Symmetric Norm Inequalities And Positive Semi-Definite Block-Matrices

Antoine Mhanna

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Antoine Mhanna[1]

1 Dept of Mathematics, Lebanese University, Hadath, Beirut, Lebanon.

tmhanat@yahoo.com

Abstract

For positive semi-definite block-matrix M, we say that M is P.S.D. and we write $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}^+_{n+m}$, with $A \in \mathbb{M}^+_n$, $B \in \mathbb{M}^+_m$. The focus is on studying the consequences of a decomposition lemma due to C. Bourrin and the main result is extending the class of P.S.D. matrices M written by blocks of same size that satisfies the inequality: $\|M\| \le \|A + B\|$ for all symmetric norms.

Keywords : Matrix Analysis, Hermitian matrices, symmetric norms.

1 Introduction

Let A be an $n \times n$ matrix and F an $m \times m$ matrix, $(m > n)$ written by blocks such that A is a diagonal block and all entries other than those of A are zeros, then the two matrices have the same singular values and for all unitarily invariant norms $\|A\| = \|F\| = \|A \oplus 0\|$, we say then that the symmetric norm on $\mathbb{M}_m$ induces a symmetric norm on $\mathbb{M}_n$, so for square matrices we may assume that our norms are defined on all spaces $\mathbb{M}_n$, $n \ge 1$. The spectral norm is denoted by $\|.\|_s$, the Frobenius norm by $\|.\|_{(2)}$, and the Ky Fan $k-$norms by $\|.\|_k$. Let $\mathbb{M}^+_n$ denote the set of positive and semi-definite part of the space of $n \times n$ complex matrices and M be any positive semi-definite block-matrices; that is, $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}^+_{n+m}$, with $A \in \mathbb{M}^+_n$, $B \in \mathbb{M}^+_m$.

2 Decomposition of block-matrices

Lemma 2.1. *For every matrix M in $\mathbb{M}^+_{n+m}$ written in blocks, we have the decomposition:*
$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}U^* + V\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}V^*$
for some unitaries $U, V \in \mathbb{M}_{n+m}$.

Proof. Factorize the positive matrix as a square of positive matrices:

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} \cdot \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix}$$

we verify that the right hand side can be written as $T^*T + S^*S$ so :

$$\begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} \cdot \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} = \underbrace{\begin{pmatrix} C & 0 \\ Y^* & 0 \end{pmatrix}}_{T^*} \cdot \underbrace{\begin{pmatrix} C & Y \\ 0 & 0 \end{pmatrix}}_{T} + \underbrace{\begin{pmatrix} 0 & Y \\ 0 & D \end{pmatrix}}_{S^*} \cdot \underbrace{\begin{pmatrix} 0 & 0 \\ Y^* & D \end{pmatrix}}_{S}.$$

Since $TT^* = \begin{pmatrix} CC+YY^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $SS^* = \begin{pmatrix} 0 & 0 \\ 0 & Y^*Y+DD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ and AA^* is unitarily congruent to A^*A for any square matrix A, the lemma follows. $\square$

Remark 1. *As a consequence of this lemma we have:*

$$\|M\| \leq \|A\| + \|B\|$$

for all symmetric norms.

Equations involving unitary matrices are called unitary orbits representations. Recall that if $A \in \mathbb{M}_n$, $R(A) = \dfrac{A+A^*}{2}$ and $I(A) = \dfrac{A-A^*}{2i}$.

Corollary 2.1. *For every matrix in $\mathbb{M}^+_{2n}$ written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U\begin{pmatrix} \frac{A+B}{2} - R(X) & 0 \\ 0 & 0 \end{pmatrix}U^* + V\begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} + R(X) \end{pmatrix}V^*$$

for some unitaries $U, V \in \mathbb{M}_{2n}$.

Proof. Let $J = \dfrac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ where I is the identity of $\mathbb{M}_n$, J is a unitary matrix, and we have:

$$J\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}J^* = \underbrace{\begin{pmatrix} \frac{A+B}{2} - R(X) & \frac{A-B}{2} + \frac{X^*-X}{2} \\ \frac{A-B}{2} - \frac{X-X^*}{2} & \frac{A+B}{2} + R(X) \end{pmatrix}}_{N}$$

Now we factorize N as a square of positive matrices:

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = J^* \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} \cdot \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} J$$

and let:

$$\begin{aligned} \delta &= J^* \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} L+M^* & M+F \\ M^*-L & F-M \end{pmatrix} \\ \psi &= \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} J &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} L+M & M-L \\ F+M^* & F-M^* \end{pmatrix} \end{aligned}$$

A direct computation shows that:

$$\begin{aligned} \delta.\psi &= \frac{1}{2} \begin{pmatrix} {\scriptstyle (L+M^*)(L+M)+(M+F)(F+M^*)} & {\scriptstyle (L+M^*)(M-L)+(M+F)(F-M^*)} \\ & \\ {\scriptstyle (M^*-L)(L+M)+(F-M)(F+M^*)} & {\scriptstyle (M^*-L)(M-L)+(F-M)(F-M^*)} \end{pmatrix} \\ &= \Gamma^*\Gamma + \Phi^*\Phi \qquad (1) \end{aligned}$$

where: $\Gamma = \dfrac{1}{\sqrt{2}} \begin{pmatrix} L+M & M-L \\ 0 & 0 \end{pmatrix}$, and $\Phi = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ F+M^* & F-M^* \end{pmatrix}$ to finish notice that for any square matrix A, A^*A is unitarily congruent to AA^* and, $\Gamma\Gamma^*$, $\Phi\Phi^*$ have the required form. $\square$

The previous corollary implies that $\frac{A+B}{2} \geq R(X)$ and $\frac{A+B}{2} \geq -R(X)$.

Corollary 2.2. *For every matrix in $\mathbb{M}_{2n}^+$ written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} + I(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - I(X) \end{pmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{n+m}$.

Proof. The proof is similar to Corollary 2.1, we have: $J_1 \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} J_1^* = \begin{pmatrix} A & iX \\ -iX^* & B \end{pmatrix}$ where $J_1 = \begin{pmatrix} I & 0 \\ 0 & -iI \end{pmatrix}$, and

$$K = JJ_1 \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} J_1^* J^* = \begin{pmatrix} \frac{A+B}{2} + I(X) & * \\ & \\ * & \frac{A+B}{2} - I(X) \end{pmatrix}$$

here (*) means an unspecified entry, the proof is similar to that in Corollary 2.1 but for reader's convenience we give the main headlines: first factorize K as a square of positive matrices; that is, $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = J_1^* J^* L^2 J J_1$ next decompose L^2 as in Lemma 2.1 to obtain

$$M = J_1^* J^* (T^*T + S^*S) JJ_1 = J_1^* J^* (T^*T) JJ_1 + J_1^* J^* (S^*S) JJ_1$$

where $TT^* = \begin{pmatrix} \frac{A+B}{2} + I(X) & 0 \\ 0 & 0 \end{pmatrix}$ and $SS^* = \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - I(X) \end{pmatrix}$ finally the congruence property completes the proof. $\square$

The existence of unitaries U and V in the decomposition process need not to be unique as one can take the special case; that is, M any diagonal matrix with diagonal entries equals a nonnegative number k, explicitly $M = kI = U\left(\frac{k}{2}I\right)U^* + V\left(\frac{k}{2}I\right)V^*$ for any U and V unitaries.

Remark 2. *Notice that from the Courant-Fischer theorem if $A, B \in \mathbb{M}_n^+$, then the eigenvalues of each matrix are the same as the singular values and $A \leq B \Longrightarrow \|A\|_k \leq \|B\|_k$, for all $k = 1, \cdots, n$, also $A < B \Longrightarrow \|A\|_k < \|B\|_k$, for all $k = 1, \cdots, n$.*

Corollary 2.3. *For every matrix in $\mathbb{M}_{2n}^+$ written in blocks of the same size, we have:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \leq \frac{1}{2}\left\{U\begin{pmatrix} A+B+|X-X^*| & 0 \\ 0 & 0 \end{pmatrix}U^* + V\begin{pmatrix} 0 & 0 \\ 0 & A+B+|X-X^*| \end{pmatrix}V^*\right\}$$

for some unitaries $U, V \in \mathbb{M}_{n+m}$.

Proof. This a consequence of the fact that $I(X) \leq |I(X)|$. $\square$

3 Symmetric Norms and Inequalities

In [1] they found that if X is hermitian then

$$\|M\| \leq \|A+B\| \tag{2}$$

for all symmetric norms. It has been given counter-examples showing that this does not necessarily holds if X is a normal but not Hermitian matrix, the main idea of this section is to give examples and counter-examples in a general way and to extend the previous inequality to a larger class of P.S.D. matrices written by blocks satisfying (2).

Theorem 3.1. *If A and B are positive definite matrices of same size. Then*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} > 0 \Longleftrightarrow A \geq XB^{-1}X^*$$

Proof. Write $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = \begin{pmatrix} I & XB^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - XB^{-1}X^* & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ XB^{-1} & I \end{pmatrix}$ where I is the identity matrix, and that complete the proof since for any matrix A,

$$A \geq 0 \Longleftrightarrow X^*AX \geq 0, \ \forall X.$$

□

Theorem 3.2. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be any square matrix written by blocks of same size, if* $AC = CA$ *then* $\det(M) = \det(AD - CB)$

Proof. Suppose first that A is invertible, let us write M as

$$M = \begin{pmatrix} Z & 0 \\ V & I \end{pmatrix} \begin{pmatrix} I & E \\ 0 & F \end{pmatrix} \tag{3}$$

upon calculation we find that: $Z = A$, $V = C$, $E = A^{-1}B$, $F = D - CA^{-1}B$ taking the determinant on each side of (3) we get:

$$\det(M) = \det(A(D - CA^{-1}B)) = \det(AD - CB)$$

the result follows by a continuity argument since the Determinant function is a continuous function. □

Given the matrix $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ a matrix in $\mathbb{M}_{2n}^+$ written by blocks of same size, we know that it M is not P.S.D., to see this notice that all the 2×2 extracted principle submatrices of M are P.S.D if and only if $X = 0$ and A is positive semi-definite. Even if a proof of this exists and would take two lines, it is quite nice to see a different constructive proof, a direct consequence of Lemma 2.1.

Theorem 3.3. *Given* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *a matrix in* $\mathbb{M}_{2n}^+$ *written in blocks of same size:*

1. *If* $\begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ *is positive semi-definite,* $I(X) > 0$ *or* $I(X) < 0$*, then there exist a matrix* Y *such that* $M = \begin{pmatrix} A & Y \\ Y^* & 0 \end{pmatrix}$ *is positive semi-definite and:*

$$\left\| \begin{pmatrix} A & Y \\ Y^* & 0 \end{pmatrix} \right\| > \|A\| \tag{4}$$

for all symmetric norms.

2. *If* $\begin{pmatrix} 0 & X \\ X^* & B \end{pmatrix}$ *is positive semi-definite,* $I(X) > 0$ *or* $I(X) < 0$ *then there exist a matrix* Y *such that* $M = \begin{pmatrix} 0 & Y \\ Y^* & B \end{pmatrix}$ *is positive semi-definite and:*

$$\left\| \begin{pmatrix} 0 & Y \\ Y^* & B \end{pmatrix} \right\| > \|B\| \tag{5}$$

The same result holds if we replaced $I(X)$ *by* $R(X)$ *because* $\begin{pmatrix} A & iX \\ -iX^* & B \end{pmatrix}$ *is unitarily congruent to* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$.

Proof. Without loss of generality we can consider $I(X) > 0$ cause $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ and $\begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix}$ are unitarily congruent, we will show the first statement as the second one has a similar proof, from Corollary 2.2 we have:

$$\begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix} \geq U \begin{pmatrix} \frac{A}{2} & 0 \\ 0 & 0 \end{pmatrix} U^* + U \begin{pmatrix} I(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} \end{pmatrix} V^*$$

Since $\begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ is congruent to $L = \begin{pmatrix} A & lX \\ lX^* & 0 \end{pmatrix}$ for any $l \in \mathbb{C}$, L is P.S.D. A is a fixed matrix, we have $\left\| U \begin{pmatrix} \frac{A}{2} & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} \end{pmatrix} V^* \right\|_k = \beta \|A\|_k$ for some $\beta \leq 1$ finally we set $Y = lX$ where $l \in \mathbb{R}$ is large enough to have $\|M\|_k > \|A\|_k$, $\forall k$ thus $\|M\| > \|A\|$ for all symmetric norms. □

Notice that there exist a permutation matrix P such that $P \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & X^* \\ X & A \end{pmatrix}$ and since $I(X) > 0$ if and only if $I(X^*) < 0$, the two assertions of Theorem 3.3 are equivalent up to a permutation similarity.

Corollary 3.1. *If* $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$, A *a positive semi-definite matrix, and we have one of the following conditions:*

1. $R(X) > 0$

2. $R(X) < 0$

3. $I(X) > 0$

4. $I(X) < 0$

Then M can't be positive semi-definite.

Proof. By Remark 1 any positive semi-definite matrix M written in blocks must satisfy $\|M\| \le \|A\| + \|B\|$ for all symmetric norms which is not the case of the matrix M constructed in Theorem 3.3. $\square$

Finally we get:

Theorem 3.4. *If $X \neq 0$ and $B = 0$, $A \geq 0$, the matrix $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ cannot be positive semi-definite.*

Proof. Suppose the converse, so $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ is positive semi-definite, without loss of generality the only case we need to discuss is when $R(X)$ has positive and negative eigenvalues, by Corollary 2.1 we can write:

$$M = U \begin{pmatrix} \frac{A}{2} - R(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} + R(X) \end{pmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{2n}$. Now if $R(X)$ has $-\alpha$ the smallest negative eigenvalue $R(X) + (\alpha + \epsilon)I > 0$ consequently the matrix

$$H = U \begin{pmatrix} \frac{A}{2} - R(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} + (\alpha+\epsilon)I + R(X) + (\alpha+\epsilon)I \end{pmatrix} V^* \tag{6}$$

$$= \begin{pmatrix} A + 2(\alpha+\epsilon)I & X + (\alpha+\epsilon)I \\ (X + (\alpha+\epsilon)I)^* & 0 \end{pmatrix} \tag{7}$$

is positive semi-definite with $R(Y) > 0$, where $Y = X + (\alpha + \epsilon)I$, by Corollary 3.1 this is a contradiction. $\square$

A natural question would be how many are the nontrivial P.S.D.matrices written by blocks ? The following lemma will show us how to construct some of them.

Lemma 3.1. *Let A and B be any $n \times n$ positive definite matrices, then there exist an integer $t \geq 1$ such that the matrix $F_t = \begin{pmatrix} tA & X \\ X^* & tB \end{pmatrix}$ is positive definite.*

Proof. Recall from Theorem 3.1 that F_1 is positive definite if and only if $A > XB^{-1}X^*$, which is equivalent to $x^*Ax > x^*XB^{-1}X^*x$ for all $x \in \mathbb{C}^n$. Set $f(x) := x^*Ax$ and $g(x) := x^*XB^{-1}X^*x$ and let us suppose, to the contrary, that there exist a vector z such that $f(z) \leq g(z)$ since $f(x)$ and $g(x)$ are homogeneous functions of degre $d = 2$ over $\mathbb{R}$ if $f(x) \geq g(x)$ for all x such that $\|x\|_s = 1$ then $f(x) \geq g(x)$ for any $x \in \mathbb{C}^n$. So let us set $K = \max_{\|x\|_s=1} g(x)$, and $L = \min_{\|x\|_s=1} f(x)$ since $g(x)$ and $f(x)$ are continuous functions and $\{x; \|x\|_s = 1\}$ is compact, there exist a vector w respectively v such that $K = g(w)$, respectively $L = f(v)$. Now choose $t \geq 1$ such that $tf(v) > \dfrac{g(w)}{t}$, to obtain

$$x^*(tA)x \geq v^*(tA)v > w^*X(tB)^{-1}X^*w \geq x^*X(tB)^{-1}X^*x$$

for all x such that $\|x\|_s = 1$, thus $x^*(tA)x > x^*X(tB)^{-1}X^*x$ for any $x \in \mathbb{C}^n$ which completes the proof. $\square$

Theorem 3.5. *Let $A = diag(\lambda_1, \cdots, \lambda_n)$, $B = diag(\nu_1, \cdots, \nu_n)$ and $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ a given positive semi-definite matrix. If X^* commute with A and X^*X equals a diagonal matrix, then*

$$\|M\| \leq \|A + B\|$$

for all symmetric norms. The same inequality holds if X commute with B and XX^* is diagonal.

Proof. It suffices to prove the inequality for the Ky Fan $k-$norms $k = 1, \cdots, n$, let $P = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ where I_n is the identity matrix of order n, since $\begin{pmatrix} B & X^* \\ X & A \end{pmatrix} = P \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} P^{-1}$ and $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ have same singular values, we will discuss only the first case; that is, when X^* commute with A and X^*X is diagonal, as the second case will follows. Let $D := X^*X = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$, as X^* commute with A, from Theorem 3.2 we conclude that the eigenvalues of $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ are the roots of

$$\det((A - \mu I_n)(B - \mu I_n) - D) = 0$$

Equivalently the eigenvalues are all the solutions of the n equations:

$$\begin{array}{rrcl}
1) & (\lambda_1-\mu)(\nu_1-\mu)-d_1 & = & 0 \\
2) & (\lambda_2-\mu)(\nu_2-\mu)-d_2 & = & 0 \\
3) & (\lambda_3-\mu)(\nu_3-\mu)-d_3 & = & 0 \\
\vdots & & \vdots & \\
i) & (\lambda_i-\mu)(\nu_i-\mu)-d_i & = & 0 \\
\vdots & & \vdots & \\
n) & (\lambda_n-\mu)(\nu_n-\mu)-d_n & = & 0
\end{array}$$

Each equation is of 2^{nd} degree, if we denote by a_i and b_i the two solutions of the i^{th} equation we deduce that:

$$\begin{array}{rcl}
a_1+b_1 & = & \lambda_1+\nu_1 \\
a_2+b_2 & = & \lambda_2+\nu_2 \\
& \vdots & \\
a_n+b_n & = & \lambda_n+\nu_n
\end{array}$$

But

$$A+B=\begin{pmatrix} \lambda_1+\nu_1 & 0 & \cdots & 0 \\ 0 & \lambda_2+\nu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n+\nu_n \end{pmatrix}$$

and each diagonal entry of $A+B$ is equal the sum of two nonegative eigenvalues of M, thus we have necessarily: $\|M\|_k \leq \|A+B\|_k$ for all $k=1,\cdots,n$ which completes the proof. $\square$

Example 3.1. *Let*

$$M_x=\begin{pmatrix} x & 0 & \frac{i}{2} & 0 \\ 0 & \frac{99}{100} & 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 & \frac{99}{100} & 0 \\ 0 & \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

If $\frac{3}{10} \leq x \leq \frac{1}{2}$, *$M_x$ is positive definite and we have:*

$$\|M_x\| \leq \|A+B\| \tag{8}$$

for all symmetric norms, where $A=\begin{pmatrix} x & 0 \\ 0 & \frac{99}{100} \end{pmatrix}$ *and* $B=\begin{pmatrix} \frac{99}{100} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. *If M_x is positive definite for* $x=\frac{3}{10}$ *then M_x is P.D. for all* $x>\frac{3}{10}$. *The eigenvalues of* $M_{\frac{3}{10}}$ *which are* *the same as the singular values of* $M_{\frac{3}{10}}$ *are:*

$$\lambda_1 = \frac{149}{200} + \frac{\sqrt{12401}}{200} \approx 1.301 \tag{9}$$

$$\lambda_2 = \frac{129}{200} + \frac{\sqrt{14761}}{200} \approx 1.25 \tag{10}$$

$$\lambda_3 = \frac{149}{200} - \frac{\sqrt{12401}}{200} \approx 0.188 \tag{11}$$

$$\lambda_4 = \frac{129}{200} - \frac{\sqrt{14761}}{200} \approx 0.0375 \tag{12}$$

And the (8) *inequality follows from Theorem* 3.5*.*

Let us study the commutation condition in Theorem 3.5. First notice that any square matrix $X = (x_{ij}) \in \mathbb{M}_n$ will commute with $A = \text{diag}(a_1, \cdots, a_n)$ if and only if :

$$Y' = \begin{pmatrix} x_{1,1}a_1 & x_{1,2}a_2 & \cdots & x_{1,n}a_n \\ x_{2,1}a_1 & x_{2,2}a_2 & \cdots & x_{2,n}a_n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}a_1 & x_{n,2}a_2 & \cdots & x_{n,n}a_n \end{pmatrix} = \begin{pmatrix} x_{1,1}a_1 & x_{1,2}a_1 & \cdots & x_{1,n}a_1 \\ x_{2,1}a_2 & x_{2,2}a_2 & \cdots & x_{2,n}a_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}a_n & x_{n,2}a_n & \cdots & x_{n,n}a_n \end{pmatrix} = Y$$

An (i, j) entry of Y' is equal to that of Y if and only if $x_{i,j}a_j = x_{i,j}a_i$, i.e. either $a_i = a_j$ or $x_{i,j} = 0$.

Corollary 3.2. *Let* $A = diag(\lambda_1, \cdots, \lambda_n)$*,* $B = diag(\nu_1, \cdots, \nu_n)$ *and* $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *a given positive semi-definite matrix. If* X^* *commute with* A*, or* X *commute with* B*, then* $\|M\| \leq \|A + B\|$ *for all symmetric norms.*

Proof. As in Theorem 3.5, we will assume without loss of generality that X^* commute with A, as the other case is similar. If X^* is diagonal the result follows from Theorem 3.5, suppose there is an off diagonal entry $x_{i,j}$ of X^* different from 0, from the commutation condition we have $a_i = a_j$ and the same goes for all such entries, of course if $AX = XA$ then

$$PAXP^{-1} = PXAP^{-1} = PAP^{-1}PXP^{-1} = PXP^{-1}PAP^{-1} = PXAP^{-1}$$

Take P to be the permutation matrix that will order the same diagonal entries of A in a one diagonal block and keeps the matrix B the same, since M is Hermitian so is PMP^{-1}

because we can consider the permutation matrix as a product of transposition matrices $P_1, \cdots, P_n$ wich are orthogonal; in other words

$$PMP^{-1} = P_1 P_2 \cdots P_n M P_n^T \cdots P_2^T P_1^T.$$

Consequently $P^T = P^{-1}$ for any permutation matrix and $\|M\| = \|PMP^T\|$ for all symmetric norms. If $H = PMP^T$, $D := PX$ and X_i is some $i \times i$ extracted submatrix of X^*, we will have the block written matrix

$$H = \begin{pmatrix} PAP^T & PX \\ X^* P^T & B \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} aI_i & O_j & \cdots & O_s \\ O_i & bI_j & \cdots & O_s \\ \vdots & \vdots & \ddots & \vdots \\ O_i & O_j & \cdots & rI_s \end{pmatrix} & \begin{pmatrix} X_i^* & O_i & \cdots & O_i \\ O_j & X_j^* & \cdots & O_j \\ \vdots & \vdots & \ddots & \vdots \\ O_s & O_s & \cdots & X_s^* \end{pmatrix} \\ \begin{pmatrix} X_i & O_j & \cdots & O_s \\ O_i & X_j & \cdots & O_s \\ \vdots & \vdots & \ddots & \vdots \\ O_i & O_j & \cdots & X_s \end{pmatrix} & \begin{pmatrix} \nu_1 & 0 & \cdots & 0 \\ 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu_n \end{pmatrix} \end{pmatrix}$$

where we denoted the diagonal matrix of order i whose diagonal entries are equal to a by aI_i and the zero block of order i by O_i. Let us calculate the roots of the characteristic polynomial of H; that is, the roots of

$$\det\left(\begin{pmatrix} (a-\lambda)I_i & O_j & \cdots & O_s \\ O_i & (b-\lambda)I_j & \cdots & O_s \\ \vdots & \vdots & \ddots & \vdots \\ O_i & O_j & \cdots & (r-\lambda)I_s \end{pmatrix} \begin{pmatrix} \nu_1-\lambda & 0 & \cdots & 0 \\ 0 & \nu_2-\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu_n-\lambda \end{pmatrix} - D^* D \right) = 0$$

we translate this to a system of blocks, while each eigenvalue of H, which is the same as its singular value, will verify one of the following equations:

$$\begin{array}{lll} 1) \quad \det\left((a-\lambda)I_i)\left(\begin{pmatrix} \nu_1-\lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i-\lambda \end{pmatrix} \right) - X_i^* X_i \right) & = & 0 \\ 2) \quad \det\left((b-\lambda)I_j)\left(\begin{pmatrix} \nu_{i+1}-\lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_{i+j}-\lambda \end{pmatrix} \right) - X_j^* X_j \right) & = & 0 \\ \vdots & & \vdots \\ c) \quad \det\left((r-\lambda)I_s)\left(\begin{pmatrix} \nu_{n-s}-\lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_n-\lambda \end{pmatrix} \right) - X_s^* X_s \right) & = & 0 \end{array} \qquad (T)$$

where c is the number of diagonal blocks we have. Let us have a closer look to any of the equations above, without loss of generality we will take the first one, the same will hold for the others, notice that all eigenvalues λ are nonnegative and we have

$$M_1 = \begin{pmatrix} aI_i & X_i^* \\ X_i & \begin{pmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} C_1 & X_i^* \\ X_i^* & K_1 \end{pmatrix}$$

is positive semi-definite because it's eigenvalues are a subset of those of M. The key idea is that for this matrix $\|C_1+K_1\| = \|C_1\|+\|K_1\|$ for all symmetric norms. where $C_1 = aI_i$ and $K = \begin{pmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i \end{pmatrix}$. Now back to the system (T) we associate like we did to M_1 each equation whose number is i to a positive semi-definite matrix M_i to obtain by Remark 1

$$\begin{aligned}
\|M_1\|_k &\leq \left\| aI_i + \begin{pmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i \end{pmatrix} \right\|_k &&= \|aI_i\|_k + \left\| \begin{pmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i \end{pmatrix} \right\|_k \\
\|M_2\|_k &\leq \left\| bI_j + \begin{pmatrix} \nu_{i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_{i+j} \end{pmatrix} \right\|_k &&= \|bI_j\|_k + \left\| \begin{pmatrix} \nu_{i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_{i+j} \end{pmatrix} \right\|_k \\
&\vdots &&\vdots \\
\|M_c\|_k &\leq \left\| rI_s + \begin{pmatrix} \nu_{n-s} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_n \end{pmatrix} \right\|_k &&= \|rI_s\|_k + \left\| \begin{pmatrix} \nu_{n-s} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_n \end{pmatrix} \right\|_k
\end{aligned}$$

for all k, but the order of the entries of B are arbitrary chosen, thus from Theorem 3.5 $\|M\|_k \leq \|A+B\|_k$ for all $k = 1, \cdots, n$ and that completes the proof. □

Corollary 3.3. *Let* $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *be a positive semi-definite matrix written by blocks. There exist a unitary* V *and a unitary* U *such that*

$$\left\| \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \right\| \leq \|UAU^* + VBV^*\| := \|A\| + \|B\|$$

for all symmetric norms.

Proof. Let U and V be two unitary matrix such that $UAU^* = D_o$ and $VBV^* = G_o$ where D_o and G_o are two diagonal matrices having the same ordering o, of eigenvalues with respect to their indexes i.e., if $\lambda_n \leq \cdots \leq \lambda_1$ are the diagonal entries of D_o, and $\nu_n \leq \cdots \leq \nu_1$ are those of G_o, then if λ_i is in the (j,j) position then ν_i will be also. Consequently $\|UAU^* + VBV^*\| = \|D_o + G_o\| = \|D_o\| + \|G_o\| = \|A\| + \|B\|$, for all the Ky-Fan $k-$norms and thus for all symmetric norms. To complete the proof notice that if $T = UXV^*$ and Q is the unitary matrix $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, by Remark 1

$$\left\| \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \right\| = \left\| Q \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} Q^* \right\| = \left\| \begin{pmatrix} D_o & T \\ T^* & G_o \end{pmatrix} \right\| \leq \|D_o\| + \|G_o\| \tag{13}$$

for all symmetric norms. □

Theorem 3.6. *Let* $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$, *if* X *is normal,* X^* *commute with* A *and* X *commute with* B, *then we have* $\|M\| \leq \|A+B\|$ *for all symmetric norms.*

Proof. We consider first that the normal matrix X^* has all of its eigenvalues distinct, by Theorem **??** and the normality condition, there exist a unitary matrix U such that U^*AU and U^*X^*U are both diagonal. A direct computation shows that:

$$\begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} U^*AU & U^*XU \\ U^*X^*U & U^*BU \end{pmatrix} = \mathcal{G}.$$

Now U^*XU also commute with U^*BU, since U^*XU is diagonal and all of its diagonal entries are distinct by Remark **??** U^*BU must be also diagonal, applying Theorem 3.5 to the matrix $\mathcal{G}$ yields to:

$$\left\| \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \right\| = \|\mathcal{G}\| \leq \|U^*AU + U^*BU\| = \|A+B\|,$$

for all symmetric norms. The inequality holds for any X normal by a continuity argument. $\square$

Lemma 3.2. *Let*

$$N = \begin{pmatrix} \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} & D \\ D^* & \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \end{pmatrix}$$

where $a_1, \cdots, a_n$ *respectively* $b_1, \cdots, b_n$ *are nonnegative respectively negative real numbers,* $A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$ *and* D *is any diagonal matrix, then nor* N *neither* $-N$ *is positive semi-definite. Set* $(d_1, \cdots, d_n)$ *as the diagonal entries of* D^*D, *if* $a_i + b_i \geq 0$ *and* $a_i b_i - d_i < 0$ *for all* $i \leq n$, *then* $\|N\| > \|A+B\|$. *for all symmetric norms*

Proof. The diagonal of N has negative and positive numbers, thus nor N neither $-N$ is positive semi-definite, now any two diagonal matrices will commute, in particular D^* and A, by applying Theorem 3.2 we get that the eigenvalues of N are the roots of

$$\det((A - \mu I_n)(B - \mu I_n) - D^*D) = 0$$

Equivalently the eigenvalues are all the solutions of the n equations:

$$\begin{array}{lrcl}
1) & (a_1-\mu)(b_1-\mu)-d_1 & = & 0 \\
2) & (a_2-\mu)(b_2-\mu)-d_2 & = & 0 \\
3) & (a_3-\mu)(b_3-\mu)-d_3 & = & 0 \\
\vdots & & \vdots & \\
i) & (a_i-\mu)(b_i-\mu)-d_n & = & 0 \\
\vdots & & \vdots & \\
n) & (a_n-\mu)(b_n-\mu)-d_n & = & 0
\end{array} \qquad (S)$$

Let us denote by x_i and y_i the two solutions of the i^{th} equation then:

$$\begin{array}{rcrcl}
x_1+y_1 & = & a_1+b_1 & \geq & 0 \\
x_2+y_2 & = & a_2+b_2 & \geq & 0 \\
\vdots & & \vdots & & \\
x_n+y_n & = & a_n+b_n & \geq & 0
\end{array} \qquad \begin{array}{rcrcl}
x_1y_1 & = & a_1b_1-d_1 & < & 0 \\
x_2y_2 & = & a_2b_2-d_2 & < & 0 \\
\vdots & & & & \vdots \\
x_ny_n & = & a_nb_n-d_n & < & 0
\end{array}$$

This implies that each equation of (S) has one negative and one positive solution, their sum is positive, thus the positive root is bigger or equal than the negative one. Since $A+B = \begin{pmatrix} a_1+b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n+b_n \end{pmatrix}$, summing over indexes we see that $\|N\|_k > \|A+B\|_k$ for $k=1,\cdots,n$ which yields to $\|N\| > \|A+B\|$ for all symmetric norms $\square$

It seems easy to construct examples of non P.S.D matrices N written in blocks such that $\|N\|_s > \|A+B\|_s$, let us have a look of such inequality for P.S.D. matrices.

Example 3.2. *Let*

$$C = \begin{pmatrix} \frac{4}{3} & 0 & 1 & -1 \\ 0 & 1 & 0 & \frac{1}{5} \\ 1 & 0 & \frac{3}{2} & 0 \\ -1 & \frac{1}{5} & 0 & 2 \end{pmatrix} = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

where $A = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix}$. *Since the eigenvalues of C are all positive with* $\lambda_1 \approx 3.008,\ \lambda_2 \approx 1.7,\ \lambda_3 \approx 0.9,\ \lambda_4 \approx 0.089$, *$C$ is positive definite and we verify that*

$$3.008 \approx \|C\|_s > \|A+B\|_s = 3$$

Example 3.3. *Let*

$$N_y = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

where $A = \begin{pmatrix} 2 & 0 \\ 0 & y \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. *The eigenvalues of* N_y *are the numbers:* $\lambda_1 = 4,\ \lambda_2 = 1,\ \lambda_3 = y,\ \lambda_4 = 0,$ *thus if* $y \geq 0$, N_y *is positive semi-definite and for all* y *such that* $0 \leq y < 1$ *we have*

1. $4 = \|N_y\|_s > \|A + B\|_s = 3$

2. $16 + y^2 + 1 = \|N\|_{(2)}^2 > \|A + B\|_{(2)}^2 = 4(3 + y) + y^2 + 1$

References

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