms.org`
[6]Laboratoire des Sciences du Climat et de l'Environnement

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References

[1] J.-W. Bao, J.-M. Wilczak, J.K. Choi, and L.H. Kantha. Numerical simulations of air-sea interaction under high wind conditions using a coupled model : A study of hurricane development. *Mon. Weather Rev.*, 128(7):2190–2210, 2000.

[2] E. Blayo and L. Debreu. Nesting ocean models. In E. Chassignet and J. Verron, editors, *An Integrated View of Oceanography: Ocean Weather Forecasting in the 21st Century*. Kluwer, 2006.

[3] E. Blayo, L. Halpern, and C. Japhet. Optimized Schwarz waveform relaxation algorithms with nonconforming time discretization for coupling convection-diffusion problems with discontinuous coefficients. In *Domain decomposition methods in science and engineering XVI*, volume 55 of *Lect. Notes Comput. Sci. Eng.*, pages 267–274. Springer, Berlin, 2007.

[4] F.O. Bryan, B.G. Kauffman, W. G. Large, and P. R. Gent. The NCAR CSM flux coupler. Technical Report NCAR/TN-424+STR, NCAR, 1996.

[5] X.-C. Cai and M. Sarkis. Local multiplicative schwarz algorithms for steady and unsteady convection-diffusion equations. *East-West J. Numer. Math.*, 6:27–41, 1998.

[6] J. M. Connors and B. Ganis. Stability of algorithms for a two domain natural convection problem and observed model uncertainty. *Comput. Geosci.*, 15(509 - 527), 2011.

[7] G. Danabasoglu, W. G. Large, J. J. Tribbia, P. R. Gent, B.P. Briegleb, and J.C. McWilliams. Diurnal coupling in the tropical oceans of CCSM3. *J. Climate*, 19(11):2347–2365, 2006.

[8] M.J. Gander. Schwarz methods over the course of time. *Electron. Trans. Numer. Anal.*, 31:228–255, 2008.

[9] M. B. Giles. Stability analysis of numerical interface conditions in fluid-structure thermal analysis. *Int. J. Numer. Meth. Fluids*, 25(4):421–436, 1997.

[10] D. E. Keyes et al. Multiphysics simulations: challenges and opportunities. *Int. J. High Performance Computing Applications*, 27:5–83, 2013.

[11] W. G. Large. Surface fluxes for practitioners of global ocean data assimilation. In E. Chassignet and J. Verron, editors, *An Integrated View of Oceanography: Ocean Weather Forecasting in the 21st Century*. Kluwer, 2006.

[12] F. Lemarié, L. Debreu, and E. Blayo. Optimal control of the convergence rate of Schwarz waveform relaxation algorithms. *Lecture Notes in Computational Science and Engineering*, 91:599–606, 2013.

[13] F. Lemarié, L. Debreu, and E. Blayo. Toward an optimized global-in-time Schwarz algorithm for diffusion equations with discontinuous and spatially variable coefficients, part 1 : the constant coefficients case. *Electron. Trans. Numer. Anal.*, 40:148–169, 2013.

[14] F. Lemarié, L. Debreu, and E. Blayo. Toward an optimized global-in-time Schwarz algorithm for diffusion equations with discontinuous and spatially variable coefficients, part 2 : the variable coefficients case. *Electron. Trans. Numer. Anal.*, 40:170–186, 2013.

[15] F. Lemarié, P. Marchesiello, L. Debreu, and E. Blayo. Sensitivity of ocean-atmosphere coupled models to the coupling method : example of tropical cyclone Erica. Research report RR-8651, INRIA, December 2014. Available at `http://hal.inria.fr/hal-00872496/PDF/RR-8651.pdf`.

[16] J.-L. Lions, R. Temam, and S. Wang. Mathematical theory for the coupled atmosphere-ocean models. *J. Math. Pures Appl.*, 74:105–163, 1995.

[17] O. S. Madsen. A realistic model of the wind-induced Ekman boundary layer. *J. Phys. Oceanogr.*, 7:248 – 255, 1977.

[18] N. Perlin, E. D. Skyllingstad, R. M. Samelson, and P. L. Barbour. Numerical simulation of air-sea coupling during coastal upwelling. *J. Phys. Oceanogr.*, 37(8):2081–2093, 2007.