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Position of the maximum in a sequence with geometric distribution

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As a sequel to [1], the position of the maximum in a geometrically distributed sample is investigated. Samples of length n are considered, where the maximum is required to be in the first d positions. The probability that the maximum occurs in the first d positions is sought for d dependent on n (as opposed to d fixed in [1]). Two scenarios are discussed. The first is when $d = \alpha n$ for $0 < \alpha \leq 1$, where Mellin transforms are used to obtain the asymptotic results. The second is when $1 \leq d = o(n)$.

Keywords: Mellin transforms, generating functions, geometric distribution.

1 Introduction

Consider a word whose letters are natural numbers. Assume that these letters occur independently and with geometric probability. So for $p + q = 1$, each letter j appears in the word with probability pq^{j-1} . We write $Q := q^{-1}$ and $L := \log Q$.

We address the question: “What is the probability that the maximum in a word occurs in the first d positions?” We take words of length n and require $d \leq n$. Previously (in [1]), d was considered fixed. Now, d is allowed to grow with n . First, we assume that d is proportional to n and then we consider the case when d is $o(n)$ (but at least 1). The latter produces the same solutions as when d is fixed (see [1]).

We distinguish between two cases: ‘strict’ and ‘weak’. A ‘strict’ maximum never recurs, whereas a ‘weak’ maximum can recur any number of times. These apply separately to the two parts of the word (the first d letters, and the remaining $n - d$ letters). This means that in total, there are four different cases to be dealt with: (strict, strict), (weak, strict), (strict, weak) and (weak, weak), where the first entry refers to the first d letters in the word, and the second entry to the rest of the word. The results in the (strict, strict) case for d fixed will hold for the other scenarios too (i.e., it is not required that d is independent of n). This is because the relevant calculations in [1] still go through when we take the limits as $n \rightarrow \infty$. This is not true for the remaining three cases which are dealt with in Section 3.

2 Results

Theorem 1 *The probability that the maximum value in a word of length n is in the first d positions (for $d = \alpha n$) is*

$$\text{Max}_{(w,s)}(n) \sim \frac{1}{L} \log \left(\frac{1}{1 - \alpha(Q-1)} \right) + \frac{1}{L} (\psi_0(n(1 - \alpha p)) - \psi_0(n)),$$

$$\text{Max}_{(s,w)}(n) \sim \frac{\alpha(Q-1)}{L(1 + \alpha(Q-1))} (1 + \psi(n(q + p\alpha))),$$

$$\text{Max}_{(w,w)}(n) \sim \frac{\log(1 + \alpha(Q-1))}{L} + \frac{1}{L} \left(\psi_0(n) - \psi_0 \left(n \left(\frac{q + \alpha p}{q} \right) \right) \right),$$

as $n \rightarrow \infty$ where $Q := q^{-1}$, $L := \log Q$, and the fluctuations are defined by

$$\psi(x) := \sum_{k \neq 0} \Gamma(1 - \chi_k) e^{2k\pi i x}, \quad \text{for } k \in \mathbb{Z},$$

and

$$\psi_0(x) := \sum_{k \neq 0} \Gamma(-\chi_k) e^{2k\pi i \log_Q x}, \quad \text{for } k \in \mathbb{Z}.$$

Note that we write \mathbf{i} rather than the i we use as an index.

Theorem 2 *The probability that the maximum value in a word of length n is in the first d positions, for $1 \leq d = o(n)$, is*

$$\begin{aligned} \text{Max}_{(w,s)}(n) &\sim \frac{(1-Q^{-1})d}{Ln} (1 + \psi(n)), \\ \text{Max}_{(s,w)}(n) &\sim \frac{(Q-1)d}{Ln} (1 + \psi(n)), \\ \text{Max}_{(w,w)}(n) &\sim \frac{(Q-1)d}{Ln} (1 + \psi(n)), \end{aligned}$$

as $n \rightarrow \infty$, where $\psi(x) = \sum_{k \neq 0} \Gamma(1 - \chi_k) e^{2k\pi i \log_Q x}$, $k \in \mathbb{Z}$.

3 Suppose $d = \alpha n$

Suppose we now consider $d = \alpha n$ where $0 < \alpha \leq 1$. The ‘ d fixed’ results from [1] continue to hold only in the (strict, strict) case. For the other three cases we make use of a technique from complex analysis called the ‘Mellin’ transform. The rules used below can be found in [2] and [5], among others. The first case is done in greater detail than the other two, as a similar process is used in each case.

3.1 Case (weak, strict), for $d = \alpha n$

For this scenario (finding the probability with which the maximum, k , occurs in the first d letters of a word) we require that there is at least one k in the first d places, and possibly more. However, the rest of the word may only have letters from the set $\{1, \dots, k-1\}$. Now suppose that $d = \alpha n$, for $0 < \alpha \leq 1$. The generating function is explained in [1], yielding the following coefficients (with d replaced by αn).

$$f_{(w,s)}(n) := \sum_{k \geq 1} \sum_{i=0}^{\alpha n - 1} (1 - q^{k-1})^{i+n-\alpha n} p q^{k-1} (1 - q^k)^{\alpha n - 1 - i}. \quad (1)$$

Now note that

$$\sum_{i=0}^{\alpha n - 1} \left(\frac{1 - q^{k-1}}{1 - q^k} \right)^i = \frac{(1 - q^k)^{\alpha n} - (1 - q^{k-1})^{\alpha n}}{(1 - q^k)^{\alpha n - 1} q^{k-1} (1 - q)}, \quad (2)$$

and thus, since $p = 1 - q$ and $(1 - a)^n \sim e^{-an}$ for small a ,

$$\begin{aligned} f_{(w,s)}(n) &= \sum_{k \geq 1} (1 - q^{k-1})^{n(1-\alpha)} [(1 - q^k)^{\alpha n} - (1 - q^{k-1})^{\alpha n}] \\ &\sim \sum_{k \geq 1} [e^{-nq^{k-1}(1-\alpha p)} - e^{-nq^{k-1}}]. \end{aligned}$$

We are now in a position to use Mellin transforms to find an approximation. We shift the fundamental strip from $\langle 0, \infty \rangle$ to $\langle -1, 0 \rangle$ and define a new function:

$$f_{ws}(x) := \sum_{k \geq 1} [(e^{-nq^{k-1}(1-\alpha p)} - 1) - (e^{-nq^{k-1}} - 1)]. \quad (3)$$

Then the Mellin transform of this function is:

$$\begin{aligned} f_{ws}^*(s) &= \sum_{k \geq 1} [(q^{k-1}(1 - \alpha p))^{-s} \Gamma(s) - (q^{k-1})^{-s} \Gamma(s)] \\ &= \sum_{k \geq 1} q^{s(1-k)} \Gamma(s) [(1 - \alpha p)^{-s} - 1] \\ &= q^s \Gamma(s) [(1 - \alpha p)^{-s} - 1] \sum_{k \geq 1} (q^{-s})^k \\ &= \Gamma(s) [(1 - \alpha p)^{-s} - 1] \frac{1}{1 - q^{-s}}, \quad \text{for } \Re(s) < 0, \end{aligned} \quad (4)$$

where $\Re(s)$ represents the real part of the complex number s . The reason for shifting the fundamental strip is that the transform exists in the intersection of the domain of convergence of the generalised Dirichlet series and the fundamental strip of $f^*(s)$. The intersection $\langle -\infty, 0 \rangle \cap \langle 0, \infty \rangle$ is empty, but with the shift we have a final fundamental strip of $\langle -1, 0 \rangle$. We choose a value inside this, say $-\frac{1}{2}$, with which to perform our inverse Mellin transform:

$$f_{ws}(x) = \frac{1}{2\pi i} \int_{(-\frac{1}{2})} \Gamma(s) [(1 - \alpha p)^{-s} - 1] \frac{1}{1 - q^{-s}} x^{-s} ds. \quad (5)$$

The notation $(-\frac{1}{2})$ under the integral sign means an integral from $-\frac{1}{2} - i\infty$ to $-\frac{1}{2} + i\infty$. This can be approximated by moving the contour to the right (and thus collecting negative residues) since we are interested in x large. The first poles we encounter are the simple pole at $s = 0$ (which would be a double pole except that one cancels with the factor $(1 - \alpha p)^{-s} - 1$) and the simple poles at $s = \chi_k := \frac{2k\pi i}{L}$, $k \in \mathbb{Z} \setminus \{0\}$ where $L := \log Q$. The former contributes the main term and the rest contribute the fluctuations which are comparatively extremely small. As $s \rightarrow 0$,

$$\Gamma(s) \sim \frac{1}{s},$$

$$(1 - \alpha p)^{-s} - 1 \sim 1 - s \log(1 - \alpha p) - 1 = -s \log(1 - \alpha p),$$

$$\frac{1}{1 - q^{-s}} = \frac{1}{1 - e^{-s \log q}} \sim \frac{1}{1 - (1 - s \log q)} = \frac{1}{s \log q},$$

and

$$x^{-s} = e^{-s \log x} \sim 1.$$

Thus the negative residue is

$$-[s^{-1}] \frac{1}{s} (-s \log(1 - \alpha p)) \frac{1}{s \log q} = \frac{\log(1 - \alpha p)}{\log q}.$$

We also have simple poles at $s = \chi_k$, for $k \neq 0$. Let $\varepsilon := s - \chi_k$ then expanding around $\varepsilon = 0$ gives

$$\frac{1}{1 - q^{-s}} = \frac{1}{1 - q^{-\chi_k - \varepsilon}} = \frac{1}{1 - q^{-\varepsilon}} = \frac{1}{1 - e^{-\varepsilon \log q}} \sim \frac{1}{1 - (1 - \varepsilon \log q)} = \frac{1}{\varepsilon \log q}.$$

So the negative residues for all non-zero k are

$$\begin{aligned} \sum_{k \neq 0} (-1) [\varepsilon^{-1}] \Gamma(\chi_k) [(1 - \alpha p)^{-\chi_k} - 1] \frac{1}{\varepsilon \log q} x^{-\chi_k} &= \frac{1}{L} \sum_{k \neq 0} \Gamma(\chi_k) [(1 - \alpha p)^{-\chi_k} - 1] x^{-\chi_k} \\ &= \frac{1}{L} \sum_{k \neq 0} \Gamma(\chi_k) [e^{-\chi_k \log(1 - \alpha p)} - 1] e^{-\chi_k \log x} \\ &= \frac{1}{L} \sum_{k \neq 0} \Gamma(\chi_k) [e^{-\chi_k \log(x(1 - \alpha p))} - e^{-\chi_k \log x}] \\ &= \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) [e^{2k\pi i \log_Q(x(1 - \alpha p))} - e^{2k\pi i \log_Q x}]. \end{aligned}$$

Now put these together to get the probability of having a weak maximum in the first d positions which does not repeat in the rest of the word and where $d = \alpha n$ grows with n as $n \rightarrow \infty$:

$$\begin{aligned} &\frac{\log(1 - \alpha p)}{\log q} + \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) [e^{2k\pi i \log_Q(n(1 - \alpha p))} - e^{2k\pi i \log_Q n}] \\ &= \frac{\log(1 - \alpha(1 - Q^{-1}))}{-L} + \frac{1}{L} (\psi_0(n(1 - \alpha p)) - \psi_0(n)) \end{aligned} \quad (6)$$

for $Q = q^{-1}$, $L = \log Q$ and $\psi_0(x)$ as in Theorem 1. Note that if $\alpha = 1$ then $d = n$ and the main term yields a probability of 1, confirming our result on a word with no restrictions.

3.2 Case (strict, weak), for $d = \alpha n$

In this case, we allow the maximum, k , to occur only once in the first d letters, but any number of times in the rest of the word. Again, [1] provides us with the generating function whose coefficients are

$$\begin{aligned} f_{(s,w)}(n) &= \sum_{k \geq 1} \alpha n p q^{k-1} (1 - q^{k-1})^{\alpha n - 1} (1 - q^k)^{n(1-\alpha)} \\ &\sim \sum_{k \geq 1} \alpha n p q^{k-1} e^{-nq^{k-1}(q+p\alpha)}. \end{aligned} \quad (7)$$

If we define the function

$$f_{sw}(x) := \sum_{k \geq 1} \alpha x p q^{k-1} e^{-xq^{k-1}(q+p\alpha)},$$

then the Mellin transform will be

$$\begin{aligned} f_{sw}^*(s) &= \sum_{k \geq 1} \alpha p q^{k-1} (q^{k-1})^{-(s+1)} (q + p\alpha)^{-(s+1)} \Gamma(s+1) \\ &= \alpha p (q + p\alpha)^{-(s+1)} \Gamma(s+1) \frac{1}{1 - q^{-s}}, \quad \text{for } \Re(s) < 0, \end{aligned} \quad (8)$$

and the fundamental strip is the overlap of the interval $(-\infty, 0)$ and the fundamental strip of xe^{-x} which is $\langle -1, \infty \rangle$, i.e., $\langle -1, 0 \rangle$. Hence we pick our contour integral from $-\frac{1}{2} - i\infty$ to $-\frac{1}{2} + i\infty$, and perform the inverse Mellin transform to get:

$$f_{sw}(x) = \frac{1}{2\pi i} \int_{(-\frac{1}{2})} \alpha p (q + p\alpha)^{-(s+1)} \Gamma(s+1) \frac{1}{1 - q^{-s}} x^{-s} ds. \quad (9)$$

By moving the contour to the right, the first poles we reach are at $s = 0$ and $s = \chi_k$, $k \neq 0$. For the main term, the negative residue is

$$-[s^{-1}] \alpha p (q + p\alpha)^{-1} \Gamma(1) \frac{1}{s \log q} = \frac{\alpha p}{L(q + p\alpha)}.$$

The fluctuations come from the negative residues of the poles at $s = \chi_k$, $k \neq 0$. Let $\varepsilon := s - \chi_k$, then around $\varepsilon = 0$ we get $\frac{1}{1 - q^{-s}} \sim \frac{1}{\varepsilon \log q}$, and so these poles contribute:

$$-\sum_{k \neq 0} [\varepsilon^{-1}] \alpha p (q + p\alpha)^{-(\chi_k+1)} \Gamma(\chi_k + 1) \frac{1}{\varepsilon \log q} x^{-\chi_k} = \frac{\alpha p}{L(q + p\alpha)} \sum_{k \neq 0} \Gamma(1 - \chi_k) e^{2k\pi i \log_Q(x(q+p\alpha))}.$$

This gives a total probability in the (strict, weak) case asymptotic to

$$\frac{\alpha(Q-1)}{L(1 + \alpha(Q-1))} (1 + \psi(n(q+p\alpha))), \quad (10)$$

as $n \rightarrow \infty$ where $\psi(x) = \sum_{k \neq 0} \Gamma(1 - \chi_k) e^{2k\pi i \log_Q x}$.

(For $\alpha = 1$, the dominant term gives $\frac{p}{L}$, which is the same as the probability of having one winner among n players in a game where each player tosses a coin until a head appears, and the winner is the player who takes the longest to toss a head, see [3].)

3.3 Case (weak, weak), for $d = \alpha n$

Here, the maximum can recur anywhere, having first appeared at least once in the first d letters. From [1], and by (2), we can approximate as in the (w,s) case to get:

$$f_{(w,w)}(n) \sim \sum_{k \geq 1} [(e^{-nq^k} - 1) - (e^{-nq^{k-1}(q+p\alpha)} - 1)]. \quad (11)$$

We define an exact function in terms of x to be:

$$f_{ww}(x) := \sum_{k \geq 1} [(e^{-xq^k} - 1) - (e^{-xq^{k-1}(q+p\alpha)} - 1)].$$

The transform of this function is

$$\begin{aligned} f_{ww}^*(s) &= \sum_{k \geq 1} [q^{-sk} \Gamma(s) - (q^{k-1})^{-s} (q + \alpha p)^{-s} \Gamma(s)] \\ &= \Gamma(s) [1 - q^s (q + \alpha p)^{-s}] \frac{1}{q^s - 1}, \end{aligned} \quad (12)$$

which exists in the strip $\langle -1, 0 \rangle$. We can thus rewrite $f_{ww}(x)$ as a contour integral

$$f_{ww}(x) = \frac{1}{2\pi i} \int_{(-\frac{1}{2})} \Gamma(s) [1 - q^s (q + \alpha p)^{-s}] \frac{1}{q^s - 1} x^{-s} ds. \quad (13)$$

The relevant simple poles occur at $s = 0$ and $s = \chi_k, k \neq 0$. The negative residue at $s = 0$ is

$$-[s^{-1}] \frac{1}{s} \log \left(\frac{q + \alpha p}{q} \right) \frac{1}{s \log q} = \frac{1}{L} \log \left(\frac{q + \alpha p}{q} \right), \quad (14)$$

which, as in the (weak, strict) case, is one for $\alpha = 1$. For the poles at $s = \chi_k$, let $\varepsilon := s - \chi_k$. Then expanding around $\varepsilon = 0$ gives $\frac{1}{q^s - 1} \sim \frac{1}{\varepsilon \log q}$ and so the negative residues are

$$-\sum_{k \neq 0} [\varepsilon^{-1}] \Gamma(\chi_k) [1 - q^{\chi_k} (q + \alpha p)^{-\chi_k}] \frac{1}{\varepsilon \log q} x^{-\chi_k} = \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) \left[e^{2k\pi i \log_Q x} - e^{2k\pi i \log_Q(x(\frac{q+\alpha p}{q}))} \right]. \quad (15)$$

By summing (14) and (15) and replacing x by n the asymptotic result for the (weak, weak) case in Theorem 1 is found.

4 Suppose $1 \leq d = o(n)$

Initially, in this section d was considered to be dependent on n according to the relationship $d = \alpha n^\gamma$, where $0 < \gamma < 1$, and $0 < \alpha \leq 1$. Mellin transforms were used to obtain the same results as found in the d fixed case (see [1]). However, it was then found (as suggested by a referee) that in fact any d such that $1 \leq d = o(n)$ will produce the same results. The explanation is given below.

We show that the results are the same as when d is fixed by referring back to the step in the calculations for d fixed (see [1]) where the $d = \alpha n$ calculations failed. The important stage is when the main term of the probability is given by the expression

$$\sum_{i=0}^{d-1} \sum_{l=0}^{d-1-i} \binom{d-1-i}{l} (-1)^l \frac{Q^{-l}(1-Q^{-1})}{L} \frac{1}{(l+1+N) \binom{N+l}{l}}. \quad (16)$$

This is the (weak, strict) case, but the others are similar. Since N grows like n , we use n instead of N for simplicity. For d fixed, it can be seen that the $l = 0$ term dominates, since each term in the sum on l is of order $O(\frac{1}{n^{l+1}})$. For d proportional to n , each term in the inner sum is of order $O(\frac{1}{n})$, so none clearly dominates, and Mellin transforms are required to find the result (see Section 3). But what if $d = \alpha n^\gamma$ for $0 < \gamma < 1$, or even $d = \frac{n}{\log n}$?

Suppose we let $f(n) = o(n)$ for some $f(n)$ such that $f(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then we can write $d = \frac{n}{f(n)} (= o(n))$. In general, a typical term in the sum on l is of order $O(\frac{1}{n f^l(n)})$. For $l = 0$, we again have an order of $O(\frac{1}{n})$. In fact, this term can be expressed as $\frac{c}{n}$ where c is a constant. This will dominate all other terms, since even the infinite sum on l (a geometric series) is:

$$\frac{1}{n} \sum_{l=1}^{\infty} (f(n))^{-l} = \frac{1}{n(f(n) - 1)} = o\left(\frac{1}{n}\right). \quad (17)$$

The results in Theorem 2 follow from the above.

5 Conclusion

Table 1 is a summary of results from this paper. (The d fixed results are from [1]). This table shows only the dominant term for the results, expressed in terms of Q .

Case	(s,s)	(w,s)	(s,w)	(w,w)
$1 \leq d = o(n)$	$\frac{(1-Q^{-1})d}{Ln}$	$\frac{(1-Q^{-1})d}{Ln}$	$\frac{(Q-1)d}{Ln}$	$\frac{(Q-1)d}{Ln}$
$d = \alpha n$	$\frac{(1-Q^{-1})\alpha}{L}$	$\frac{\log(1-\alpha(1-Q^{-1}))}{-L}$	$\frac{\alpha(Q-1)}{L(1+\alpha(Q-1))}$	$\frac{\log(1+\alpha(Q-1))}{L}$

Tab. 1: Table of results for the two categories

If we consider α small (i.e., close to zero) in the second category, we should get similar solutions to category one (in which d is always small relative to n for n large). We thus determine what these dominant terms look like asymptotically as $\alpha \rightarrow 0$. We use the approximations $\log(1+x) \sim x$ and $\frac{1}{1-x} \sim 1$ as $x \rightarrow 0$ ([4]). Suppose $d = \alpha n$, then for the (weak, strict) case, we have

$$\frac{\log(1-\alpha(1-Q^{-1}))}{\log Q^{-1}} \sim \frac{-\alpha(1-Q^{-1})}{\log Q^{-1}} = \frac{\alpha(1-Q^{-1})}{L}.$$

For the (strict, weak) case, we find that

$$\frac{\alpha(Q-1)}{L(1+\alpha(Q-1))} \sim \frac{\alpha(Q-1)}{L},$$

and for the (weak, weak) case,

$$\frac{\log(1+\alpha(Q-1))}{L} \sim \frac{\alpha(Q-1)}{L}.$$

By replacing each α by $\frac{d}{n}$, it can be seen that each of these corresponds to the d fixed case in Table 1 above.

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References

- [1] M. Archibald. Restrictions on the position of the maximum/minimum in a geometrically distributed sample. In *Mathematics and Computer Science III: Algorithms, Trees, Combinatorics and Probabilities*, September 2004.
- [2] P. Flajolet and R. Sedgewick. Analytic combinatorics, symbolic combinatorics (chapters I-IX). <http://pauillac.inria.fr/algo/flajolet/Publications/books.html>, November 2004.
- [3] P. Kirschenhofer and H. Prodinger. The number of winners in a discrete geometrically distributed sample. *Annals in Applied Probability*, 6:687–694, 1996.
- [4] R. Sedgewick and P. Flajolet. *An Introduction to the Analysis of Algorithms*. Addison-Wesley, 1996.
- [5] W. Szpankowski. *Average Case Analysis of Algorithms on Sequences*. John Wiley and Sons, New York, 2001.