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We present the main results of a study for the existence of vacant and occupied unbounded connected components in a non-homogeneous Poisson blob process. The method used in the proofs is a multi-scale percolation comparison.

Keywords: Poisson blob model, continuum percolation, phase transition, multi-scale percolation

1 Introduction

One of the most well known examples of phenomena that introduces and motivates the study of continuum percolation is the process of the ground getting wet during a period of rain. At each point hit by a raindrop, one sees a circular wet patch. Right after the rain begins to fall what one sees is a small wet region inside a large dry region. At some instant, so many raindrops have hit the ground that the situation changes from that to a small dry region inside a large wet region. Typically, the parameter in which there is a phase transition behaviour is the density of the raindrops.

Continuum percolation models in which each point of a two-dimensional homogeneous Poisson point process is the centre of a disk of given (or random) radius r, have been extensively studied. In this note we present phase transition results for a sequence of Poisson point process which defines Poisson Boolean models and whose rates depend on the past. In order to prove our results we rely on a multi-scale percolation structure. General reference for percolation and continuum percolation are the books of Grimmett [2] and Meester and Roy [3]. A nice example of the use of multi-scale percolation technique can be found is Fontes *et al* [1].

2 Model and phase transition results

Let $\beta > 0$ be fixed number. Define $A_0 = \emptyset$. Having defined the sets $A_0, A_1, \ldots, A_n$, define the process X_{n+1} as the non-homogeneous Poisson point process with intensity function given by:

$$f_{n+1}(x) = \exp(-\beta|B(x, n+1) \setminus \cup_{k=0}^{n} A_k|) \tag{1}$$

where $B(a, r)$ is the square of length r having centre at a and $|C|$ is the area (Lebesgue measure) of the set C.

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Let $\{x_i^{(n+1)} : i \geq 1\}$ be the set of points from the process X_{n+1}. Define the set $A_{n+1} = \cup_{i=1}^{\infty} B(x_i^{(n+1)}, n+1)$ as the random set covered by the boxes from the process X_{n+1}. Define the total covered set $A_\infty = \bigcup_{n=0}^{\infty} A_n$.

The fundamental question in continuum percolation theory is about the existence of unbounded connected components. That is why we ask the following questions about the random set A_∞ and its complement, the set A_∞^c. Let A be the component of A_∞ which contains the origin. If the origin is not contained in A_∞, this is the empty set. Define

$$\theta(\beta) = \mathbb{P}_\beta(A \text{ is unbounded}). \tag{2}$$

It is clear that $\theta(\beta)$ is a decreasing function of β. Hence define the critical parameter β_c as follows:

$$\beta_c = \sup\{\beta > 0 : \theta(\beta) > 0\}. \tag{3}$$

The following result holds

Theorem 1.

$$0 < \beta_c < \infty.$$

Similar questions can also be asked the complement set of A_∞. Define C as the component of $(A_\infty)^c$ which contain the origin. Define the vacant percolation probability as

$$\theta^\star(\beta) = \mathbb{P}_\beta(C \text{ is unbounded}). \tag{4}$$

In this case, we have that $\theta^\star(\beta)$ is an increasing function of β. Hence define the critical parameter $\beta_c^\star$ as follows:

$$\beta_c^\star = \inf\{\beta > 0 : \theta^\star(\beta) > 0\}. \tag{5}$$

We also prove the following theorem

Theorem 2.

$$0 < \beta_c^\star < \infty.$$

It is clear that X_1 is actually an homogeneous Poisson process with intensity $\exp(-\beta|B(0,1)|)$. Thus, A_1 will contain the covered set of a Poisson Boolean model with radius random variable being degenerate at $1/2$ and intensity $\exp(-\beta|B(0,1)|)$. Thus, if $\exp(-\beta|B(0,1)|) > \lambda_c$, the probability that the origin is contained in an unbounded component of A_1 is positive, where λ_c is the critical intensity of the Poisson Boolean model with radius being degenerate at $1/2$. Therefore, we have $\theta(\beta) > 0$ for this β. Hence, we have that $\beta_c > 0$. A similar argument also holds for $\beta_c^\star$ and we can easily show that, $\beta_c^\star > 0$.

This is an announcement of results from a joint work with P. Ferrari, L. Fontes, S. Popov and A. Sarkar. The proofs rely on a multi-scale comparison argument to prove that the probability of certain events related to the existence of an unbounded connected component is exponentially close to 1 for large values of β.

References

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