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# Reconstruction Thresholds on Regular Trees

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We consider the model of *broadcasting on a tree*, with binary state space, on the infinite rooted tree  $\mathbb{T}^k$  in which each node has  $k$  children. The root of the tree takes a random value 0 or 1, and then each node passes a value independently to each of its children according to a  $2 \times 2$  transition matrix  $\mathbf{P}$ . We say that *reconstruction is possible* if the values at the  $d$ th level of the tree contain non-vanishing information about the value at the root as  $d \rightarrow \infty$ . Extending a method of Brightwell and Winkler, we obtain new conditions under which reconstruction is impossible, both in the general case and in the special case  $p_{11} = 0$ . The latter case is closely related to the *hard-core model* from statistical physics; a corollary of our results is that, for the hard-core model on the  $(k + 1)$ -regular tree with activity  $\lambda = 1$ , the unique simple invariant Gibbs measure is extremal in the set of Gibbs measures, for any  $k \geq 2$ .

**Keywords:** broadcasting on a tree, reconstruction, hard-core model, Gibbs measure, extremality

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## 1 Introduction

### 1.1 Broadcasting on a tree

We consider a model of a broadcasting on the rooted tree  $\mathbb{T}^k$ , in which every node has  $k$  children.

Let  $\mathbf{P} = \{p_{ij}, i, j = 0, 1\}$  be a  $2 \times 2$  stochastic matrix, which we regard as a transition matrix on the set  $\{0, 1\}$ . Each node  $u \in \mathbb{T}^k$  will carry a value  $\phi(u) \in \{0, 1\}$ , generated as follows. The root takes value 0 with probability  $\pi_0 = p_{10}/(p_{01} + p_{10})$  and value 1 with probability  $\pi_1 = 1 - \pi_0$ . Thereafter the configuration on  $\mathbb{T}^k$  is generated recursively; if a node has value  $i \in \{0, 1\}$ , each of its  $k$  children takes the value 0 with probability  $p_{i0}$  and the value 1 with probability  $p_{i1}$ , all choices being made independently.

We write  $\phi = \{\phi(u), u \in \mathbb{T}^k\}$  for a configuration on the whole tree, and denote by  $\mu$  the probability measure on  $\{0, 1\}^{\mathbb{T}^k}$  resulting from this broadcasting construction.

For a node  $u \in \mathbb{T}^k$ , let  $\mathbb{T}^k(u)$  be the subtree consisting of  $u$  and all its descendants. By the choice of  $\pi_0$ , we have a translation invariance property for  $\mu$ ; namely that  $\mu(\phi(u) = 0) = \pi_0$  for every  $u \in \mathbb{T}^k$ , and so for any  $u, v \in \mathbb{T}^k$ , the configurations on  $\mathbb{T}^k(u)$  and  $\mathbb{T}^k(v)$  have the same distribution, under a natural mapping between the subtrees  $\mathbb{T}^k(u)$  and  $\mathbb{T}^k(v)$ .

We are interested in the following question of *reconstruction*: for  $d \geq 1$ , how much information about the value at node  $u$  is given by the values of the  $d$ th generation of its descendants?

Questions of this sort arise in several contexts – for example genetics, communication theory and statistical physics – and have been quite widely studied in the last few years; see Mossel [Mos03] for a survey, and [EKPS00, BRZ95, Iof96, KMP01, Mos01, MP03, BW03, JM03] for a variety of approaches to this

sort of model (which can of course be considerably generalised from our particular setting of a binary state space and a regular tree).

The question above can be made precise in several (often equivalent) ways. We use the following formulation.

Let  $\mathbb{W}_d(u)$  be the set of descendants of  $u$  at distance exactly  $d$  from  $u$ . For a set  $S \subseteq \mathbb{T}^k$ , write  $\sigma(S)$  for the  $\sigma$ -algebra of events which depend only on the values  $\{\phi(u), u \in S\}$ .

Define the random variable

$$A(d, u) = \mu(\phi(u) = 0 | \sigma(\mathbb{W}_d(u))),$$

that is, the conditional probability that the value at  $u$  is 0, given only the information from the  $d$ th generation of its descendants.

From the independence structure given by the broadcasting construction, additional knowledge of any information from nodes beyond the  $d$ th generation does not change the conditional distribution of the value of  $u$ ; that is,

$$A(d, u) = \mu\left(\phi(u) = 0 \mid \sigma\left(\bigcup_{d'=d}^{\infty} \mathbb{W}_{d'}(u)\right)\right).$$

Of course, if  $d_1 > d_2$ , then

$$\sigma\left(\bigcup_{d'=d_1}^{\infty} \mathbb{W}_{d'}(u)\right) \subseteq \sigma\left(\bigcup_{d'=d_2}^{\infty} \mathbb{W}_{d'}(u)\right),$$

so by the backwards martingale convergence theorem (see e.g. Section 14.4 of [Wil91]), we have that  $A(d, u) \rightarrow A(u)$  a.s. as  $d \rightarrow \infty$ , where

$$A(u) = \mu(\phi(u) = 0 | \mathcal{T}(u));$$

here  $\mathcal{T}(u)$  is the *tail  $\sigma$ -algebra* of descendants of  $u$ , defined by

$$\mathcal{T}(u) = \bigcap_{d=1}^{\infty} \sigma\left(\bigcup_{d'=d}^{\infty} \mathbb{W}_{d'}(u)\right).$$

By the translation invariance property above, the random variable  $A(u)$  has the same distribution for all  $u \in \mathbb{T}^k$ .

*Definition:* We say that *reconstruction is impossible* (for a given  $\mathbf{P}$  and  $k$ ) if the random variable  $A(u)$  is almost surely constant, and otherwise that *reconstruction is possible*.

A complete answer to the question of when reconstruction is possible is currently only known for the case where  $\mathbf{P}$  is symmetric. Then let  $p_{00} = p_{11} = 1 - \varepsilon$ ; reconstruction is possible if and only if  $k(1 - 2\varepsilon)^2 > 1$  (see for example [BRZ95, EKPS00, Iof96]).

In general, however, there are gaps between the best known necessary and sufficient conditions for reconstruction to be possible. In this paper we give new conditions on  $\mathbf{P}$  under which we show that reconstruction is impossible.

In Proposition 4.1 of [MP03], Mossel and Peres show that reconstruction is impossible whenever

$$\frac{(p_{00} - p_{10})^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \leq \frac{1}{k}. \quad (1)$$

We improve the bound to give the following condition:

**Theorem 1.** *Reconstruction is impossible whenever*

$$\left(\sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}}\right)^2 \leq \frac{1}{k}. \quad (2)$$

A calculation (see Section 4) shows that the LHS of (2) is always less than or equal to that of (1), with equality in the following special cases: (i)  $\mathbf{P}$  is symmetric; (ii)  $p_{ij} = 0$  for some  $i, j$ ; (iii)  $p_{00} = p_{10}$ ,  $p_{01} = p_{11}$ . Note that for symmetric  $\mathbf{P}$ , (2) becomes the condition that  $k(1 - 2\varepsilon)^2 \leq 1$ , and our proof of Theorem 1 gives another proof that reconstruction is impossible under this condition.

We then focus on the special case where  $p_{11} = 0$  (of course, the case  $p_{00} = 0$  is analogous). This case is closely related to the *hard-core model* from statistical physics, and has been recently studied by Brightwell and Winkler [BW03] and Rozikov and Suhov [RS03]. Certain specific properties in this case allow a more sophisticated argument which gives a much better condition than is obtained by putting  $p_{11} = 0$  in Theorem 1.

## 1.2 Hard-core model

In this section we state our result for the case  $p_{11} = 0$  and explain the correspondence with the hard-core model on a regular tree.

Following [BW03], we parametrise  $\mathbf{P}$  by the quantity  $w > 0$ , setting

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+w} & \frac{w}{1+w} \\ 1 & 0 \end{pmatrix}, \quad (3)$$

or equivalently by the quantity  $\lambda = w(1+w)^k > 0$ , whose significance we explain later; note that the correspondence between  $\lambda > 0$  and  $w > 0$  is one-to-one and monotonic.

Let  $\lambda_c = \lambda_c(k)$  be the infimum of the set of  $\lambda$  such that reconstruction is possible. It follows from Proposition 12 of [Mos01] that in fact reconstruction is possible for any  $\lambda > \lambda_c$  (so that  $\lambda_c$  is also the supremum of the set of  $\lambda$  such that reconstruction is impossible).

Brightwell and Winkler [BW03] show that, as  $k \rightarrow \infty$ ,

$$\frac{1 + o(1)}{\ln k} \leq \lambda_c(k) \leq (\ln k)^2(1 + o(1)). \quad (4)$$

We improve the lower bound to give the following:

**Theorem 2.**  $\lambda_c(k) > e - 1$  for all  $k$ .

(For the equivalent threshold value  $w_c$  with  $w_c(1 - w_c)^k = \lambda_c$ , one can deduce that  $w_c(k) > (\ln k - \ln \ln k)/k$  for all  $k$ ).

We will now describe the correspondence between the broadcasting model and the hard-core model, and explain (without proofs) the significance of Theorem 2 for the hard-core model on the  $(k+1)$ -regular tree. For more details on the correspondence between the two models, see also [BW03] and its references.

We denote the  $(k+1)$ -regular tree by  $\tilde{\mathbb{T}}^k$ . We can still regard  $\tilde{\mathbb{T}}^k$  as a rooted tree, in which the root node has  $k+1$  children and every other node has  $k$  children. We can then carry out the broadcasting construction on  $\tilde{\mathbb{T}}^k$  in exactly the same way as we did on  $\mathbb{T}^k$ ; now the root has  $k+1$  rather than  $k$  children, but the values at these  $k+1$  children are chosen i.i.d. according to the value at the root and the transition matrix  $\mathbf{P}$  just as before. We will write  $\tilde{\mu}$  for the probability measure on  $\{0, 1\}^{\tilde{\mathbb{T}}^k}$  resulting from this construction.

The independence structure of the random walk implies that the measure  $\tilde{\mu}$  is *simple*, by which we mean that, for any  $u$ , the configurations

$$\{\phi(v), v \in C_1(u)\}, \dots, \{\phi(v), v \in C_{k+1}(u)\}$$

are mutually independent given  $\phi(u)$ , where the  $C_i$  are the connected components of  $\tilde{\mathbb{T}}^k \setminus \{u\}$ . Although we have defined  $\tilde{\mu}$  in an asymmetric way, it's also the case that it is *invariant*, in the sense that it is preserved by any automorphism of  $\tilde{\mathbb{T}}^k$ . In particular, the choice of the root is not important.

To introduce the hard-core model, we first consider the case of a finite graph with node-set  $S$  (and some neighbour relation).

We can identify a configuration  $\phi \in \{0, 1\}^S$  with the subset  $I_\phi := \{u \in S : \phi(u) = 1\}$  of  $S$ .

A set  $I \subseteq S$  is called an *independent set* if no two neighbours in the graph are both members of  $I$ .

The *hard-core measure* on  $S$  with *activity*  $\lambda > 0$  is the probability measure  $\nu$  on  $\{0, 1\}^S$  such that

$$\nu(I_\phi \text{ is an independent set}) = 1,$$

and such that for an independent set  $I_0$ ,  $\nu(I_\phi = I_0)$  is proportional to  $\lambda^{|I_0|}$ . Thus in fact

$$\nu(\phi = \phi_0) = Z_\lambda^{-1} \lambda^{|I_{\phi_0}|} 1(I_{\phi_0} \text{ is an independent set}),$$

where we have the normalising factor

$$Z_\lambda = \sum_{\phi_0: I_{\phi_0} \text{ is independent}} \lambda^{|I_{\phi_0}|}.$$

When  $\lambda = 1$ ,  $I_\phi$  has the uniform distribution over the set of independent subsets of  $S$ .

An equivalent characterisation is that  $\nu$  is the unique probability measure such that, for any  $\phi_0 \in \{0, 1\}^S$  and any  $u \in S$ ,

$$\nu(\phi(u) = 1 \mid \phi(v) = \phi_0(v) \text{ for all } v \neq u) = \frac{\lambda}{1 + \lambda} 1\{I_{\phi_0} \cup \{u\} \text{ is independent}\} \quad (5)$$

The condition (5) makes sense equally when  $S$  is infinite, except that (since conditional probabilities are only well defined up to almost sure equality) we should now only demand the condition holds for  $\nu$ -almost all  $\phi_0$ . Putting  $S = \tilde{\mathbb{T}}^k$ , we say that a probability measure  $\nu$  satisfying (5) (for all  $u \in \tilde{\mathbb{T}}^k$  and  $\nu$ -almost all  $\phi_0$ ) is a *Gibbs measure* for the hard-core model on  $\tilde{\mathbb{T}}^k$  with activity  $\lambda$ .

It is quite straightforward to show that the measure  $\tilde{\mu}$  defined above by the broadcasting construction with  $\mathbf{P}$  as in (3) is a Gibbs measure for the hard-core model with activity  $\lambda$ . However, now that the state space is infinite, it's no longer the case that such a measure need be unique. In fact, there is a critical point  $\lambda'_c = \lambda'_c(k) = k^k / (k-1)^{(k+1)}$  (identified by Kelly [Kel85]); for  $\lambda \leq \lambda'_c$ ,  $\tilde{\mu}$  is the *only* Gibbs measure, whereas for  $\lambda > \lambda'_c$ , there are others. Nevertheless, for any  $\lambda$ , the measure  $\tilde{\mu}$  is the only *simple invariant* Gibbs measure; (this can be deduced, for example, from Theorem 4.1 of [Zac83] – see also Section 5 of that paper for relevant discussion).

The set of Gibbs measures forms a simplex; that is, any mixture of Gibbs measures is also a Gibbs measure, and in particular there is a set of *extremal* Gibbs measures such that every Gibbs measure is expressible in a unique way as a mixture of extremal measures. For  $\lambda > \lambda'_c$ , we can therefore ask whether the measure  $\tilde{\mu}$  is extremal (equivalently, not expressible as a mixture of other Gibbs measures).

It turns out that  $\tilde{\mu}$  is extremal at activity  $\lambda$  if and only if reconstruction is impossible for the corresponding broadcasting model on  $\mathbb{T}^k$  with transition matrix  $\mathbf{P}$ . (This is a consequence of the general fact that a Gibbs measure is extremal iff it is trivial on the tail  $\sigma$ -algebra, and of the independence structure given by the broadcasting constructions of  $\mu$  and  $\tilde{\mu}$ ). Hence the reconstruction threshold  $\lambda_c$  defined after (3) is also the *extremality* threshold for  $\tilde{\mu}$ ; Theorem 2 therefore shows that whenever  $\lambda \leq e - 1$ , the unique simple invariant Gibbs measure  $\tilde{\mu}$  for the hard-core model with activity  $\lambda$  is extreme, for any  $k$ .

In particular,  $\tilde{\mu}$  is extreme in the special case  $\lambda = 1$  for any  $k$ .

### 1.3 Outline of proof

Our proof of Theorems 1 and 2 is in the same spirit as the proof by Brightwell and Winkler of the lower bound in (4) for the hard-core model [BW03].

We first develop a coupling between the distributions of the random variable  $A(u)$  conditioned on two different events, with certain additional properties beyond those used in [BW03]. We then use this coupling to establish a recursion linking the distribution of  $A(u)$  to those of  $A(u_1), \dots, A(u_k)$ , where  $u_1, \dots, u_k$  are the children of  $u$ . (Of course, we already know from the translation invariance property described in Section 1.1 that all of these distributions are the same). If the recursion relation is contractive in a suitable sense, we obtain that  $A(u)$  must be a.s. constant.

In Section 2, we first prove a lemma on conditional probabilities in a more general setting. Specialising to our context, we obtain the existence of a coupling of a pair of random variables  $A_0, A_1$  with the following properties:

- (i) The distribution of  $A_0$  is the distribution of  $A(u)$  under  $\mu$  conditioned on the event  $\{\phi(u) = 0\}$ ;
- (ii) The distribution of  $A_1$  is the distribution of  $A(u)$  under  $\mu$  conditioned on the event  $\{\phi(u) = 1\}$ ;
- (iii) With probability 1, either  $A_0 = A_1$  or  $A_1 \leq \pi_0 \leq A_0$ ;
- (iv) If  $A_0 = A_1$  with probability 1, then both are equal to  $\pi_0$  with probability 1, and so also  $A(u) = \pi_0$  a.s. under  $\mu$ .

We develop the recursion relations and complete the proofs in Section 3.

The full properties of the coupling are only needed in the hard-core case, where a particular convexity property holds for the recursion relations. The argument in the general case is not as powerful, and rather than all of property (iii) above, we use only that  $A_1 \leq A_0$  with probability 1. Restricting the bound in Theorem 1 to the case  $p_{11} = 0$  gives a much weaker bound than that in Theorem 2 (in fact, one obtains only the bound  $\lambda_c \geq \lambda'_c$  where  $\lambda'_c$  is the threshold for the uniqueness of the Gibbs measure; this bound is obvious in the context of the hard-core model since if the Gibbs measure is unique it is trivially extreme).

## 2 Conditioned conditional probabilities

We first consider the setting of a general probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $B \in \mathcal{F}$  be an event with probability  $\pi_0 = 1 - \pi_1$ , and suppose  $0 < \pi_0 < 1$ . Write  $B^C$  for the complement of  $B$ . Let  $\mathcal{G}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$ .

We consider the random variable  $\mathbb{P}(B|\mathcal{G})$  (which is the  $\mathcal{G}$ -measurable random variable, unique up to almost sure equality, such that for all  $D \in \mathcal{G}$

$$\mathbb{P}(D \cap B) = \int_D \mathbb{P}(B|\mathcal{G})(\omega) d\mathbb{P}(\omega). \tag{6}$$

See for example Chapter 9 of [Wil91] for background on conditional probabilities).

**Lemma 3.** *Suppose  $0 \leq p_0 \leq p_1 \leq 1$ , and that  $D \in \mathcal{G}$  with*

$$\mathbb{P}(B|\mathcal{G})(\omega) \in [p_0, p_1] \text{ for all } \omega \in D. \quad (7)$$

Then

$$\frac{\pi_1}{\pi_0} \frac{p_0}{1-p_0} \mathbb{P}(D|B^C) \leq \mathbb{P}(D|B) \leq \frac{\pi_1}{\pi_0} \frac{p_1}{1-p_1} \mathbb{P}(D|B^C).$$

*Proof.* From (6) and (7) we have

$$p_0 \mathbb{P}(D) \leq \mathbb{P}(D \cap B) \leq p_1 \mathbb{P}(D)$$

and

$$(1-p_1) \mathbb{P}(D) \leq \mathbb{P}(D \cap B^C) \leq (1-p_0) \mathbb{P}(D).$$

Combining these we get

$$\frac{p_0}{1-p_0} \mathbb{P}(D \cap B^C) \leq \mathbb{P}(D \cap B) \leq \frac{p_1}{1-p_1} \mathbb{P}(D \cap B^C).$$

Since  $\mathbb{P}(D|B) = \mathbb{P}(D \cap B)/\pi_0$  and  $\mathbb{P}(D|B^C) = \mathbb{P}(D \cap B^C)/\pi_1$ , the result follows.  $\square$

In particular, if  $J$  is a subset of the interval  $[0, \pi_0)$  then we can set  $D = \{\omega : \mathbb{P}(B|\mathcal{G})(\omega) \in J\}$  to obtain

$$\mathbb{P}\left\{\mathbb{P}(B|\mathcal{G}) \in J|B\right\} \leq \mathbb{P}\left\{\mathbb{P}(B|\mathcal{G}) \in J|B^C\right\},$$

while if  $J \subseteq (\pi, 1]$  then the inequality is reversed. In each case equality holds only if both sides are 0. Hence:

**Corollary 4.** *There exists a coupling of two random variables  $Y_0$  and  $Y_1$ , such that  $Y_0$  has the distribution of  $\mathbb{P}(B|\mathcal{G})$  conditioned on  $B$  occurring, such that  $Y_1$  has the distribution of  $\mathbb{P}(B|\mathcal{G})$  conditioned on  $B$  not occurring, and such that:*

(i) *whenever  $Y_0 < \pi_0$ , then  $Y_1 = Y_0$ , and*

(ii) *whenever  $Y_1 > \pi_0$ , then  $Y_1 = Y_0$ .*

Therefore either  $Y_0 = Y_1$  or  $Y_1 \leq \pi_0 \leq Y_0$ .

Also the distributions of  $Y_0$  and  $Y_1$  are identical iff  $Y_0 = Y_1 = \pi_0$  with probability 1, or equivalently iff  $\mathbb{P}(B|\mathcal{G}) = \pi_0$  with probability 1.

Applying this result with  $B = \{\phi(u) = 0\}$ , with  $\mathcal{G} = \mathcal{T}(u)$ , with  $\mathbb{P} = \mu$  and so with  $A(u) = \mathbb{P}(B|\mathcal{G})$ , we obtain the coupling of  $A_0, A_1$  with the properties claimed in Section 1.3.

### 3 Recurrences for likelihood ratios

Let  $u \in \mathbb{T}^k$  and let  $\mathbf{y}$  be a configuration on the set  $\mathbb{W}_d(u)$  (the descendants of  $u$  at distance exactly  $d$ ).

For  $S \subset \mathbb{T}^k$ , write  $\phi \downarrow_S$  for the configuration  $\phi$  restricted to  $S$ .

Define the “likelihood functions”

$$\begin{aligned} q^{(0)}(d, \mathbf{y}) &= \mu(\phi \downarrow_{\mathbb{W}_d(u)} = \mathbf{y} \mid \phi(u) = 0) \\ q^{(1)}(d, \mathbf{y}) &= \mu(\phi \downarrow_{\mathbb{W}_d(u)} = \mathbf{y} \mid \phi(u) = 1). \end{aligned}$$

For  $i = 0, 1$ , the function  $q^{(i)}(d, \mathbf{y})$  gives the probability of observing the configuration  $\mathbf{y}$  on the set of descendants of  $u$  at distance  $d$ , given that the value at  $u$  itself is  $i$ . (Note that because of the translation invariance property noted in Section 1.1, the choice of  $u$  is not important).

Define also the “likelihood ratio” function

$$q(d, \mathbf{y}) = \frac{q^{(0)}(d, \mathbf{y})}{q^{(1)}(d, \mathbf{y})}.$$

Let  $d \geq 2$  and let the children of  $u$  be  $u_1, \dots, u_k$ . A configuration  $\mathbf{y}$  on  $\mathbb{W}_d(u)$  corresponds to a set of configurations  $\mathbf{y}_1, \dots, \mathbf{y}_k$  on  $\mathbb{W}_{d-1}(u_1), \dots, \mathbb{W}_{d-1}(u_k)$ . We then have

$$\begin{aligned} q^{(0)}(d, \mathbf{y}) &= \prod_{j=1}^k [p_{00}q^{(0)}(d-1, \mathbf{y}_j) + p_{01}q^{(1)}(d-1, \mathbf{y}_j)] \\ q^{(1)}(d, \mathbf{y}) &= \prod_{j=1}^k [p_{10}q^{(0)}(d-1, \mathbf{y}_j) + p_{11}q^{(1)}(d-1, \mathbf{y}_j)] \end{aligned}$$

and so

$$\begin{aligned} q(d, \mathbf{y}) &= \prod_{j=1}^k \left\{ \frac{p_{00}q^{(0)}(d-1, \mathbf{y}_j) + p_{01}q^{(1)}(d-1, \mathbf{y}_j)}{p_{10}q^{(0)}(d-1, \mathbf{y}_j) + p_{11}q^{(1)}(d-1, \mathbf{y}_j)} \right\} \\ &= \left( \frac{p_{00}}{p_{10}} \right)^k \prod_{j=1}^k \left\{ 1 + \frac{c_0 - c_1}{q(d-1, \mathbf{y}_j) + c_1} \right\}, \end{aligned} \tag{8}$$

where we define

$$c_0 = \frac{p_{01}}{p_{00}}, \quad c_1 = \frac{p_{11}}{p_{10}}.$$

Define also  $a(d, \mathbf{y}) = \mu(\phi(u) = 0 \mid \phi \downarrow_{\mathbb{W}_d(u)} = \mathbf{y})$ . The function  $a$  gives the conditional probability that the value at the node  $u$  is 0, given a configuration on the set of descendants of  $u$  at distance  $d$ . We have

$$\begin{aligned} a(d, \mathbf{y}) &= \frac{\pi_0 q^{(0)}(d, \mathbf{y})}{\pi_0 q^{(0)}(d, \mathbf{y}) + \pi_1 q^{(1)}(d, \mathbf{y})} \\ &= \frac{1}{1 + \frac{\pi_1}{\pi_0 q(d, \mathbf{y})}}. \end{aligned} \tag{9}$$



Returning to the random variable  $A(d, u) = \mu(\phi(u) = 0 | \sigma(\mathbb{W}_d(u)))$  defined in Section 1, we have

$$A(d, u) = a(d, \phi \downarrow_{\mathbb{W}_d(u)}),$$

that is, the function  $a(d, \cdot)$  applied to the actually observed values of the configuration  $\phi$  on  $\mathbb{W}_d(u)$ .

Similarly define

$$Q(d, u) = q(d, \phi \downarrow_{\mathbb{W}_d(u)}).$$

From (9), we have

$$A(d, u) = \frac{1}{1 + \frac{\pi_1}{\pi_0 Q(d, u)}}, \quad Q(d, u) = \frac{\pi_1}{\pi_0} \left( \frac{1}{1 - A(d, u)} - 1 \right).$$

Recalling  $A(d, u) \rightarrow A(u)$  a.s., we have  $Q(d, u) \rightarrow Q(u)$  a.s., where

$$Q(u) = \frac{\pi_1}{\pi_0} \left( \frac{1}{1 - A(u)} - 1 \right).$$

From (8) we get

$$Q(d, u) = \left( \frac{p_{00}}{p_{10}} \right)^k \prod_{j=1}^k \left( 1 + \frac{c_0 - c_1}{Q(d-1, u_j) + c_1} \right),$$

and, taking  $d \rightarrow \infty$ ,

$$Q(u) = \left( \frac{p_{00}}{p_{10}} \right)^k \prod_{j=1}^k \left( 1 + \frac{c_0 - c_1}{Q(u_j) + c_1} \right).$$

Now put

$$L(u, d) = \ln Q(u, d)$$

and

$$\begin{aligned} L(u) &= \ln Q(u) \\ &= \ln \left[ \frac{\pi_1}{\pi_0} \left( \frac{1}{1 - A(u)} - 1 \right) \right]. \end{aligned}$$

We have  $L(u, d) \rightarrow L(u)$  a.s. as  $d \rightarrow \infty$ , and

$$L(u) = k \ln \left( \frac{p_{00}}{p_{10}} \right) + \sum_{j=1}^k \ln \left( 1 + \frac{c_0 - c_1}{\exp(L(u_j)) + c_1} \right). \quad (10)$$

Since  $L(u)$  can be written as a strictly increasing function of  $A(u)$ , with  $A(u) = \pi_0$  corresponding to  $L(u) = 0$ , we can translate the coupling of  $A_0, A_1$  described in Section 1.3 and proved in Section 2 into a coupling of two random variables  $L_0, L_1$  with the following properties:

- (i) The distribution of  $L_0$  is the distribution of  $L(u)$  conditioned on the event  $\{\phi(u) = 0\}$ ;
- (ii) The distribution of  $L_1$  is the distribution of  $L(u)$  conditioned on the event  $\{\phi(u) = 1\}$ ;
- (iii) With probability 1, either  $L_0 = L_1$  or  $L_1 \leq 0 \leq L_0$ ;
- (iv) If  $L_0 = L_1$  with probability 1, then both are equal to 0 with probability 1, and then also  $A(u) = \pi_0$  with probability 1.

So to conclude that  $A(u)$  is a.s. constant, it's enough to show that  $\mathbb{E}(L_0 - L_1) = 0$ .

Returning to (10), note that  $\exp(L(u_j)) \geq 0$ , and so the quantity inside the second logarithm is always at least  $\min(c_0/c_1, 1) > 0$ . Thus the distribution of  $L(u)$  has compact support; hence the same is true for  $L_0$  and  $L_1$ , and certainly  $\mathbb{E}|L_0| < \infty$ ,  $\mathbb{E}|L_1| < \infty$ .

Again let  $u_1, \dots, u_k$  be the children of a node  $u$ . From the broadcasting construction we get the following information.

Conditional on  $\phi(u) = 0$ :

the  $\phi(u_j)$ ,  $j = 1, \dots, k$  are i.i.d. taking value 0 w.p.  $p_{00}$  and value 1 w.p.  $p_{01}$ . Then the  $L(u_j)$  are i.i.d., and the distribution of each is a mixture of the distribution of  $L_0$  (with weight  $p_{00}$ ) and the distribution of  $L_1$  (with weight  $p_{01}$ ).

Conditional on  $\phi(u) = 1$ :

the  $\phi(u_j)$ ,  $j = 1, \dots, k$  are i.i.d. taking value 0 w.p.  $p_{10}$  and value 1 w.p.  $p_{11}$ . Then the  $L(u_j)$  are i.i.d., and the distribution of each is a mixture of the distribution of  $L_0$  (with weight  $p_{10}$ ) and the distribution of  $L_1$  (with weight  $p_{11}$ ).

Hence from (10),

$$\mathbb{E}L_0 = -k \ln \left( \frac{p_{00}}{p_{10}} \right) + k \left[ p_{00} \mathbb{E} \ln \left( 1 + \frac{c_0 - c_1}{\exp(L_0) + c_1} \right) + p_{01} \mathbb{E} \ln \left( 1 + \frac{c_0 - c_1}{\exp(L_1) + c_1} \right) \right].$$

and

$$\mathbb{E}L_1 = -k \ln \left( \frac{p_{00}}{p_{10}} \right) + k \left[ p_{10} \mathbb{E} \ln \left( 1 + \frac{c_0 - c_1}{\exp(L_0) + c_1} \right) + p_{11} \mathbb{E} \ln \left( 1 + \frac{c_0 - c_1}{\exp(L_1) + c_1} \right) \right].$$

Subtracting and using the fact that  $p_{00} - p_{10} = p_{11} - p_{01}$ , we obtain

$$\mathbb{E}(L_0 - L_1) = k \mathbb{E} [f(L_0) - f(L_1)], \tag{11}$$

where

$$\begin{aligned} f(x) &= (p_{11} - p_{01}) \ln \left( 1 + \frac{c_0 - c_1}{e^x + c_1} \right) \\ &= \frac{c_1 - c_0}{(1 + c_0)(1 + c_1)} \ln \left( 1 + \frac{c_0 - c_1}{e^x + c_1} \right). \end{aligned} \tag{12}$$

### 3.1 General case

If  $c_0 = c_1$ , then the function  $f$  defined at (12) is constant. In that case, (11) shows that  $\mathbb{E}(L_0 = L_1) = 0$ , and reconstruction is impossible.

So assume that  $c_0 \neq c_1$ . Then  $f$  is strictly increasing, and one obtains that

$$f'(x) = \frac{(c_1 - c_0)^2}{(1 + c_0)(1 + c_1)} \frac{e^x}{(e^x + c_0)(e^x + c_1)}. \quad (13)$$

Putting  $y = e^x$  and taking the reciprocal, one can find the value of  $x$  maximising (13) by finding the value of  $y \geq 0$  minimising  $(y + c_0)(y + c_1)y^{-1}$ . One obtains  $y = e^x = (c_0c_1)^{1/2}$ , and so

$$\begin{aligned} \sup_x f'(x) &= \frac{(c_1 - c_0)^2}{(1 + c_0)(1 + c_1)} \frac{(c_0c_1)^{1/2}}{((c_0c_1)^{1/2} + c_0)((c_0c_1)^{1/2} + c_1)} \\ &= \frac{(c_1 - c_0)^2}{(1 + c_0)(1 + c_1)} \frac{1}{(\sqrt{c_1} + \sqrt{c_0})^2} \\ &= \frac{(\sqrt{c_1} - \sqrt{c_0})^2}{(1 + c_0)(1 + c_1)} \\ &= \left( \sqrt{\frac{1}{1 + c_0} \frac{c_1}{1 + c_1}} - \sqrt{\frac{c_0}{1 + c_0} \frac{1}{1 + c_1}} \right)^2 \\ &= \left( \sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}} \right)^2. \end{aligned} \quad (14)$$

Since we know that  $L_1 \leq L_0$  with probability 1, we then have that

$$0 \leq f(L_0) - f(L_1) \leq \sup_x f'(x)(L_0 - L_1)$$

with equality on the RHS iff  $L_0 = L_1$ , with probability 1. Hence, from (11),

$$\mathbb{E}(L_0 - L_1) \leq k \sup_x f'(x) \mathbb{E}(L_0 - L_1),$$

with equality iff both sides are 0. So to show that  $\mathbb{E}(L_0 - L_1) = 0$ , and therefore that reconstruction is impossible, it's enough to show that  $k \sup_x f'(x) \leq 1$ . Using (14), we see that (2) indeed implies that reconstruction is impossible, and the proof of Theorem 1 is done.

### 3.2 Hard-core case

Recall that

$$\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+w} & \frac{w}{1+w} \\ 1 & 0 \end{pmatrix};$$

we have also that  $\pi_0 = (1 + w)/(1 + 2w)$ ,  $\pi_1 = w/(1 + 2w)$ , and  $c_0 = w$ ,  $c_1 = 0$ . We have also defined  $\lambda = w(1 + w)^k$ . Equation (12) now becomes

$$f(x) = -\frac{w}{1 + w} \ln(1 + we^{-x}),$$

and now the function  $f$  is concave as well as strictly increasing. Hence in particular, if  $x_0 < x_1$  and  $y_0 < y_1$  with  $x_0 \leq y_0$  and  $x_1 \leq y_1$ , then

$$0 \leq \frac{f(y_1) - f(y_0)}{y_1 - y_0} \leq \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \quad (15)$$

Recurrence (10) now becomes

$$L(u) = -k \ln(1+w) + \sum_{j=1}^k \ln(1 + we^{-L(u_j)}),$$

and so in particular  $L(u)$  is always greater than or equal to  $-k \ln(1+w)$ ; the same is therefore true of  $L_0$  and  $L_1$  also. Combining this with property (iii) of the coupling described after (10), we have that, with probability 1, either  $L_0 = L_1$  or  $-k \ln(1+w) \leq L_1 \leq 0 \leq L_0$ . Thus, using (15), we obtain that with probability 1

$$\begin{aligned} 0 \leq f(L_0) - f(L_1) &\leq (L_0 - L_1) \frac{f(0) - f(-k \ln(1+w))}{0 - (-k \ln(1+w))} \\ &= (L_0 - L_1) \frac{w}{1+w} \frac{-\ln(1+w) + \ln(1+\lambda)}{k \ln(1+w)}, \\ &= (L_0 - L_1) \frac{w}{k(1+w)} \left( \frac{\ln(1+\lambda)}{\ln(1+w)} - 1 \right), \end{aligned} \quad (16)$$

where we have used

$$f(0) = -w \ln(1+w)/(1+w)$$

and

$$\begin{aligned} f(-k \ln(1+w)) &= -\frac{w}{1+w} \ln(1 + we^{k \ln(1+w)}) \\ &= -\frac{w}{1+w} \ln(1 + w(1+w)^k) \\ &= -\frac{w}{1+w} \ln(1+\lambda). \end{aligned}$$

Combining (11) and (16), we get  $0 \leq \mathbb{E}(L_0 - L_1) \leq \rho [\mathbb{E}(L_0 - L_1)]$ , where

$$\rho = \frac{w}{1+w} \left( \frac{\ln(1+\lambda)}{\ln(1+w)} - 1 \right).$$

To obtain that  $\mathbb{E}(L_0 - L_1) = 0$ , and hence that reconstruction is impossible, it's enough that  $\rho < 1$ . But

$$\begin{aligned} \rho &< \frac{w}{1+w} \frac{\ln(1+\lambda)}{\ln(1+w)} \\ &\leq \ln(1+\lambda) \sup_{w>0} \frac{w}{(1+w) \ln(1+w)} \\ &= \ln(1+\lambda), \end{aligned}$$

(since the quantity within the sup is decreasing in  $w$  and tends to 1 as  $w \downarrow 0$ ). So certainly if  $\lambda \leq e - 1$ , then  $\rho < 1$  as desired. Also  $\rho$  is continuous as a function of  $\lambda$  (or  $w$ ), so the threshold value  $\lambda_2(k)$  is in fact strictly greater than  $e - 1$ , and the proof of Theorem 2 is done.

## 4 A calculation

For completeness, we here include details to show that the LHS of (2) is always less than or equal to that of (1). Define  $b_0 = \min(p_{00}, p_{11}) \leq \max(p_{00}, p_{11}) = b_1$ . We obtain

$$\begin{aligned} \min\{p_{00} + p_{10}, p_{01} + p_{11}\} &= \min\{b_0 + 1 - b_1, b_1 + 1 - b_0\} \\ &= b_0 + 1 - b_1 \\ &= (1 - b_0)(1 - b_1) + b_0b_1 + 2b_0(1 - b_1) \\ &\leq (1 - b_0)(1 - b_1) + b_0b_1 + 2\sqrt{b_0b_1}\sqrt{(1 - b_1)(1 - b_0)} \end{aligned} \quad (17)$$

$$= \left[ \sqrt{b_0b_1} + \sqrt{(1 - b_1)(1 - b_0)} \right]^2. \quad (18)$$

(The inequality in (17) follows since  $b_0 \leq b_1$  and  $1 - b_1 \leq 1 - b_0$ ; equality holds if  $b_0 = b_1$  or if one of  $b_0$  and  $b_1$  is 0 or 1).

Then, starting from the LHS of (2) and using (18),

$$\begin{aligned} \left( \sqrt{p_{00}p_{11}} - \sqrt{p_{01}p_{10}} \right)^2 &= \left[ \sqrt{b_0b_1} - \sqrt{(1 - b_1)(1 - b_0)} \right]^2 \\ &\leq \left[ \sqrt{b_0b_1} - \sqrt{(1 - b_1)(1 - b_0)} \right]^2 \frac{\left[ \sqrt{b_0b_1} + \sqrt{(1 - b_1)(1 - b_0)} \right]^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \\ &= \frac{[b_0b_1 - (1 - b_1)(1 - b_0)]^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \\ &= \frac{(b_0 - (1 - b_1))^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}} \\ &= \frac{[\pm(p_{00} - p_{10})]^2}{\min\{p_{00} + p_{10}, p_{01} + p_{11}\}}, \end{aligned}$$

which gives the LHS of (1) as required.

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