The number of distinct part sizes of some multiplicity in compositions of an Integer. A probabilistic Analysis

Guy Louchard

Discrete Mathematics and Theoretical Computer Science **AC**, 2003, 155–170

The number of distinct part sizes of some multiplicity in compositions of an Integer. A probabilistic Analysis

Guy Louchard

Université Libre de Bruxelles, Département d'Informatique, CP 212, Boulevard du Triomphe, B-1050 Bruxelles, Belgium
louchard@ulb.ac.be

Random compositions of integers are used as theoretical models for many applications. The degree of distinctness of a composition is a natural and important parameter. A possible measure of distinctness is the number X of distinct parts (or components). This parameter has been analyzed in several papers. In this article we consider a variant of the distinctness: the number $X(m)$ of distinct parts of multiplicity m that we call the m-distinctness. A first motivation is a question asked by Wilf for random compositions: what is the asymptotic value of the probability that a randomly chosen part size in a random composition of an integer ν has multiplicity m. This is related to $\mathbb{E}(X(m))$, which has been analyzed by Hitczenko, Rousseau and Savage. Here, we investigate, from a probabilistic point of view, the first full part, the maximum part size and the distribution of $X(m)$. We obtain asymptotically, as $\nu \to \infty$, the moments and an expression for a continuous distribution φ, the (discrete) distribution of $X(m,\nu)$ being computable from φ.

Keywords: Mellin transforms, urns models, Poissonization, saddle point method, generating functions

1 Introduction

Let us first recall some well-known results. Let us consider the composition of an integer ν, i.e. $\nu = \sum_i^N x_i, \quad x_i$: integer > 0. Considering all compositions as equiprobable, we know (see [HL01]) that the number of parts N is asymptotically Gaussian, $\nu \to \infty$:

$$N \sim \mathcal{N}\left(\frac{\nu}{2}, \frac{\nu}{4}\right), \tag{1}$$

and that the part sizes are asymptotically id $\mathrm{GEOM}(1/2)$ and *independent*. Consider now n random variables (R.V.), $\mathrm{GEOM}(1/2)$ and define the indicator R.V.[†]

$$Y_i := [\![\text{value } i \text{ appears among these } n \text{ R.V.}]\!]$$

Then, asymptotically, $n \to \infty$, the Y_i are independent. The first empty part value, i.e the first k such that $Y_k = 0$, is of order $O(\log n)$. Here and in the sequel, $\log := \log_2, L := \ln 2$. Similarly, the maximum part

[†] Here we use the indicator function notation proposed by Knuth et al. [GKP89].

size is also of order $O(\log n)$, as well as the number Y of distinct values (part sizes): $Y = \sum_1^\infty Y_i$. The asymptotic distributions and moments of these R.V. are also given in [HL01]. We know (see Hwang and Yeh [HY97]) that

$$\mathbb{E}(Y) \sim \log n + \gamma/L - 1/2 + \beta(\log n) + O(1/n)$$

where β is a small periodic function of $\log n$, and the distribution of Y is highly concentrated around its mean, with a $O(1)$ range. All these distributions depend on $\log n$. Hence, with (1), the same R.V. related to ν are asymptotically equivalent by replacing $\log n$ by $\log \nu - 1$ (see [HL01]).

In this article we consider a variant of the distinctness: the number $X(m)$ of distinct parts of multiplicity m that we call the m-distinctness. A first motivation is a question asked by Wilf for random compositions: what is the asymptotic value of the probability $P(m,\nu)$ that a randomly chosen part size in a random composition of an integer ν has multiplicity m. (The corresponding problem for random partitions has been analyzed in Corteel et al. [CPSW99]). Of course, here,

$$P(m,\nu) = \mathbb{E}(X(m,\nu)/Y(\nu)),$$

where we explicitly show the dependence on ν. But, as already mentioned, $Y(\nu)$ has asymptotically the same distribution as Y (with $\log n$ replaced by $\log \nu - 1$). On the other side, Y is highly concentrated around its mean . Hence, asymptotically, as shown in Hitczenko, Savage [HS99] and Hitczenko et al [HRS02], for $m = O(1)$,

$$P(m,v) \sim \mathbb{E}(X(m,\nu))/\mathbb{E}(Y(\nu)).$$

Here, we investigate, from a probabilistic point of view, the first full part, the maximum part size and the distribution of $X(m,\nu)$. We obtain asymptotically, as $\nu \to \infty$, the moments and an expression for a continuous distribution φ, the (discrete) distribution of $X(m,\nu)$ being computable from φ. We will see that, again, all asymptotic distributions for some multiplicity m depend only on $\log n$. Hence, the same R.V. related to ν are again simply obtained by replacing $\log n$ by $\log \nu - 1$. The paper is organized as follows: in Section 2, we consider a fixed multiplicity $m = O(1)$. We analyze the moments, the first full part, the maximum part size, and the distribution of $X(m)$. Section 3 is devoted to large multiplicity m. Section 4 concludes the paper. Due to length constraints, some proofs have been briefly presented.

In this section, we are interested in the properties of the R.V.:
$X_i(m) := [\![$value i appears among the n GEOM$(1/2)$ R.V. with multiplicity m, for fixed $m = O(1)]\!]$.
Of course,

$$\Pr[X_i(m) = 1] = \binom{n}{m}(1/2^i)^m(1-1/2^i)^{n-m}. \tag{2}$$

We immediately see that the dominant range is given by $i = \log n + O(1)$. To the left and the right of this range, $\Pr[X_i(m) = 1] \sim 0$. Within the range, $\Pr[X_i(m) = 1]$ is asymptotically equivalent to a Poisson distribution:

$$\Pr[X_i(m) = 1] \sim \frac{1}{m!}(n/2^i)^m \exp(-n/2^i),$$

and, with $X(m) := \sum_1^\infty X_i(m)$,

$$\mathbb{E}(X(m)) \sim G(n,m),$$

where, using the "sum splitting technique " as described in Knuth [Knu73], p.131,

$$G(n,m) := \frac{1}{m!}\sum_{i=1}^{\infty}(n/2^i)^m \exp(-n/2^i),$$

which, for large n, can be analyzed using Mellin transforms: see Flajolet et al. [FGD95]. It is well known that the dominant value is given by some constant. The oscillatory part has a very small amplitude, usually of order 10^{-5}. Indeed, set $f(y) := y^m e^{-y}$. We obtain

$$G(n,m) = \frac{1}{m!}\sum_{i=1}^{\infty} f(n/2^i),$$

the Mellin transform of which is

$$G^*(s) = \frac{\Gamma(m+s)}{m!}\frac{2^s}{1-2^s},$$

defined in the fundamental strip $< -m, 0 >$. To the right of this strip, the poles of $G^*(s)$ are a simple pole at $s = 0$, and simple poles at $s = \chi_k := 2k\pi i/L (k \neq 0)$. The singular expansion of $G^*(s)$ is given by ‡

$$G^*(s) \asymp \left[\frac{\Gamma(m)}{Lm!s}\right] + \sum_{k\neq 0}\frac{\Gamma(m+\chi_k)}{Lm!(s-\chi_k)}.$$

This leads, by converse mapping, to

$$G(n,m) \sim \frac{1}{mL} + \beta_0(\log n) + O(1/n), \tag{3}$$

where β_0 is a small periodic function of $\log n$:

$$\beta_0(\log_2 n) := \sum_{k\neq 0}\frac{\Gamma(m+\chi_k)}{Lm!}n^{-\chi_k} = \sum_{k\neq 0}\frac{\Gamma(m+\chi_k)}{Lm!}e^{-2\pi ik\log n}.$$

In the sequel, $\beta.(\log n)$ will always denote (small) periodic functions. As $n \sim \mathcal{N}(\frac{\nu}{2}, \frac{\nu}{4})$, we just have to replace $\log n$ by $\log \nu - 1$. So we recover the mean already computed in Hitczenko and Savage, [HS99] and Hitczenko , Rousseau and Savage, [HRS02]. To compute all moments, we must check that the X_i are asymptotically independent. We could proceed as was done in [HL01] for the Y_i, but we follow here another route. Let us consider $\Pi_n = \mathbb{E}(z^X)$. We obtain

Theorem 1.1.

$$\Pi_n \sim \prod_{l=1}^{\infty}\left[\left(1-\frac{1}{m!}(n/2^l)^m e^{-n/2^l}\right) + z\frac{1}{m!}(n/2^l)^m e^{-n/2^l}\right], n \to \infty.$$

Proof. We use an urn model, as in Sevastyanov and Chistyakov, [SČ64] and Chistyakov, [Chi67], and the Poissonization method (see, for instance Jacquet and Szpankowski [JS98] for a general survey). If we Poissonize, with parameter τ , the number of balls (i.e the number n of R.V. here), the generating function of X_l is given from (2), by

$$\left(1-\frac{1}{m!}(\tau/2^l)^m e^{-\tau/2^l}\right) + z\frac{1}{m!}(\tau/2^l)^m e^{-\tau/2^l},$$

‡ The symbol $\asymp$ is used to denote the fact that two functions are of the same asymptotic order.

and we have independency of cells occupation. This leads to

$$e^{-\tau}\sum_n \frac{\tau^n}{n!}\Pi_n = \prod_{l=1}^{\infty}\left[\left(1-\frac{1}{m!}(\tau/2^l)^m e^{-\tau/2^l}\right)+z\frac{1}{m!}(\tau/2^l)^m e^{-\tau/2^l}\right].$$

Hence, by Cauchy, we obtain $\Pi_n = \frac{n!}{2\pi i}\int_\Gamma \exp\{nf(\tau)\}d\tau/\tau$, where Γ is inside the analyticity domain of the integrand and encircles the origin, and

$$f(\tau) := -\log\tau+\tau/n+\frac{1}{n}\sum_{l=1}^{\infty}\ln\left[\left(1-\frac{1}{m!}(\tau/2^l)^m e^{-\tau/2^l}\right)+z\frac{1}{m!}(\tau/2^l)^m e^{-\tau/2^l}\right].$$

By standard saddle-point method (see, for instance, Flajolet and Sedgewick, [FS94]), we look for τ^* such that $f'(\tau^*)=0$, with

$$f'(\tau) = -1/\tau+1/n-\frac{z-1}{n\tau}\sum_{l=1}^{\infty}\frac{(\tau/2^l)^{m+1}-m(\tau/2^l)^m}{m!\exp(\tau/2^l)-(\tau/2^l)^m+z(\tau/2^l)^m}.$$

But, again by Mellin, for fixed $z>0$,

$$\sum_{l=1}^{\infty}\frac{(\tau/2^l)^{m+1}-m(\tau/2^l)^m}{m!\exp(\tau/2^l)-(\tau/2^l)^m+z(\tau/2^l)^m} \sim C+\beta.(\log\tau).$$

with

$$C := \int_0^\infty \frac{y^{m+1}-my^m}{m!\exp(y)-y^m+zy^m}dy/L.$$

Hence $\tau^* \sim n+C$. It is easily checked that $C=0$. Finally, $\Pi_n \sim \frac{n!e^{nf(\tau^*)}}{\sqrt{2\pi}\tau^*\sqrt{nf''(\tau^*)}}$, and, by Stirling, we easily derive the theorem. □

Theorem 1.1 confirms the asymptotic independence assumption.

1.1 The moments of $X(m)$

We now have all necessary ingredients to compute the moments. The variance of $X(m)$ is now easily derived: we obtain, by Mellin,

$$\begin{aligned}
\mathrm{VAR}(X(m)) &\sim \frac{1}{m!}\sum_1^\infty (n/2^i)^m\exp(-n/2^i)\left[1-\frac{1}{m!}(n/2^i)^m\exp(-n/2^i)\right]\\
&\sim \int_0^\infty e^{-y}\frac{y^m}{m!}(1-e^{-y}\frac{y^m}{m!})\frac{dy}{Ly}+\beta_1(\log_2 n)\\
&= \frac{1}{mL}-\frac{(2m-1)!}{Lm!^2 2^{2m}}+\beta_1(\log_2 n).
\end{aligned}$$

The other moments can be derived as follows. We obtain, setting $z = e^s$,

$$\begin{aligned}
\ln(\Pi_n) \sim S_2 &= \sum_{l=1}^{\infty} \ln\left[1 + (e^s - 1)\frac{1}{m!}(n/2^l)^m \exp(-n/2^l)\right] \\
&= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}(e^s-1)^i V_i}{i}, \text{ with} \\
V_i &:= \sum_{l=1}^{\infty} \left[\frac{1}{m!}(n/2^l)^m\right]^i \exp(-in/2^l).
\end{aligned}$$

The centered moments of $X(m)$ can be obtained by analyzing

$$S_3 := \exp[S_2 - sV_1].$$

Again, by Mellin, we obtain

$$V_i \sim B_i + \beta_i(\log n),$$

with

$$B_i = \int_0^{\infty} \left[\frac{y^m}{m!}\right]^i e^{-iy} \frac{dy}{Ly} = \frac{(im-1)!}{m!^i L i^{im}},$$

and finally, the centered moments are given by

$$\begin{aligned}
\tilde{\sigma}^2 := \mathrm{VAR}(X(m)) &\sim \frac{1}{mL} - \frac{(2m)!}{2Lm!^2 2^{2m} m}, \\
\tilde{\mu}_3 := \mu_3(X(m)) &\sim \frac{1}{mL} - \frac{3(2m)!}{2Lm!^2 2^{2m} m} + \frac{2(3m)!}{3Lm!^3 3^{3m} m}, \\
\tilde{\mu}_4 := \mu_4(X(m)) &\sim \frac{1}{mL} + \frac{3}{m^2L^2} - \frac{3(4m)!}{2Lm!^4 4^{4m} m} + \frac{4(3m)!}{Lm!^3 3^{3m} m} \\
&- \frac{7(2m)!}{2Lm!^2 2^{2m} m} - \frac{3(2m)!}{3L^2 m!^2 2^{2m} m^2} + \frac{3(2m)!^2}{4L^2 m!^4 2^{4m} m^2}.
\end{aligned}$$

The neglected terms are made of periodic functions $\beta_\cdot(\log n)$ and of $O(\frac{1}{n})$ contributions.

Again, the centered moments (of order ≥ 2) of X related to a composition of ν are given by the same expressions.

For $n = 20000, m = 2$, we have done a simulation (of $T = 4000$ sets). We obtain the results of Table 1 (the probability related moments are explained later on). For an easy comparison, we give here only four significant digits.

1.2 The maximum part size of multiplicity m

The maximum part size $\mathcal{M}_n(m)$ of multiplicity m is such that

$$\Pr(\mathcal{M}_n(m) < k) \sim \prod_{i=k}^{\infty} \left[1 - \frac{1}{m!}(n/2^i)^m \exp(-n/2^i)\right].$$

	Theoretical asymptotic value	Observed value	Probability related value
mean	.7213 ...	.7345...	.7214...
variance	.5861 ...	.5945...	.5863...
μ_3	.3750...	.3752 ...	.3752...
μ_4	1.1197...	1.1341...	1.1198...

Tab. 1: Moments, $n = 20000, m = 2$.

Set $\eta := Lk - \ln n$. This leads, with $\eta = O(1)$, to

$$\Pr(\mathcal{M}_n(m) < k) \sim \varphi_1(m,\eta),$$

with

$$\varphi_1(m,\eta) = \prod_{j=0}^{\infty}\left[1 - \frac{1}{m!}e^{-m(\eta+Lj)}e^{-e^{-(\eta+Lj)}}\right].$$

Figure 1 gives $\varphi_1(m,\eta)$ for $m = 1,\dots,4$, bottom to top. It appears that for $\eta \to -\infty, \varphi_1(m,\eta)$ seems to converge to some value, which of course corresponds to

$$P(m,0) := \Pr(X(m) = 0),$$

but a closer view reveals the usual fluctuations, shown in Figure 2, for $m = 2$. Set $\psi(n) := \log n - \lfloor \log n \rfloor$ (fractional part). With $\eta = L(-6 - \psi(20000))$, we obtain $P(2,0) = .4489079864\dots$, which will be compared later on with a direct expression.

Similarly, we derive

$$\Pr(\mathcal{M}_n(m) = k-1) \sim \varphi_2(m,\eta) = \varphi_1(m,\eta)e^{-m(\eta-L)}e^{-e^{-(\eta-L)}}/m!.$$

Figure 3 gives $\varphi_2(m,\eta)$ for $m = 1,\dots,4$, (more and more concentrated as m increases).

Our simulation for $n = 20000, m = 2$ of $T = 4000$ sets leads to Figure 4 (φ_1, observed = circle, asymptotic = line) and Figure 5 (φ_2, observed = circle, asymptotic = line). Again, for compositions, we replace $\log n$ by $\log \nu - 1$.

1.3 First full part value of multiplicity m

Another variable of interest is the first k such that $X_k = 1$, i.e we are interested in the probability

$$\Pr[X_i = 0, i = 1\cdots k-1, X_k = 1].$$

Note that this is the opposite situation of the Y_k case (see [HL01]), where we looked for the *first* k such that $Y_k = 0$. The probability is asymptotically given by

$$\prod_{i=1}^{k-1}\left[1 - \frac{1}{m!}(n/2^i)^m \exp(-n/2^i)\right]\frac{1}{m!}(n/2^k)^m \exp(-n/2^k).$$

Again, we set $\eta := Lk - \ln n$. This leads asymptotically, with $\eta = O(1)$ to

$$\Pr[X_i = 0, i = 1 \cdots k-1, X_k = 1] \sim \varphi_3(m, \eta),$$

with

$$\begin{aligned}
\varphi_3(m,\eta) &= \varphi_4(m,\eta)\frac{1}{m!}e^{-m\eta}e^{-e^{-\eta}}, \\
\varphi_4(m,\eta) &= \prod_{j=1}^{\infty}\left[1 - \frac{1}{m!}e^{-m(\eta-Lj)}e^{-e^{-(\eta-Lj)}}\right].
\end{aligned}$$

Again, for compositions, we replace $\log n$ by $\log \nu - 1$. Figure 6 gives $\varphi_4(2,\eta)$ and Figure 7 gives $\varphi_4(2,\eta)$ for large values of η. Again, this is oscillating and corresponds to $P(2,0)$.

1.4 Asymptotic distribution of $X(m)$

The analysis is rather similar to the one we used in [Lou87] and [HL01]. First of all we have, for any fixed $k = O(\log n)$,

$$P(m,0) \sim \varphi_4(\eta)\varphi_1(\eta).$$

Let us choose $k = \lfloor \log n \rfloor$. This leads to $\eta = -L\psi(n)$ and we obtain a periodic function of ψ:

$$P(m,0) \sim \varphi_4[-L\psi(n)]\varphi_1[-L\psi(n)],$$

shown in Figure 10 for $m = 2$. For $n = 20000, m = 2$, the numerical value of $P(2,0)$ is exactly the same as before. Now we turn to $P(m, j) := \Pr(X(m) = j)$. We take advantage of the fact that all urns are empty *before* the first occupied urn, $k-1$ say. Then, again with $\eta := Lk - \ln n$,

$$\begin{aligned}
P(m,1) &\sim \sum_k \varphi_3(\eta - L)\varphi_1(\eta), \\
P(m,2) &\sim \sum_k \varphi_3(\eta - L)\varphi_1(\eta)\sum_{r_1 \geq k}\left\{\frac{1}{m!}(n/2^{r_1})^m \exp(-n/2^{r_1}) \Big/ \left[1 - \frac{1}{m!}(n/2^{r_1})^m \exp(-n/2^{r_1})\right]\right\},
\end{aligned}$$

and more generally,

$$\begin{aligned}
P(m,u+1) &\sim \sum_k \varphi_3(\eta - L)\varphi_1(\eta). \\
&\quad . \sum [\![r_1 > r_2 > .. > r_u, r_j \geq k]\!] \prod_{i=1}^{u}\left\{\frac{1}{m!}(n/2^{r_i})^m \exp(-n/2^{r_i}) \Big/ \left[1 - \frac{1}{m!}(n/2^{r_i})^m \exp(-n/2^{r_i})\right]\right\}
\end{aligned}$$

Now we set $r_i = k + w_i, l = k - \lfloor \log n \rfloor$ and we finally derive the following theorem

Theorem 1.2. *Set* $\psi(n) := \log n - \lfloor \log n \rfloor$, *then*

$$P(m,u+1) \sim \sum_{l=-\infty}^{\infty} \varphi_5[L(l - \psi(n))],$$

with

$\varphi_5(\eta) = \varphi_3(\eta - L)\varphi_1(\eta)$.

$$.\sum[\![w1 > w2 > \ldots > w_u, w_j \geq 0]\!]\prod_{i=1}^{u}\left\{\frac{1}{m!}e^{-m(\eta+Lw_i)}e^{-e^{-(\eta+Lw_i)}} \Big/ \left[1 - \frac{1}{m!}e^{-m(\eta+Lw_i)}e^{-e^{-(\eta+Lw_i)}}\right]\right\}$$

Note that, for compositions, we obtain asymptotically $\psi(n) = \psi(\nu)$. We get again periodic function of $\psi(n)$. We give in Figure 11 and Figure 12 the sums $\sum_{i=0}^{3} P(2,i), \sum_{i=0}^{4} P(2,i)$. The effect of computing $P(2,i)$ with bounded indices (we limit the values of w_u to 16) becomes apparent at the 10^{-7} precision.

Figure 13 gives $P(m,i), m = 1,\ldots,4$, (from top to bottom to the right of $i = 2$). The distributions become more concentrated as m increases.

Finally, we compare the observed distribution of $X(2)$ with the asymptotic one in Figure 14 (observed = circle, asymptotic = line). Apart from $i = 0$ the fi t is quite good. The "Probability related values" moments given in Table 1 are computed with the distribution $P(2,i)$.

2 Large multiplicity *m*

2.1 Fixed number of parts *n*

It is now clear that large m are related to small integer values i. More precisely, the number M_i of integers equal to i is asymptotically given by a Gaussian:

$$\Pr(M_i = m) \sim \exp\{-(m - n/2^i)^2/[2n/2^i(1 - 1/2^i)]\}/\sqrt{2\pi n/2^i(1 - 1/2^i)}. \tag{4}$$

The means $n/2^i, i = 1,2,\ldots$ are given by $n/2, n/4,\ldots$, separated by $n/4, n/8,\ldots$ which shows that the Gaussians (4) are asymptotically exponentially distinct in the sense that some common intervals, for instance $m \in [3n/2^{i+2} - n/2^{i+3}..3n/2^{i+2} + n/2^{i+3}]$ have asymptotically small probability measures. So for any large value m , *only one* value

$$i = \text{round}[\log(n/m)] \tag{5}$$

is related to m and $X(m)$ has only two possible values: $\{0,1\}$. The following events are equivalent: $[\![X_i(m) = 1]\!] \equiv [\![M_i = m]\!]$. The probability (4) is small, of order at most $O(1/\sqrt{m})$. Figure 15 gives $\Pr(X_i(m) = 1)$ for $n = 2000$ (fi rst three ranges, $i = 1,2,3$) and Figure 16 gives the corresponding distribution functions, together with the observed values provided by a simulation of $T = 2000$ sets (observed = circle, asymptotic = line).

An interesting check would be to recover the dominant term of the mean of $Y : \mathbb{E}(Y) \sim \log n$. Choose $\tilde{j} := \alpha \log n, 0 < \alpha < 1$ which corresponds, by (5), to $\tilde{m} = n^{1-\alpha}$. For each $i \leq \tilde{j}$, by Euler–McLaurin,

$$\sum_{m=\lfloor 3n/2^{i+2}\rfloor}^{\lfloor 3n/2^{i+1}\rfloor} \exp\{-(m - n/2^i)^2/[2n/2^i(1 - 1/2^i)]\}/\sqrt{2\pi n/2^i(1 - 1/2^i)} \sim 1,$$

and this contributes to $\mathbb{E}(Y)$ by $S_1 = \tilde{j}$. On the other side, each $m < \tilde{m}$ contributes, by (3), with $\frac{1}{mL}$, with a total contribution

$$S_2 = \frac{1}{L}\sum_{1}^{\tilde{m}} 1/m \sim \frac{1}{L}\ln \tilde{m}.$$

The quantity $S_1 + S_2 \sim \log n$ as expected.

2.2 Composition of ν.

Now the number of parts N is such that (see(1))

$$N \sim \mathcal{N}\left(\frac{\nu}{2}, \frac{\nu}{4}\right).$$

We obtain

$$\mathbb{E}(M_k) = \frac{\nu}{2}\frac{1}{2^k}. \tag{6}$$

The asymptotic distribution of M_k is obtained as follows. We derive, setting $\tilde{M}_k := (M_k - n/2^k)/\sqrt{\nu}$,

$$\begin{aligned}
\mathbb{E}\left[\exp[iM_k\theta/\sqrt{\nu}]\right] &= \mathbb{E}\left[\exp[in\theta/(\sqrt{\nu}2^k) + i\tilde{M}_k\theta]\right] \\
&\sim \mathbb{E}\left[\exp[in\theta/(\sqrt{\nu}2^k) - \theta^2 n/(2\nu 2^k)(1-1/2^k)]\right] \\
&\sim \exp\left[i\nu\theta/(2\sqrt{\nu}2^k) - \nu\theta^2/(2\nu 2.2^k)(1-1/2^k) + \nu/8[i\theta/(\sqrt{\nu}2^k) - \theta^2/(2\nu 2^k)(1-1/2^k)]^2\right] \\
&\sim \exp\left[i\theta\sqrt{\nu}/(2.2^k) - \theta^2/2[1/(4.4^k) + 1/(2.2^k)(1-1/2^k)]\right], \nu \to \infty.
\end{aligned}$$

The first term confirms (6). The second term shows that

$$M_k \sim \mathcal{N}\left(\frac{\nu}{2}\frac{1}{2^k}, \nu\sigma_m^2\right),$$

with

$$\sigma_m^2 = 1/(4.4^k) + 1/(2.2^k)(1-1/2^k).$$

The conclusions of Sec. 2.2 are still valid.

3 Conclusion

Using various techniques from analysis and probability theory, we have analyzed the stochastic properties of the m-distinctness of random compositions. An interesting open problem would be be to extend our results to the Carlitz compositions, where two successive parts are different (see [LP02]).

Acknowledgements

The pertinent comments of the referees led to substantial improvements in the presentation.

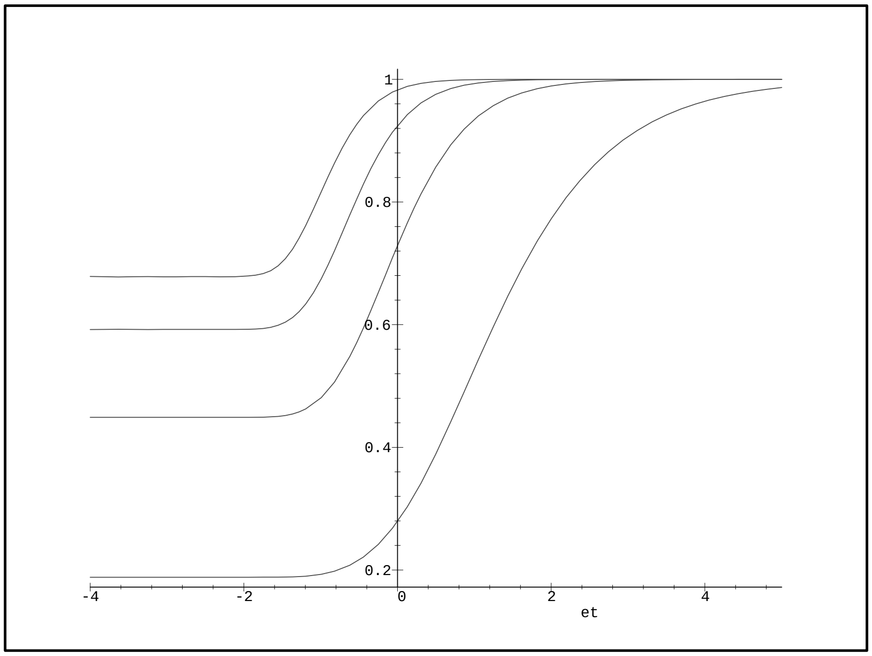

Fig. 1: $\varphi_1(m,\eta)$ for $m = 1,\ldots,4$, bottom to top

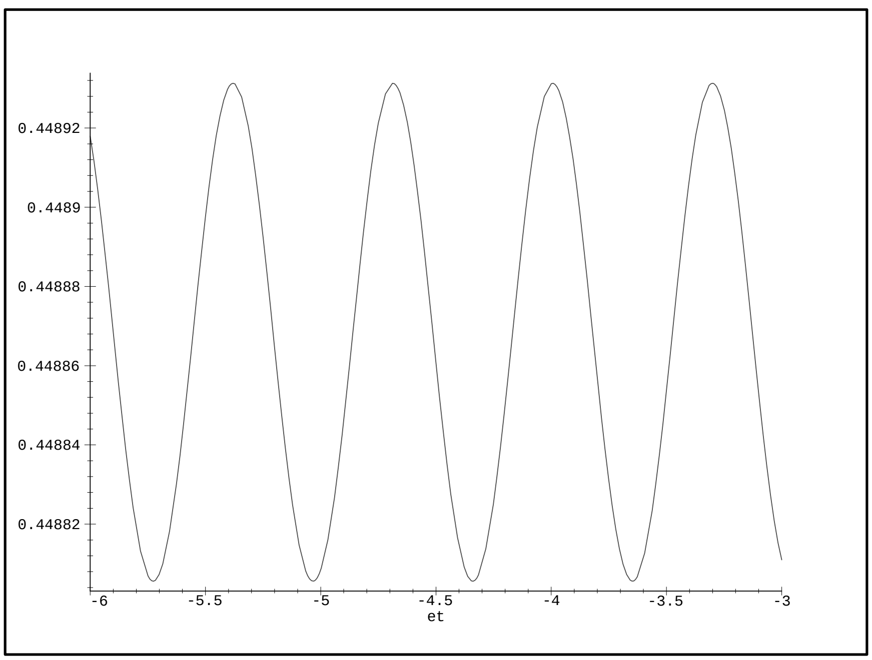

Fig. 2: $\varphi_1(2,\eta)$ for large negative values of η

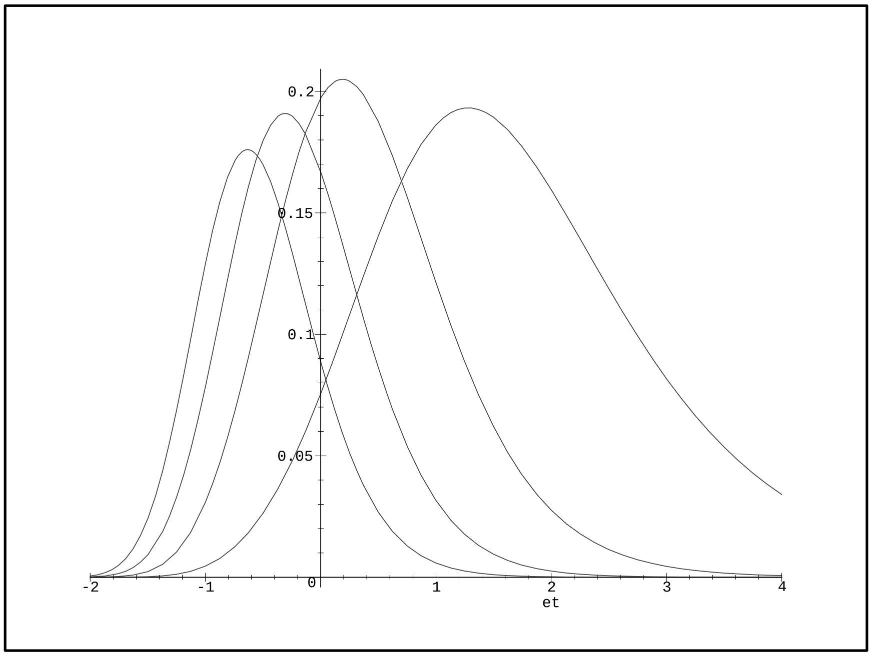

Fig. 3: $\varphi_2(m,\eta)$ for $m = 1,\ldots,4$

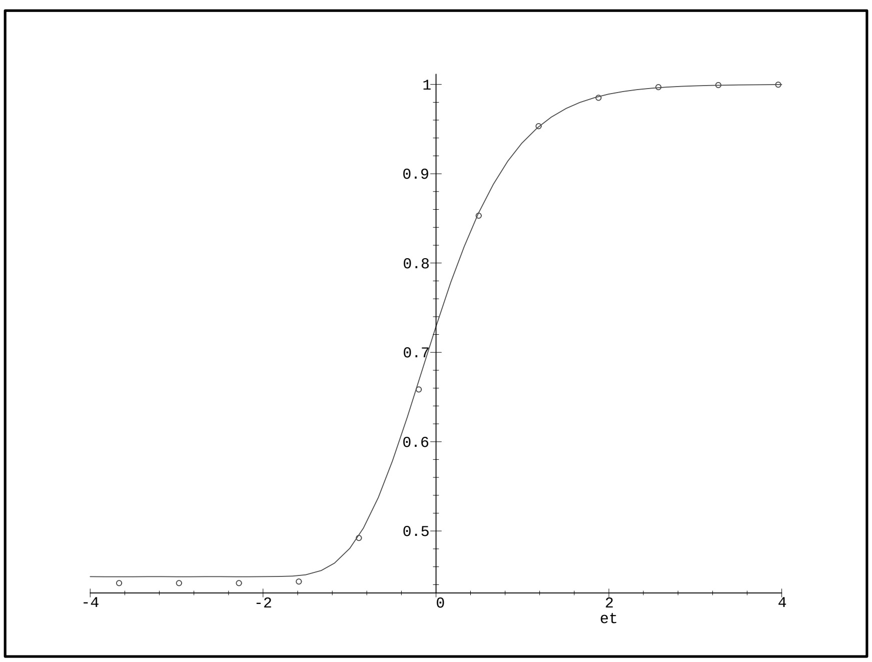

Fig. 4: Maximum part size distribution function ($m = 2$, observed = circle, asymptotic = line)

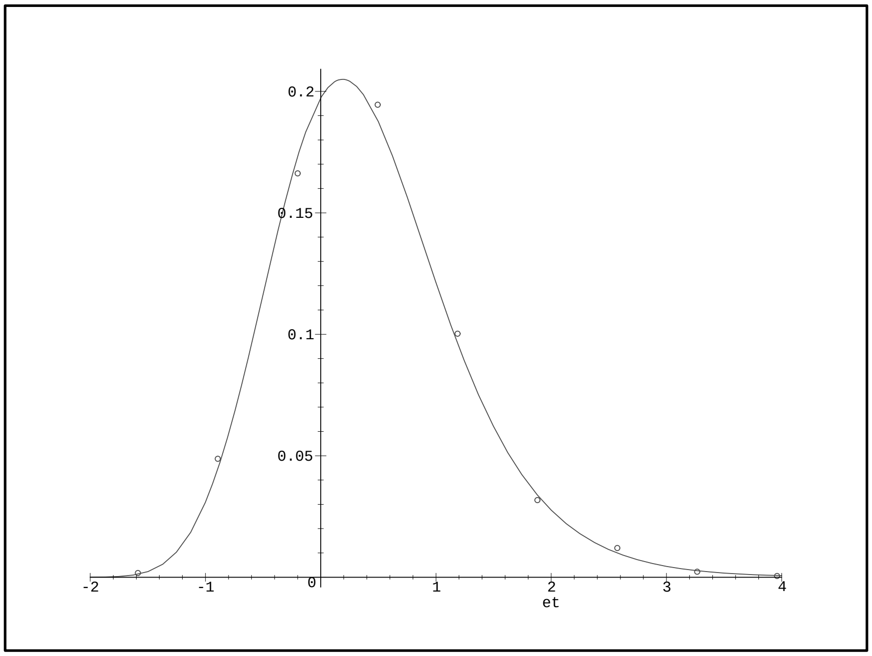

Fig. 5: Maximum part size distribution ($m = 2$, observed = circle, asymptotic = line)

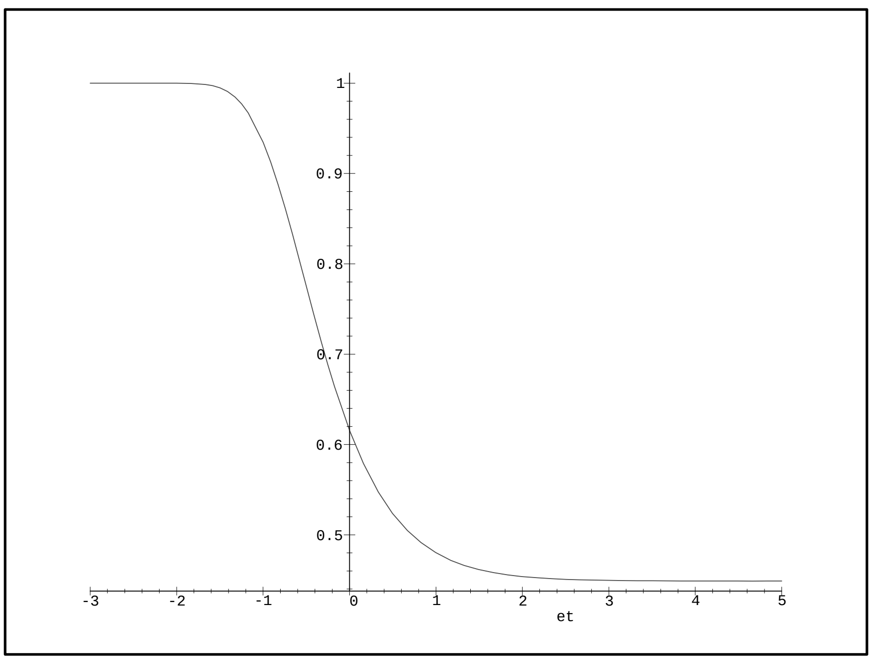

Fig. 6: $\varphi_4(2, \eta)$

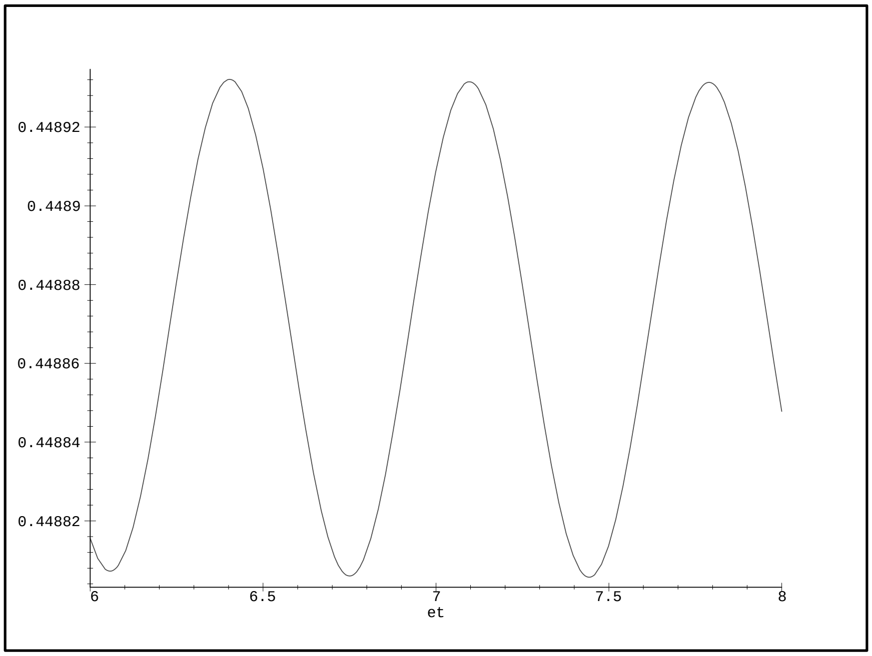

Fig. 7: $\varphi_4(2,\eta)$ for large values of η

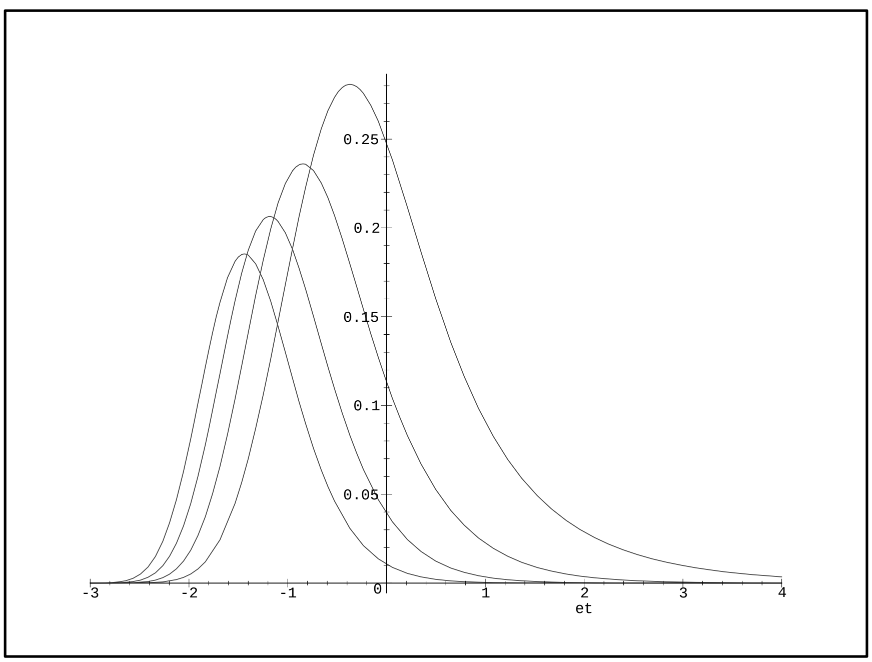

Fig. 8: $\varphi_3(2,\eta)$ for $m = 1,\ldots,4$.

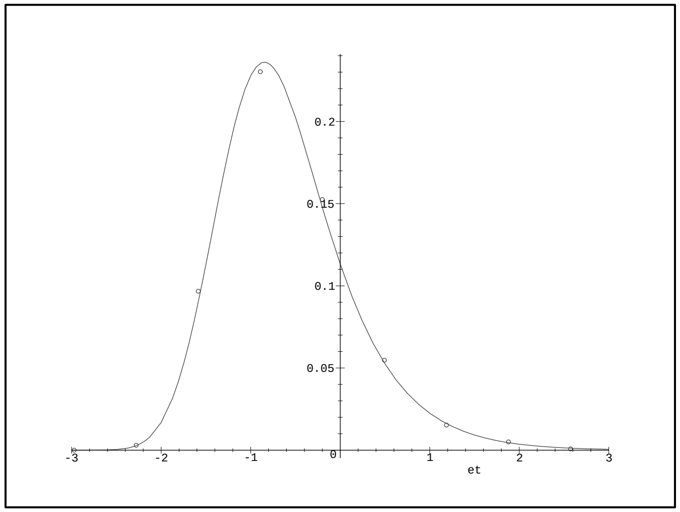

Fig. 9: First full part distribution ($m = 2$, observed = circle, asymptotic = line)

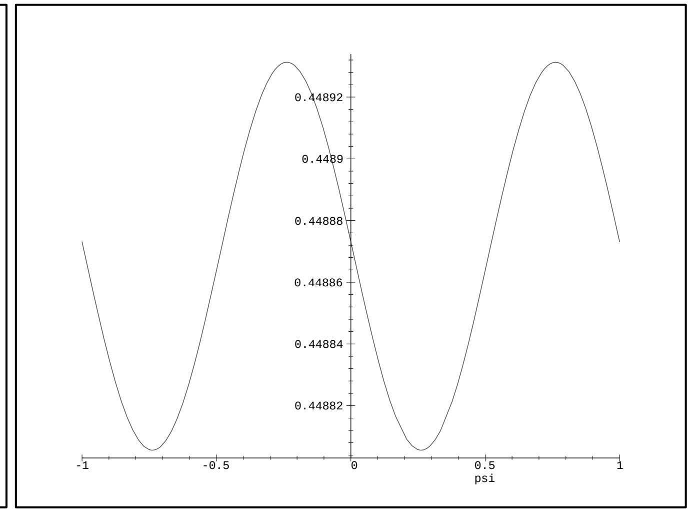

Fig. 10: $P(2,0)$ as a function of ψ

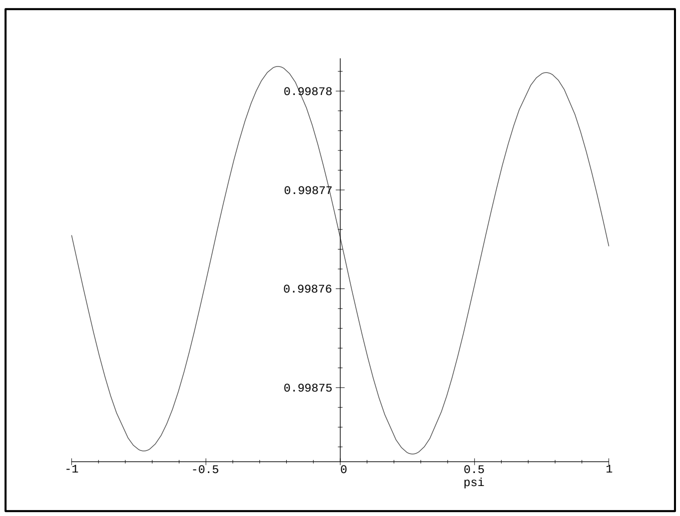

Fig. 11: $\sum_{i=0}^{3} P(2,i)$

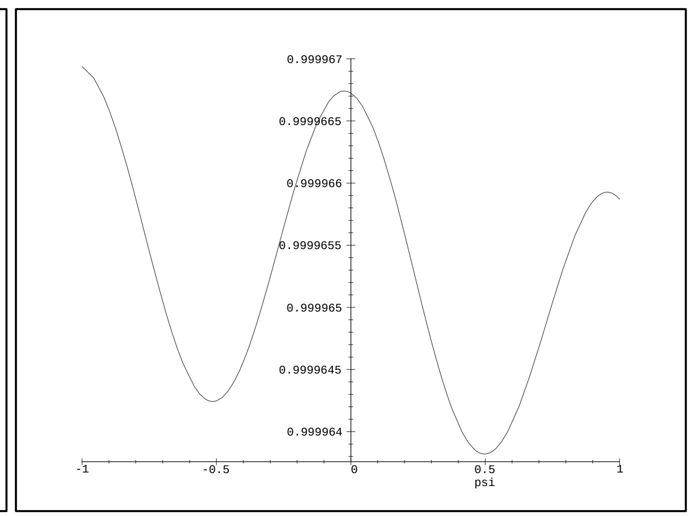

Fig. 12: $\sum_{i=0}^{4} P(2,i)$

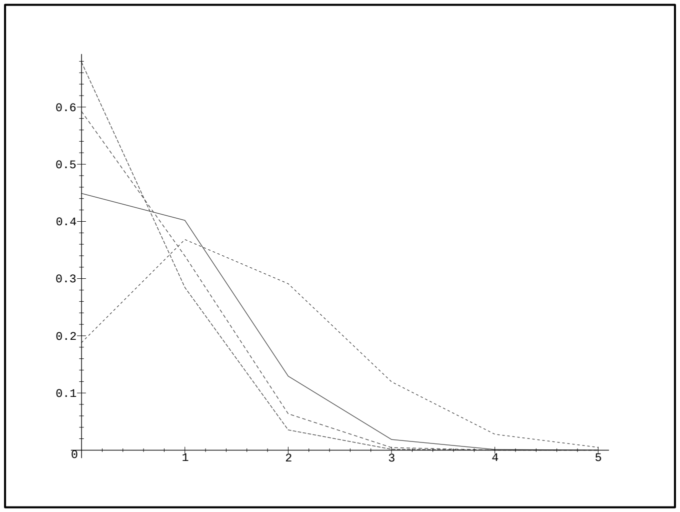

Fig. 13: $P(m,i), m = 1,\dots,4$

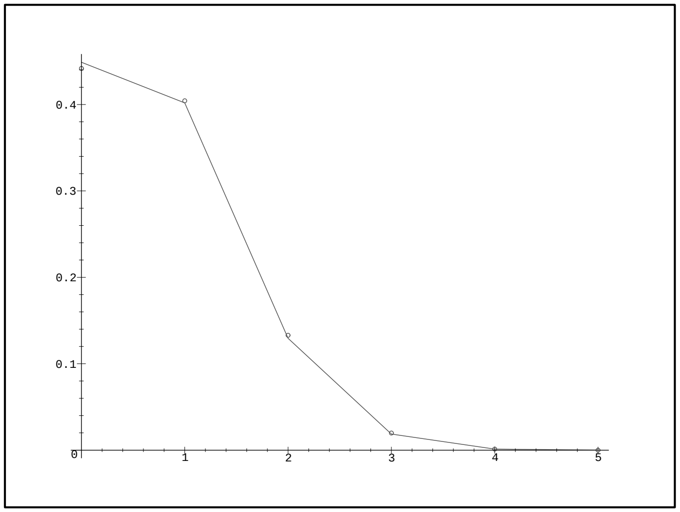

Fig. 14: Distribution of $X(2)$ (observed = circle, asymptotic = line)

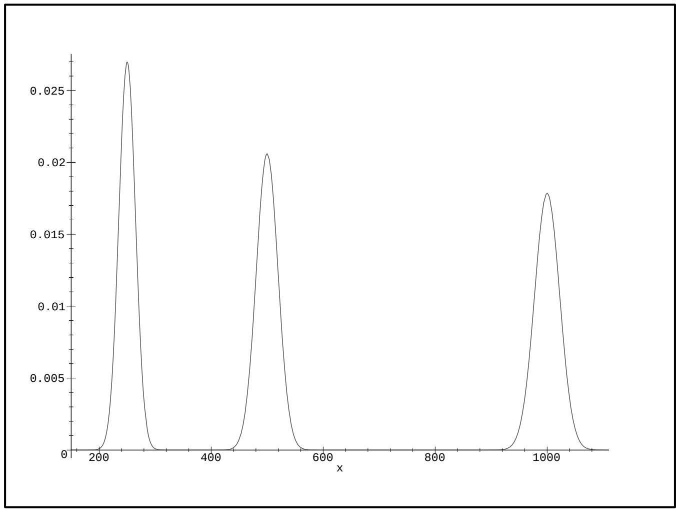

Fig. 15: $\Pr(X_i(m) = 1), i = 1,\dots,3$

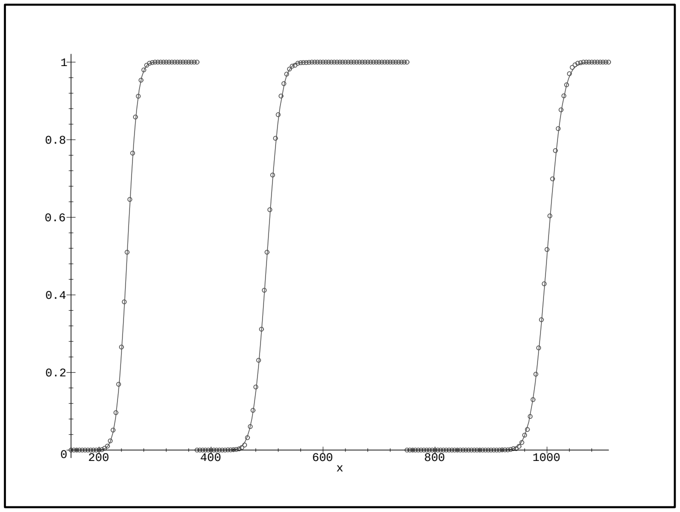

Fig. 16: Distribution function of $M_i, i = 1,\dots,3$ (observed = circle, asymptotic = line)

References

[Chi67] V. P. Chistyakov. Discrete limit distributions in the problem of balls falling in cells with arbitrary probabilities. *Math. Notes*, 1:6–11, 1967.

[CPSW99] Sylvie Corteel, Boris Pittel, Carla D. Savage, and Herbert S. Wilf. On the multiplicity of parts in a random partition. *Random Structures Algorithms*, 14(2):185–197, 1999.

[FGD95] Philippe Flajolet, Xavier Gourdon, and Philippe Dumas. Mellin transforms and asymptotics: harmonic sums. *Theoret. Comput. Sci.*, 144(1-2):3–58, 1995. Special volume on mathematical analysis of algorithms.

[FS94] P. Flajolet and R. Sedgewick. Analytic combinatorics – symbolic combinatorics: Saddle point asymptotics. Book in preparation. See also Technical Report 2376, INRIA, 1994.

[GKP89] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete mathematics*. Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1989. A foundation for computer science.

[HL01] Paweł Hitczenko and Guy Louchard. Distinctness of compositions of an integer: a probabilistic analysis. *Random Structures Algorithms*, 19(3-4):407–437, 2001. Analysis of algorithms (Krynica Morska, 2000).

[HRS02] Paweł Hitczenko, Cecil Rousseau, and Carla D. Savage. A generating functionology approach to a problem of Wilf. *J. Comput. Appl. Math.*, 142(1):107–114, 2002. Probabilistic methods in combinatorics and combinatorial optimization.

[HS99] P. Hitczenko and C.D. Savage. On the multiplicity of parts in a random composition of a large integer. Technical report. Avalaible at http://www.csc.ncsu.edu/faculty/savage/, 1999.

[HY97] H.-K. Hwang and Y.-N. Yeh. Measures of distinctness for random partitions and compositions of an integer. *Adv. in Appl. Math.*, 19(3):378–414, 1997.

[JS98] Philippe Jacquet and Wojciech Szpankowski. Analytical de-Poissonization and its applications. *Theoret. Comput. Sci.*, 201(1-2):1–62, 1998.

[Knu73] Donald E. Knuth. *The art of computer programming. Volume 3*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1973. Sorting and searching, Addison-Wesley Series in Computer Science and Information Processing.

[Lou87] G. Louchard. Exact and asymptotic distributions in digital and binary search trees. *RAIRO Inform. Théor. Appl.*, 21(4):479–495, 1987.

[LP02] Guy Louchard and Helmut Prodinger. Probabilistic analysis of Carlitz compositions. *Discrete Math. Theor. Comput. Sci.*, 5(1):71–95 (electronic), 2002.

[SČ64] B. A. Sevast′janov and V. P. Čistjakov. Asymptotic normality in the classical problem of balls. *Theory of Probability and Applications*, 9:198–211, 1964.