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Connected τ -critical hypergraphs of minimal size

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A hypergraph \mathcal{H} is τ -critical if $\tau(\mathcal{H} - E) < \tau(\mathcal{H})$ for every edge $E \in \mathcal{H}$, where $\tau(\mathcal{H})$ denotes the transversal number of \mathcal{H} . It can be shown that a connected τ -critical hypergraph \mathcal{H} has at least $2\tau(\mathcal{H}) - 1$ edges; this generalises a classical theorem of Gallai on χ -vertex-critical graphs with connected complements. In this paper we study connected τ -critical hypergraphs \mathcal{H} with exactly $2\tau(\mathcal{H}) - 1$ edges. We prove that such hypergraphs have at least $2\tau(\mathcal{H}) - 1$ vertices, and characterise those with $2\tau(\mathcal{H}) - 1$ vertices using a directed odd ear decomposition of an associated digraph. Using Seymour's characterisation of χ -critical 3-chromatic square hypergraphs, we also show that a connected square hypergraph \mathcal{H} with fewer than $2\tau(\mathcal{H})$ edges is τ -critical if and only if it is χ -critical 3-chromatic. Finally, we deduce some new results on χ -vertex-critical graphs with connected complements.

Keywords: τ -critical hypergraph, χ -critical 3-chromatic hypergraph

1 Introduction

A hypergraph \mathcal{H} is a finite set of finite non-empty sets called the *edges* of \mathcal{H} . The *vertices* of \mathcal{H} are the elements of the set $V(\mathcal{H}) = \bigcup_{E \in \mathcal{H}} E$. A set $T \subseteq V(\mathcal{H})$ is a *transversal* (also *vertex cover* or *blocking set*) of \mathcal{H} if $T \cap E \neq \emptyset$ for every $E \in \mathcal{H}$. The smallest cardinality of a transversal of \mathcal{H} is the *transversal number* $\tau(\mathcal{H})$. A *k-colouring* of a hypergraph \mathcal{H} is an assignment of at most k colours to $V(\mathcal{H})$ such that no edge is monochromatic. The chromatic number $\chi(\mathcal{H})$ is the smallest k such that \mathcal{H} admits a k -colouring. A hypergraph \mathcal{H} is τ -critical (resp. χ -critical) if $\tau(\mathcal{H} - E) < \tau(\mathcal{H})$ (resp. $\chi(\mathcal{H} - E) < \chi(\mathcal{H})$) for every $E \in \mathcal{H}$.

A number of authors have studied τ -critical hypergraphs; see for example [1, 2, 3]. It is trivial to verify that a hypergraph is τ -critical if and only if all its components are τ -critical. So what can be said about *connected* τ -critical hypergraphs? In particular, it seems natural to ask what is the minimal possible number of edges in a connected τ -critical hypergraph.

We first present a sharp lower bound on the number of edges in a connected τ -critical hypergraph, and then investigate the cases where equality is attained. We exhibit a surprising connection with χ -critical 3-chromatic square hypergraphs studied by Seymour [7], and show how our results relate to the work of Gallai [4] on χ -vertex-critical graphs with connected complements.

2 Main results

The following two results were proved in [9]. (A hypergraph is a *star* if all its edges have a common vertex.)

Theorem 1 *If \mathcal{H} is a connected τ -critical hypergraph, then for every $E \in \mathcal{H}$ the edges of $\mathcal{H} - E$ can be partitioned into $\tau(\mathcal{H}) - 1$ stars of size at least two.*

Corollary 2 *If \mathcal{H} is a connected τ -critical hypergraph, then $|\mathcal{H}| \geq 2\tau(\mathcal{H}) - 1$.*

The bound in Corollary 2 is sharp, as can be seen by considering odd cycles. Hypergraphs attaining equality in Corollary 2 are called *minimal connected τ -critical hypergraphs*. We might hope that such hypergraphs would be of an analysable form. Indeed, since a partition into 2-stars of a hypergraph corresponds to a matching of its line graph, Theorem 1 implies the following useful result. (A graph G *factor-critical* if $G - x$ has a perfect matching, for every $x \in V(G)$.)

Corollary 3 *If \mathcal{H} is a minimal connected τ -critical hypergraph, then $L(\mathcal{H})$ is factor-critical.*

Lovász [5] proved that every factor-critical graph has an *odd ear decomposition*: it can be built up from a single vertex by successively attaching the end vertices of odd paths. So by Corollary 3 the line graph of a minimal connected τ -critical hypergraph has an odd ear decomposition. This fact can be used to prove the following two results.

Theorem 4 *If \mathcal{H} is a minimal connected τ -critical hypergraph, then \mathcal{H} has a system of distinct representatives.*

Corollary 5 *If \mathcal{H} is a minimal connected τ -critical hypergraph, then $|V(\mathcal{H})| \geq |\mathcal{H}|$.*

Again, considering odd cycles shows that the bound in Corollary 5 is sharp. A hypergraph with an equal number of edges and vertices is said to be *square*. As might be expected, the minimal connected τ -critical hypergraphs which are square have particularly nice properties. Indeed, they can be characterised in terms of an odd ear decomposition of an associated digraph.

With any digraph D we can associate the hypergraph $\mathcal{H}_D = \{\{x\} \cup N^+(x) \mid x \in V(D)\}$ where $N^+(x)$ denotes the set of outneighbours of x in D . Note that \mathcal{H}_D is square and has a system of distinct representatives. Conversely, if \mathcal{H} is a square hypergraph with a system of distinct representatives $f : \mathcal{H} \rightarrow V(\mathcal{H})$, then $\mathcal{H} = \mathcal{H}_D$, where D is the digraph with vertex set $V(\mathcal{H})$ and arc set $\{(x, y) \mid x \in V(\mathcal{H}), y \in f^{-1}(x) \setminus \{x\}\}$.

A *directed odd ear* with respect to a digraph D consists of a directed odd path such that the two end vertices are in $V(D)$ but no internal vertices belong to $V(D)$. A *directed odd ear decomposition* of a digraph D is a sequence D_0, \dots, D_p of digraphs such that D_0 is a single vertex, $D_p = D$, and for $i = 1, \dots, p$, D_i is obtained from D_{i-1} by adding a directed odd ear joining two not necessarily distinct vertices of D_{i-1} .

Seymour [7] proved that a square hypergraph \mathcal{H} is χ -critical 3-chromatic if and only if $\mathcal{H} = \mathcal{H}_D$, where D is a strongly connected digraph with no directed even circuits. The following result can be proved using Seymour's theorem, Corollary 3 and Theorem 4. The *absorption number* $\beta(D)$ of a digraph D is the minimal size of a set $S \subseteq V(D)$ such that every $x \in V(D) \setminus S$ has an outneighbour in S .

Theorem 6 *For any square hypergraph \mathcal{H} , the following conditions are equivalent:*

1. \mathcal{H} is minimal connected τ -critical;
2. \mathcal{H} is χ -critical 3-chromatic and $|\mathcal{H}| < 2\tau(\mathcal{H})$;
3. $\mathcal{H} = \mathcal{H}_D$, where D has a directed odd ear decomposition, contains no directed even circuits and $|V(D)| < 2\beta(D)$.

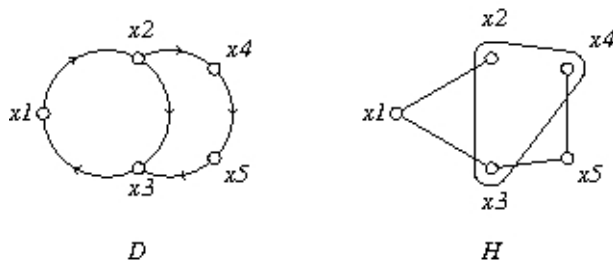


Fig. 1: $\mathcal{H} = \mathcal{H}_D$, where D has a directed odd ear decomposition and contains no directed even circuits; the associated square hypergraph \mathcal{H}_D is minimal connected τ -critical by Theorem 6.

Let \mathcal{H}^* denote the vertex-edge dual of \mathcal{H} . The following result can be proved using Corollary 5 and Theorem 6.

Corollary 7 *If \mathcal{H} is a minimal connected τ -critical hypergraph, then so is \mathcal{H}^* if and only if \mathcal{H} is square.*

3 Application to χ -vertex-critical graphs

A hypergraph has the *Helly property* if all its intersecting partial hypergraphs are stars. There is a useful link between the chromatic number of graphs and the transversal number of Helly hypergraphs. Namely, given a graph G , let $\mathcal{A}(G)$ be the hypergraph formed with the maximal independent sets of G , and denote its dual by $\mathcal{A}^*(G)$. It is not difficult to check that $\mathcal{A}^*(G)$ has the Helly property and $\chi(G) = \tau(\mathcal{A}^*(G))$. A graph G is χ -vertex-critical if $\chi(G - x) < \chi(G)$, for every vertex $x \in V(G)$; note that a graph G is χ -vertex-critical if and only if $\mathcal{A}^*(G)$ is τ -critical. Hence the restriction to Helly hypergraphs of Corollary 2 is equivalent to the following classical result of Gallai [4], also proved in [6, 8].

Theorem 8 (Gallai 1963) *A χ -vertex-critical graph G with a connected complement has at least $2\chi(G) - 1$ vertices.*

The restriction to Helly hypergraphs of Corollary 5 is equivalent to the following result.

Theorem 9 *A χ -vertex-critical graph G with a connected complement and $2\chi(G) - 1$ vertices has at least $2\chi(G) - 1$ maximal independent sets.*

Finally, Theorem 6 implies the following.

Theorem 10 *If G is a graph with a connected complement, $2\chi(G) - 1$ vertices and $2\chi(G) - 1$ maximal independent sets, then G is χ -vertex-critical if and only if $\mathcal{A}^*(G)$ is a χ -critical 3-chromatic Helly hypergraph.*

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