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► **To cite this version:**

Gyula Pap. Packing non-returning A-paths algorithmically. 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), 2005, Berlin, Germany. pp.139-144, 10.46298/dmtcs.3399 . hal-01184355

**HAL Id: hal-01184355**

**<https://inria.hal.science/hal-01184355>**

Submitted on 14 Aug 2015

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# Packing non-returning $A$ -paths algorithmically

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In this paper we present an algorithmic approach to packing  $A$ -paths. It is regarded as a generalization of Edmonds' matching algorithm, however there is the significant difference that here we do not build up any kind of alternating tree. Instead we use the so-called 3-way lemma, which either provides augmentation, or a dual, or a subgraph which can be used for contraction. The method works in the general setting of packing non-returning  $A$ -paths. It also implies an ear-decomposition of criticals, as a generalization of the odd ear-decomposition of factor-critical graph.

**Keywords:**  $A$ -paths, matching

## 1 Introduction

The paper is devoted to the problem of packing fully node-disjoint non-returning  $A$ -paths in a graph  $G = (V, E)$ . Given a graph and a subset  $A \subseteq V$ , a path is said to be an  $A$ -path if its ends are two distinct nodes in  $A$ . Packing fully node-disjoint  $A$ -paths reduces to maximum matching in an auxiliary graph, see T. Gallai (3). The special case  $A = V$  is in fact equivalent to maximum matching. W. Mader considered a more difficult problem. We are given a subset  $A \subseteq V$  with a partition  $\mathcal{A}$ . An  $A$ -path is called an  $\mathcal{A}$ -path if its ends are in two distinct members of  $\mathcal{A}$ . Mader (5) gave a min-max formula for the maximum number of fully node-disjoint  $\mathcal{A}$ -paths. A polynomial time algorithm to find these paths was given by L. Lovász using his matroid parity apparatus. Matroid parity is still a challenging topic in combinatorial optimization. If a problem turns out to be an instance for matroid parity, this does not necessarily imply a polynomial time algorithm or a good characterization. Lovász disentangled some technical details to construct an algorithm, see (4). Later, A. Schrijver gave a funny reduction to linear matroid parity – which by itself also implies an algorithm. It was a challenge to construct directly an algorithm for packing  $\mathcal{A}$ -paths. Such an algorithm was given by Chudnovsky et al. (2). They in fact work with the concept of non-zero  $A$ -paths, which is a generalization of  $\mathcal{A}$ -paths, see also (1). The main goal of this paper is to construct an algorithm which presents the “dual” in a more structured form. Our method implies an ear-decomposition of “criticals” – this generalizes the ear-decomposition of factor-critical graphs.

Maximum matching is a special case of the problem discussed in this paper, let us briefly sketch how the method works for maximum matching. For a given matching  $M \subseteq E$  in  $G$ , we call an odd cycle  $C \subseteq E$  an  $M$ -alternating odd cycle if  $|C \cap M| = (|C| - 1)/2$  and  $C$  is incident to an  $M$ -exposed node. The following lemma can be proved directly, a proof “on the level of bipartite matching” can be given. In

<sup>†</sup>Research is supported by OTKA grants T 037547 and TS 049788, by European MCRIN Adonet, Contract Grant No. 504438 and by the Egerváry Research Group of the Hungarian Academy of Sciences. e-mail: gyuszk@cs.elte.hu

fact, Edmonds' alternating forests provide an alternative proof of this lemma. Our crucial observation is that a matching algorithm can be constructed by only using the below lemma as a black box. This black box is regarded as a compact formulation of some consequences of alternating forests. However, one can also give a short, inductive proof without alternating forests.

**Lemma 1.1 (3-Way Lemma for Matching)** *Given an undirected graph  $G$  with a matching  $M$ , then at least one of the following alternatives holds:*

1. *There is a matching  $N$  with  $|N| = |M| + 1$ .*
2. *There is a matching  $N$  with  $|N| = |M|$  and an  $N$ -alternating odd cycle in  $G$ .*
3. *There is a vector  $c \in \{0, 1, 2\}^V$  such that the weight of any edge is at least 2, and the sum of its entries is exactly  $2|M|$ .*

This lemma allows us to interpret of Edmonds' algorithm as follows. Consider a matching  $M$  in graph  $G$ , try Lemma 1.1. Alternative 1 gives an augmentation, alternative 3 verifies optimality. Alternative 2 provides an odd cycle for contraction. Contraction of an alternating odd cycle has the property that augmentation, or a Berge-Tutte-dual in  $G/C$  can be expanded to  $G$ .

## 2 Packings in p-graphs — Definitions

The most important notion in this paper is a **permutation labeled graph** or **p-graph**, for short. A p-graph comes in the form of  $G, A, \omega, \pi$ , where  $G$  is a graph,  $A$  is a set of nodes,  $\pi$  are edge-labels. This notion provides a generalization of some well-known packing problems – matching, node-disjoint  $A$ -paths, non-zero  $A$ -paths. The motivation for this version is that important reduction principles used by our algorithm stay within the concept of a p-graph, but does not stay within well-known previous concepts. The precise definition of a p-graph is formulated as follows.

Let  $G = (V, E)$  be an undirected graph with node-set  $V$ , edge-set  $E$  with a reference orientation. Let  $A \subseteq V$  be a fixed set of **terminals**. Let  $\Omega$  be an arbitrary **set of “potentials”** and let **jj, JJ** be called **Jolly Joker** (some imaginary labels). Let  $\omega : A \rightarrow \Omega$  define the **potential of origin** for the terminals. Let  $\pi : E \rightarrow S(\Omega) \cup \{\mathbf{JJ}\}$  where  $S(\Omega)$  is the set of all permutations of  $\Omega$ . For an edge  $ab = e \in E$ , let  $\pi(e, a) := \pi(e)$  and  $\pi(e, b) := \pi^{-1}(e)$  be the **mapping of potential** on edge  $ab$ . (We use  $\circ$  for the composition of permutations. We define  $\mathbf{JJ}^{-1} := \mathbf{JJ} \circ \pi := \pi \circ \mathbf{JJ} := \mathbf{JJ}$  and  $\mathbf{JJ}(\omega) := \pi(\mathbf{jj}) := \mathbf{jj}$  for any  $\pi \in S(\Omega) \cup \{\mathbf{JJ}\}$  and for any  $\omega \in \Omega \cup \{\mathbf{jj}\}$ .) A **walk** in  $G$  is a sequence of nodes and edges, say  $W = (v_0, e_0, v_1, e_1, \dots, e_{k-1}, v_k)$  where  $e_i = v_i v_{i+1}$  or  $e_i = v_{i+1} v_i$  for all  $0 \leq i \leq k-1$ .  $W$  is called an  **$A$ -walk** in  $G$  if  $v_0, v_k \in A$  and  $v_j \notin A$  (for  $j \neq 0, k$ ).  $\chi_W \in \mathbb{N}^V$  denotes the **traversing multiplicity vector** of walk  $W$ , defined by  $\chi_W(v) := |\{j : v_j = v\}|$ . A walk  $W$  is called a **path** if  $\chi_W \leq \mathbf{1}$ . We will usually use letters  $P, R$  for paths. For an  $A$ -walk let  $\pi(W) := \pi(e_0, v_0) \circ \pi(e_1, v_1) \circ \dots \circ \pi(e_{k-1}, v_{k-1})$  define the **mapping of potentials on  $W$** .  $W$  is called **non-returning** if  $\pi(W)(\omega(v_0)) \neq \omega(v_k)$ . (Hence, an empty  $A$ -walk (having a single node and no edge) is not considered to be non-returning. Notice, if  $W$  traverses any edge with label **JJ**, then  $W$  is non-returning.) A family  $\mathcal{P}$  of fully node-disjoint non-returning  $A$ -paths is called a **packing**.  $\nu = \nu(G) = \nu(G, A, \omega, \pi)$  denotes the **maximum cardinality of a packing**. Also, a “node-capacited packing problem” can be defined. Consider a function  $b \in \mathbb{N}^V$  of **node capacities**. A family  $\mathcal{W}$  of  $A$ -walks (we allow walks to be taken multiply) is called a  **$b$ -packing** if  $\sum_{W \in \mathcal{W}} \chi_W \leq b$ . Let  $\nu_b = \nu_b(G) = \nu_b(G, A, \omega, \pi)$  denotes the maximum cardinality of a  $b$ -packing.  $b = \mathbf{1}$  defines packings,  $b = \mathbf{2}$  defines 2-packings.

### 3 Min-max Theorems for packings

For a set  $F \subseteq E$  of edges, let  $A^F := A \cup V(F)$ .  $F$  is called  **$A$ -balanced** if  $\omega$  can be extended to a function  $\omega^F : A^F \rightarrow \Omega$  s.t. each edge  $ab \in F$  is  $\omega^F$ -**balanced** – i.e.  $\pi(ab, a)(\omega^F(a)) = \omega^F(b)$ . Let  $c_{\text{odd}}(G, A)$  be the number of components in  $G$  having an odd number of nodes in  $A$  – these will be called **odd components of  $G, A$** . Let  $c_1(G, A)$  be the number of nodes in  $A$  which are isolated nodes of  $G$ .

**Theorem 3.1** *In a  $p$ -graph the maximum cardinality of a packing is determined by*

$$\nu(G, A, \omega, \pi) = \min_{F, X} |X| + \frac{1}{2} (|A^F - X| - c_{\text{odd}}(G - F - X, A^F - X)) , \quad (1)$$

where the minimum is taken over an  $A$ -balanced edge-set  $F$  and a set  $X \subseteq V$ .

**Theorem 3.2** *In a  $p$ -graph the maximum cardinality of a 2-packing is determined by*

$$\nu_2(G, A, \omega, \pi) = \min_{F, X} 2|X| + |A^F - X| - c_1(G - F - X, A^F - X) , \quad (2)$$

where the minimum is taken over an  $A$ -balanced edge-set  $F$  and a set  $X \subseteq V$ .

In Theorem 3.2 we do not count odd components to determine a maximum 2-packing, this indicates that 2-packings are simpler than packings. A similar relation there is between matchings and 2-matchings, the latter admitting a reduction to bipartite matching, Kőnig's Theorem. The following theorem is in fact a reformulation of Theorem 3.2, here we formulate a Kőnig-type condition for 2-packings.

**Theorem 3.3** *In a  $p$ -graph the maximum cardinality of a 2-packing is determined by*

$$\nu_2(G, A, \omega, \pi) = \min \|c\| , \quad (3)$$

where  $\|c\| := \sum_{v \in V} c(v)$  and the minimum is taken over **2-covers**  $c$ , i.e. vectors  $c \in \{0, 1, 2\}^V$  such that  $c \cdot \chi_W \geq 2$  for any non-returning  $A$ -walk.

### 4 Contraction of dragons

A path  $P$  is called a **half- $A$ -path** if it starts in a terminal  $s \in A$ , ends in a node  $t \in V$  and  $V(P) \cap A = \{s\}$ . We say  $P$  **ends in  $t$  with potential**  $\pi(P)(\omega(s))$ . Consider a node  $v \in V$  and a potential  $\omega_0 \in \Omega \cup \{\mathbf{jj}\}$ . We say a node  $v$  is  **$\omega_0$ -reachable** (or  $\omega_0$  is reachable at  $v$ ), if there is a pair  $\mathcal{P}, P_v$  such that  $P_v$  is a half- $A$ -path ending in  $v$  with  $\omega_0$ , and  $\mathcal{P}$  is a packing of  $\nu$  non-returning  $A$ -paths each of which is fully node-disjoint from  $P_v$ . We say a node is **reachable** if it is  $\omega_0$ -reachable for some  $\omega_0 \in \Omega \cup \{\mathbf{jj}\}$ .  $v$  is called **uniquely reachable** if it is  $\omega_0$ -reachable only with a single element  $\omega_0 \neq \mathbf{jj}$ . Otherwise – if  $v$  is **jj-reachable** or there are at least two different elements of  $\Omega$  which are reachable at  $v$ , then  $v$  is called **multiply reachable**. The definition implies that a reachable terminal is uniquely reachable. We call a  $p$ -graph  $G$  a **dragon** if  $|A| = 2\nu + 1$  and every node is reachable. A  $p$ -graph is called **critical** if it is a dragon such that every non-terminal is multiply reachable. (The notion of criticals is analogue to the notion used in (1). The notion of dragons should be considered as a weak version of criticality.) Let us use the expression **odd cycle** for  $p$ -graphs s.t.  $G = (V, E)$  is an odd cycle,  $A = V$ , and all the edges in  $E$  give one-edge non-returning  $A$ -walks (which are in fact non-returning  $A$ -paths except for 1-edge odd cycles). A  $p$ -graph with  $V = \{a, b\}$ ,  $E = \{ab\}$ ,  $A = \{a\}$  is called a **rod**.

**Claim 4.1** *Odd cycles and rods are dragons.* □

A crucial lemma is the following, saying that the min-max formula holds for dragons.

**Lemma 4.2 (A dragon has a special dual)** *Suppose a  $G$  is a dragon with exactly its nodes in  $V_1$  being uniquely reachable, say  $v \in V_1$  is  $\omega'(v)$ -reachable. Let  $F := \{e \in E[V_1] : e \text{ is } \omega'\text{-balanced}\}$ . Then  $2\nu = |V_1| - c(G - F, V_1)$ .*

The notion “reachability” is in fact motivated by the goal to define the contraction of dragon subgraphs.

**Definition 4.3 (Contraction of a dragon)** *Consider a set  $Z \subseteq V$  such that  $G[Z]$  is dragon. We define the contracted p-graph on  $G/Z$  as follows. Let  $Z_1$  be the uniquely reachable nodes in  $G[Z]$ , say  $a \in Z_1$  is  $\omega_a$ -reachable. Let  $A/Z := A - Z + \{Z\}$ . Let  $\Omega' := \Omega + \bullet$  for some new element  $\bullet \notin \Omega$ . Let  $\omega_Z(s) := \omega(s)$  for all  $s \in A/Z - \{Z\}$ , and let  $\omega_Z(\{Z\}) := \bullet$ . We define  $\pi_Z(e)$  by the following case splitting. If  $e$  is disjoint from  $Z$ , then we define  $\pi_Z(e)$  by extending  $\pi(e)$  to  $\Omega'$  by mapping  $\bullet$  to  $\bullet$ . For an edge  $ab$  with  $a \in Z_1$ ,  $b \notin Z$  we label its image  $\{Z\}b$  s.t.  $\pi_Z(\{Z\}b)(\{Z\}) = \pi(ab)(\omega_a)$ . For an edge  $ab$  with  $a \in Z - Z_1$ ,  $b \notin Z$  we define let  $\pi_Z(\{Z\}b) := \mathbf{JJ}$ .*

We define the **contraction of a node-disjoint family  $\mathcal{Z}$  of dragons**  $G/\mathcal{Z}, A/\mathcal{Z}, \omega_{\mathcal{Z}}, \pi_{\mathcal{Z}}$  by contracting the dragons in  $\mathcal{Z}$  one-by-one. By definition, a contraction has the following properties.

**Claim 4.4 (Expansion of a packing)** *From any packing in  $G/\mathcal{Z}$  one can construct a packing in  $G$  which exposes the same number of terminals.*

**Claim 4.5 (Pre-image of a dragon)** *The pre-image of a dragon  $Z_1$  in  $G/\mathcal{Z}$  is dragon. (Thus,  $\mathcal{Z}/Z_1 := \{Z : Z \in \mathcal{Z}, \{Z\} \notin Z_1\} \cup \{\text{the pre-image of } Z_1\}$  is a finer node-disjoint family of dragons.)*

## 5 The 3-Way Lemma and the algorithm

Our main tool in the algorithm is the 3-Way Lemma for packings. Consider a packing  $\mathcal{P}$  in  $G$  and a dragon  $Z$  in  $G$ . We say  $\mathcal{P}$  is **equipped with  $Z$**  if  $\mathcal{P}$  consists of some paths disjoint from  $V(Z)$  and exactly  $\nu(G[Z]) = (|A \cap V(Z)| - 1)/2$  paths inside  $Z$ .

**Lemma 5.1 (The 3-way Lemma)** *Consider a p-graph with a packing  $\mathcal{P}$ . Then at least one of the following alternatives holds:*

1. *There is a packing  $\mathcal{R}$  with  $|\mathcal{R}| = |\mathcal{P}| + 1$ .*
2. *There is a packing  $\mathcal{R}$  s.t.  $|\mathcal{R}| = |\mathcal{P}|$ , and is equipped with a rod or an odd cycle.*
3. *There is a 2-cover  $c$  such that  $2|\mathcal{P}| = \|c\|$ . (I.e. a verifying 2-cover for  $2 \times \mathcal{P}$ )*

The 3-Way Lemma is applied sequentially in the algorithm to construct sequences of contractions. A **sequence of contractions** is a sequence  $(\mathcal{Z}_1, G_1, \mathcal{P}_1, \mathcal{R}_1, S_1), \dots, (\mathcal{Z}_m, G_m, \mathcal{P}_m, \mathcal{R}_m, S_m), (\mathcal{Z}_{m+1}, G_{m+1}, \mathcal{P}_{m+1})$  with  $m \geq 0$ , and the following properties.  $\mathcal{Z}_0 = \emptyset$ , and  $\mathcal{Z}_i$  is a node-disjoint family of dragons in  $G$ .  $G_i = (V_i, E_i) := G/\mathcal{Z}_i$ .  $G_i[S_i]$  is an odd cycle or a rod, where  $S_i \subseteq V_i$ .  $\mathcal{R}_i$  is a packing in  $G_i$  which is equipped with  $S_i$ .  $\mathcal{P}_{i+1} := \mathcal{R}_i/S_i$ ,  $\mathcal{Z}_{i+1} := \mathcal{Z}_i/S_i$  for  $i = 1, \dots, m$ . Each  $\mathcal{P}_i, \mathcal{R}_i$  leaves the same number of terminals uncovered.

The proof of Theorem 3.1 and the algorithm relies on the following key observation, which provides a tool to construct a verifying pair. It says that from a 2-packing verification in a contraction we can construct a packing verification in the original p-graph.

**Lemma 5.2 (Constructing a verifying pair)** *Suppose we have a sequence of contractions, and a 2-cover  $c$  in  $G_{m+1}$  with  $2|\mathcal{P}_{m+1}| = |c|$ . Then for all  $i$ ,  $\mathcal{P}_i$  is a maximum packing in  $G_i$  and one can construct a verifying pair for  $\mathcal{P}_i$ .*

Now we are in position to sketch the algorithm. Our algorithm has an input of a  $p$ -graph  $G$  and a packing  $\mathcal{P}$ . The output is either a larger packing, or a verifying pair for  $\mathcal{P}$ . The algorithm starts off with initiating the trivial sequence of contractions,  $m = 0$ . In a general step, apply Lemma 5.1 to  $G_{m+1}, \mathcal{P}_{m+1}$ ! If alternative 1 holds, then by Claim 4.4 one can construct a packing in  $G$  larger than  $\mathcal{P}$ . If alternative 2 holds, then by Claim 4.5 one can construct a longer sequence of contractions. If alternative 3 holds, then by Claim 5.2  $\mathcal{P}$  is maximum, and a verifying pair can be constructed. Full proofs are given in (7). Detailed analysis of the algorithm implies that dragons have a so-called dragon-decomposition.

**Definition 5.3** *A dragon-decomposition is given by a forest  $F \subseteq E$  which has the following properties.*

1. *The components of forest  $(V(F) \cup A, F)$  are exactly  $\{F_a : \text{for each } a \in A\}$  s.t. for each  $a \in A$  we have  $A \cap V(F_a) = \{a\}$ .*
2. *Let  $\omega^F : V(F) \cup A \rightarrow \Omega$  be the (uniquely defined) function s.t. each edge in  $F$  is  $\omega^F$ -balanced. Let  $F'$  be the set of  $\omega^F$ -balanced edges. Let  $\mathcal{K}$  is the family of components of  $G - F'$ .  $F/\mathcal{K}$  is a tree.*
3.  *$K, V(F) \cap V(K), \omega^F, \pi$  is critical.*

**Lemma 5.4** *Dragons are exactly those  $p$ -graphs which have a dragon-decomposition.  $V(F) \cup A$  is exactly the set of uniquely reachable nodes.*

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