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# Connected $\tau$ -critical hypergraphs of minimal size

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A hypergraph  $\mathcal{H}$  is  $\tau$ -critical if  $\tau(\mathcal{H} - E) < \tau(\mathcal{H})$  for every edge  $E \in \mathcal{H}$ , where  $\tau(\mathcal{H})$  denotes the transversal number of  $\mathcal{H}$ . It can be shown that a connected  $\tau$ -critical hypergraph  $\mathcal{H}$  has at least  $2\tau(\mathcal{H}) - 1$  edges; this generalises a classical theorem of Gallai on  $\chi$ -vertex-critical graphs with connected complements. In this paper we study connected  $\tau$ -critical hypergraphs  $\mathcal{H}$  with exactly  $2\tau(\mathcal{H}) - 1$  edges. We prove that such hypergraphs have at least  $2\tau(\mathcal{H}) - 1$  vertices, and characterise those with  $2\tau(\mathcal{H}) - 1$  vertices using a directed odd ear decomposition of an associated digraph. Using Seymour's characterisation of  $\chi$ -critical 3-chromatic square hypergraphs, we also show that a connected square hypergraph  $\mathcal{H}$  with fewer than  $2\tau(\mathcal{H})$  edges is  $\tau$ -critical if and only if it is  $\chi$ -critical 3-chromatic. Finally, we deduce some new results on  $\chi$ -vertex-critical graphs with connected complements.

**Keywords:**  $\tau$ -critical hypergraph,  $\chi$ -critical 3-chromatic hypergraph

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## 1 Introduction

A hypergraph  $\mathcal{H}$  is a finite set of finite non-empty sets called the *edges* of  $\mathcal{H}$ . The *vertices* of  $\mathcal{H}$  are the elements of the set  $V(\mathcal{H}) = \bigcup_{E \in \mathcal{H}} E$ . A set  $T \subseteq V(\mathcal{H})$  is a *transversal* (also *vertex cover* or *blocking set*) of  $\mathcal{H}$  if  $T \cap E \neq \emptyset$  for every  $E \in \mathcal{H}$ . The smallest cardinality of a transversal of  $\mathcal{H}$  is the *transversal number*  $\tau(\mathcal{H})$ . A *k-colouring* of a hypergraph  $\mathcal{H}$  is an assignment of at most  $k$  colours to  $V(\mathcal{H})$  such that no edge is monochromatic. The chromatic number  $\chi(\mathcal{H})$  is the smallest  $k$  such that  $\mathcal{H}$  admits a  $k$ -colouring. A hypergraph  $\mathcal{H}$  is  $\tau$ -critical (resp.  $\chi$ -critical) if  $\tau(\mathcal{H} - E) < \tau(\mathcal{H})$  (resp.  $\chi(\mathcal{H} - E) < \chi(\mathcal{H})$ ) for every  $E \in \mathcal{H}$ .

A number of authors have studied  $\tau$ -critical hypergraphs; see for example [1, 2, 3]. It is trivial to verify that a hypergraph is  $\tau$ -critical if and only if all its components are  $\tau$ -critical. So what can be said about *connected*  $\tau$ -critical hypergraphs? In particular, it seems natural to ask what is the minimal possible number of edges in a connected  $\tau$ -critical hypergraph.

We first present a sharp lower bound on the number of edges in a connected  $\tau$ -critical hypergraph, and then investigate the cases where equality is attained. We exhibit a surprising connection with  $\chi$ -critical 3-chromatic square hypergraphs studied by Seymour [7], and show how our results relate to the work of Gallai [4] on  $\chi$ -vertex-critical graphs with connected complements.

## 2 Main results

The following two results were proved in [9]. (A hypergraph is a *star* if all its edges have a common vertex.)

**Theorem 1** *If  $\mathcal{H}$  is a connected  $\tau$ -critical hypergraph, then for every  $E \in \mathcal{H}$  the edges of  $\mathcal{H} - E$  can be partitioned into  $\tau(\mathcal{H}) - 1$  stars of size at least two.*

**Corollary 2** *If  $\mathcal{H}$  is a connected  $\tau$ -critical hypergraph, then  $|\mathcal{H}| \geq 2\tau(\mathcal{H}) - 1$ .*

The bound in Corollary 2 is sharp, as can be seen by considering odd cycles. Hypergraphs attaining equality in Corollary 2 are called *minimal connected  $\tau$ -critical hypergraphs*. We might hope that such hypergraphs would be of an analysable form. Indeed, since a partition into 2-stars of a hypergraph corresponds to a matching of its line graph, Theorem 1 implies the following useful result. (A graph  $G$  *factor-critical* if  $G - x$  has a perfect matching, for every  $x \in V(G)$ .)

**Corollary 3** *If  $\mathcal{H}$  is a minimal connected  $\tau$ -critical hypergraph, then  $L(\mathcal{H})$  is factor-critical.*

Lovász [5] proved that every factor-critical graph has an *odd ear decomposition*: it can be built up from a single vertex by successively attaching the end vertices of odd paths. So by Corollary 3 the line graph of a minimal connected  $\tau$ -critical hypergraph has an odd ear decomposition. This fact can be used to prove the following two results.

**Theorem 4** *If  $\mathcal{H}$  is a minimal connected  $\tau$ -critical hypergraph, then  $\mathcal{H}$  has a system of distinct representatives.*

**Corollary 5** *If  $\mathcal{H}$  is a minimal connected  $\tau$ -critical hypergraph, then  $|V(\mathcal{H})| \geq |\mathcal{H}|$ .*

Again, considering odd cycles shows that the bound in Corollary 5 is sharp. A hypergraph with an equal number of edges and vertices is said to be *square*. As might be expected, the minimal connected  $\tau$ -critical hypergraphs which are square have particularly nice properties. Indeed, they can be characterised in terms of an odd ear decomposition of an associated digraph.

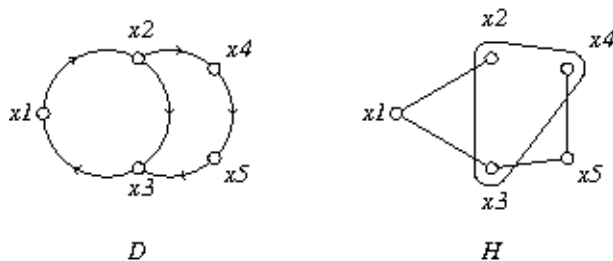
With any digraph  $D$  we can associate the hypergraph  $\mathcal{H}_D = \{\{x\} \cup N^+(x) \mid x \in V(D)\}$  where  $N^+(x)$  denotes the set of outneighbours of  $x$  in  $D$ . Note that  $\mathcal{H}_D$  is square and has a system of distinct representatives. Conversely, if  $\mathcal{H}$  is a square hypergraph with a system of distinct representatives  $f : \mathcal{H} \rightarrow V(\mathcal{H})$ , then  $\mathcal{H} = \mathcal{H}_D$ , where  $D$  is the digraph with vertex set  $V(\mathcal{H})$  and arc set  $\{(x, y) \mid x \in V(\mathcal{H}), y \in f^{-1}(x) \setminus \{x\}\}$ .

A *directed odd ear* with respect to a digraph  $D$  consists of a directed odd path such that the two end vertices are in  $V(D)$  but no internal vertices belong to  $V(D)$ . A *directed odd ear decomposition* of a digraph  $D$  is a sequence  $D_0, \dots, D_p$  of digraphs such that  $D_0$  is a single vertex,  $D_p = D$ , and for  $i = 1, \dots, p$ ,  $D_i$  is obtained from  $D_{i-1}$  by adding a directed odd ear joining two not necessarily distinct vertices of  $D_{i-1}$ .

Seymour [7] proved that a square hypergraph  $\mathcal{H}$  is  $\chi$ -critical 3-chromatic if and only if  $\mathcal{H} = \mathcal{H}_D$ , where  $D$  is a strongly connected digraph with no directed even circuits. The following result can be proved using Seymour's theorem, Corollary 3 and Theorem 4. The *absorption number*  $\beta(D)$  of a digraph  $D$  is the minimal size of a set  $S \subseteq V(D)$  such that every  $x \in V(D) \setminus S$  has an outneighbour in  $S$ .

**Theorem 6** *For any square hypergraph  $\mathcal{H}$ , the following conditions are equivalent:*

1.  $\mathcal{H}$  is minimal connected  $\tau$ -critical;
2.  $\mathcal{H}$  is  $\chi$ -critical 3-chromatic and  $|\mathcal{H}| < 2\tau(\mathcal{H})$ ;
3.  $\mathcal{H} = \mathcal{H}_D$ , where  $D$  has a directed odd ear decomposition, contains no directed even circuits and  $|V(D)| < 2\beta(D)$ .



**Fig. 1:**  $\mathcal{H} = \mathcal{H}_D$ , where  $D$  has a directed odd ear decomposition and contains no directed even circuits; the associated square hypergraph  $\mathcal{H}_D$  is minimal connected  $\tau$ -critical by Theorem 6.

Let  $\mathcal{H}^*$  denote the vertex-edge dual of  $\mathcal{H}$ . The following result can be proved using Corollary 5 and Theorem 6.

**Corollary 7** *If  $\mathcal{H}$  is a minimal connected  $\tau$ -critical hypergraph, then so is  $\mathcal{H}^*$  if and only if  $\mathcal{H}$  is square.*

### 3 Application to $\chi$ -vertex-critical graphs

A hypergraph has the *Helly property* if all its intersecting partial hypergraphs are stars. There is a useful link between the chromatic number of graphs and the transversal number of Helly hypergraphs. Namely, given a graph  $G$ , let  $\mathcal{A}(G)$  be the hypergraph formed with the maximal independent sets of  $G$ , and denote its dual by  $\mathcal{A}^*(G)$ . It is not difficult to check that  $\mathcal{A}^*(G)$  has the Helly property and  $\chi(G) = \tau(\mathcal{A}^*(G))$ . A graph  $G$  is  $\chi$ -vertex-critical if  $\chi(G - x) < \chi(G)$ , for every vertex  $x \in V(G)$ ; note that a graph  $G$  is  $\chi$ -vertex-critical if and only if  $\mathcal{A}^*(G)$  is  $\tau$ -critical. Hence the restriction to Helly hypergraphs of Corollary 2 is equivalent to the following classical result of Gallai [4], also proved in [6, 8].

**Theorem 8 (Gallai 1963)** *A  $\chi$ -vertex-critical graph  $G$  with a connected complement has at least  $2\chi(G) - 1$  vertices.*

The restriction to Helly hypergraphs of Corollary 5 is equivalent to the following result.

**Theorem 9** *A  $\chi$ -vertex-critical graph  $G$  with a connected complement and  $2\chi(G) - 1$  vertices has at least  $2\chi(G) - 1$  maximal independent sets.*

Finally, Theorem 6 implies the following.

**Theorem 10** *If  $G$  is a graph with a connected complement,  $2\chi(G) - 1$  vertices and  $2\chi(G) - 1$  maximal independent sets, then  $G$  is  $\chi$ -vertex-critical if and only if  $\mathcal{A}^*(G)$  is a  $\chi$ -critical 3-chromatic Helly hypergraph.*

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