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In this paper, we provide the second part of the study of the pseudo-permutations. We first derive a complete analysis of the pseudo-permutations, based on hyperplane arrangements, generalizing the usual way of translating the permutations. We then study the module of the pseudo-permutations over the symmetric group and provide the characteristics of this action.

Keywords: Hyperplane Arrangements, Symmetric Group, Permutations, *q*-analogs

1 Introduction

Representation and reasoning about time-dependent information is an active and important research area in Artificial Intelligence and in Computer Science. Vilain and Kautz's point algebra (see [VK86]) is one of the most influential models used for representing and reasoning with qualitative time information, because the reasoning tasks are solvable in polynomial time in that model, contrary to more powerful models, as Allen's Interval Algebra (see [All81]) in which the reasoning tasks are computationally intractable in general. The problem is the following: one considers a set of events which happen at certain dates, and wants to use this information to solve a problem, take a decision. However, it is often not meaningful *when* the events occur, while the relevant informations are the *temporal relations* between events: did event i happened before, during or after event j ? In this context, it is natural to represent the temporal relations between n events by an ordered sequence of nonempty parts of $[\![1,n]\!]$ such that each integer appears exactly once. If i is in a part of $[\![1,n]\!]$ which appears in the sequence before the part which contains j, then the event i happened before the event j. If they appear in the same part, they occured simultaneously. For example, the sequence $\{1\}\{3,4\}\{2\}$ means that event 1 occured first, event 2 occured last, and events 3 and 4 occured at the same time. In the following, we will call such a sequence a *pseudo-permutation of order n* (in the example, $n = 4$), we will use parentheses instead of braces, and we will remove unnecessary comas. Therefore, we will write the example as $(1)(34)(2)$.

Having defined the pseudo-permutations, one can study their combinatorial properties. They appear to have a very rich structure and many interesting properties that were first presented in [KLNPS00]. Among those, one can see that their enumeration is related to Eulerian numbers (see [Com70, FS70]), and that an inversion table associated with each pseudo-permutation can be defined. This induces a partial order on pseudo-permutations compatible with the inversion tables and we will see that this order is a lattice, as it is the case for the usual permutations. One can also define the descents of a pseudo-permutation and prove that this notion has the same properties as the usual descents on permutations. About the lattice structure, one can also see that the set can be divided in connected components that are hypercubes, the dimensions of which can naturally be interpretated in combinatorial terms.

The aim of this paper is to show new developments among pseudo-permutations and show how one can generalize usual properties of permutations to the case of pseudo-permutations.

The paper is structured as follows. We first recall the definitions and the first properties of the pseudo-permutations (Section 2). In the next section, we first recall the usual interpretation of the symmetric group as a Coxeter group (Section 3.1) and then present our geometrical interpretation of the pseudo-permutations (Section 3.2) and derive a formula enumerating the pseudo-permutations by length and dimension (Section 3.3). We then concentrate on the set of the pseudo-permutations as a module over the symmetric group and compute its characteristics (Section 4).

2 Backgroung

In this section, we recall the basic definitions and properties of the pseudo-permutations.

2.1 Preliminaries

We will use the following standard notations. Let us denote by $[\![i,j]\!]$ the set $\{k \in \mathbb{N} \mid i \leq k \leq j\}$ and by $\mathfrak{S}_n$ the set of all the permutations of order n, *i.e.*, the set of all sequences of n integers which contain each integer in $[\![1,n]\!]$ exactly once.

We recall that $\lambda = (\lambda_1, \dots, \lambda_k)$ is a *partition* of an integer n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\sum_i \lambda_i = n$. We say that k is the *length* of the partition and that n is the *weight* of the partition.

To simplify the notation of some formulas, we will use the classical convention to shorten the expression of a partition like this: if λ is composed of k_1 times the same part λ_1, then k_2 times another part λ_2, *etc.*, and finally k_p times the same part λ_p, we will write

$$\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_p^{k_p}).$$

For example, the partition $\lambda = (3,3,2,1,1,1,1)$ can be shortened as $\lambda = (3^2, 2, 1^4)$.

We recall that $I = (i_1, \dots, i_p)$ is a *composition* of an integer n if the sum of the i_j's is equal to n. We say that k is the *length* of the composition and that n is the *weight* of the composition.

2.1.1 Multinomial coefficients and q-multinomials

The multinomial coefficients are very classical numbers and the reader can refer to [Com70] for a complete overview about these. In this paper, we will only recall their usual definition. Let $(n_1, \dots, n_k)$ be a composition of n. Then their corresponding multinomial coefficient is defined as

$$\binom{n}{n_1, \dots, n_k} = \frac{n!}{(n_1)! \cdots (n_k)!}$$

Notice that when $k = 2$, we find the usual binomial coefficients.

It is also possible to define a q-analog of this formula simply following the q-analogs of the integers. Let us define the q-analog of the integer n as

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{i=0}^{n-1} q^i.$$

Notice that $[n]_1 = n$. Then one can define the factorial of $[n]_q$ as

$$([n]_q)! = \prod_{i=1}^{n} ([i]_q)!.$$

Finally, one defines the q-multinomial coefficient as

$$\begin{bmatrix} n \\ n_1, \dots, n_k \end{bmatrix}_q = \frac{([n]_q)!}{([n_1]_q)! \cdots ([n_k]_q)!}$$

Notice that this formula give always rise to a polynomial in q and that one recovers the usuel multinomial coefficient for $q = 1$.

2.2 Pseudo-permutations

We recall that a *partition of a set S* is a set of subsets S_1, S_2, ..., S_k of S such that $\cup_{i=1}^{k} S_i = S$ and for all $i \neq j$, $S_i \cap S_j = \emptyset$. The S_i are then the *parts* of the partition. A partition is *ordered* if we consider a sequence of subsets instead of a set of subsets.

Let n be an integer. The set $\mathfrak{P}(n)$ of the pseudo-permutations of order n is the set of sequences of non-empty parentheses such that each integer in $[\![1,n]\!]$ appears exactly once. In other words, the set $\mathfrak{P}(n)$ is the set of ordered partitions of $[\![1,n]\!]$ with nonempty parts. For example, here is the complete set $\mathfrak{P}(3)$:

$$\begin{aligned} \mathfrak{P}(3) = \{ \quad & (1)(2)(3),\ (1)(3)(2),\ (2)(1)(3),\ (2)(3)(1),\ (3)(1)(2),\ (3)(2)(1), \\ & (1)(23),\ (2)(13),\ (3)(12),\ (23)(1),\ (13)(2),\ (12)(3),\ (123) \ \} . \end{aligned}$$

Since the order inside the parentheses is irrelevant, we will generally write the integers in the parentheses in increasing order.

Definition 1 Given an element $\mathfrak{s}$ of $\mathfrak{P}(n)$, one can then define its *horizontal reading*, denoted as $\mu(\mathfrak{s})$, which is the permutation obtained by removing all the parentheses, reading the integers in the parentheses in increasing order.

As for the usual permutations, one can generate a graph which vertices are the elements of $\mathfrak{P}(n)$ and which edges are defined according to the following operators:

- The operator M_i acts on the i-th and the $i+1$-th parentheses of a pseudo-permutation as follows: if each element of the i-th parenthese is smaller than all the elements of the $(i+1)$-th, then one can merge these two parentheses into one single parenthese which contains the union of the elements of these two parentheses.

- The operator $S_{i,j}$ acts on the i-th parenthese of a pseudo-permutation as follows: it splits this parenthese into two parentheses, the *second* one containing the j smallest elements of the initial parenthese and the first one containing the others.

For example, one can apply to the permutation $\sigma = (7)(3)(568)(12)(4)$ the operators M_2, M_4, $S_{3,1}$, $S_{3,2}$ and $S_{4,1}$:

$$\begin{aligned}\sigma \quad &\xrightarrow{M_2} (7)(3568)(12)(4)\\ &\xrightarrow{M_4} (7)(3)(568)(124)\\ &\xrightarrow{S_{3,1}} (7)(3)(68)(5)(12)(4)\\ &\xrightarrow{S_{3,2}} (7)(3)(8)(56)(12)(4)\\ &\xrightarrow{S_{4,1}} (7)(3)(568)(2)(1)(4)\end{aligned}$$

2.3 Basic properties of pseudo-permutations

2.3.1 Enumeration of pseudo-permutations

As defined, the set $\mathfrak{P}(n)$ is the set of ordered partitions of n elements with nonempty parts. Therefore, it satisfies the induction relation:

$$\mathrm{Card}(\mathfrak{P}(n)) = \sum_{i=0}^{n-1} \binom{n}{i} \mathrm{Card}(\mathfrak{P}(i)).$$

One can then solve this induction relation. It then comes the connection with the Eulerian numbers $A_{n,p}$ (see [FS70, Com70, GKLLRT95] for a complete overview about Eulerian numbers and their q-analogs) and one finds the relation:

$$\mathrm{Card}(\mathfrak{P}(n)) = \sum_{i=0}^{n-1} A_{n,i}\, 2^i.$$

2.3.2 Inversions of pseudo-permutations

Let n be an integer and let $\mathfrak{s}$ be an element of $\mathfrak{P}(n)$. For every pair (i,j) with $1 \le i < j \le n$, we define the value of the inversion (i,j) as follows:

- If i and j are in the same parenthese, this value is equal to $\frac{1}{2}$.
- If i and j are in distinct parentheses, this value is equal to 0 if the parenthese of i is before the parenthese of j in $\mathfrak{s}$ and it is equal to 1 in the other case.

The *table of inversions* of $\mathfrak{s}$ is then the list of the non zero-valued pairs (i,j) with $1 \le i < j \le n$ of integers given with the value of the inversion (i,j). The *number of inversions* of $\mathfrak{s}$, denoted by $I(\mathfrak{s})$ is then the sum for all $1 \le i < j \le n$ of the values of the inversions (i,j). For example, the table of inversions of the pseudo-permutation $(45)(13)(2)$ is equal to:

$$\left\{\frac{1}{2}(1,3),\ (1,4),\ (1,5),\ (2,3),\ (2,4),\ (2,5),\ (3,4),\ (3,5),\ \frac{1}{2}(4,5)\right\},$$

and its number of inversions is 8.

2.3.3 Dimension of a pseudo-permutation and enumeration

Let us now define the *dimension* (we will see later why this statistics is called like this) of a pseudo-permutation as the number of parts of this particular ordered partition of n elements.

Let $\mathfrak{P}_{n,k}$ be the set of the pseudo-permutations of n of dimension k and let $U_{n,k}$ be its cardinality. It is easy to see that these numbers satisfy the following formula:

$$U_{n,k} = \sum_{\lambda} \binom{k}{k_1, k_2, \dots, k_p} \binom{n}{\lambda_1^{k_1}, \dots, \lambda_p^{k_p}},$$

where the sum runs over all partitions $\lambda = (\lambda_1^{k_1}, \dots, \lambda_p^{k_p})$ of n of length k. We will see in Section 3.3 that it is this equation one has to generalize to get the q-enumeration of pseudo-permutations by inversions, having fixed the dimension and the order.

3 Geometrical interpretation

3.1 Geometrical background

This subsection is devoted to recall the geometrical background of this work. We will only give a summary of the main properties concerning the symmetric group, since the reader can find, emphe.g., in [Bou68, CM65, Hum90] a complete exposition of this subject.

Let E be the vector space $\mathbb{R}^n$. The symmetric group $\mathfrak{S}(n)$ acts on E as the permutation of the basis vectors $(e_1, \dots, e_n)$. Then $r_i = e_i - e_{i+1}$ is the root associated with the transposition $\sigma_i = (i, i+1)$ and the root system of the symmetric group is given by

$$\Delta = \left\{e_i - e_j, i \neq j, 1 \leq i, j \leq n\right\}.$$

The vector space V spanned by the roots is the hyperplane which equation is hence given by

$$V = \left\{ (x_1, \dots, x_n), \sum_i x_i = 0 \right\}.$$

In this hyperplane, let us define H_i as the orthogonal of r_i and then define H_i^+ (respectively H_i^-) as

$$H_i = \{v \in V, (V|r_i) = 0\}, \qquad H_i^+ = \{v \in V, (v|r_i) > 0\}, \qquad H_i^- = \{v \in V, (v|r_i) < 0\}.$$

Then the fundamental region, *i.e.*,the region associated with the identity permutation is given by

$$C_{(1,\dots,n)} = \{v \in V, \forall\, i \in [\![1, n]\!], (v|r_i) > 0\}.$$

Finally, given a permutation σ such that $\sigma = \sigma_i.\sigma'$, the cell C_σ is defined as the cell obtained by taking the orthogonal reflection by H_i of the cell $C_{\sigma'}$. In practice, one can use the following representation of the cells C_σ:

$$C_\sigma = \left\{ (x_1, x_2, \dots, x_n), \ \sum_{i=1}^{n} x_i = 0 \quad \text{and} \quad x_{\sigma(1)} > x_{\sigma(2)} > \dots > x_{\sigma(n)}. \right\}$$

For example, one can find in Figure 1 the description of the case $n = 3$, represented in the hyperplane $V = \{(x_1, x_2, x_3), x_1 + x_2 + x_3 = 0\}$.

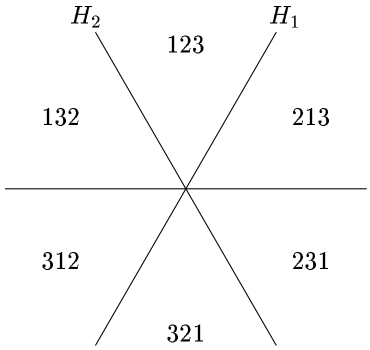

Fig. 1: The regions corresponding to the elements of $\mathfrak{S}(3)$.

3.2 Geometrical interpretation of pseudo-permutations

This subsection is devoted to study the geometrical interpretation of the pseudo-permutations. First, let $n \geq k$ be two integers and $\mathfrak{s}$ a pseudo-permutation of order n and dimension k. Let us write $\mathfrak{s}$ as

$$\mathfrak{s} = \big((a_1,a_2,\dots,a_{n_a}),(b_1,b_2,\dots,b_{n_b}),\dots(k_1,k_2,\dots,k_{n_k})\big)\,,$$

where the a_i's belong to the same parenthesis, as are the b_i's (but not in the same parenthesis as the a_i's), and so on. We then define the "cell" of $\mathfrak{s}$ as given by the following rules:

$$C_{\mathfrak{s}} = \left\{ (x_1,x_2,\dots,x_n),\ \sum_{i=1}^{n} x_i = 0 \quad \text{and} \quad x_{\sigma(a_1)} = \dots = x_{\sigma(a_{n_1})} > \dots > x_{\sigma(k_1)} = \dots = x_{\sigma(k_{n_k})}. \right\}$$

For example, one can find in Figure 2 the description of the case $n = 3$, represented in the hyperplane $V = \{(x_1,x_2,x_3),\ x_1+x_2+x_3 = 0\}$.

Remark 2 Let us notice that this definition coincides with the usual definition on permutations for the pseudo-permutations of maximal dimension, that is, when $k = n$. It corresponds to the case when each parenthesis exactly contains one integer and, in this case, the horizontal reading of a pseudo-permutation is a one-to-one correspondence that sends these particular pseudo-permutations to the usual permutations.

The dimension of a pseudo-permutation being its number of parentheses, one can see that this statistics corresponds to the smallest dimension (minus one) of a subvector space of $\mathbb{R}^n$ containing its cell. This gives a very convincing argument to explain why there does not seem to exist a simple formula for the number of pseudo-permutations of order n: this number corresponds to the union of geometrically distincts objects that do not live in the same vector spaces.

Remark 3 One can also notice that there exist a simple geometrical interpretation of the operators M_i. Indeed, the operator M_i translates as the following operations on a pseudo-permutation $\mathfrak{s}$:

- Take the closure of the cell corresponding to $\mathfrak{s}$.

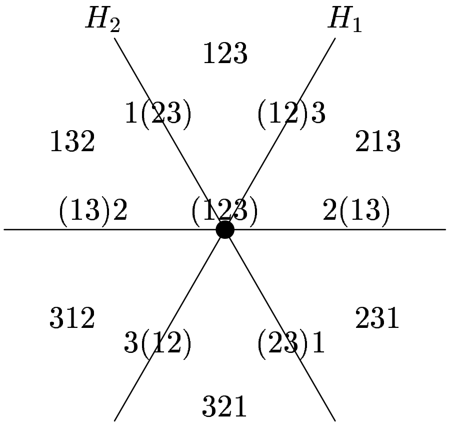

Fig. 2: The regions corresponding to the elements of $\mathfrak{P}(3)$.

- Take the closure of the cell corresponding to the pseudo-permutation obtained from $\mathfrak{s}$ by exchanging its i-th and $i+1$-th parenthesis.
- Take the intersection of these cells and consider in this intersection the part of maximum dimension.

This part exactly corresponds to the pseudo-permutation $M_i(\mathfrak{s})$.

We will now concentrate on the q-enumeration of the elements of $\mathfrak{P}(n,k)$ for all n and k.

3.3 *q-enumeration of pseudo-permutations by inversions*

In this section, we provide a formula for the q-enumeration by inversions of the elements of $\mathfrak{P}(n,k)$. Let $n \geq k$ be two integers. Let us denote by $[U_{n,k}]_q$ the q-analog of $U_{n,k}$ given by their inversions of the elements of $\mathfrak{P}(n,k)$. In other terms, one has

$$[U_{n,k}]_q = \sum_{\mathfrak{s} \in \mathfrak{P}(n,k)} q^{I(\mathfrak{s})}.$$

Theorem 4 *Let $n \geq k$ be two integers. Then one has:*

$$[U_{n,k}]_q = \sum_{\lambda} q^{\sum_i \binom{k_i}{2}} \binom{k}{k_1, k_2, \ldots, k_p} \begin{bmatrix} n \\ \lambda_1^{k_1}, \ldots, \lambda_p^{k_p} \end{bmatrix}_q,$$

where the sum runs over all partitions $\lambda = (\lambda_1^{k_1}, \ldots, \lambda_p^{k_p})$ of n of length k.

Proof — We will only give here a sketch of proof of this formula since it is only technical and uses many combinatorial properties (as the q-enumeration of permutations filling a given ribbon shape) that are not connected to the aim of this paper. Let us see how each term of the right member of the equation can be translated to explain its use. We suggest to the reader to first read Example 5 to get used to the objects presented here.

The power of q in the right-hand side of the equation only comes from the fact that, being given the partition λ, the horizontal reading is a bijection between the set of pseudo-permutations that have k_1 parentheses filled with λ_1 integers, k_2 parentheses filled with λ_2 integers *etc.*, and the permutations. This bijection does not preserve the number of inversions but preserve this number up to a constant, that corresponds to the total number of inversions within the parentheses, that is $\sum \binom{k_i}{2}$.

The first multinomial coefficient $\binom{k}{k_1,k_2,\ldots,k_p}$ comes from the fact that there are $\binom{k}{k_1,k_2,\ldots,k_p}$ different permutations of $(\lambda_1^{k_1},\ldots,\lambda_p^{k_p})$.

The second multinomial coefficient $\begin{bmatrix} n \\ \lambda_1^{k_1},\ldots,\lambda_p^{k_p} \end{bmatrix}_q$, that in fact is a q-multinomial coefficient, comes from the q-enumeration of permutations (using the bijection previously mentionned) filling all the ribbons that have k_1 rows with λ_1 cells, k_2 rows with λ_2 cells, and so on. ■

Example 5 Let us give a few examples of the previous formula. First, let us study the case $k = n$. In this case, there is only one partition of n of length k. It is the partition 1^n. So

$$[U_{n,n}]_q = q^0 \binom{n}{n} \begin{bmatrix} n \\ 1^n \end{bmatrix}_q = ([n]_q)!\ .$$

This formula is the usual formula for the q-enumeration of permutations by their inversion numbers. It is very logical to find this formula since the horizontal reading of a pseudo-permutation is a one-to-one correspondence between the pseudo-permutations of maximal dimension and the usual permutations, that preserves the number of inversions.

Let us now see the case $k = n-1$. In this case, there is only one partition of n of length k. It is $(2,1^{n-2})$. It then comes:

$$[U_{n,n-1}]_q = q^1 \binom{n-1}{1,n-2} \begin{bmatrix} n \\ 2,1^{n-2} \end{bmatrix}_q = (n-1) \times q \frac{([n]_q)!}{[2]_q}\ .$$

Let us finally see the case $k = n-2$. In this case, there are two partitions of n of length k. These are $(3,1^{n-3})$ and $(2^2,1^{n-4})$. It then comes:

$$[U_{n,n-2}]_q = (n-2) \times q^{\frac{3}{2}} \frac{([n]_q)!}{[3]_q} + \frac{(n-2)(n-3)}{2} \times q \frac{([n]_q)!}{([3]_q)^2}\ .$$

4 The pseudo-permutation $\mathfrak{S}(n)$-module

One can easily endow the pseudo-permutations with an action of the symmetric group in the following way: if $\mathfrak{s}$ is an element of $\mathfrak{P}(n)$, the symmetric group $\mathfrak{S}(n)$ acts on $\mathfrak{s}$ by permuting the integers inside $\mathfrak{s}$. For example, the permutation (23415) acts on $(15)(3)(24)$ and the result is $(25)(4)(13)$.

4.1 *Young subgroups and pseudo-permutations*

Let us recall the definition and representation theoretical properties of the Young subgroups. We will only give a summary of the main properties concerning these subgroups since the reader can find, *e.g.*, in [McD95, Ful97, FH91] a complete exposition on this subject.

Let $I = (i_1,\ldots,i_p)$ be a composition of n. The Young subgroup $\mathfrak{S}(I)$ of $\mathfrak{S}(n)$ is defined as

$$\mathfrak{S}(I) = \mathfrak{S}(i_1) \times \mathfrak{S}(i_2) \times \cdots \times \mathfrak{S}(i_p),$$

the k-th component $\mathfrak{S}(i_k)$ acting on the interval $[\![i_1 + \cdots + i_{k-1} + 1, i_1 + \cdots + i_k]\!]$. The cosets of $\mathfrak{S}(n)/\mathfrak{S}(I)$ can easily be described thanks to some special matrices. We define the set M_I of matrices $p \times n$ as the matrices composed of 0's and 1's such that there is one 1 per column and i_k 1's in the k-th row, for all k. These matrices exactly are the usual representation of the cosets.

It is now important to notice that there is a bijection between the set $\mathfrak{P}(n)$ and the union of the sets M_I, where I runs over the compositions on n, defined by putting the ones corresponding to the k-th parenthesis on the k-th row. So, this bijection leads to a bijection between the pseudo-permutations and the cosets of the Young subgroups, and this bijection is in fact a morphism of $\mathfrak{S}(n)$-modules (the bijection is compatible with the action of the symmetric group on both subgroups).

Example 6 The following matrix corresponds to the pseudo-permutation $4\binom{1}{6}\binom{2}{3}5$:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

4.2 *q*-characteristic of $\mathfrak{P}(n)$

Since the bijection previously defined is compatible with the action of the symmetric group and since both are compatible with the q-grading, the q-characteristic (by dimension) of the pseudo-permutation is the same as the q-characteristic of the cosets. This characteristic can be easily computed since it is, by definition, the characteristic of the induced representation on $\mathfrak{S}(n)$ of the trivial representation of the corresponding Young subgroup. So the characteristic for the pseudo-permutations of dimension d and weight n is

$$\sum_I S_I,$$

where I runs over the compositions of n in d parts and S_I denotes the product $S_{i_1} \cdots S_{i_d}$ of the complete symmetric functions S_i.

One can then compute the q-characteristic of $\mathfrak{P}(n)$. It comes:

$$\mathfrak{F}(\mathfrak{P}(n))_q = \sum_d \sum_{I=(i_1,\ldots,i_d)} q^d S_I.$$

Adding up all these formulas, one can get the characteristic of all the pseudo-permutations. This last formula factorizes and one gets :

$$\mathfrak{F}(\mathfrak{P})_q = \frac{1}{1 - (\sum_{i \geq 1} q^{i-1} S_i)}.$$

Conclusion

In this paper, we have shown a few new properties of a new combinatorial object. These properties are fundamental since they are generalizations of the key properties of the symmetric group. That allows us to think that many other properties of the symmetric group have a real meaning in this new context.

Another interesting direction of investigation having many applications is to consider pseudo-permutations with multiple occurences of the integers. This gives a very general and powerful model which makes it possible to represent the relations between multiple events which can have a temporal length. It seems that some new questions arise in this interesting context.

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