Variational problems on product spaces Different obstacle constraints

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Abstract

We study two principle minimizing problems, subject of different constraints. Our open sets are assumed bounded, except mentioning otherwise; precisely $\Omega =]0,1[^n \in \mathbb{R}^n, n = 1$ or $n = 2$.

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1 Introduction

Theorem 1.1 (Rellich-Kondrachov theorem)**.** *Suppose Ω is bounded and of class C^1 then, $W^{1,p} \subset L^p$ with compact injection for all p (and all n).*

Let $p \geq 2$ and $W^{1,p}(]0,1[;\mathbb{R}^2) = \{u = (u_1, u_2); u_1 \in W^{1,p}(]0,1[;\mathbb{R}), u_2 \in W^{1,p}(]0,1[;\mathbb{R})\}$

Define the functionals

1. $W^{1,p}(]0,1[;\mathbb{R}^2) \to \mathbb{R}_+$
$$u \to F(u) = \int_0^1 |u'(x)|_2^p = \int_0^1 \left(|u_1'(x)|^2 + |u_2'(x)|^2\right)^{\frac{p}{2}}$$
2. $W_0^{1,p}(\Omega;\mathbb{R}^2) \to \mathbb{R}_+$
$$u \to K(u) = \int_\Omega |\nabla u(x)|_2^p = \int_\Omega \left(|\nabla u_1(x)|_2^2 + |\nabla u_2(x)|_2^2\right)^{\frac{p}{2}}$$

Mainly, our goals are:

- show if that there exists $u_0 \in A_i$ unique such that, $G(u_0) = \inf\{G(u); u \in A_i\}$

- write the Euler-Lagrange equation satisfied by a 'smooth' u_0

Let us define the constraint sets:

1. $A_1 = \{u \in W^{1,p}(]0,1[;\mathbb{R}^2)$: $|u|_2^2 = \left(|u_1|^2 + |u_2|^2\right) = 1$ **a.e.** so $|u_i|_\infty \leq 1, u_1^2 = 1 - u_2^2; u(0) = (0,1), u(1) = (1,0)\}$

2. $A_2 = \{u \in W^{1,p}(\Omega;\mathbb{R}^2); u_1 = 0 \;\&\; u_2 = 1$ on $\partial\Omega; u_1 \in W_0^{1,p}(\Omega), |u|_2^2 = \left(|u_1|^2 + |u_2|^2\right) = 1$ **a.e.** so $|u_i|_\infty \leq 1, u_1^2 = 1 - u_2^2\}$

Note that the condition **a.e.** is implicitly important. One can notice that it could be written directly into equation $u_2 = \sqrt{1-(u_1)^2}$; without loss of generality we didn't do so. Clearly, boundary condition does not define a vector space, if $u_1(0) = 0, u_1(1) = 1$, we write $u_1 = g$ and $u_2 = 1 - g$ on $\partial\Omega$ and g may be a function defined on the open set Ω as well.

2 Solutions

Lemma 2.1. *$A_i \neq \phi$ for all i.*

Proof. For $i = 1$ consider the bounded smooth functionals

$$x \to u_1 = \begin{cases} \exp\left(\dfrac{x}{x^p - 1}\right) & \text{for any p} > 0 \text{ if } x \in [0.1[\\ 0 & \text{if not} \end{cases}$$

For $i = 2$, similarly but more explicitly we use the following proposition about partition of unity which lead to the result after a regularization process. □

Proposition 2.1. *let Ω be an open set of $\mathbb{R}^d$ and K a compact $\subset \Omega$. Then $\exists\ \Phi \in C_c(\mathbb{R}^d)$, such that*

$$0 \leq \Phi \leq 1, \qquad supp(\Phi) \subset \Omega.$$

Definition 2.1. *The p-norm on $\mathbb{R}^n$ is defined as:*

$$x = (x_1, \cdots, x_n) \in \mathbb{R}^n, p \in]0; +\infty[: x \to |x|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

and it is denoted by $|.|_p$.

Lemma 2.2. *If* $u_1 \in L^p(\Omega)$, *and* $u_2 \in L^p(\Omega)$ *then* $\left(|u_1(x)|^2 + |u_2(x)|^2\right)^{\frac{p}{2}} \in L^1(\Omega)$

Proof. Write $|u|_2 \leq C|u|_p$ for some $C > 0$. □

2.1 Existence and uniqueness

Note that product spaces such $V \times V$ are equipped with the sum norm that is $\|u\| + \|v\|$.
Usually we will study $K(u)^{\frac{1}{p}}$, and $F(u)^{\frac{1}{p}}$ as the norm L^p will appear explicitly. Before we state the main theorem, we have:

Proposition 2.2. $|v(\partial\Omega)| \leq C\|v\|_{W^{1,p}} \quad \forall v \in W^{1,p}]0,1[$
where $\partial\Omega := \{0,1\}$

Remark 2.1. *A minimizer of a positive valued function* f *is also a minimizer of* f^p *and conversely ,* $\forall p > 0$.

Theorem 2.1.

1. *There exists at least one function* $u = (u_1, u_2) \in A_1$ *solving* $F(u) = \min\limits_{w \in A_1} F(w)$.
2. *There exists at least one function* $u = (u_1, u_2) \in A_2$ *solving* $K(u) = \min\limits_{w \in A_2} K(w)$.

Proof. First $F(u)^{\frac{1}{p}}$ and $K(u)^{\frac{1}{p}}$ are both continuous convex functions thus weakly lower semi continuous. Also the constraints sets are weakly closed, in the sense that, if $u_n \rightharpoonup u$, and u_n satisfies any of the constraint, u will be as well . For the boundary condition that is $u = g$ on the boundary, choose any h satisfying same constraints, $u_n - h$ is a sequence $\in W_0^{1,p} \times W_0^{1,p}$, a convex closed subspace of $W^{1,p} \times W^{1,p}$, hence weakly closed.
For the condition of $|u_i|_\infty \leq 1$ a.e. it suffices to show that $|u_1|_\infty \leq 1$ a.e.

Take a sequence weakly convergent to u in $W^{1,p}$ by Rellich-K. Theorem we have a strong convergence at least in one of the L^p's. Thus we can extract a subsequence that converges a.e. to u. Giving that $|\Omega| < \infty$, by Egoroff theorem the a.e convergence is equivalent to uniform convergence, up to arbitrarily negligible sets. Since the set is closed for the uniform convergence, we conclude that $|u_i|_\infty \leq 1$, $i = 1, 2$ **a.e**

It could be said directly after the extraction of a subsequence a.e. convergent, that we have

$$|u_{k_j 1}|^2 + |u_{k_j 2}|^2 \to |u_1|^2 + |u_2|^2 = 1 \textbf{ a.e}$$

Remaining to show that the functionals verify a coercivity condition over the product space.

1. Set $f := \inf_{u \in A_1} F(u)$. If $f = +\infty$ we are done, suppose f is finite. Select a minimizing sequence$\{u_k\}$, then $F(u_k) \to f$ because we are in $\mathbb{R}$

   $$F(u) \geq C. \left(\|u_1\|^p_{W_0^{1,p}} + \|u_2\|^p_{W_0^{1,p}} \right) \geq CC' \left(\|u_1\|_{W_0^{1,p}} + \|u_2\|_{W_0^{1,p}} \right)^p$$

   One can verify because of boundary conditions (on A_1) that we have equivalence between the two norms $\|.\|_{W_0^{1,p}}$ and $\|.\|_{W^{1,p}}$ i.e.

   $$F(u_k) \geq \alpha \|u_k\| := \alpha \left(\|u_{k_1}\|_{W^{1,p}} + \|u_{k_2}\|_{W^{1,p}} \right)^p .$$

   This estimate implies that $\{u_k\}$ is bounded in $W^{1,p} \times W^{1,p}$. Consequently there exist a subsequence $\{u_{k_j}\}$ and a function $u \in W^{1,p} \times W^{1,p}$ such that; $u_{k_j} \rightharpoonup u$ weakly in $W^{1,p} \times W^{1,p}$, thus $F(u)$ is weakly lower semi continuous. $F(u) \leq \lim\inf_{j \to \infty} F(u_{k_j}) = f$, since $u \in A_1$ it follows that

   $$F(u) = f = \underset{u \in A_1}{min}\ F(u).$$

2. Similarly, set $m := \inf_{u\in A_2} K(u)$. If $m = +\infty$ we are done, suppose m is finite, select a minimizing sequence $\{u_k\}$, then $K(u_k) \to m$ because we are in $\mathbb{R}$.

$$K(u) = \int_\Omega |\nabla u(x)|_2^p = \int_\Omega \Big(|\nabla u_1(x)|_2^2 + |\nabla u_2(x)|_2^2\Big)^{\frac{p}{2}}$$

$$K(u) = \int_\Omega \left((\frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_1}{\partial x_2})^2 + (\frac{\partial u_2}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_2})^2 \right)^{\frac{p}{2}}$$

$$\geq C \int_\Omega (\frac{\partial u_1}{\partial x_1})^p + (\frac{\partial u_1}{\partial x_2})^p + (\frac{\partial u_2}{\partial x_1})^p + (\frac{\partial u_2}{\partial x_2})^p$$

$$\geq CC'(\|u_1\|^p_{W^{1,p}}) + C(\|\nabla\sqrt{1-{u_1}^2}\|^p_{L^p}) \tag{1}$$

Consequently

$$K(u_k) \geq min(CC', C)(\|u_{k_1}\|^p_{W^{1,p}} + \|\nabla\sqrt{1-{u_{k_1}}^2}\|^p_{L^p}), \tag{2}$$

and u_1 is bounded. But if u_1 is bounded so is u_2 and conversely for:

$$1 - \|u_1\|^2 \leq |1 - \|{u_1}^2\|| \leq \|1 - {u_1}^2\| = \|{u_2}^2\| \leq \|u_2\|^2$$

Thus we conclude that the sequence $\{u_k\}$ is bounded in $W^{1,p} \times W^{1,p}$ and the proof is similar to that of $F(u)$.

$\square$

Theorem 2.2. *The minimizing problem:* $F(u) = \min_{w\in A_1} F(w)$ *has a unique solution*

Proof. Suppose not, if u is a minimizer and a distinct minimizer v exists, $v := (v_1, v_2)$ write $w = (w_1, w_2) = \left(\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}\right)$ and recall that the Euclidean norm $|.|_2$ is strictly convex, which means that as long as

$$v' \neq \alpha.u' \text{ a.e.} \tag{3}$$

we have this strict inequality:

$$G(w)^{\frac{1}{p}} = \left[\int_0^1 |w'(x)|_2^p\right]^{\frac{1}{p}} = \left[\int_0^1 \left(\left(\frac{u_1'+v_1'}{2}\right)^2 + \left(\frac{u_2'+v_2'}{2}\right)^2\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} \tag{4}$$

$$< \frac{1}{2}\left[\int_0^1 \left(|u'|_2 + |v'|_2\right)^p\right]^{\frac{1}{p}} \tag{5}$$

$$\leq \frac{1}{2}G(u)^{\frac{1}{p}} + \frac{1}{2}G(v)^{\frac{1}{p}} \tag{6}$$

which contradicts the minimum property. This contradiction completes the proof if we showed that $v' \neq \alpha.u'$ a.e, suppose the converse and let $u = \beta v + cte$, if $u_1 = \beta_1 v_1 + cte_1$, applying boundary constraints and using Proposition 2.2 we conclude that $u_1 \neq \beta_1 v_1 + cte_1$ a.e. for any β_1 and any constant cte_1 □

3 Euler-Lagrange

Lemma 3.1. *$F(u)$ and $K(u)$ are both differentiable (C^1) on the product space except at* $(0,0)$

Proof. This follows by the regularity of the $|.|_2$ norm and derivation under integral sign. □

From this, we can compute the Euler-Lagrange equations giving the existence of a minimizer $(u_{0_1}, u_{0_2}) \neq 0$. Bearing in mind that C^1 Gateaux differentiable is the same as C^1 Frechet -differentiable. We will give the 'equation' satisfied by the 'minimizer' of $K(u)$ as it is the most general case. Fix $v \in W_0^{1,p}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$. Since $|u|_2 = 1$ a.e, we have

$$|u + \tau v|_2 \neq 0 \text{ a.e.}$$

for each sufficiently small τ by continuity. Consequently

$$v(\tau) := \frac{u + \tau v}{|u + \tau v|_2} \in A_2$$

Thus

$$k(\tau) := K(v(\tau)$$

has a minimum at $\tau = 0$, and so

$$k'(0) = 0.$$

Norms on product spaces are of course Euclidien norms, that is $|.|_2$. Matrices such the gradient matrix (here it's a 2×2 matrix) can be identified to a vector $\in \mathbb{R}^4$, and let $(.)$ denotes the usual scalar product on $\mathbb{R}^n$, by a direct computation we have:

Proposition 3.1. $v'(\tau) = \dfrac{v}{|u+\tau v|} - \dfrac{[(u+\tau v).v](u+\tau v)}{|u+\tau v|^3}$

Theorem 3.1. *Let $u \in A_2$ satisfy*

$$K(u) = \min_{w \in A_2} K(w).$$

Then

$$\int_\Omega p|Du|^{\frac{p}{2}-1}[(Du.Dv) - |Du|^2(u.v)] \tag{7}$$

for each $v \in W_0^{1,p}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$.

Proof. In fact

$$K(u) = \int_\Omega |Du|^p$$

where Du is the gradient matrix associated to u and the norm as said is the one associated to the scalar product:$< A, B >= Tr(B^t.A)$.

$$k'(0) = 0 = \int_\Omega p|Du|^{\frac{p}{2}-1}\ Du.Dv'(0) \tag{8}$$

$$= \int_\Omega p|Du|^{\frac{p}{2}-1} Du.D(v - (u.v)u)$$

Upon differentiating $|u|^2 = 1$ a.e., we have

$$(Du)^T u = 0$$

Using this fact, we then verify

$$Du.(D(u.v)u) = |Du|^2(u.v) \quad \text{a.e. in } \Omega$$

This identity employed in (7) gives (6). We leave details to the interested reader. [2]

$\square$

References

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